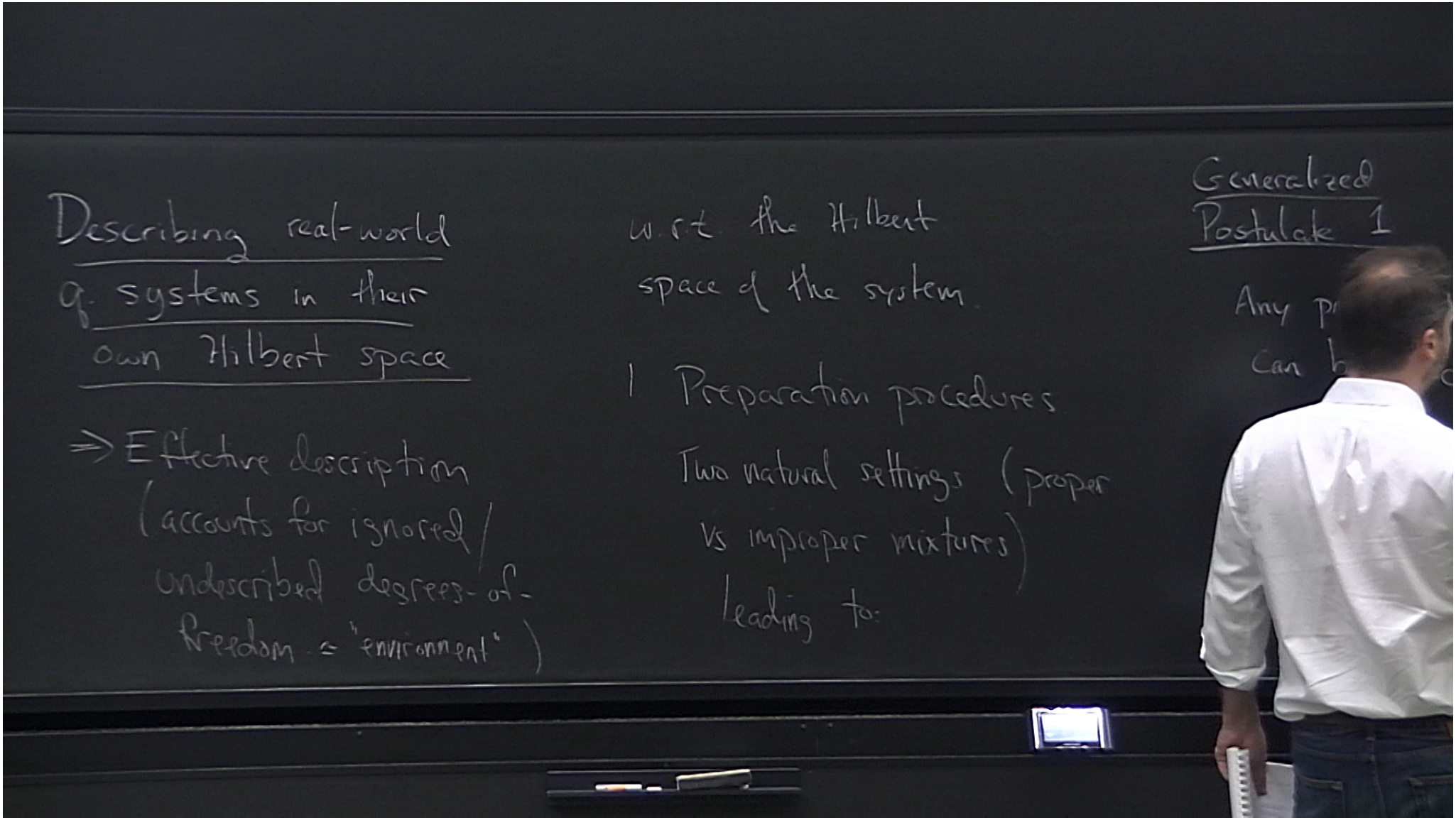
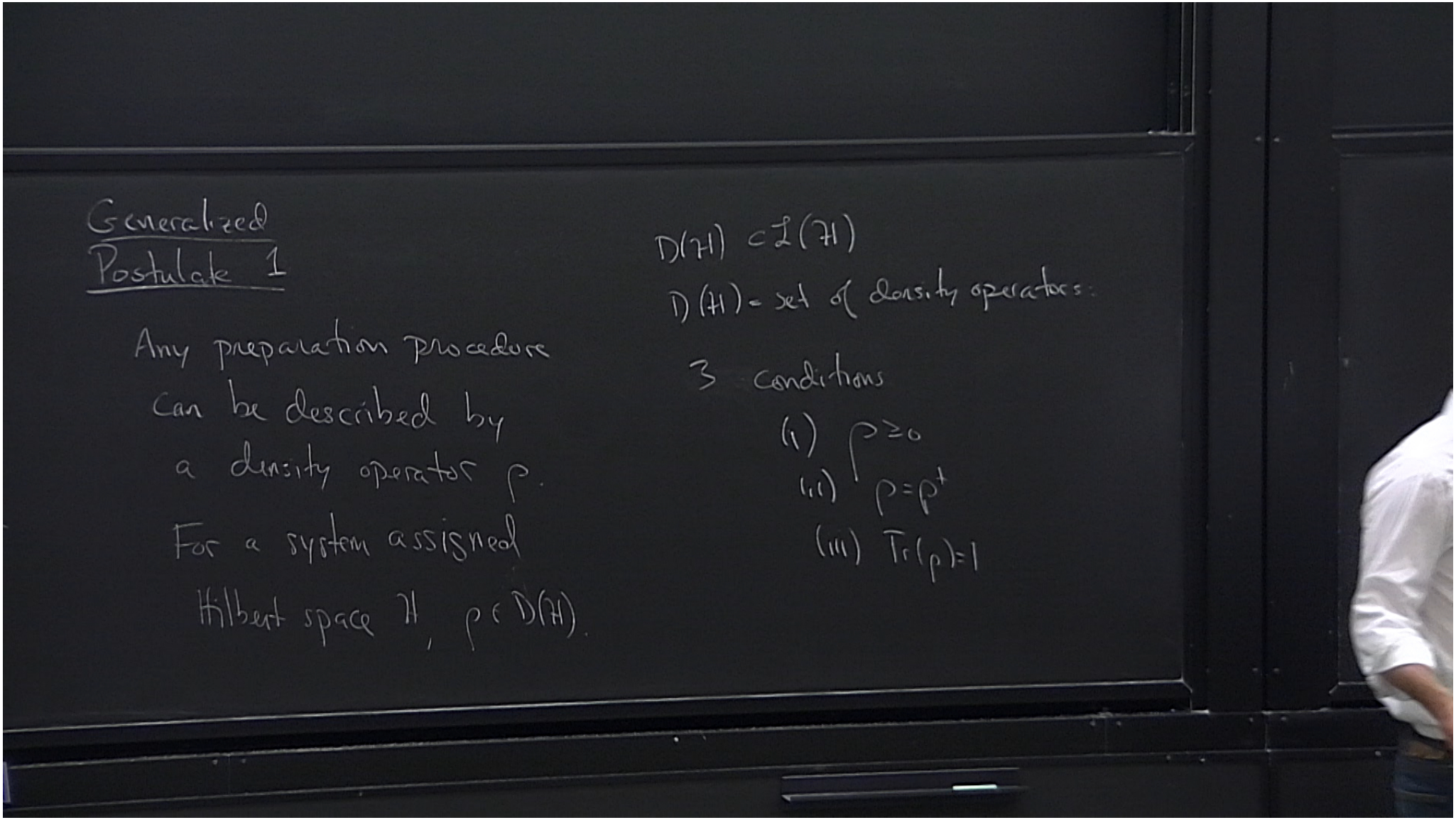
Title: PSI 17/18 - Quantum Theory - Lecture 8

Date: Sep 14, 2017 10:45 AM

URL: http://pirsa.org/17090041

Abstract:


Describing real-world q. systems in their own Hilbert space
⇒ Effective description (accounts for ignored/ undescribed degrees-of-freedom ≙ "environment")
w.r.t. the Hilbert space of the system
1 Preparation procedures
Two natural settings (proper vs improper mixtures)
leading to:
Generalized Postulate 1
Any p
Can b


Generalized
Postulate 1
Any preparation procedure
can be described by
a density operator ρ.
For a system assigned
Hilbert space H, ρ ∈ D(H).
D(H) ⊂ L(H)
D(H) = set of density operators:
3 conditions
(i) ρ ≥ 0
(ii) ρ = ρ†
(iii) Tr(ρ) = 1

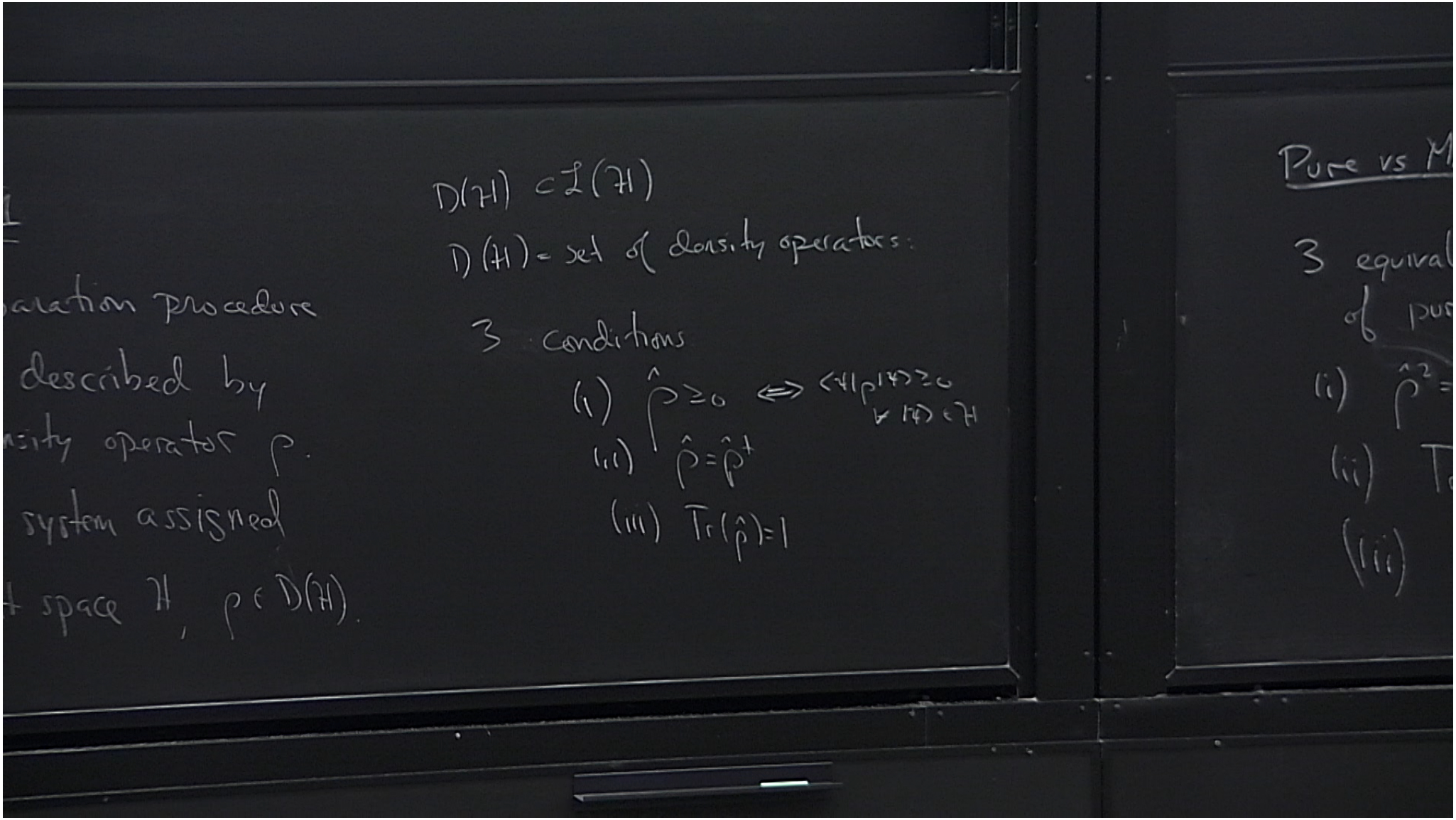
Pure vs Mixed States

3 equivalents definitions of pure states

(i) $\hat{\rho}^2 = \hat{\rho}$ (assuming $\mathrm{Tr}(\rho) = 1$)

(ii) $\mathrm{Tr}(\hat{\rho}^2) = \mathrm{Tr}(\hat{\rho}) = 1$

(iii) $\hat{\rho} = |\psi\rangle\langle\psi|$, $|\psi\rangle \in \mathcal{H}$


aration procedure
described by
sity operator ρ.
system assisned
t space H, ρ ∈ D(H).
D(H) ⊂ L(H)
D(H) = set of density operators:
3 conditions
(i) ρ̂ ≥ 0 ⇔ ⟨ψ|ρ|ψ⟩ ≥ 0 ∀ |ψ⟩ ∈ H
(ii) ρ̂ = ρ̂†
(iii) Tr(ρ̂) = 1
Pure vs M
3 equival
of pur
(i) ρ̂² =
(ii) T
(iii)

In practice, (ii)

is the most useful as a computational test/check for purity.

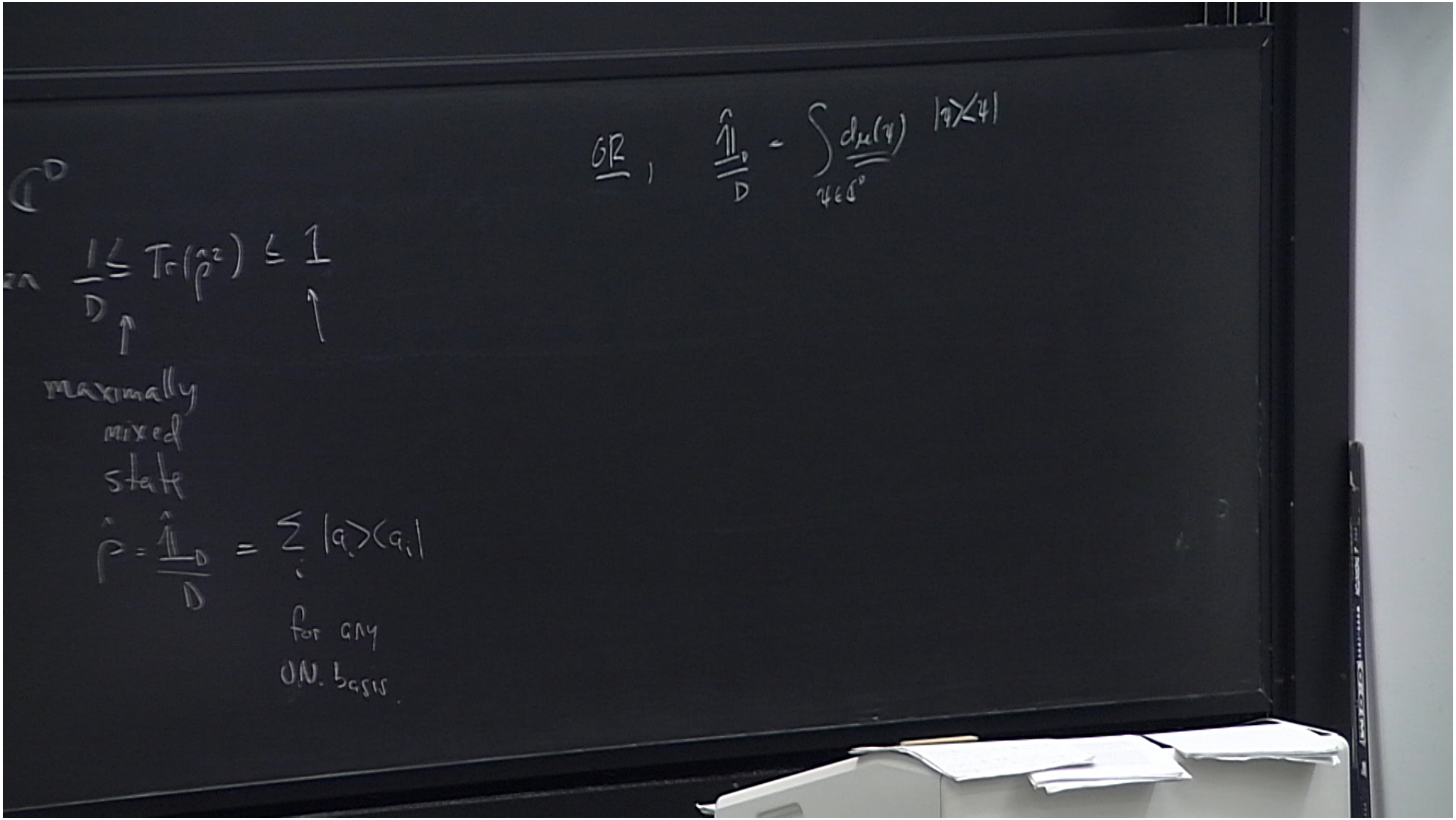
$\mathcal{H} = \mathbb{C}^D$

Then $\frac{1}{D} \leq \mathrm{Tr}(\hat{\rho}^2) \leq 1$

$\frac{1}{D}$: maximally mixed state

$\hat{\rho} = \frac{\hat{\mathbb{1}}_D}{D} = \sum_i |q_i\rangle\langle q_i|$ for any O.N. basis

GR,


$\mathbb{C}^D$
$\frac{1}{D} \leq \mathrm{Tr}(\hat{\rho}^2) \leq 1$
maximally mixed state
$\hat{\rho} = \frac{\hat{\mathbb{1}}_D}{D} = \sum_i |q_i\rangle\langle q_i|$
for any O.N. basis
GR, $\frac{\hat{\mathbb{1}}_D}{D} = \int_{\psi \in S^*} d\mu(\psi)\, |\psi\rangle\langle\psi|$

Theorem: Any (mathematical object) $\hat{\rho} \in D(\mathcal{H})$

Can be realized operationally

as a convex-combination (probabilistic mixture) of pure states.

Theorem

"Ambiguity of Mixtures"

Theorem: Any (mathematical object) $\hat{\rho} \in D(\mathcal{H})$

Can be realized operationally

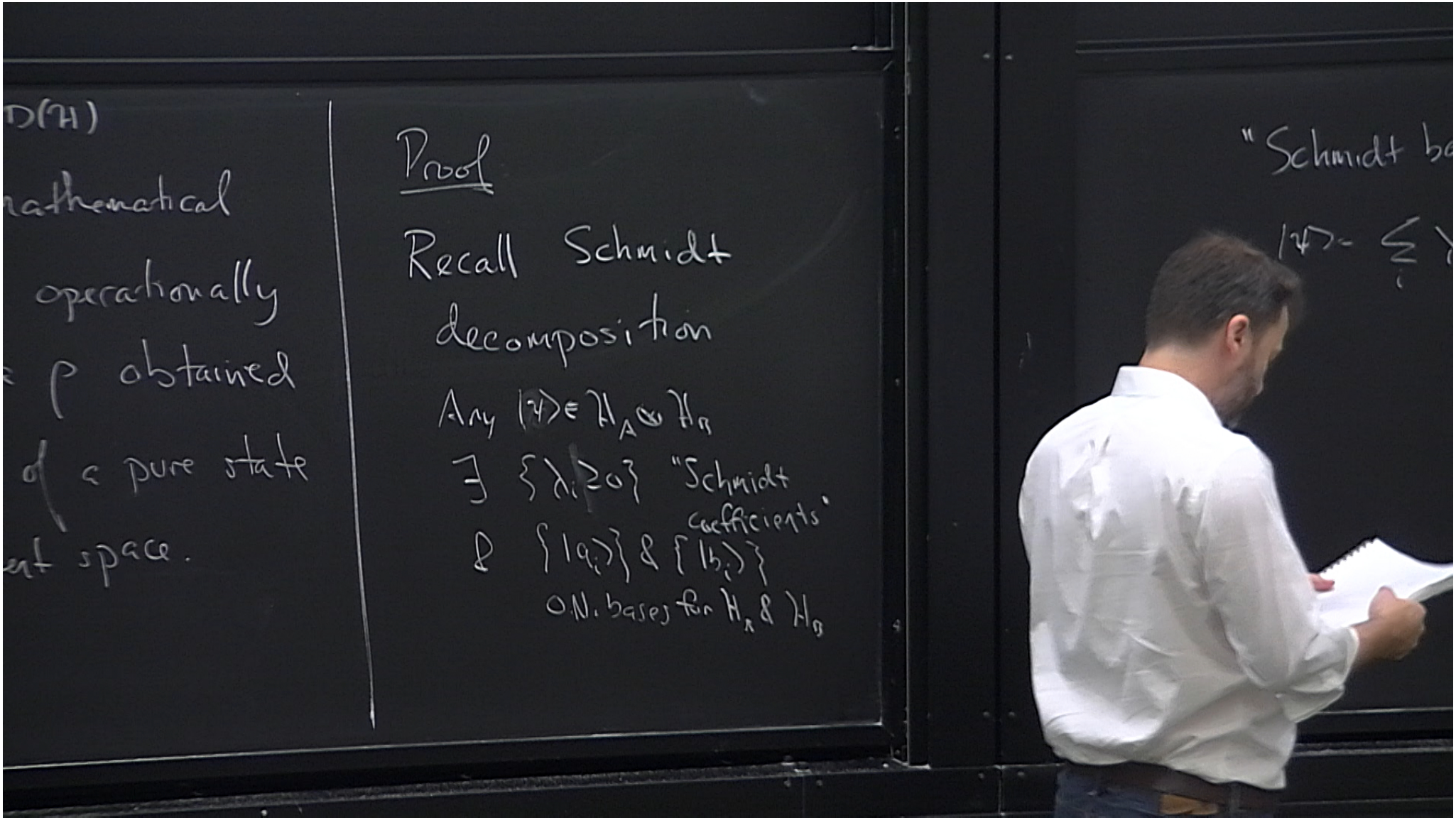
as a convex-combination (probabilistic mixture) of pure states.

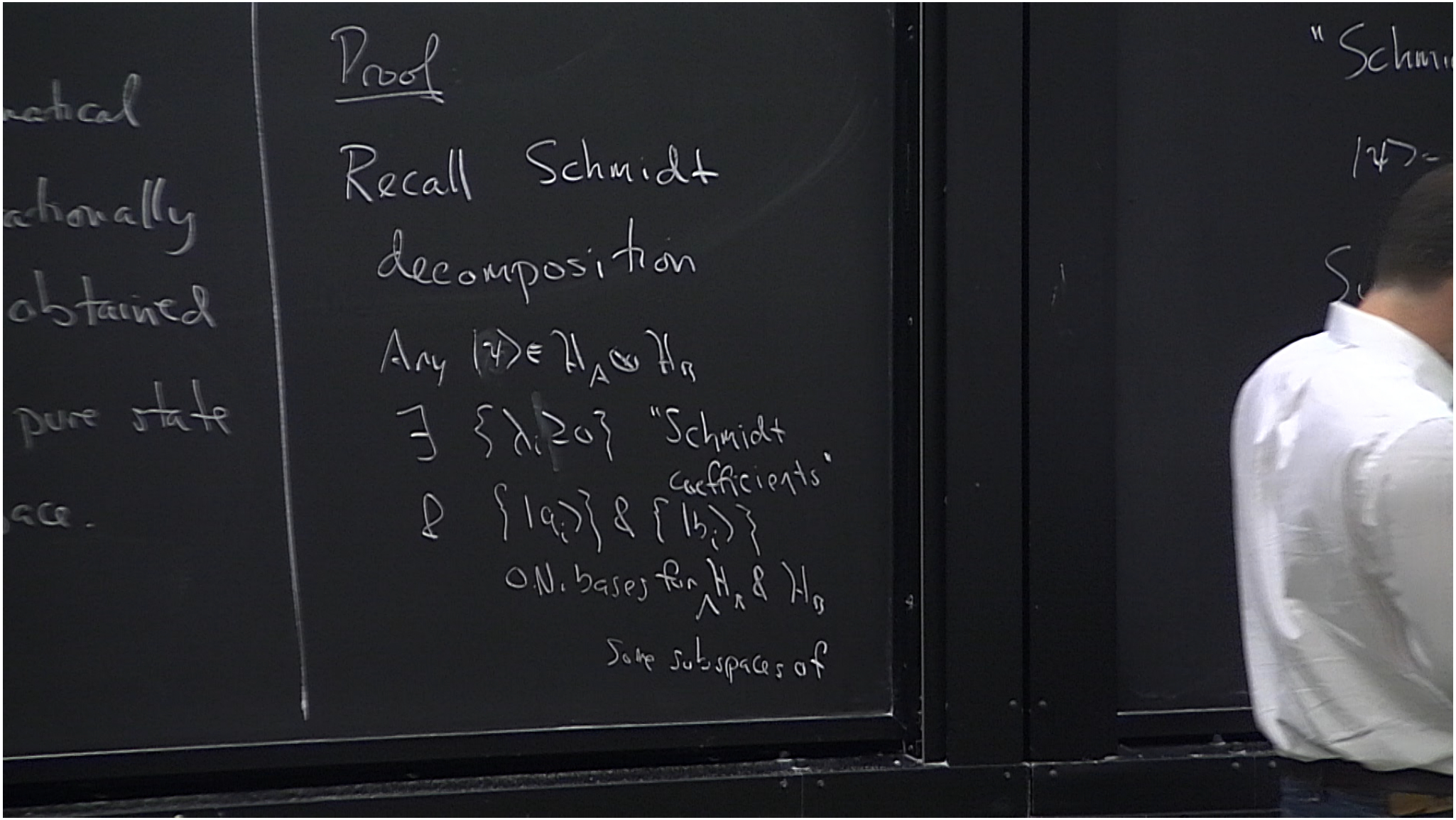
"**Ambiguity of Mixtures**": Any mixed state admits an infinite # of distinct convex decompositions into pure states.

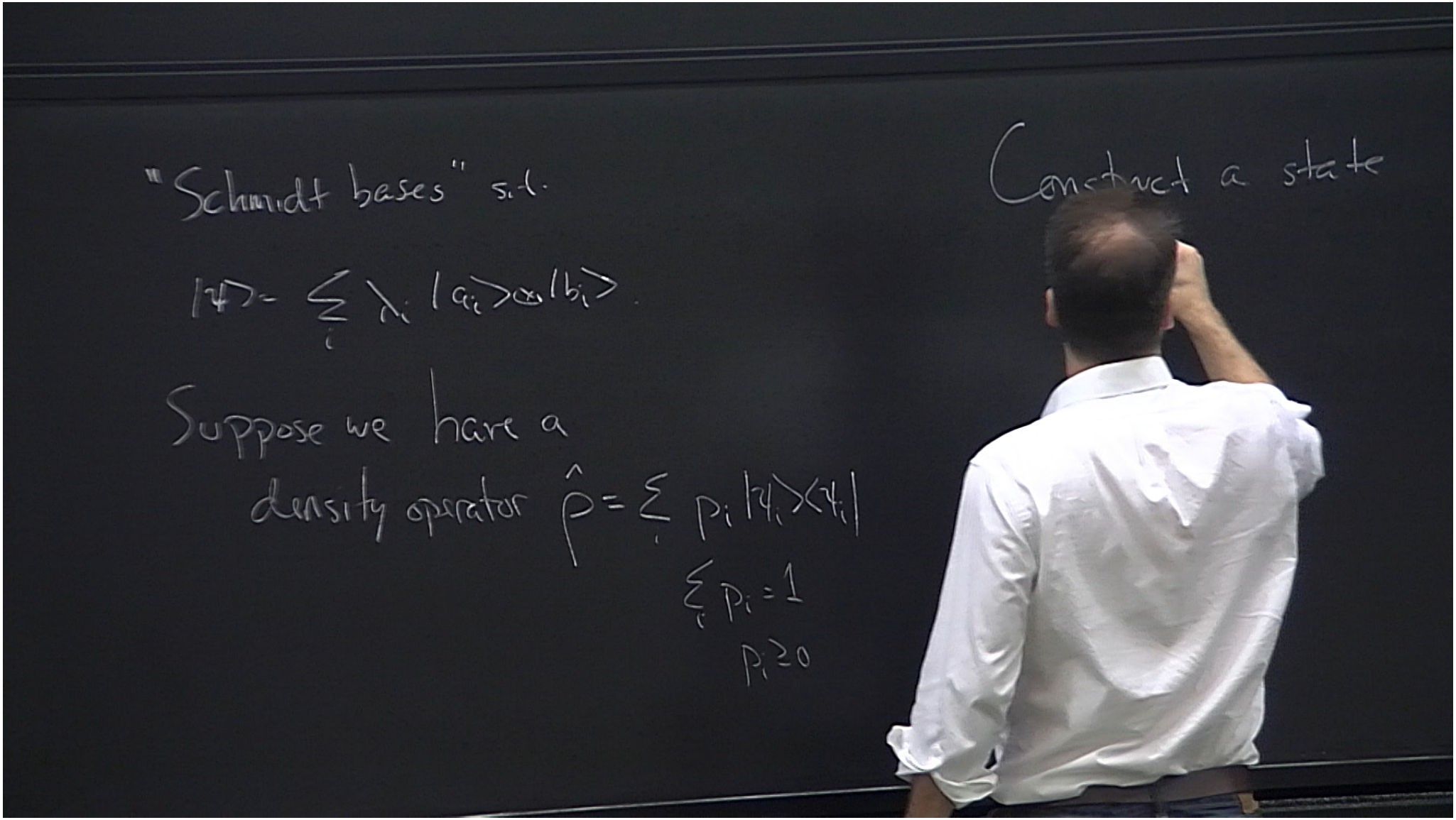
Theorem

Theorem Any density operator $\rho \in D(\mathcal{H})$ (mathematical object) can be realized operationally as the reduced state ρ obtained via partial trace of a pure state in a larger Hilbert space.

...ure states.


D(H)
mathematical
operationally
ρ obtained
of a pure state
ent space.
Proof
Recall Schmidt
decomposition
Any |ψ⟩ ∈ H_A ⊗ H_B
∃ {λ_i ≥ 0} "Schmidt coefficients"
& {|a_i⟩} & {|b_i⟩}
O.N. bases for H_A & H_B
"Schmidt ba
|ψ⟩ = Σ


Proof
Recall Schmidt
decomposition
Any $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$
$\exists \{\lambda_i \geq 0\}$ "Schmidt coefficients"
& $\{|a_i\rangle\}$ & $\{|b_i\rangle\}$
O.N. bases for $\mathcal{H}_A$ & $\mathcal{H}_B$
Some subspaces of


"Schmidt bases" s.t.
$|\psi\rangle = \sum_i \lambda_i |a_i\rangle \otimes |b_i\rangle$
Suppose we have a
density operator $\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$
$\sum_i p_i = 1$
$p_i \geq 0$
Construct a state

Construct a state

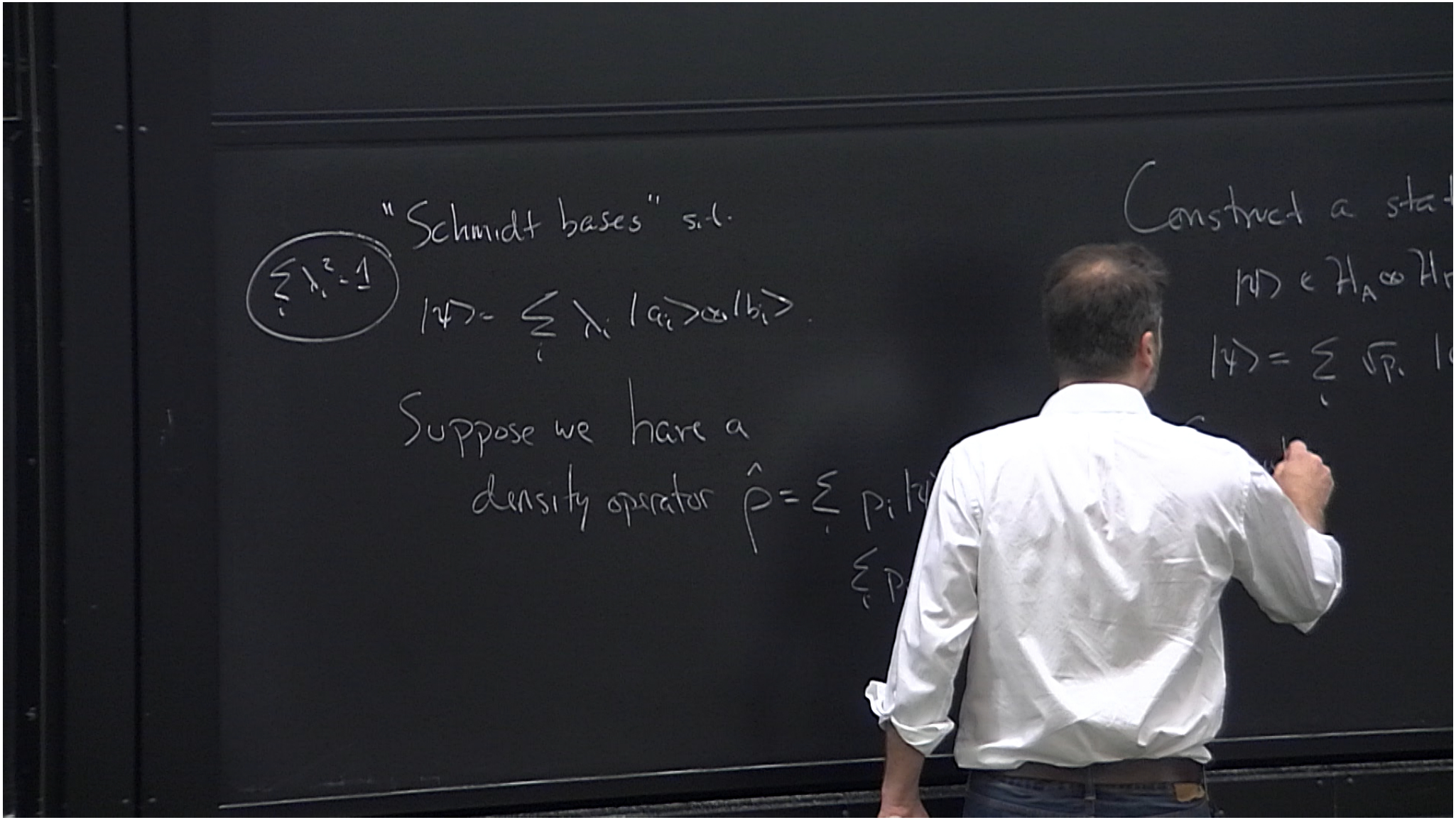
$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ s.t

$|\psi\rangle = \sum_i \sqrt{p_i}\, |a_i\rangle \otimes |b_i\rangle$

$\sum_i p_i |\psi_i\rangle\langle\psi_i|$

$\sum_i p_i = 1$

$p_i \geq 0$


"Schmidt bases" s.t.
$\sum_i \lambda_i^2 = 1$
$|\psi\rangle = \sum_i \lambda_i |a_i\rangle \otimes |b_i\rangle$
Suppose we have a
density operator $\hat{\rho} = \sum_i p_i$
Construct a
$|\psi\rangle = \sum_i \sqrt{p_i}$

Construct a state

$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ s.t

$$|\psi\rangle = \sum_{i=1}^{k \le \dim(\mathcal{H}_B)} \sqrt{p_i}\, |a_i\rangle \otimes |b_i\rangle$$

$$\rho = \sum_j p_j |a_j\rangle\langle a_j|$$

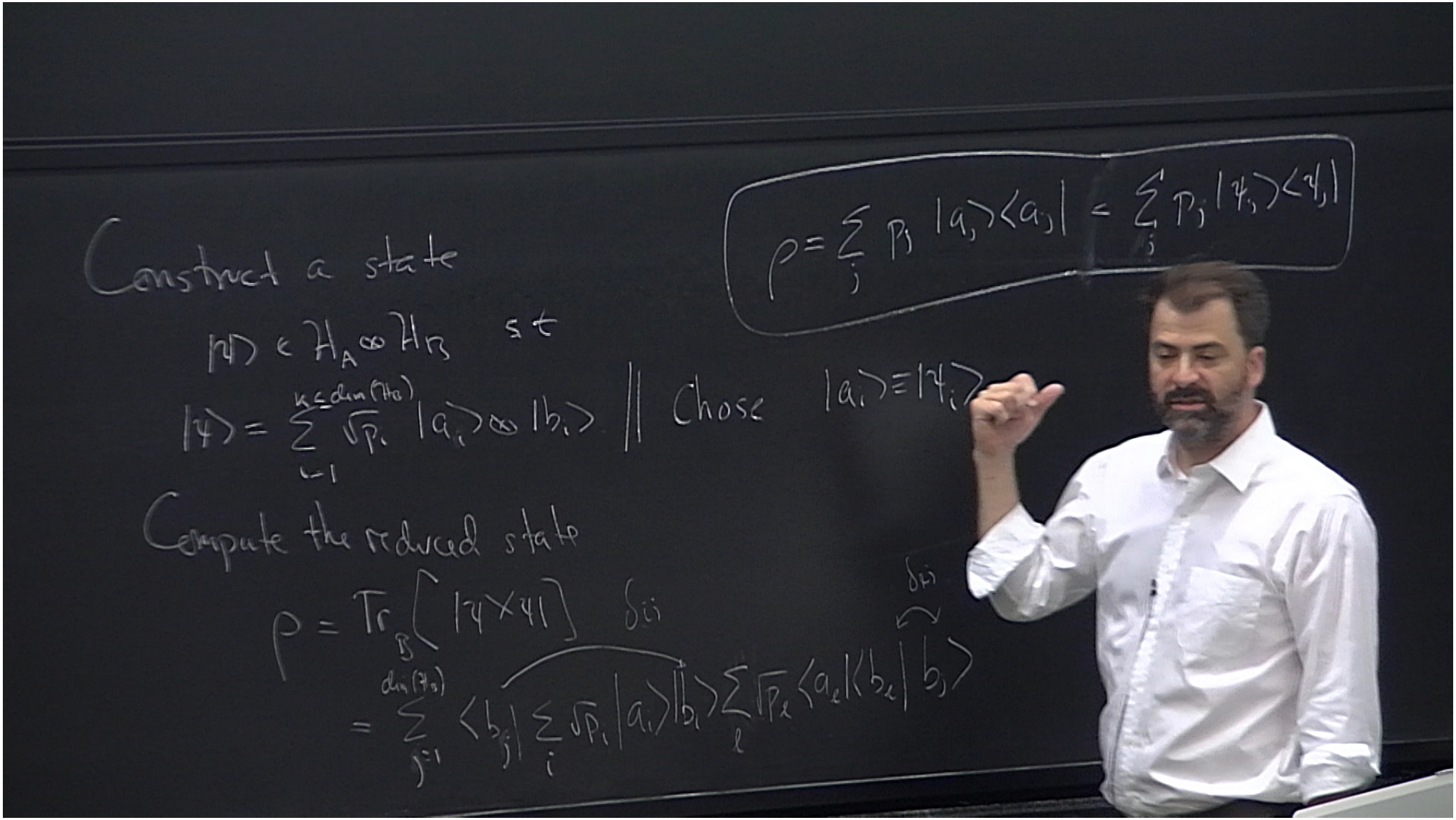
Compute the reduced state

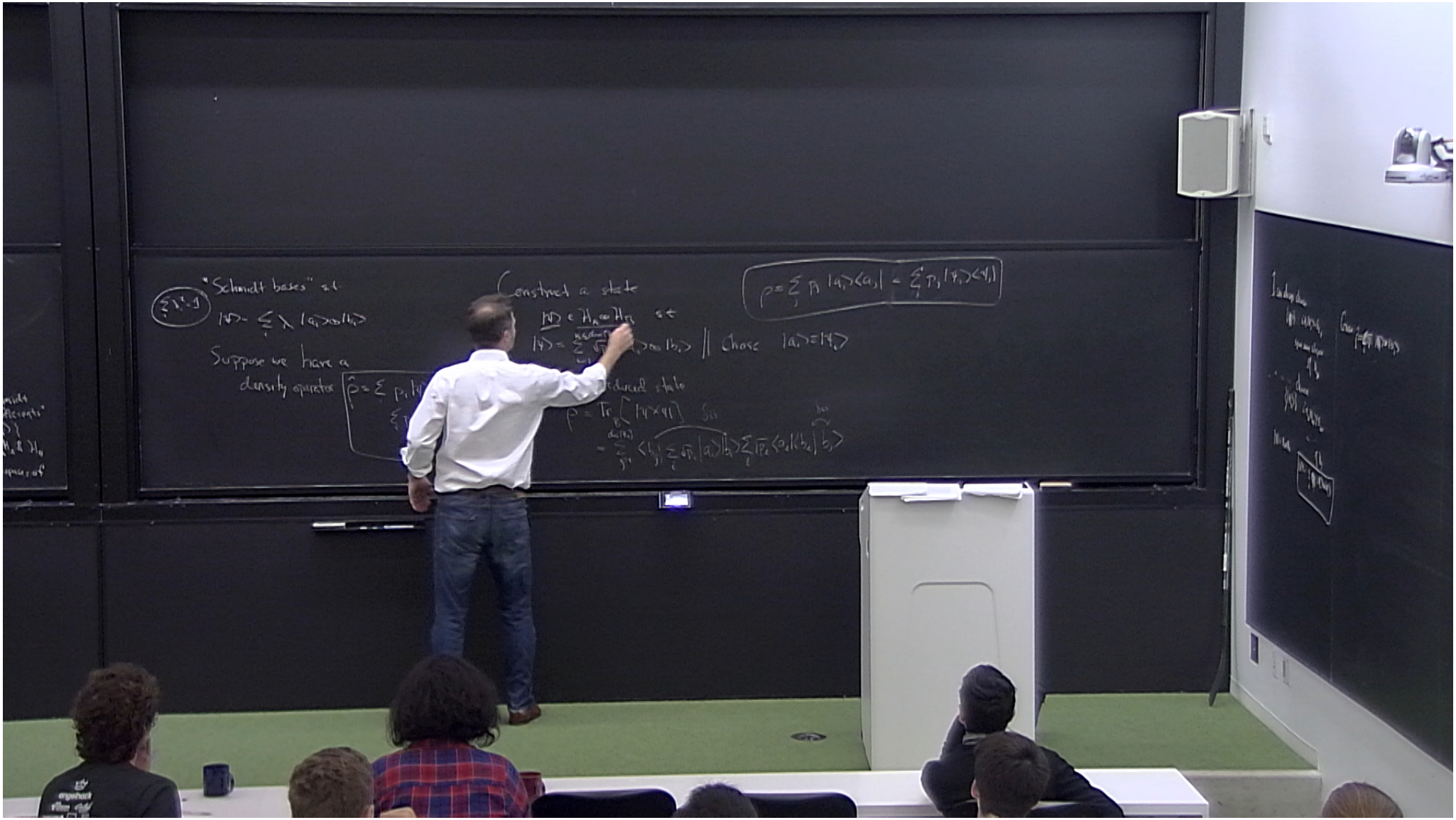
$$\rho = \mathrm{Tr}_B\left[|\psi\rangle\langle\psi| \right]$$

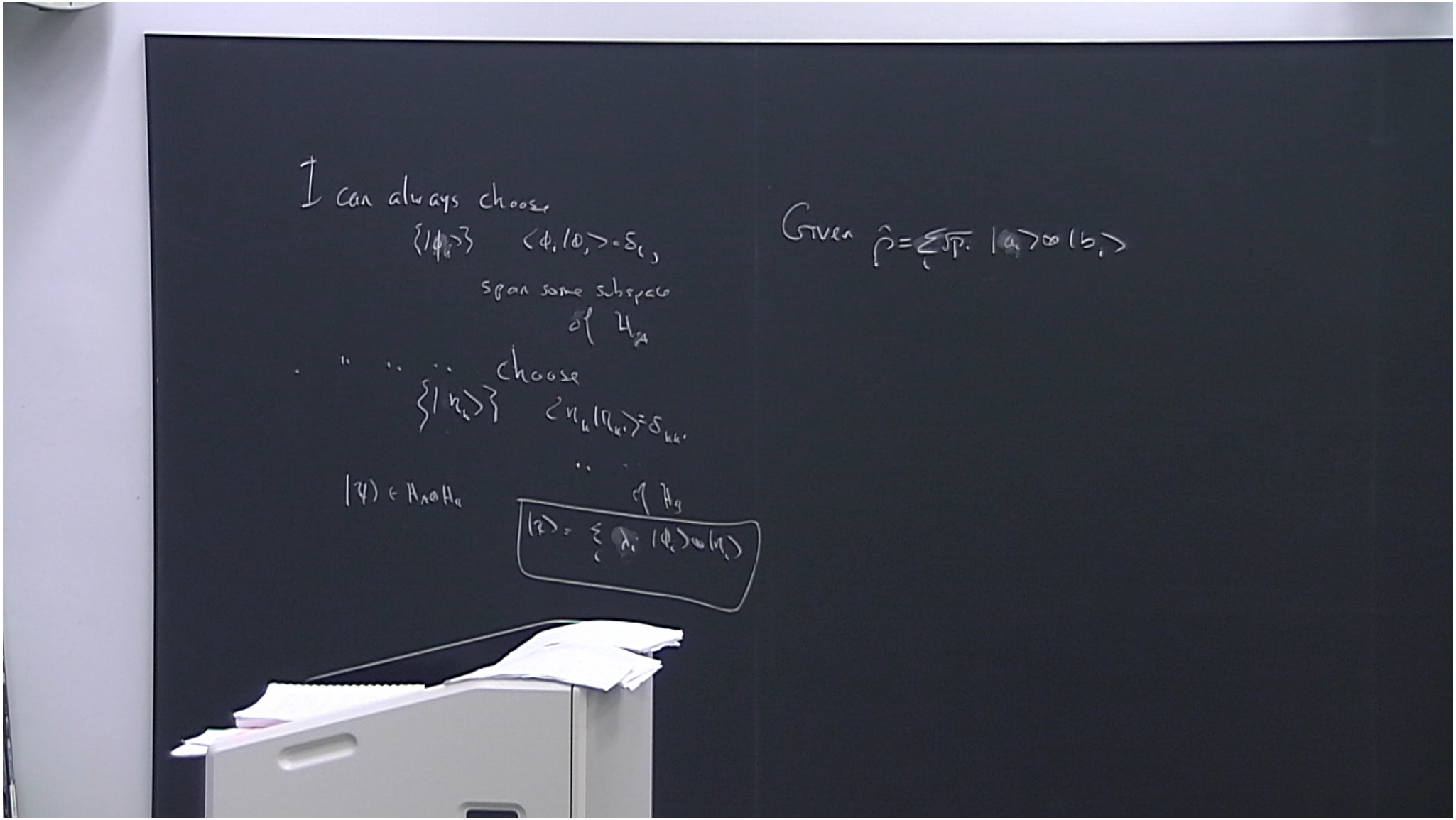
δ_{ij}

$$= \sum_{j=1}^{\dim(\mathcal{H}_B)} \langle b_j | \sum_i \sqrt{p_i} |a_i\rangle |b_i\rangle \sum_\ell \sqrt{p_\ell} \langle a_\ell | \langle b_\ell | b_j \rangle$$

$\delta_{\ell j}$


Construct a state
$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ s.t
$|\psi\rangle = \sum_{i=1}^{k \le \dim(\mathcal{H}_B)} \sqrt{p_i}\, |a_i\rangle \otimes |b_i\rangle$ // Chose $|a_i\rangle = |\psi_i\rangle$
$\rho = \sum_j p_j |a_j\rangle\langle a_j| = \sum_j p_j |\psi_j\rangle\langle\psi_j|$
Compute the reduced state
$\rho = \mathrm{Tr}_B[|\psi\rangle\langle\psi|]$
$= \sum_{j=1}^{\dim(\mathcal{H}_B)} \langle b_j| \sum_i \sqrt{p_i} |a_i\rangle |b_i\rangle \sum_\ell \sqrt{p_\ell} \langle a_\ell| \langle b_\ell| b_j\rangle$
δ_{ij}
$\delta_{\ell j}$


"Schmidt bases" s.t.
Suppose we have a
density operator
Construct a state
Reduced state
Choose


I can always choose
$\{|\phi_i\rangle\}$ $\langle\phi_i|\phi_j\rangle = \delta_{ij}$
span some subspace
of $\mathcal{H}_A$
choose
$\{|n_k\rangle\}$ $\langle n_k|n_{k'}\rangle = \delta_{kk'}$
$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$
of $\mathcal{H}_B$
$|\psi\rangle = \sum_i \lambda_i |\phi_i\rangle \otimes |n_i\rangle$
Given $\hat{\rho} = \sum_i p_i |a_i\rangle\langle b_i|$

Construct a state

$|\chi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ s.t.

$$|\chi\rangle = \sum_{i=1}^{k \le \dim(\mathcal{H}_B)} \sqrt{p_i}\, |a_i\rangle \otimes |b_i\rangle$$

|| Chose $|a_i\rangle \equiv |\psi_i\rangle$

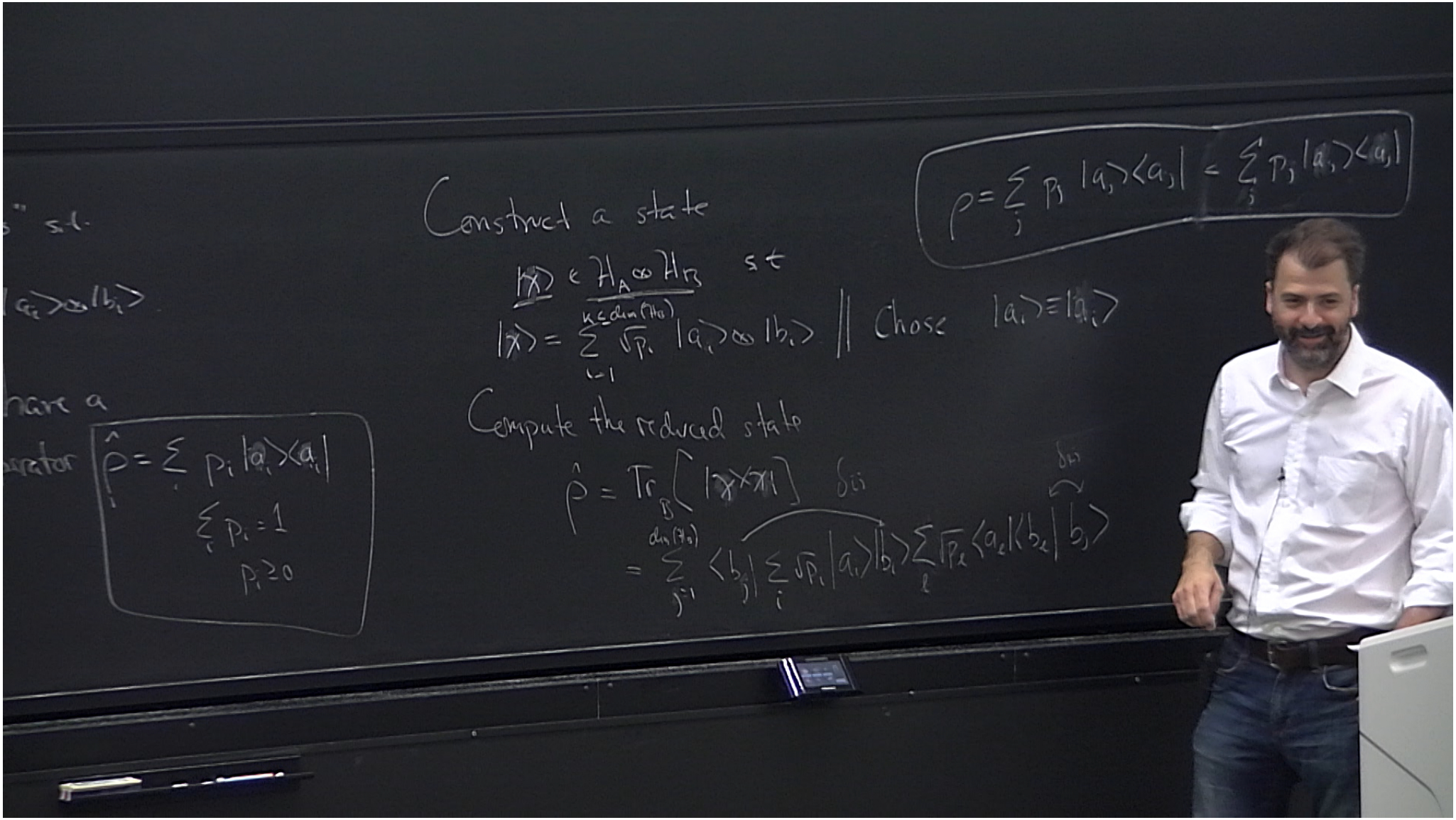
$$\rho = \sum_j p_j |a_j\rangle\langle a_j| = \sum_j$$

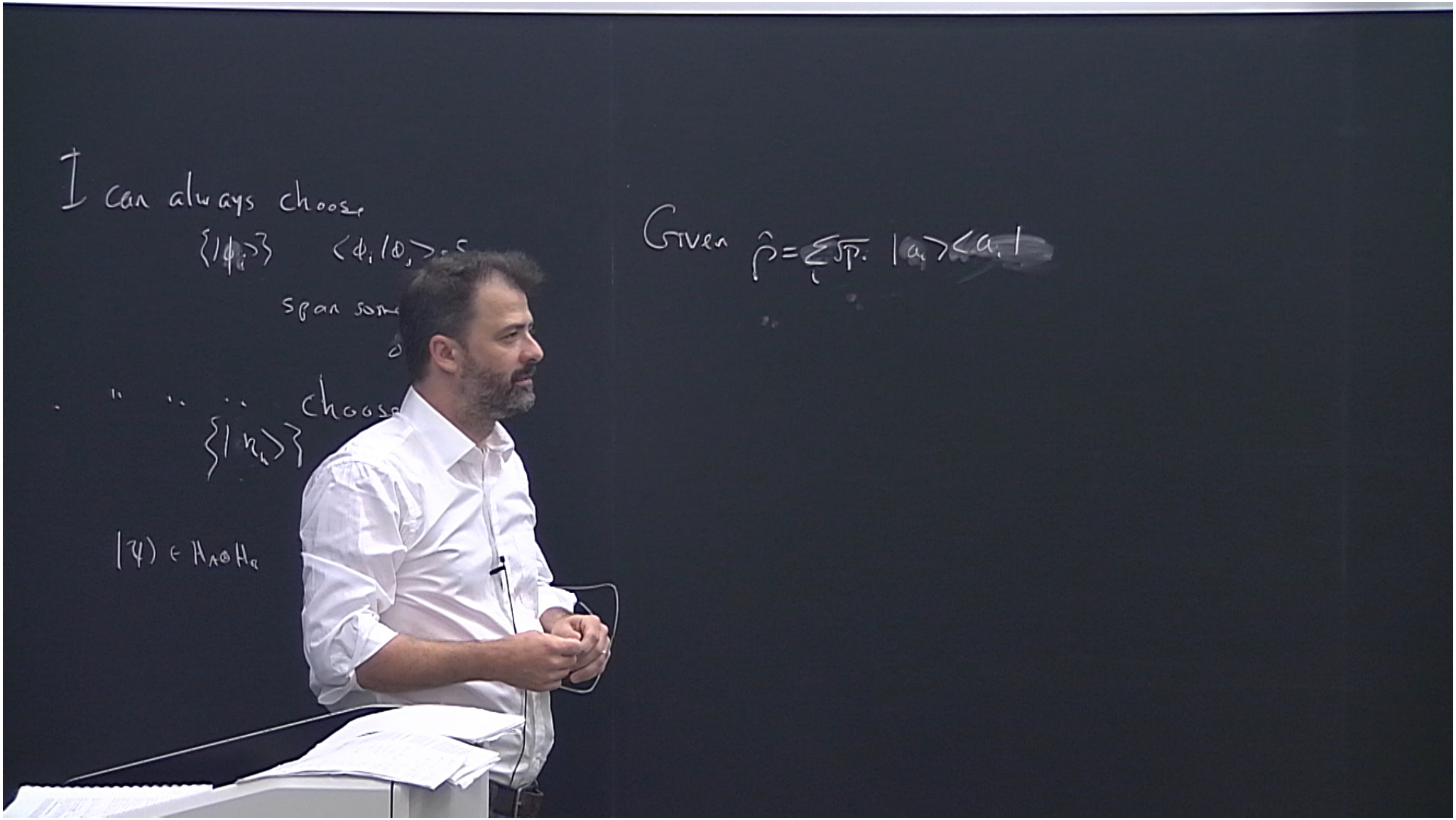
Compute the reduced state

$$\hat{\rho} = \mathrm{Tr}_B\left[|\psi\rangle\langle\psi| \right] \quad \delta_{ij}$$

$$= \sum_{j=1}^{\dim(\mathcal{H}_B)} \langle b_j | \sum_i \sqrt{p_i} |a_i\rangle |b_i\rangle \sum_\ell \sqrt{p_\ell} \langle a_\ell | \langle b_\ell | b_j \rangle$$

(δ_{ij})


Construct a state
$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ s.t
$|\psi\rangle = \sum_{i=1}^{K \le \dim(\mathcal{H}_A)} \sqrt{p_i}\,|a_i\rangle \otimes |b_i\rangle$ // Chose $|a_i\rangle \equiv |a_i\rangle$
$\rho = \sum_j p_j |a_j\rangle\langle a_j| = \sum_i p_i |a_i\rangle\langle a_i|$
Compute the reduced state
$\hat{\rho} = \mathrm{Tr}_B[|\psi\rangle\langle\psi|]$ δ_{ij}
$= \sum_{j=1}^{\dim(\mathcal{H}_B)} \langle b_j| \sum_i \sqrt{p_i}|a_i\rangle|b_i\rangle \sum_\ell \sqrt{p_\ell}\langle a_\ell|\langle b_\ell| b_j\rangle$
$\delta_{\ell j}$
$\hat{\rho} = \sum_i p_i |a_i\rangle\langle a_i|$
$\sum_i p_i = 1$
$p_i \ge 0$


I can always choose
Given $\hat{\rho} = \sum_i p_i |a_i\rangle\langle a_i|$
$|\psi\rangle \in H_A \otimes H_a$

General $|\psi\rangle = \sum_{jk} c_{jk} |\alpha_j\rangle \otimes |\beta_k\rangle$

$= \sum_j \lambda_j |a_j\rangle \otimes |b_j\rangle$

I can always choose

$\{|\phi_i\rangle\}$ $\langle\phi_i|\phi_j\rangle = \delta_{ij}$

span some subspace of $\mathcal{H}_A$

.. .. .. choose

$\{|\eta_k\rangle\}$ $\langle\eta_k|\eta_{k'}\rangle = \delta_{kk'}$

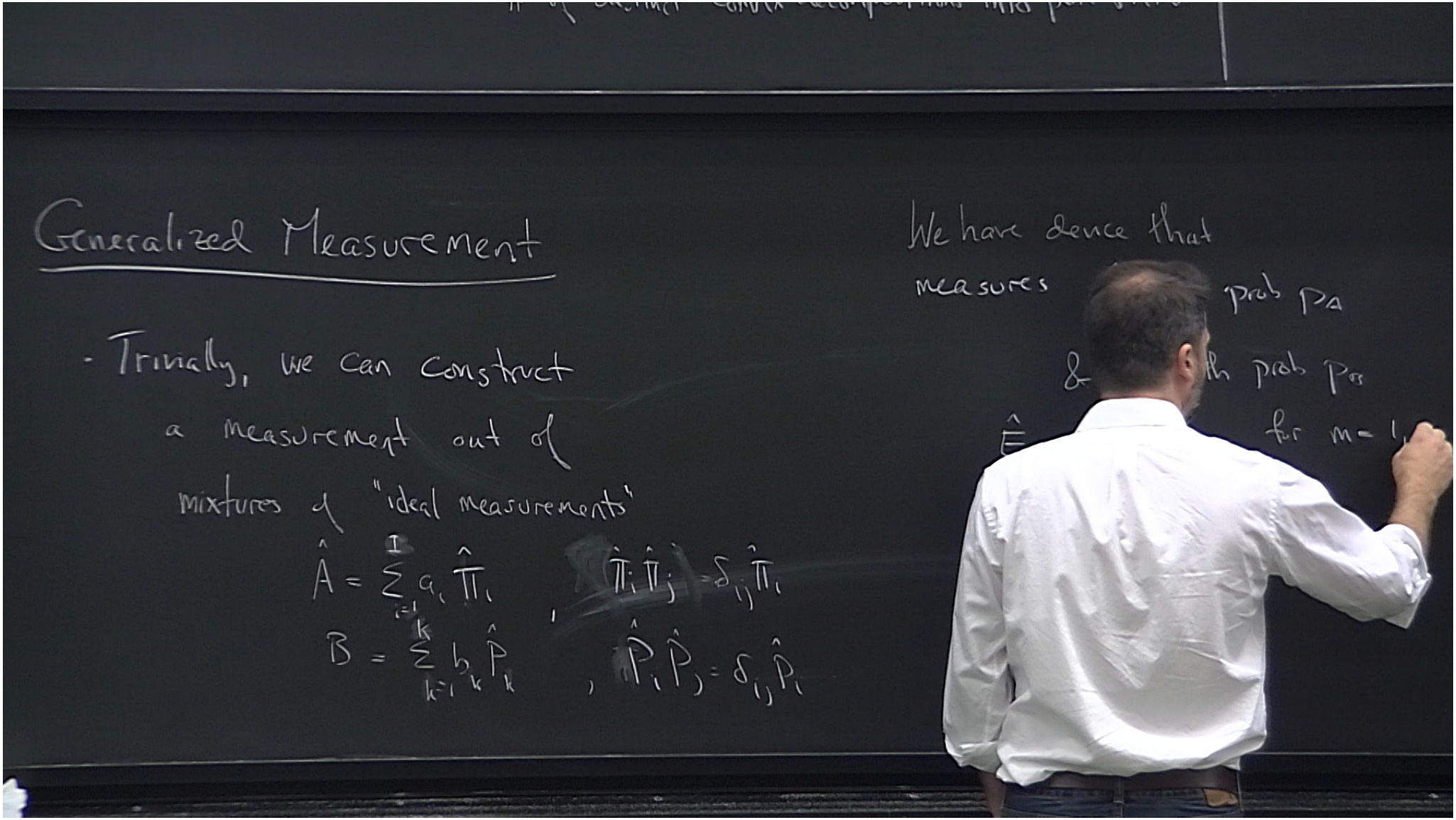
.. ..

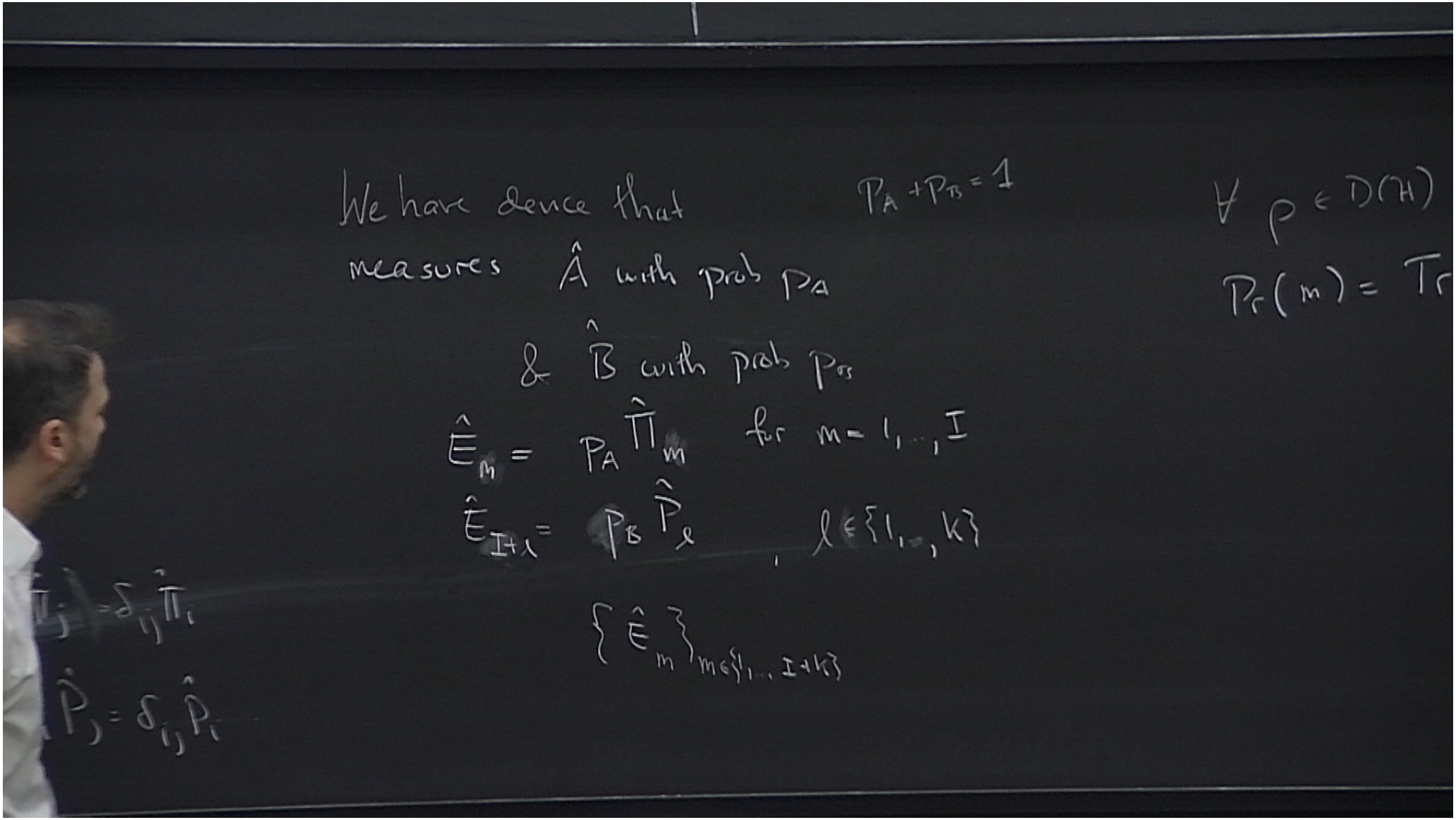
$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ of $\mathcal{H}_B$

$|\psi\rangle = \sum_i \lambda_i |\phi_i\rangle \otimes |\eta_i\rangle$

Given $\hat{\rho} = \sum_i p_i |a_i\rangle\langle a_i|$



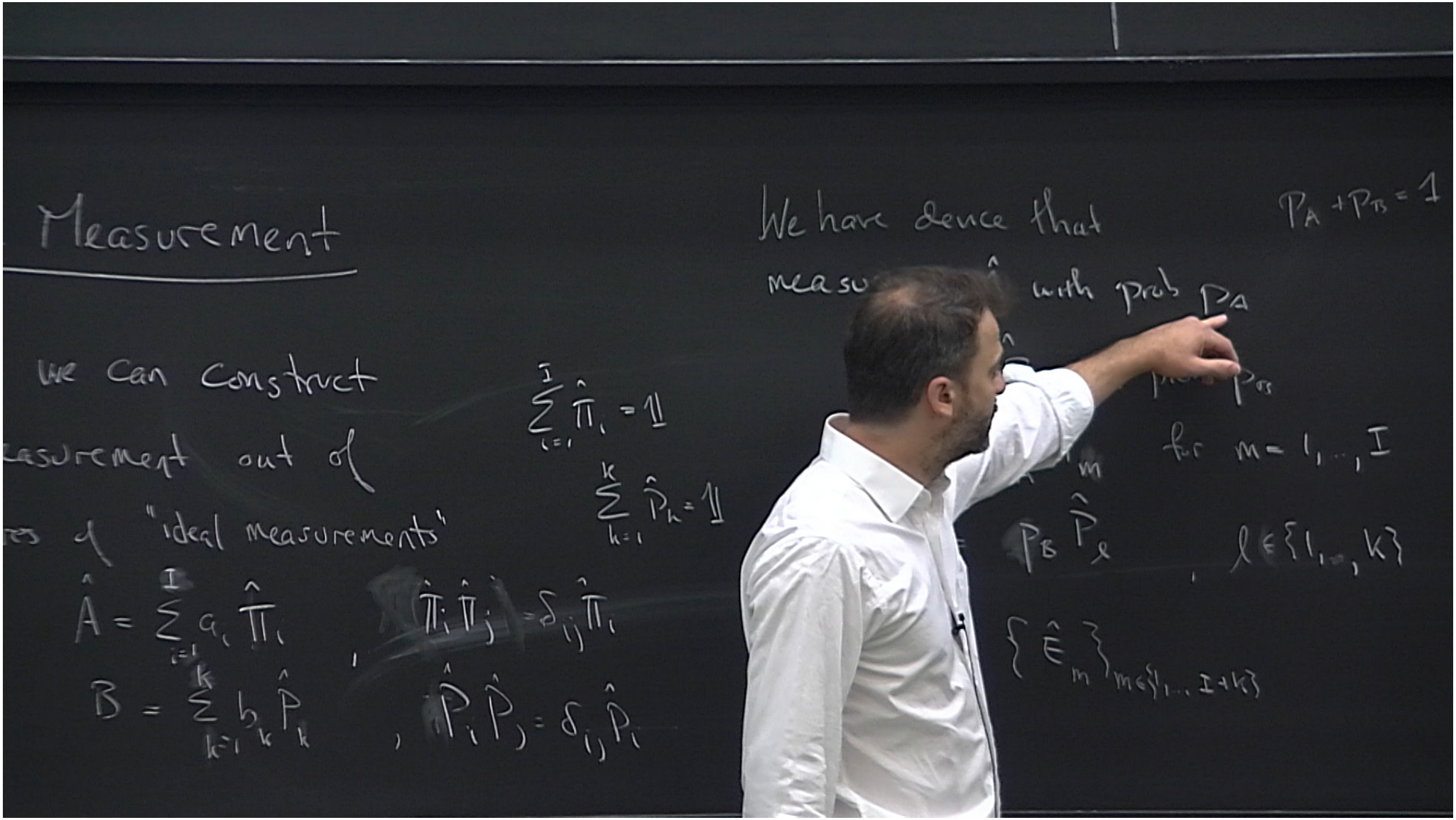

Generalized Measurement
- Trivially, we can construct a measurement out of mixtures of "ideal measurements"
$\hat{A} = \sum_{i=1}^{I} a_i \hat{\Pi}_i$, $\hat{\Pi}_i \hat{\Pi}_j = \delta_{ij} \hat{\Pi}_i$
$B = \sum_{k=1}^{K} b_k \hat{P}_k$, $\hat{P}_i \hat{P}_j = \delta_{ij} \hat{P}_i$
We have hence that
measures
prob p_A
&
prob p_B
for m =


We have hence that
measures $\hat{A}$ with prob p_A
& $\hat{B}$ with prob p_B
$p_A + p_B = 1$
$\hat{E}_m = p_A \hat{\Pi}_m$ for $m = 1, \ldots, I$
$\hat{E}_{I+l} = p_B \hat{P}_l$, $l \in \{1, \ldots, k\}$
$\{\hat{E}_m\}_{m \in \{1, \ldots, I+k\}}$
$\forall \rho \in D(\mathcal{H})$
$p_r(m) = \mathrm{Tr}$
$\hat{\Pi}_j \hat{\Pi}_i = \delta_{ij} \hat{\Pi}_i$
$\hat{P}_j \hat{P}_i = \delta_{ij} \hat{P}_i$

$\forall\ \rho \in D(\mathcal{H})$

$P_r(m) = Tr[\hat{E}_m \hat{\rho}]$

You know which measurement was performed to interpret results.


Measurement
We can Construct
"ideal measurements"
with prob
for

(ii) Ideal measurement of a composite system

$\mathcal{H}_A \otimes \mathcal{H}_B$

$\rho = \rho_A \otimes \rho_B$

Observable: $\hat{A} = \sum_i a_i \hat{P}_i \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$



$$P_r(i|\rho) = Tr\left[\hat{P}_i\, \hat{\rho}_A \otimes \hat{\rho}_B\right]$$

We want an effective description

just in $\mathcal{H}_A$

$$P_r(i) = Tr\left[\hat{E}_i\, \hat{\rho}_A\right]$$

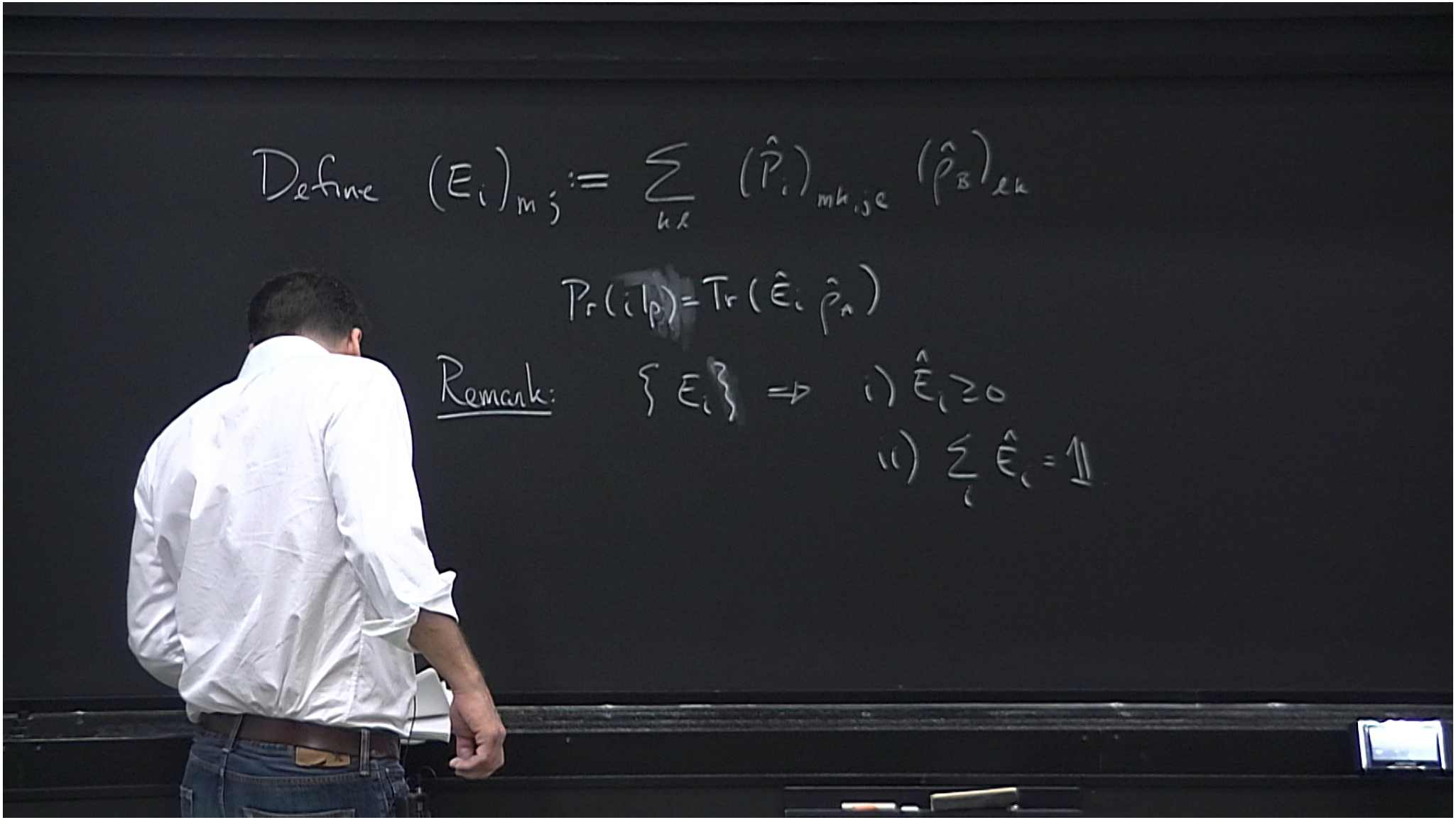
where $\hat{E}_i = Tr_B\left[\hat{P}_i\, \mathbb{1}_A \otimes \hat{\rho}_B\right]$

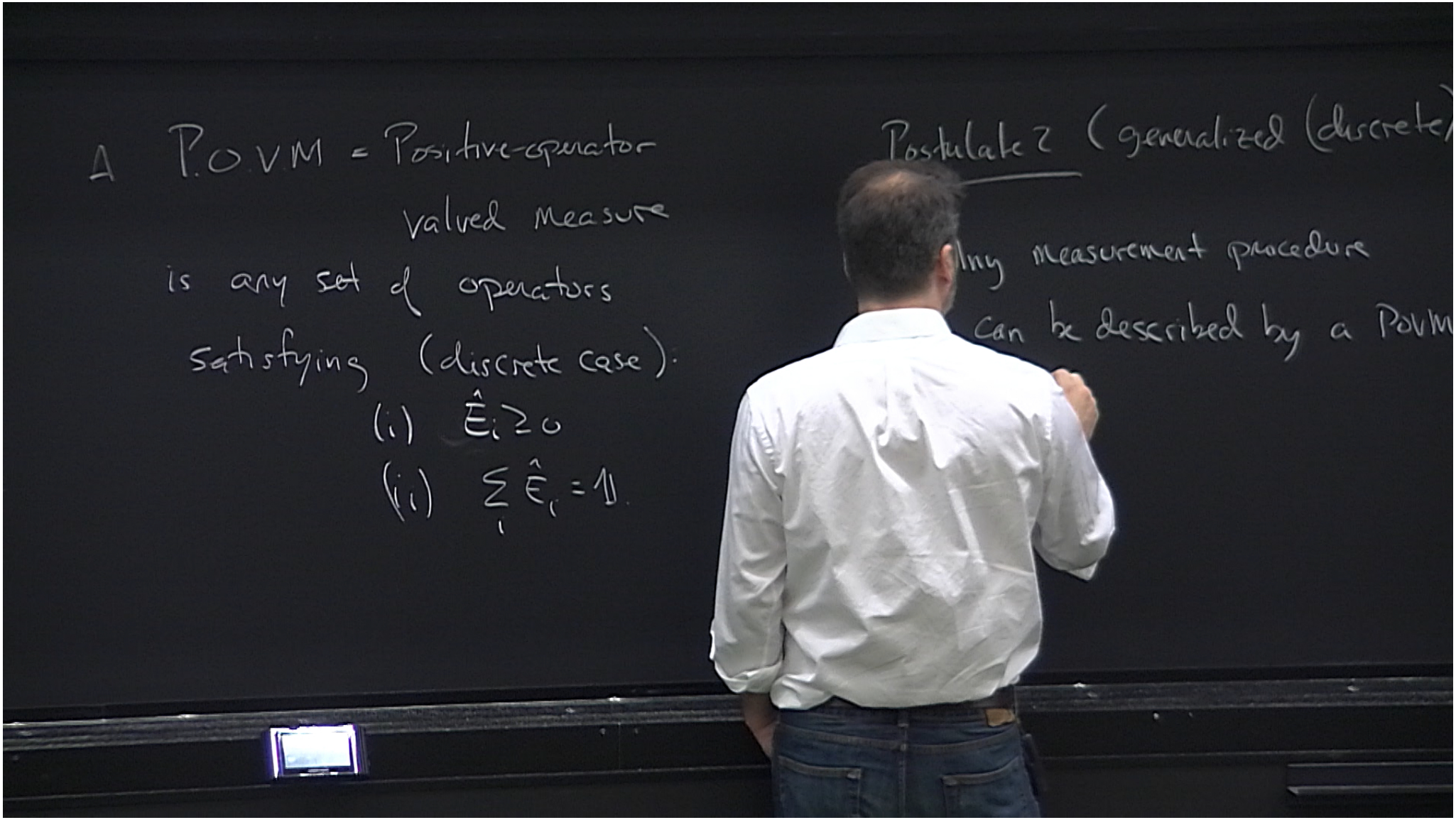
$\mathcal{H}_A \otimes \mathcal{H}_B$)

Probability of each outcome depends on ρ_B!

$$P_i(i|\rho) = (*) = \sum_{mj} \sum_{kl} (P_i)_{mk,jl} (\rho_A)_{jm} (\rho_B)_{lk}$$

$$e.g \quad (P_i)_{mk,jl} = \langle m| \otimes \langle k| \hat{P}_i |j\rangle \otimes |l\rangle$$


Define $(E_i)_{mj} := \sum_{kl} (\hat{P}_i)_{mk,jl} (\hat{\rho}_B)_{lk}$
$P_r(i|\rho) = Tr(\hat{E}_i \hat{\rho}_A)$
Remark: $\{E_i\} \Rightarrow$ i) $\hat{E}_i \geq 0$
ii) $\sum_i \hat{E}_i = \mathbb{1}$


A P.O.V.M = Positive-operator valued measure
is any set of operators satisfying (discrete case):
(i) $\hat{E}_i \geq 0$
(ii) $\sum_i \hat{E}_i = \mathbb{1}$
Postulate 2 (generalized (discrete)
ny measurement procedure
can be described by a POVM

Postulate 2 (generalized (discrete) measurement)

Any measurement procedure

can be described by a POVM $\{\hat{E}_k\}$

s.t $\Pr(k|\rho) = \mathrm{Tr}[\hat{E}_k \hat{\rho}]$

where $\rho \in D(\mathcal{H}_S)$ where $\mathcal{H}_S$ is the Hilbert space of system of interest

$E_k \in \mathcal{L}(\mathcal{H}_S)$

M = Positive-operator valued measure

...t of operators

...ns (discrete case):

(i) $\hat{E}_i \geq 0$

$\sum_i \hat{E}_i = \mathbb{1}$

Postulate 2 (generalized (discrete) measurement)

Any measurement procedure can be described by a POVM $\{\hat{E}_k\}$ s.t. $P_r(k|\rho) = Tr[\hat{E}_k \hat{\rho}]$



where $\rho \in D(\mathcal{H}_S)$ where $\mathcal{H}_S$ is the Hilbert space of system of interest

$E_k \in \mathcal{L}(\mathcal{H}_S)$