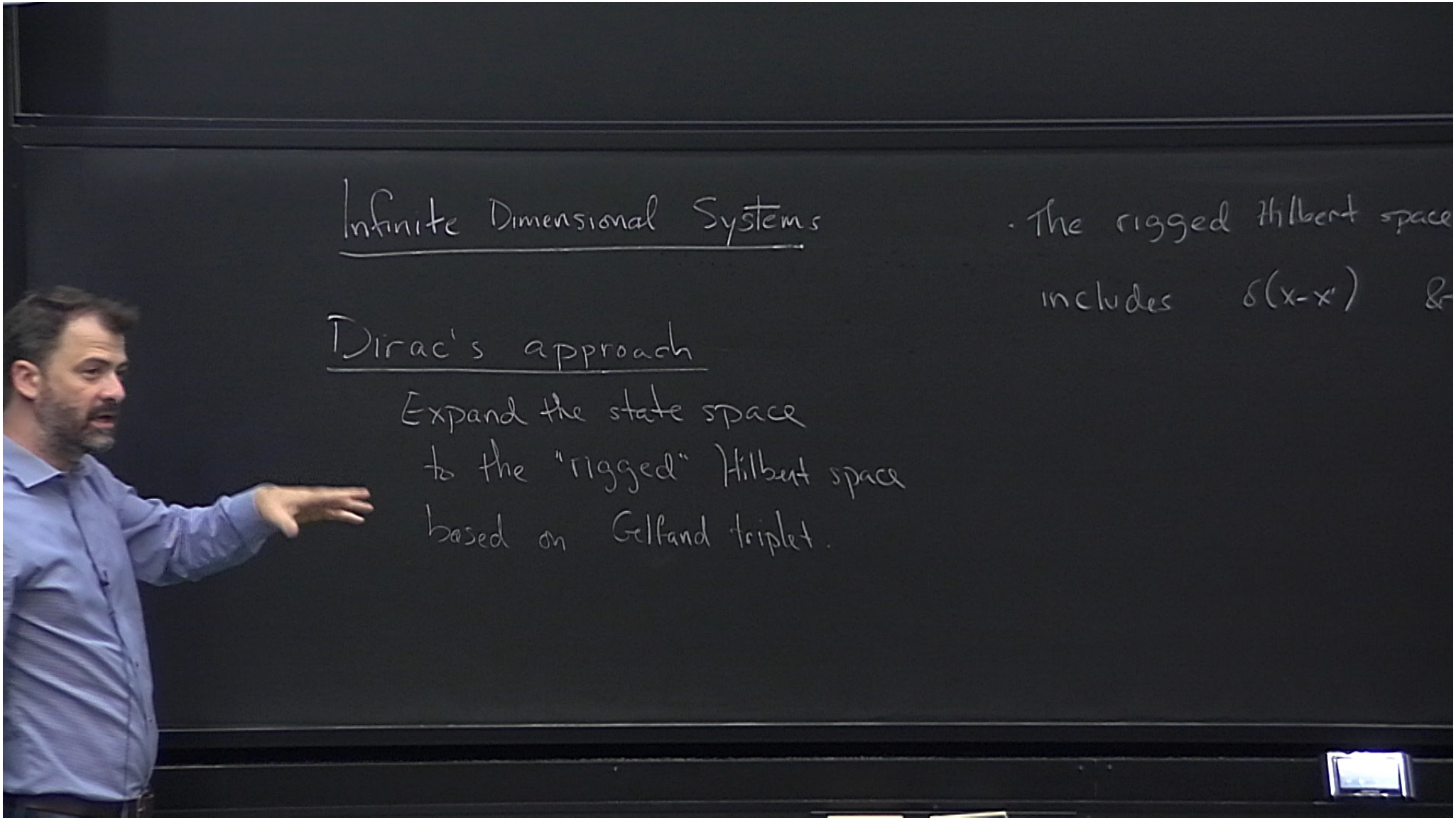
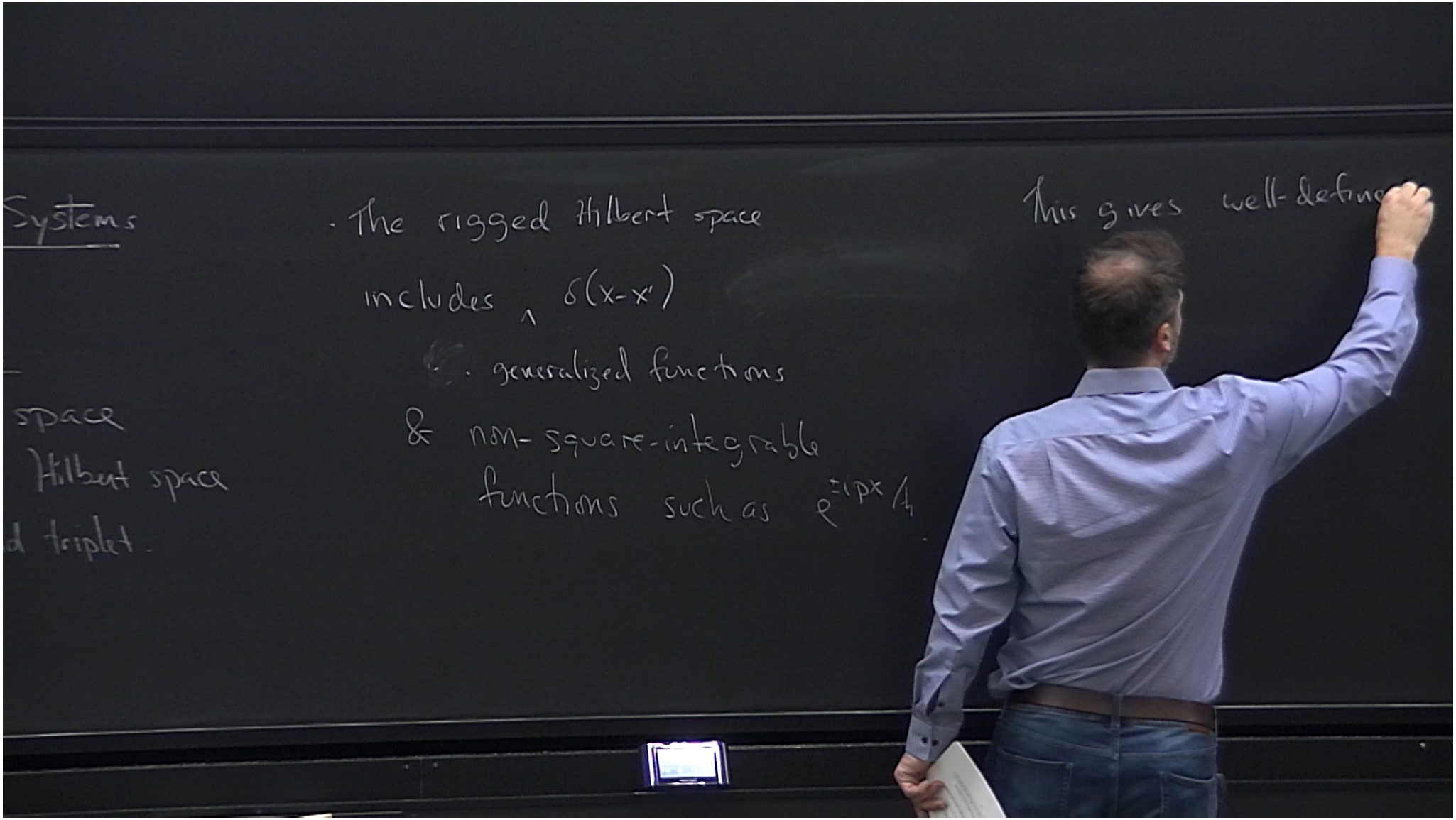
Title: PSI 17/18 - Quantum Theory - Lecture 7

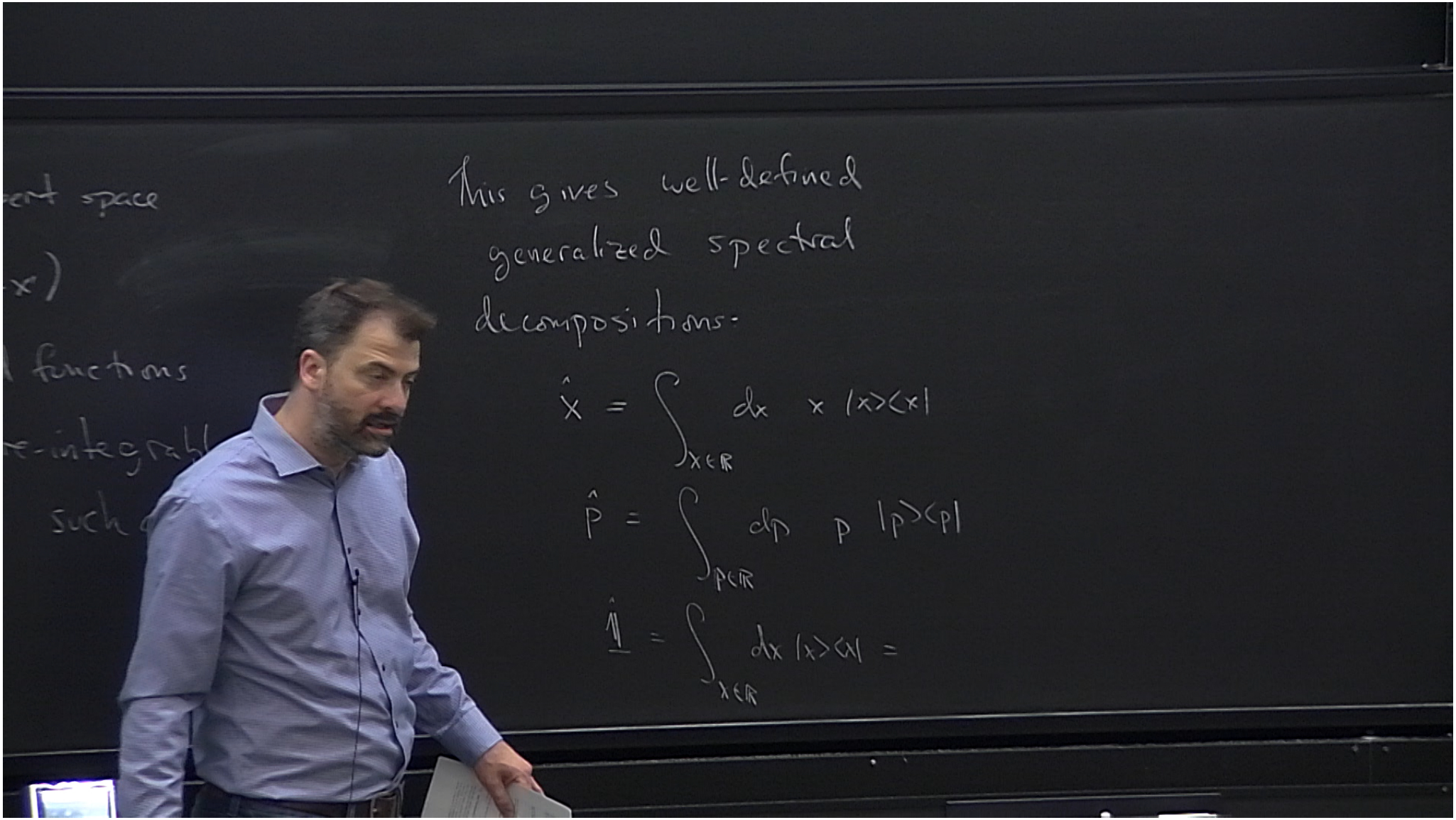
Date: Sep 13, 2017 10:45 AM

URL: http://pirsa.org/17090040

Abstract:


Infinite Dimensional Systems
Dirac's approach
Expand the state space
to the "rigged" Hilbert space
based on Gelfand triplet.
· The rigged Hilbert space
includes δ(x-x')


Systems
space
Hilbert space
d triplet.
· The rigged Hilbert space
includes $\delta(x-x')$
· generalized functions
& non-square-integrable
functions such as $e^{\pm ipx}$
This gives well-defin


This gives well-defined
generalized spectral
decompositions:
$\hat{x} = \int_{x \in \mathbb{R}} dx \; x \, |x\rangle\langle x|$
$\hat{p} = \int_{p \in \mathbb{R}} dp \; p \, |p\rangle\langle p|$
$\hat{\mathbb{1}} = \int_{x \in \mathbb{R}} dx \, |x\rangle\langle x| =$

ent space

x)

functions

e-integrable

such as $e^{\pm ipx/\hbar}$.

This gives well-defined generalized spectral decompositions:

$$\hat{X} = \int_{x\in\mathbb{R}} dx \; x \, |x\rangle\langle x|$$

$$\hat{P} = \int_{p\in\mathbb{R}} dp \; p \, |p\rangle\langle p|$$

$$\hat{\mathbb{1}} = \int_{x\in\mathbb{R}} dx \, |x\rangle\langle x| = \int_{p\in\mathbb{R}} dp \, |p\rangle\langle p|$$

where

$\langle x|x'\rangle = \delta(x-x')$

$\langle p|p'\rangle = \delta(p-p')$

$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

Generalize postulate 2 b)

$$P_r(x \in [a,b]) = Tr\left[|\psi\rangle\langle\psi| \int_a^b dx\, |x\rangle\langle x| \right]$$

$$= \int_a^b dx\, |\psi(x)|^2$$

where $\psi(x) = \langle x|\psi\rangle$

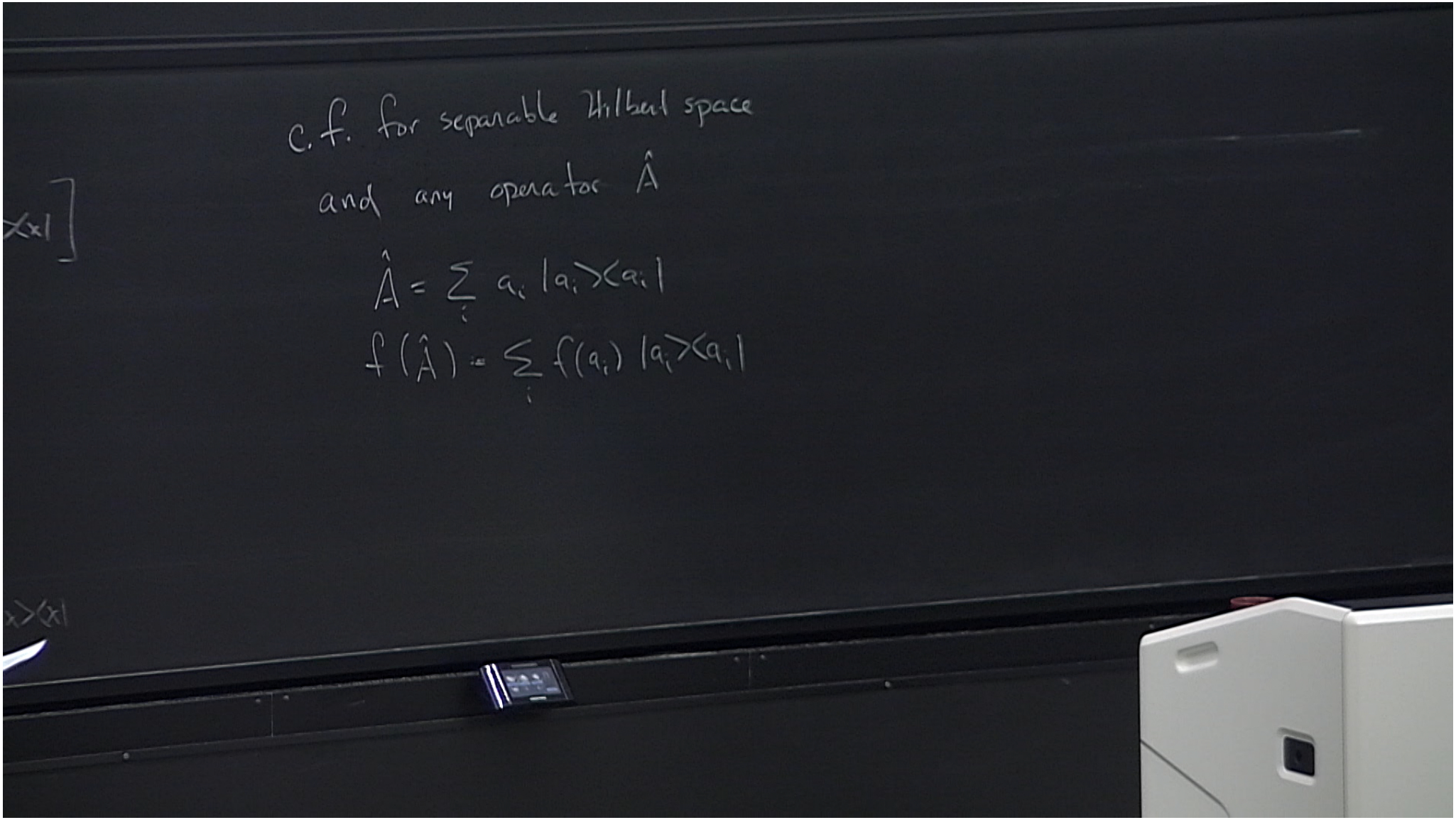
- Generalize postulate 2.b)

$$P_r(x \in [a,b]) = Tr\left[|\psi\rangle\langle\psi| \int_a^b dx\, |x\rangle\langle x| \right]$$

$$= \int_a^b dx\, |\psi(x)|^2$$

where $\psi(x) = \langle x|\psi\rangle$

- We can define any $f(\hat{x})$, function of an unbounded operator, $f(\hat{x}) = \int_{x\in\mathbb{R}} dx\, f(x)\, |x\rangle\langle x|$


c.f. for separable Hilbert space
and any operator $\hat{A}$
$\hat{A} = \sum_i a_i |a_i\rangle\langle a_i|$
$f(\hat{A}) = \sum_i f(a_i) |a_i\rangle\langle a_i|$

Von Neumann's approach

Let $\hat{A}$ be a linear operator on $\mathcal{H}$

Domain of $\hat{A}$, $D(\hat{A})$, is the set of all $|\psi\rangle$ s.t. $\hat{A}|\psi\rangle \in \mathcal{H}$

- If $\hat{A}$ is densely defined on $\mathcal{H}$, then adjoint of $\hat{A}$, $\hat{A}^\dagger$, is defined by:

(i) $D(A^\dagger)$ is all ϕ



uator on $\mathcal{H}$

, is the set of all
$\hat{A}|\psi\rangle \in \mathcal{H}$

d on $\mathcal{H}$, then adjoint
is defined by:

(i) $D(A^\dagger)$ is all $\phi \in \mathcal{H}$
for which $\exists\ \eta$ s.t
$\langle A\psi|\phi\rangle = \langle\psi|\eta\rangle$
$\forall\ \psi \in D(\hat{A})$

(ii) $A^\dagger\phi = \eta$, where η is the
unique vector s.t. $\langle A\psi|\phi\rangle = \langle\psi|\eta\rangle$
$\forall\ \psi \in D(\hat{A})$

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is all $\phi \in \mathcal{H}$

$\exists\ \eta$ s.t.

$\langle \psi | \phi \rangle = \langle \psi | \eta \rangle$

$\forall\ |\psi\rangle \in D(\hat{A})$

$= \eta$, where η is the

vector s.t. $\langle A\psi | \phi \rangle = \langle \psi | \eta \rangle$

$\forall\ |\psi\rangle \in D(\hat{A})$

Self-adjoint ~~operator~~ $\hat{A}^\dagger = \hat{A}$,

is defined by:

(i) $\hat{A}$ is Hermitian

$(A\phi, \psi) = (\phi, A\psi)$

$\forall\ \psi, \phi \in D(A)$

(ii) $D(\hat{A}) = D(\hat{A}^\dagger)$

Example: $\hat{p} = -i\frac{d}{dx}$ is self-adjoint on $L^2(\mathbb{R})$

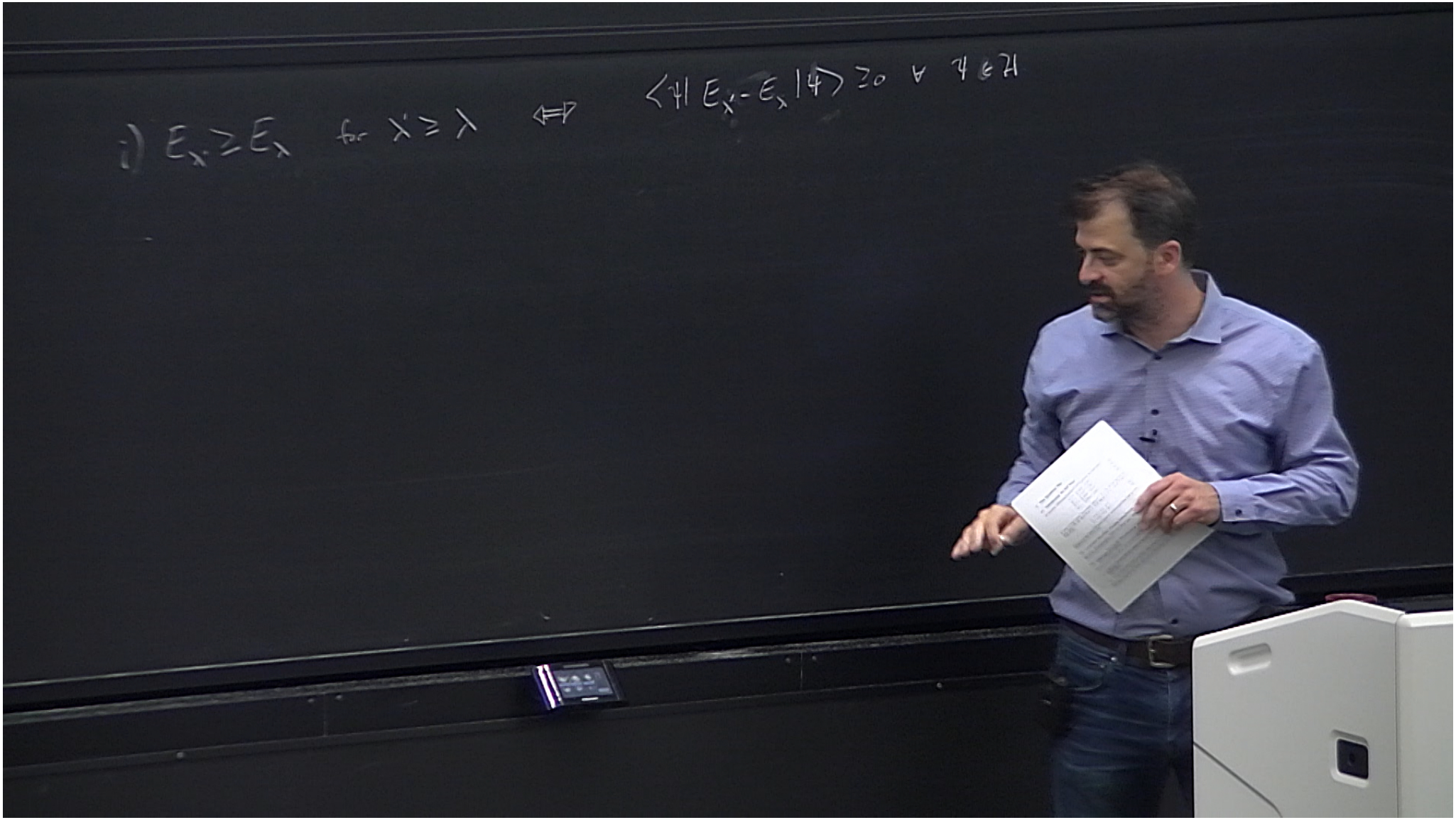
but not self-adjoint on $L^2([a,b])$

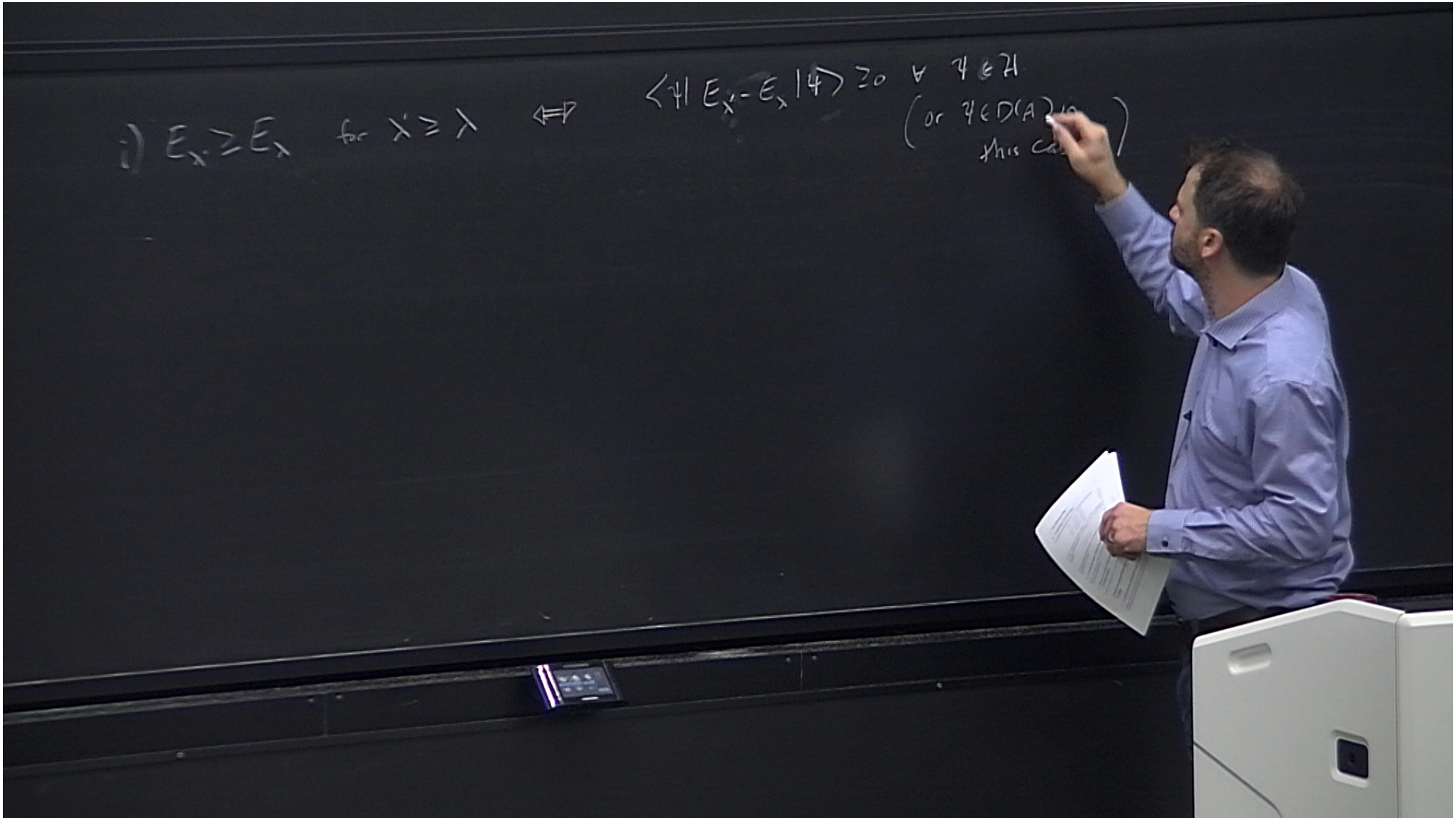
Von Neumann's generalized spectral decomposition:

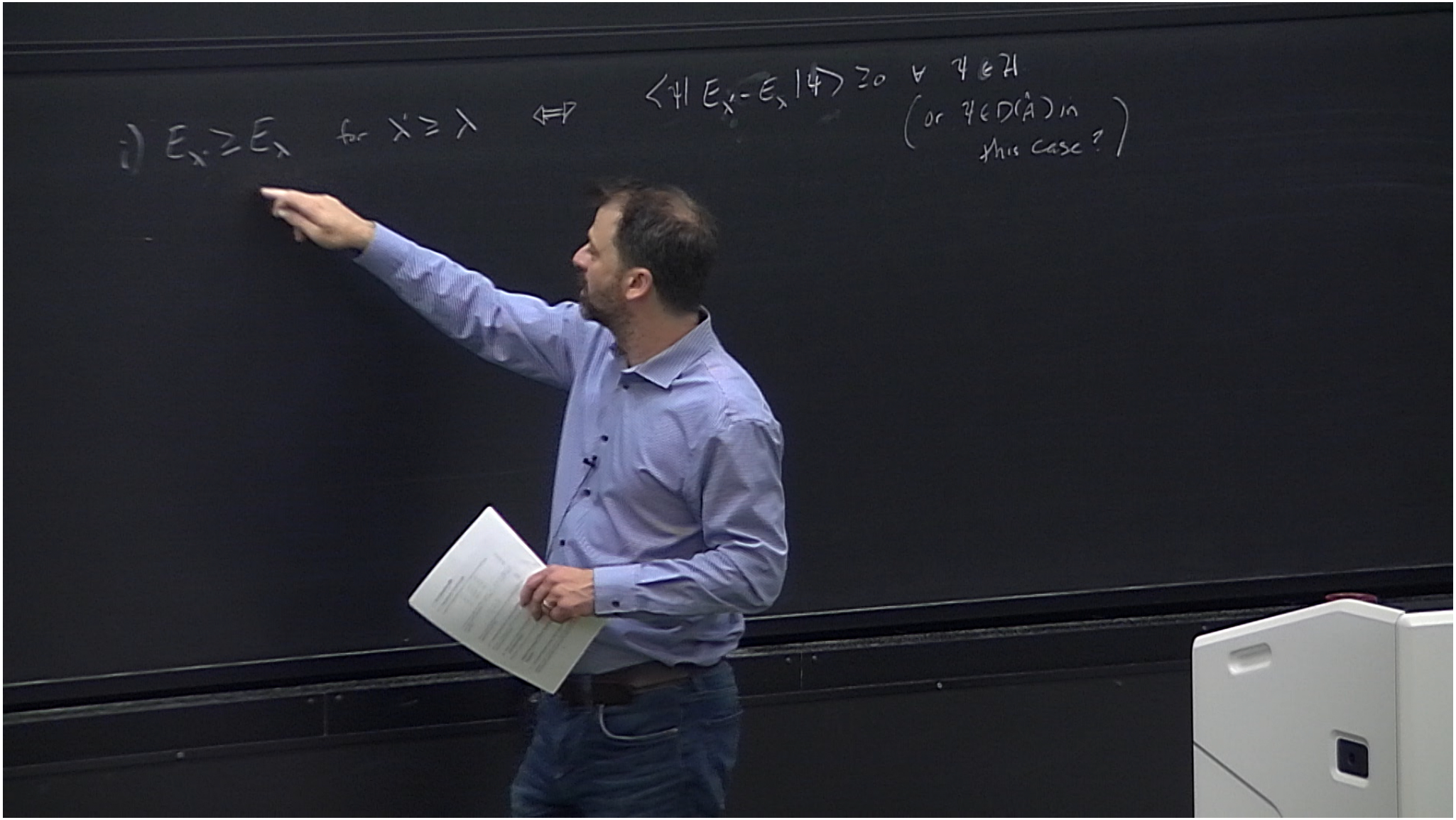
$\forall$ self-adjoint operator $\hat{A}$

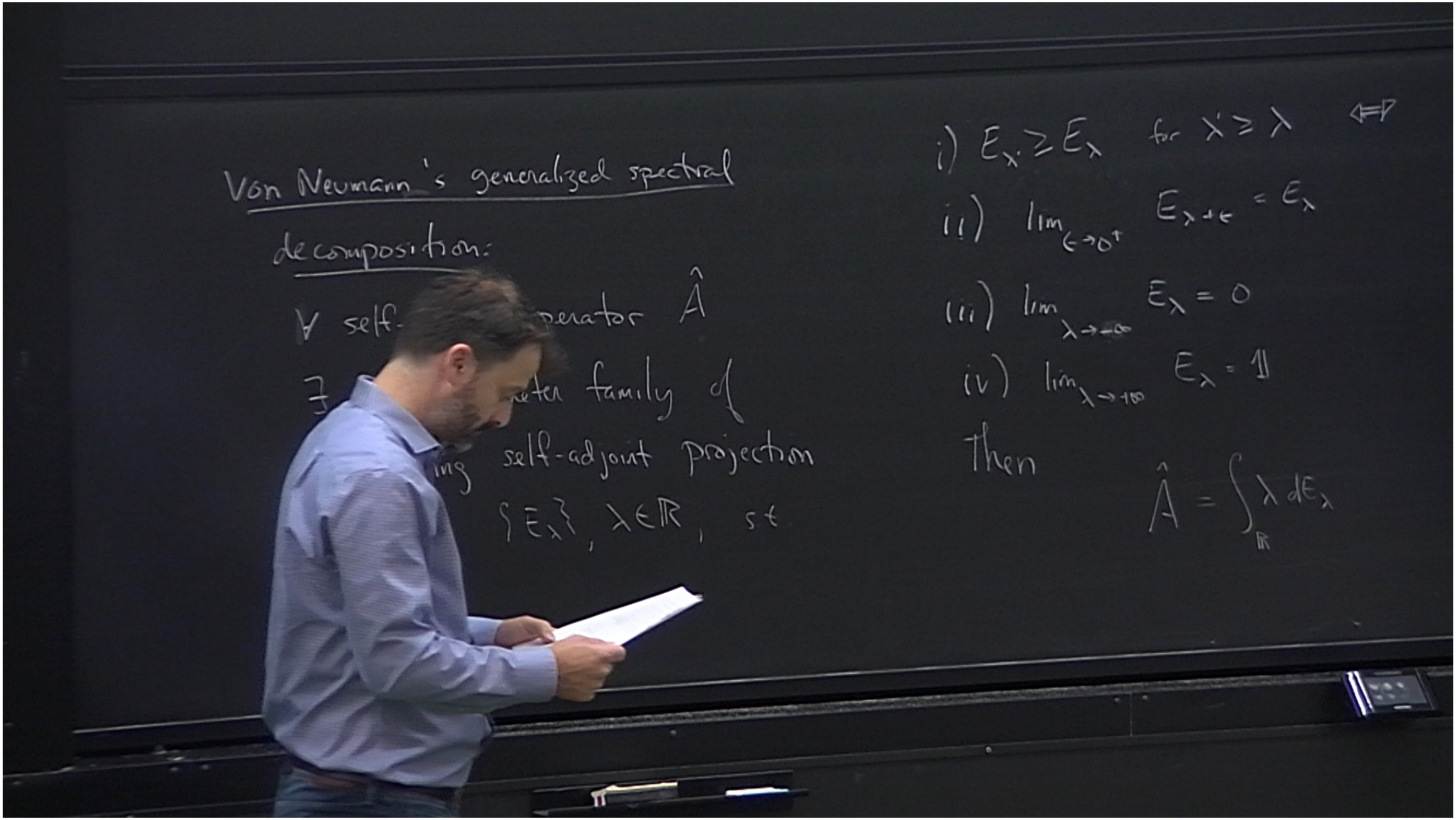
$\exists$ a one-parameter family of commuting self-adjoint projection operators $\{E_\lambda\}$, $\lambda \in \mathbb{R}$, s.t.

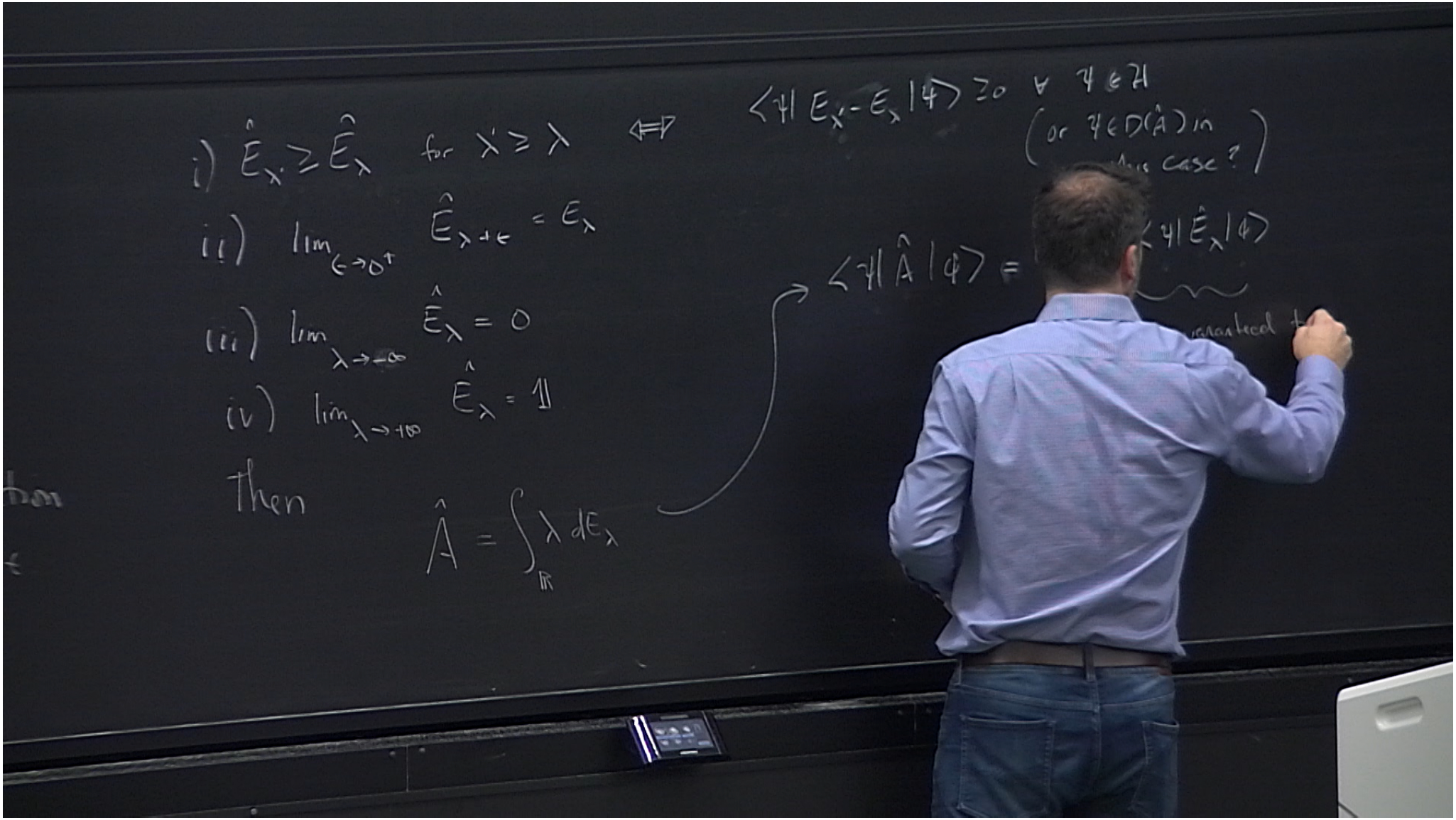
i) $E_{\lambda'} \geq E_\lambda$ for $\lambda' \geq \lambda$

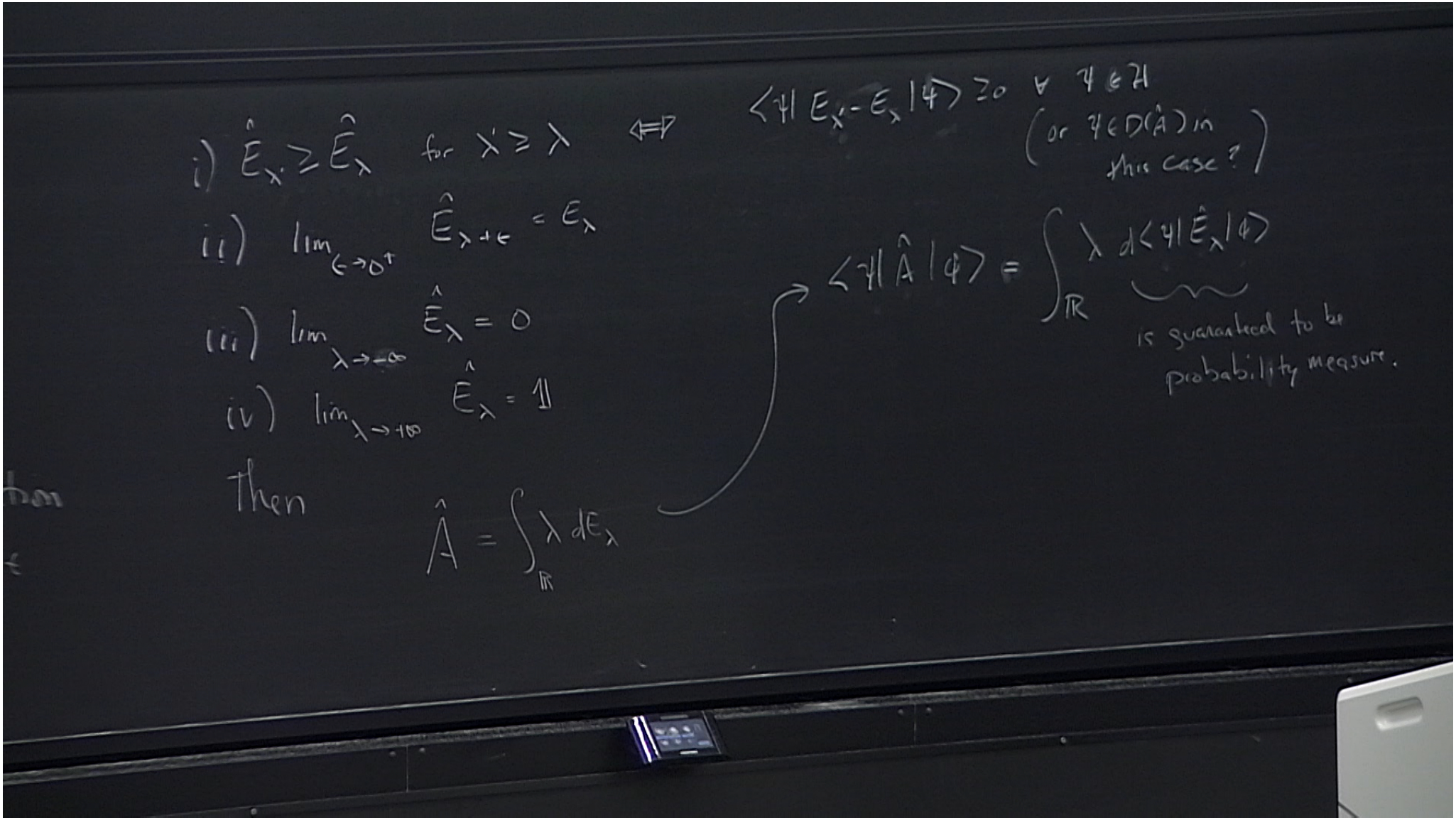

i) $E_{\lambda'} \geq E_\lambda$ for $\lambda' \geq \lambda$ $\Leftrightarrow$ $\langle \psi | E_{\lambda'} - E_\lambda | \psi \rangle \geq 0 \quad \forall \psi \in \mathcal{H}$

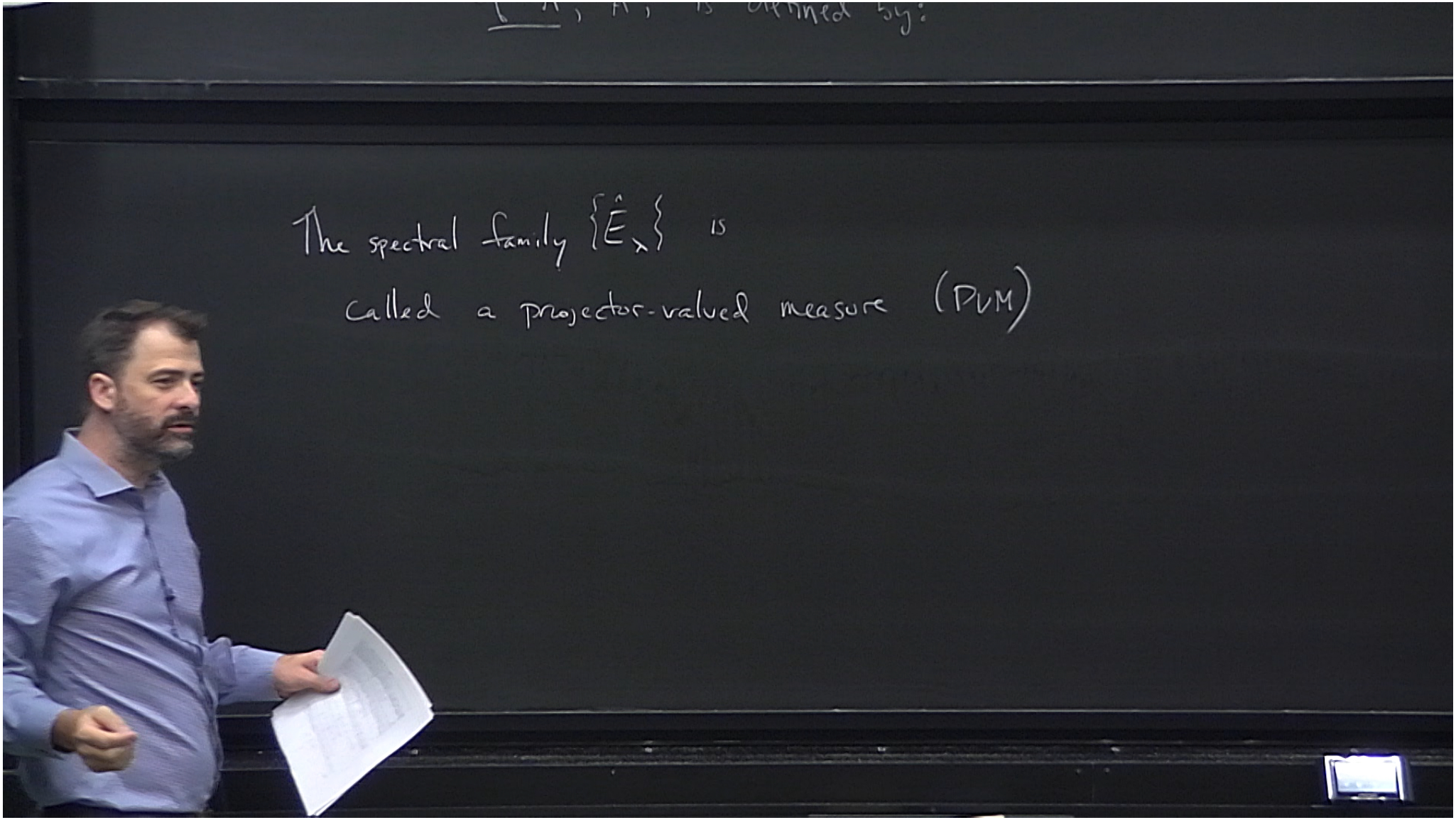

i) $E_{\lambda'} \geq E_\lambda$ for $\lambda' \geq \lambda$ $\Leftrightarrow$ $\langle \psi | E_{\lambda'} - E_\lambda | \psi \rangle \geq 0$ $\forall$ $\psi \in \mathcal{H}$
(or $\psi \in D(A)$ ...
this ca...)

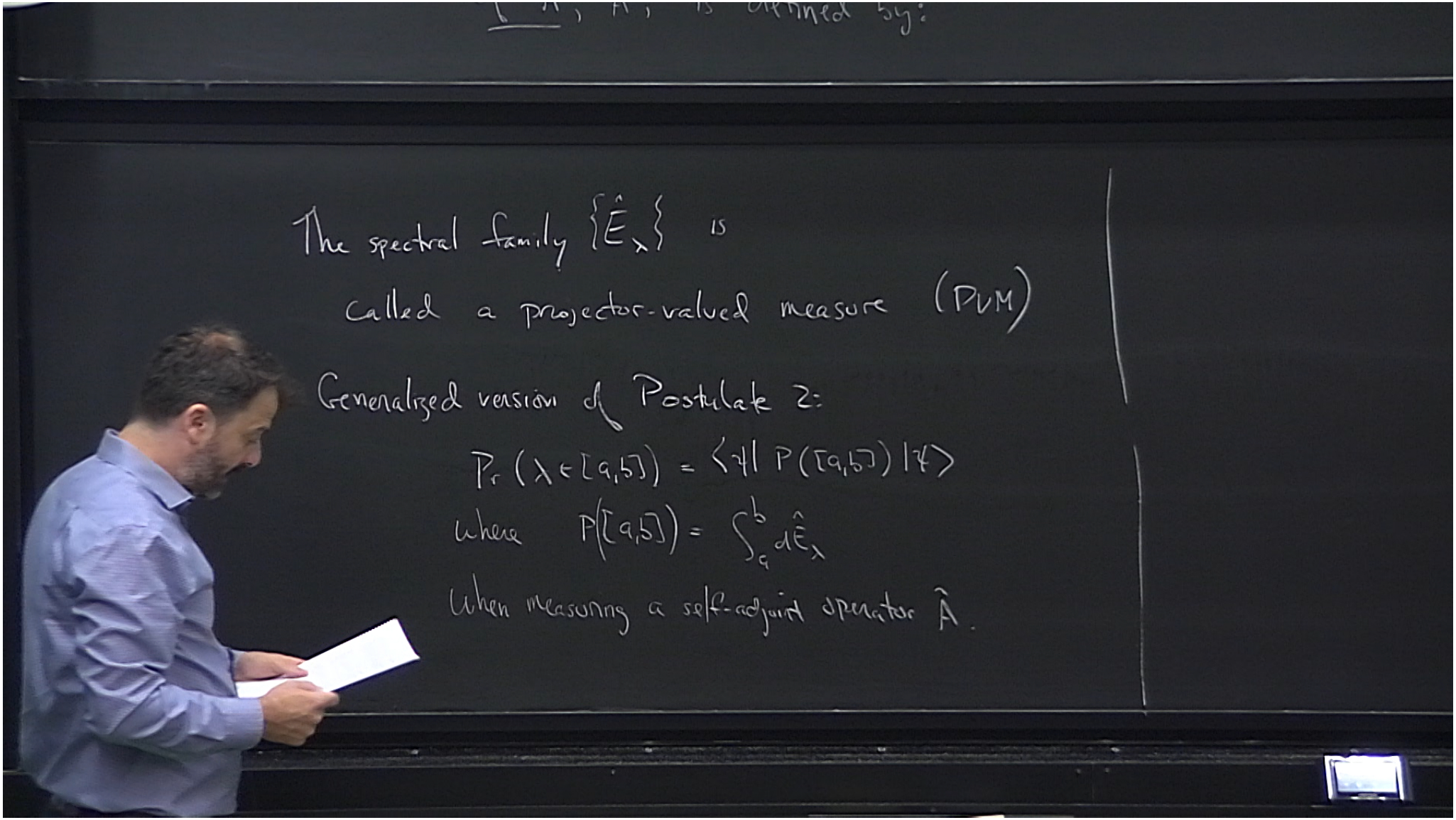

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(or $\psi \in D(\hat{A})$ in this case?)

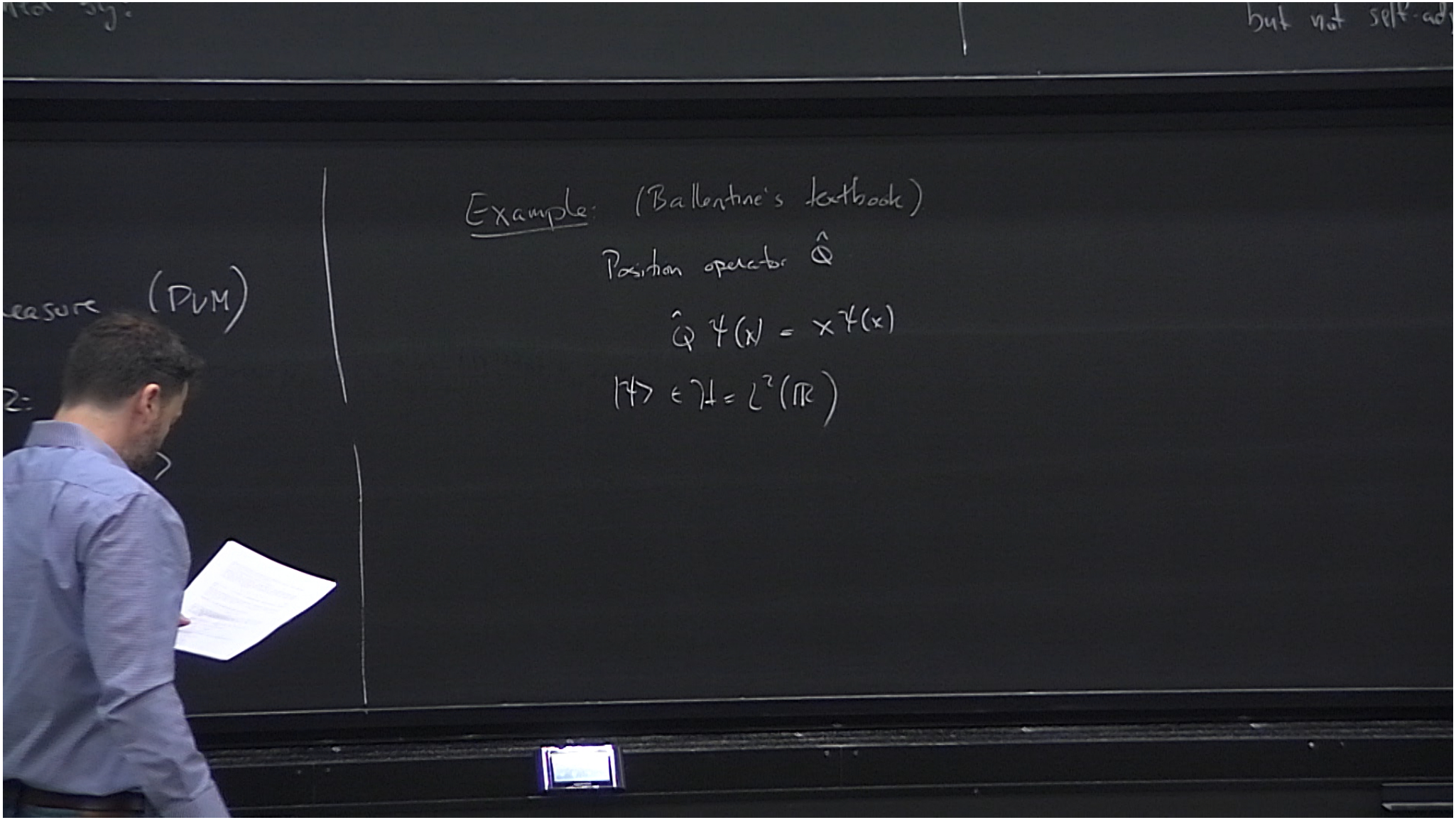

Von Neumann's generalized spectral
decomposition:
∀ self-
operator Â
∃
meter family of
ing self-adjoint projection
{E_λ}, λ∈ℝ, s.t.
i) E_λ' ≥ E_λ for λ' ≥ λ
ii) lim_{ε→0⁺} E_{λ+ε} = E_λ
iii) lim_{λ→-∞} E_λ = 0
iv) lim_{λ→+∞} E_λ = 𝟙
Then
Â = ∫_ℝ λ dE_λ

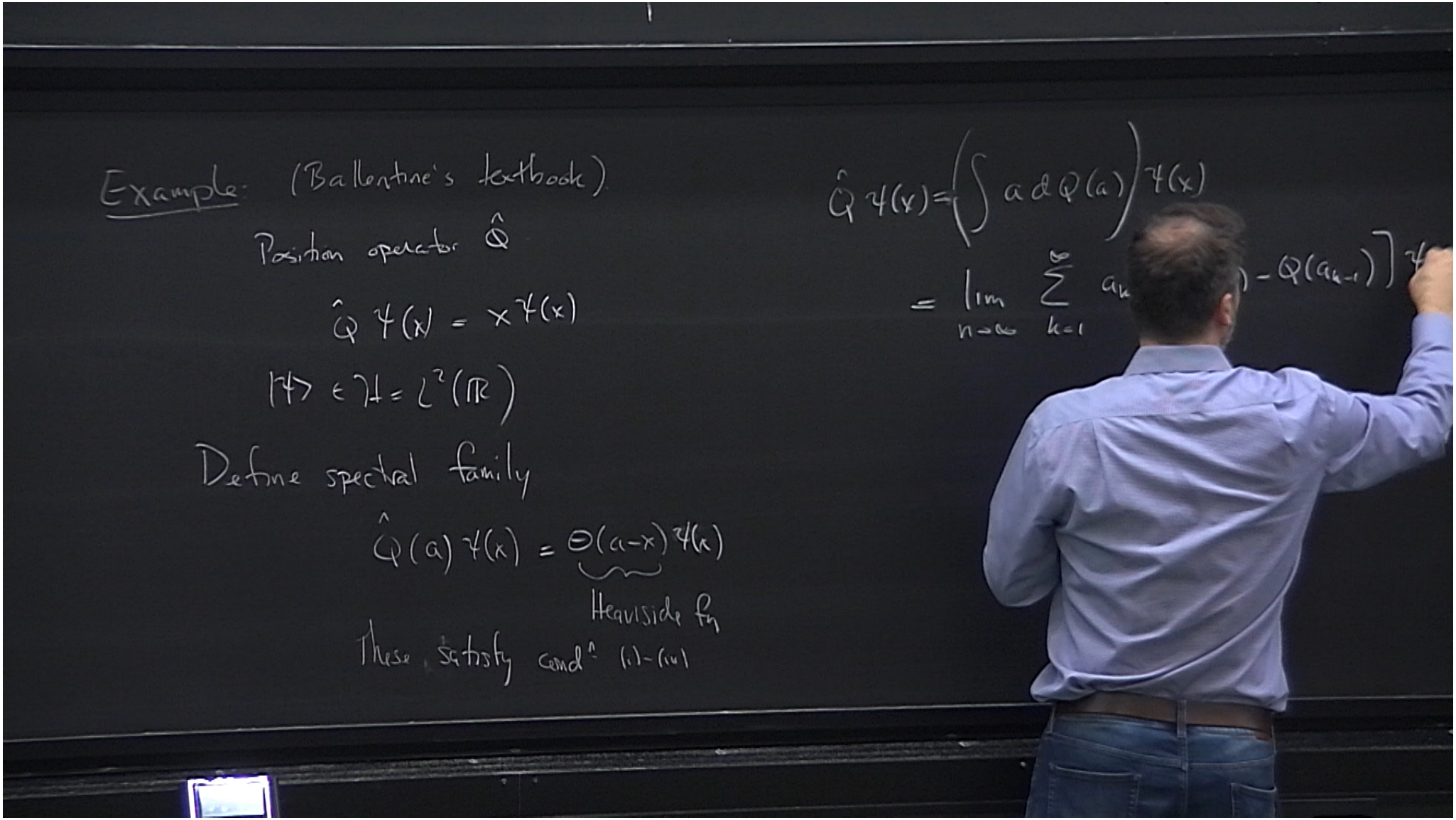

i) $\hat{E}_{\lambda'} \geq \hat{E}_\lambda$ for $\lambda' \geq \lambda$ $\Longleftrightarrow$ $\langle \psi | E_{\lambda'} - E_\lambda | \psi \rangle \geq 0$ $\forall \psi \in \mathcal{H}$
(or $\psi \in D(\hat{A})$ in this case?)
ii) $\lim_{\epsilon \to 0^+} \hat{E}_{\lambda+\epsilon} = E_\lambda$
iii) $\lim_{\lambda \to -\infty} \hat{E}_\lambda = 0$
iv) $\lim_{\lambda \to +\infty} \hat{E}_\lambda = \mathbb{1}$
Then $\hat{A} = \int_{\mathbb{R}} \lambda \, dE_\lambda$
$\langle \psi | \hat{A} | \psi \rangle =$
$\langle \psi | \hat{E}_\lambda | \psi \rangle$

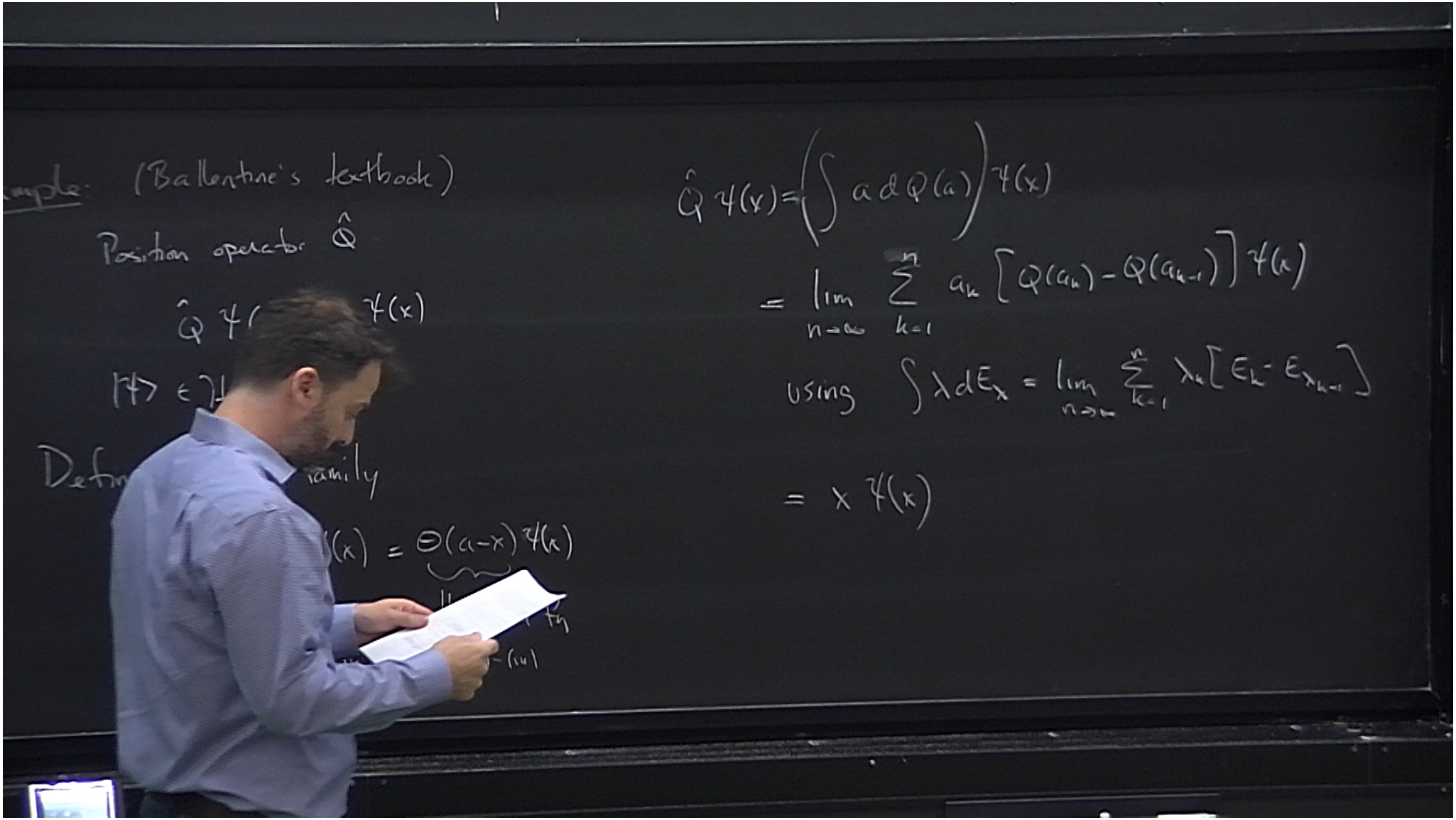

i) $\hat{E}_{\lambda'} \geq \hat{E}_\lambda$ for $\lambda' \geq \lambda$ $\Longleftrightarrow$ $\langle\psi|E_{\lambda'} - E_\lambda|\psi\rangle \geq 0$ $\forall\ \psi \in \mathcal{H}$ (or $\psi \in D(\hat{A})$ in this case?)
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Then $\hat{A} = \int_{\mathbb{R}} \lambda \, dE_\lambda$
$\langle\psi|\hat{A}|\psi\rangle = \int_{\mathbb{R}} \lambda \, d\langle\psi|\hat{E}_\lambda|\psi\rangle$
is guaranteed to be probability measure.

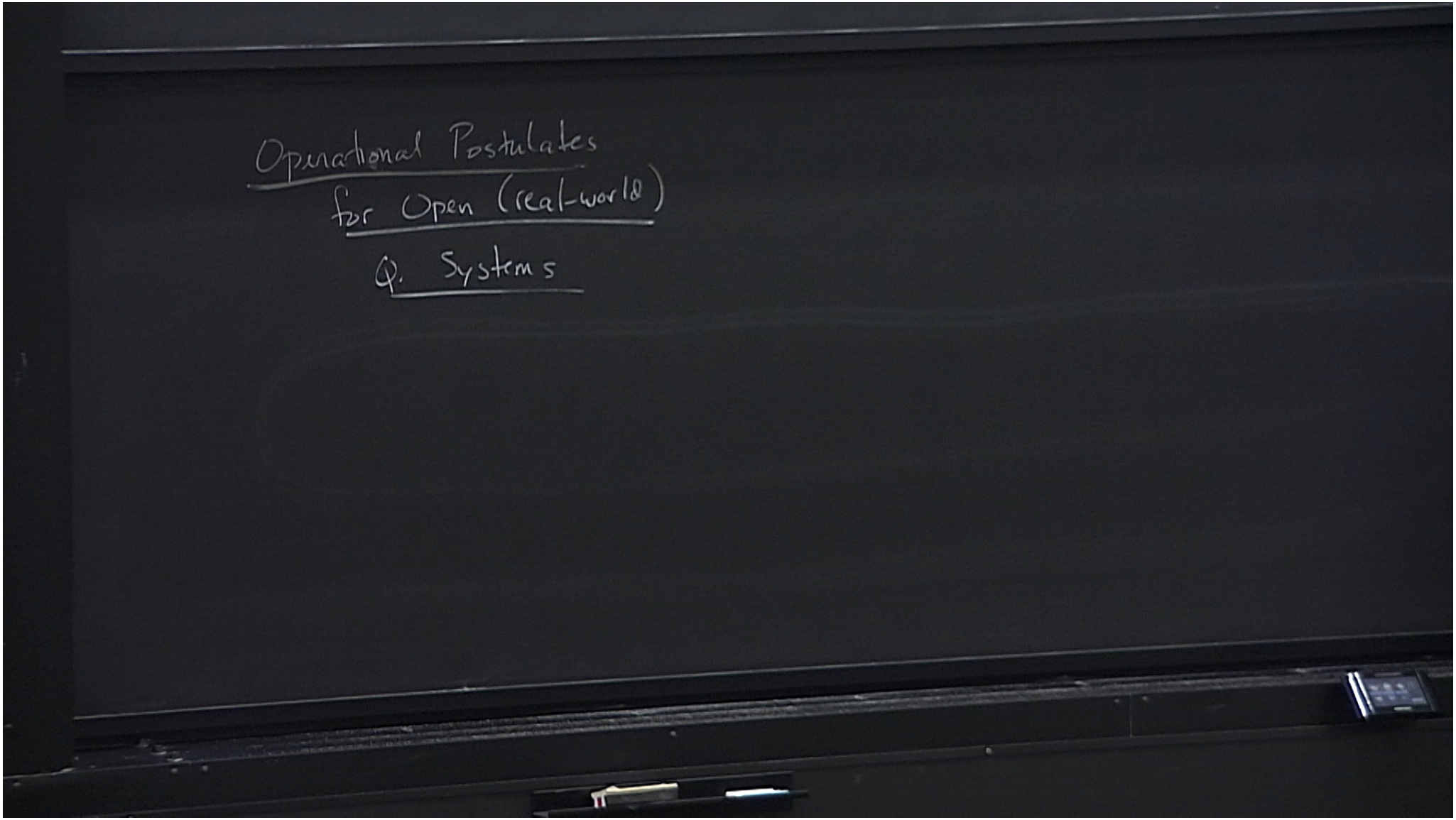

The spectral family $\{\hat{E}_\lambda\}$ is
called a projector-valued measure (PVM)

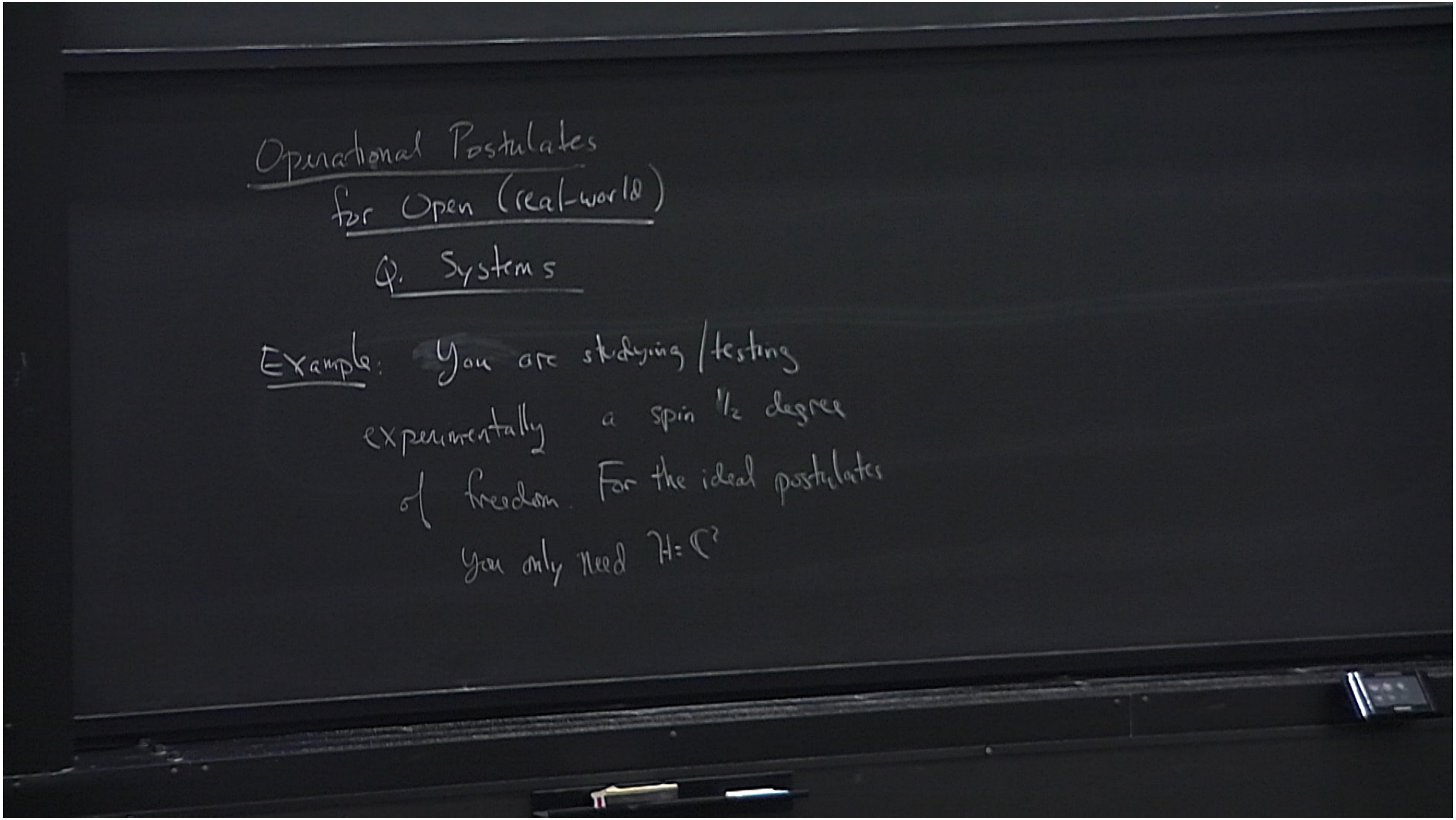

$\hat{A}$, $\hat{A}$, is defined by:
The spectral family $\{\hat{E}_\lambda\}$ is
called a projector-valued measure (PVM)
Generalized version of Postulate 2:
$P_r(\lambda \in [a,b]) = \langle \psi | P([a,b]) | \psi \rangle$
where $P([a,b]) = \int_a^b d\hat{E}_\lambda$
When measuring a self-adjoint operator $\hat{A}$.

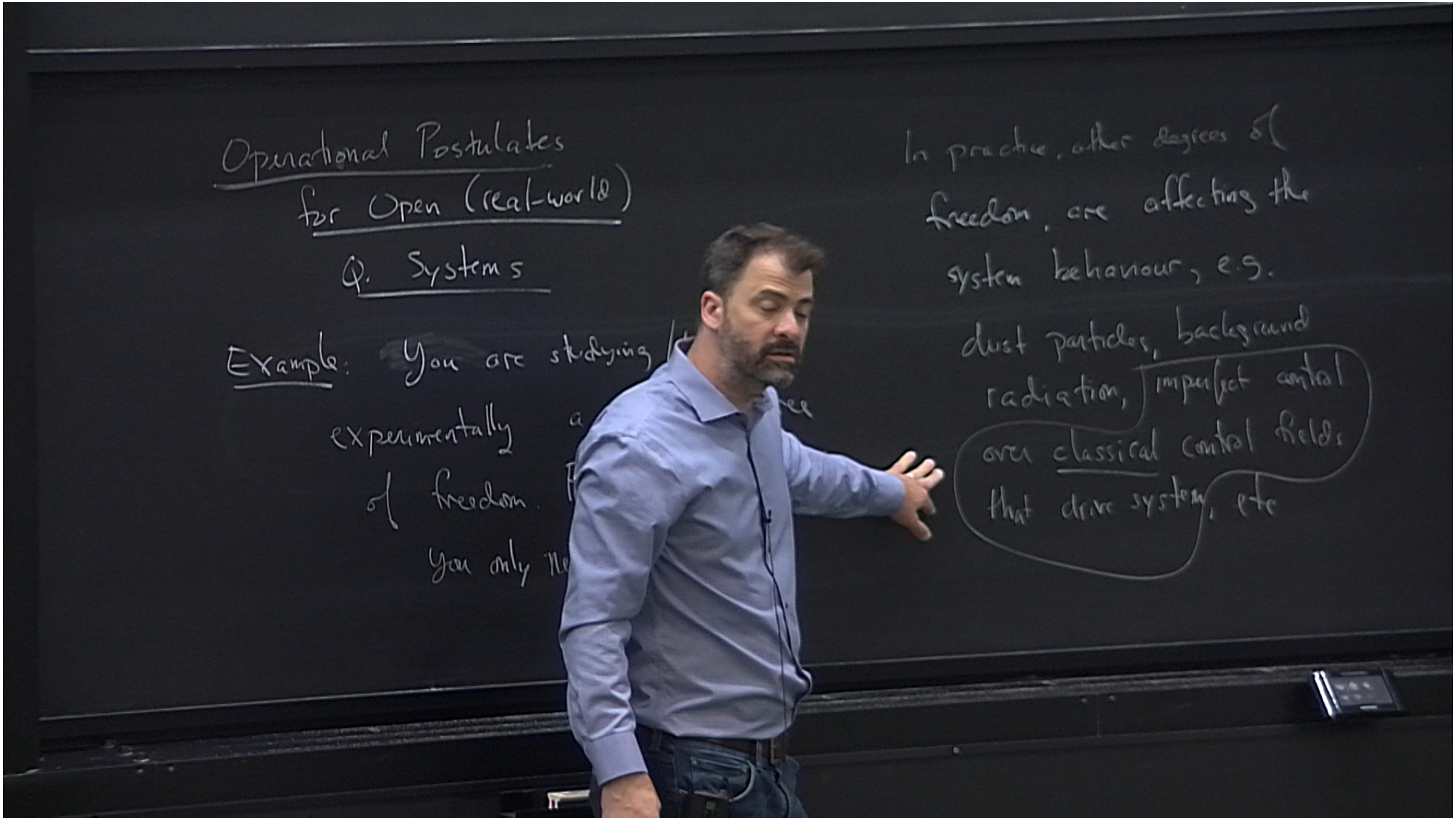

but not self-adj
Example: (Ballentine's textbook)
Position operator $\hat{Q}$
$\hat{Q}\,\psi(x) = x\,\psi(x)$
$|\psi\rangle \in \mathcal{H} = L^2(\mathbb{R})$
(PVM)

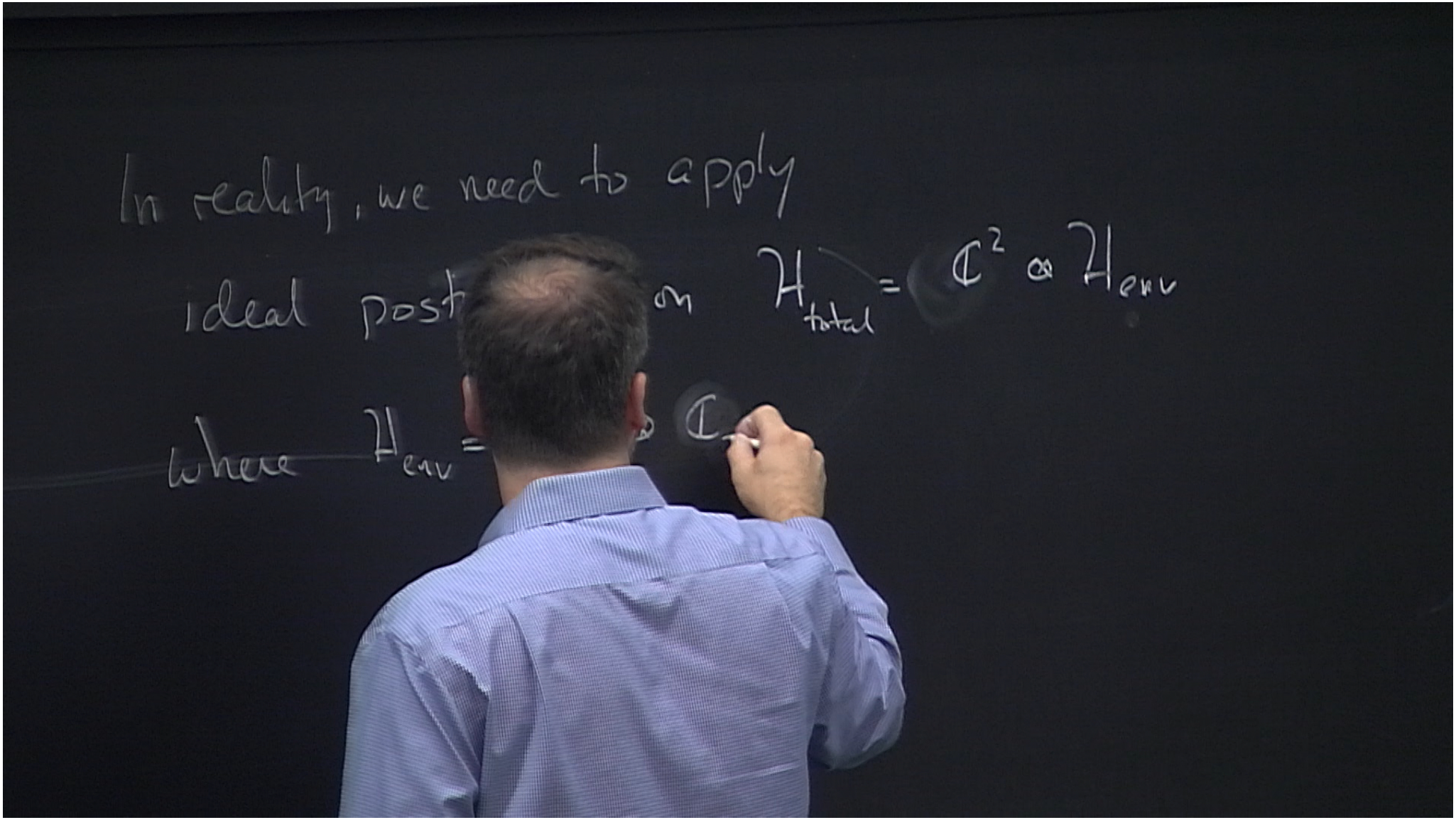

Example: (Ballentine's textbook)
Position operator $\hat{Q}$
$\hat{Q}\psi(x) = x\psi(x)$
$|\psi\rangle \in \mathcal{H} = L^2(\mathbb{R})$
Define spectral family
$\hat{Q}(a)\psi(x) = \Theta(a-x)\psi(x)$
Heaviside fn
These satisfy cond^s (i)-(iv)
$\hat{Q}\psi(x) = \left(\int a\, dQ(a)\right)\psi(x)$
$= \lim_{n\to\infty} \sum_{k=1}^{\infty} a_k$
$- Q(a_{k-1})]$

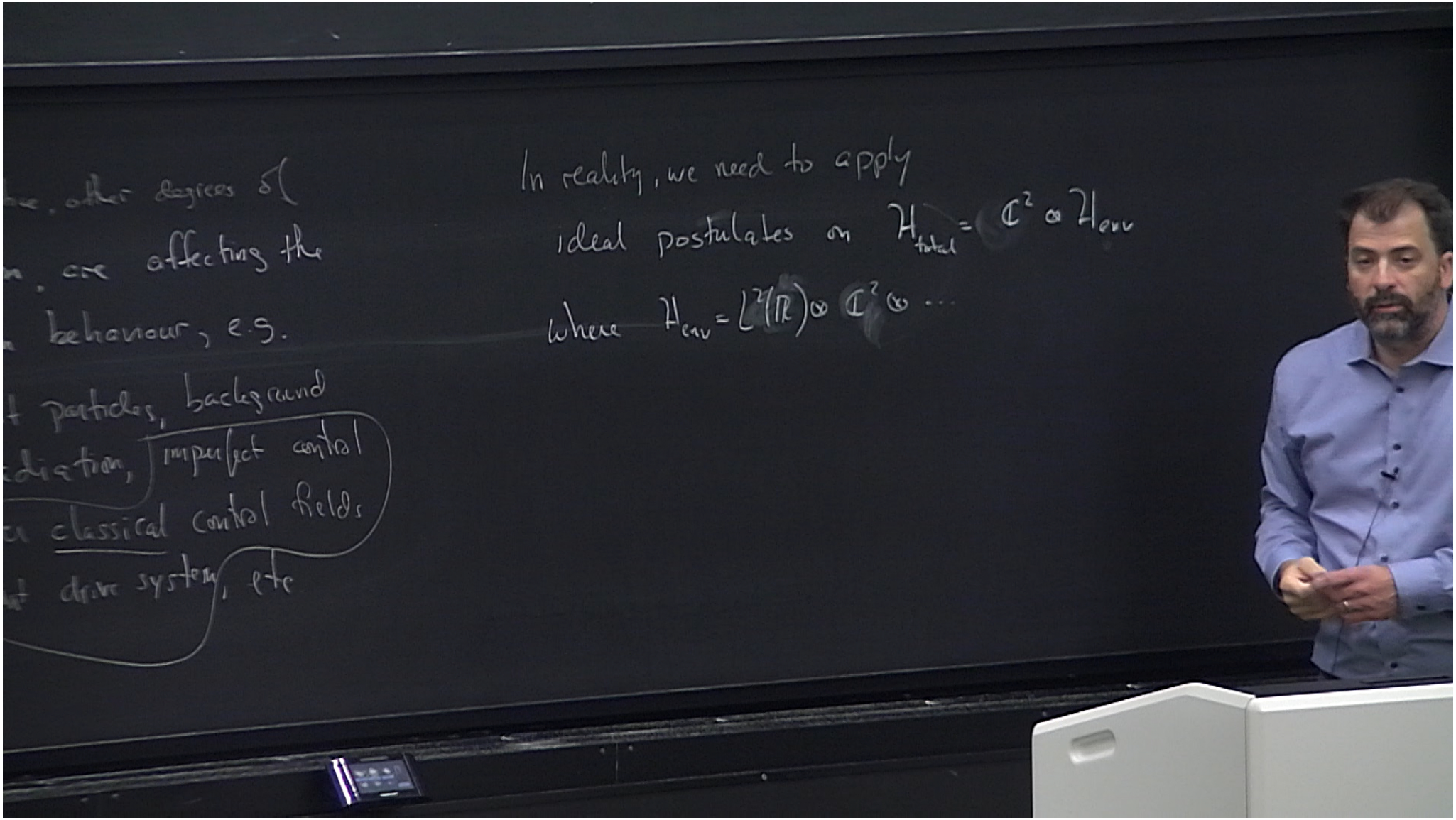

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Position operator $\hat{Q}$
$\hat{Q}\,\psi($ $\psi(x)$
$|\psi\rangle \in \mathcal{H}$
Defn
family
$(x) = \Theta(a-x)\,\psi(x)$
$\hat{Q}\,\psi(x) = \left(\int a\, dQ(a)\right)\psi(x)$
$= \lim_{n\to\infty} \sum_{k=1}^{n} a_k \left[Q(a_k) - Q(a_{k-1})\right]\psi(x)$
using $\int \lambda\, dE_x = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k \left[E_k - E_{x_{k-1}}\right]$
$= x\,\psi(x)$


Operational Postulates
for Open (real-world)
Q. Systems


Operational Postulates
for Open (real-world)
Q. Systems
Example: You are studying/testing
experimentally a spin 1/2 degree
of freedom. For the ideal postulates
you only need $\mathcal{H} = \mathbb{C}^2$


Operational Postulates
for Open (real-world)
Q. Systems
Example: You are studying
experimentally a
of freedom.
You only
In practice, other degrees of
freedom, are affecting the
system behaviour, e.g.
dust particles, background
radiation, imperfect control
over classical control fields
that drive system, etc


In reality, we need to apply
ideal post
on $\mathcal{H}_{total} = \mathbb{C}^2 \otimes \mathcal{H}_{env}$
where $\mathcal{H}_{env} =$
$\otimes \mathbb{C}$


other degrees of
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In reality, we need to apply
ideal postulates on $\mathcal{H}_{total} = \mathbb{C}^2 \otimes \mathcal{H}_{env}$
where $\mathcal{H}_{env} = L^2(\mathbb{R}) \otimes \mathbb{C}^2 \otimes \cdots$

$[A, A']$ is defined by:

Idea of Open Q Systems

⇒ Generalized structures

for Prep$\hat{}$s, Transf$\hat{}$s

& Measurements

that can be applied

on the system Hilbert space.

eg $\mathcal{H} = \mathbb{C}^2$



but not self-adj

The structures we get



Density operator

$\hat{\rho} \geq 0$

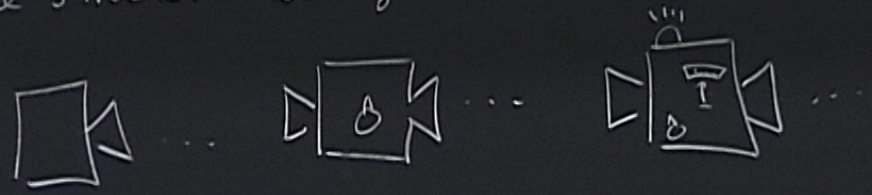
$\hat{\rho} = \hat{\rho}^\dagger$

$Tr(\hat{\rho}) = 1$

Completely Positive maps

but not self-adj

The structures we get



Density operator

$\hat{\rho} \geq 0$

$\hat{\rho} = \hat{\rho}^\dagger$

$Tr(\hat{\rho}) = 1$

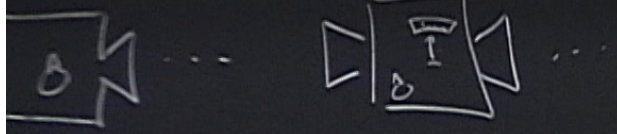
Completely Positive maps

"CP maps"

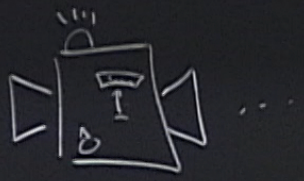
POVMs

Positive-operator valued Measures

set

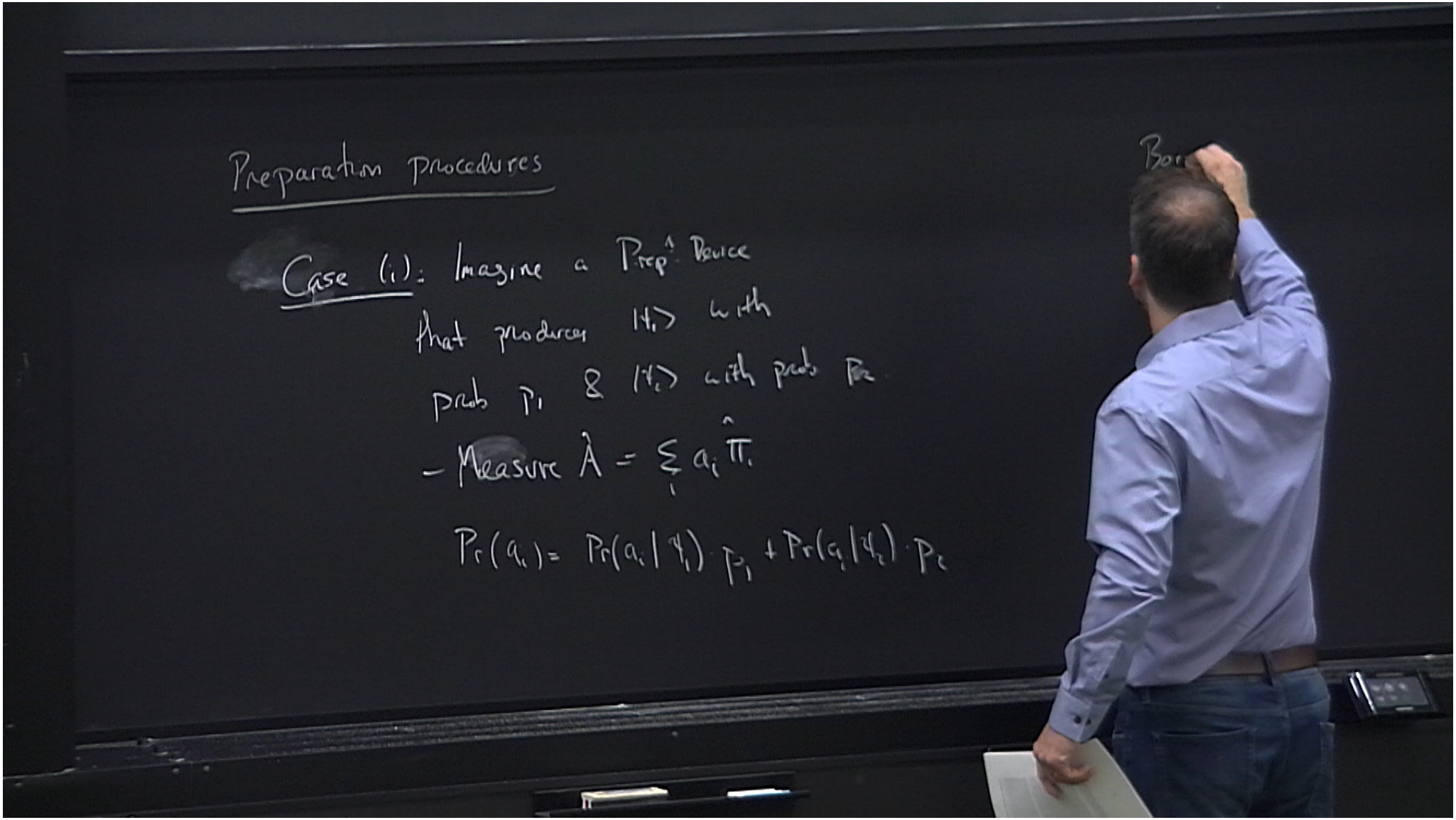


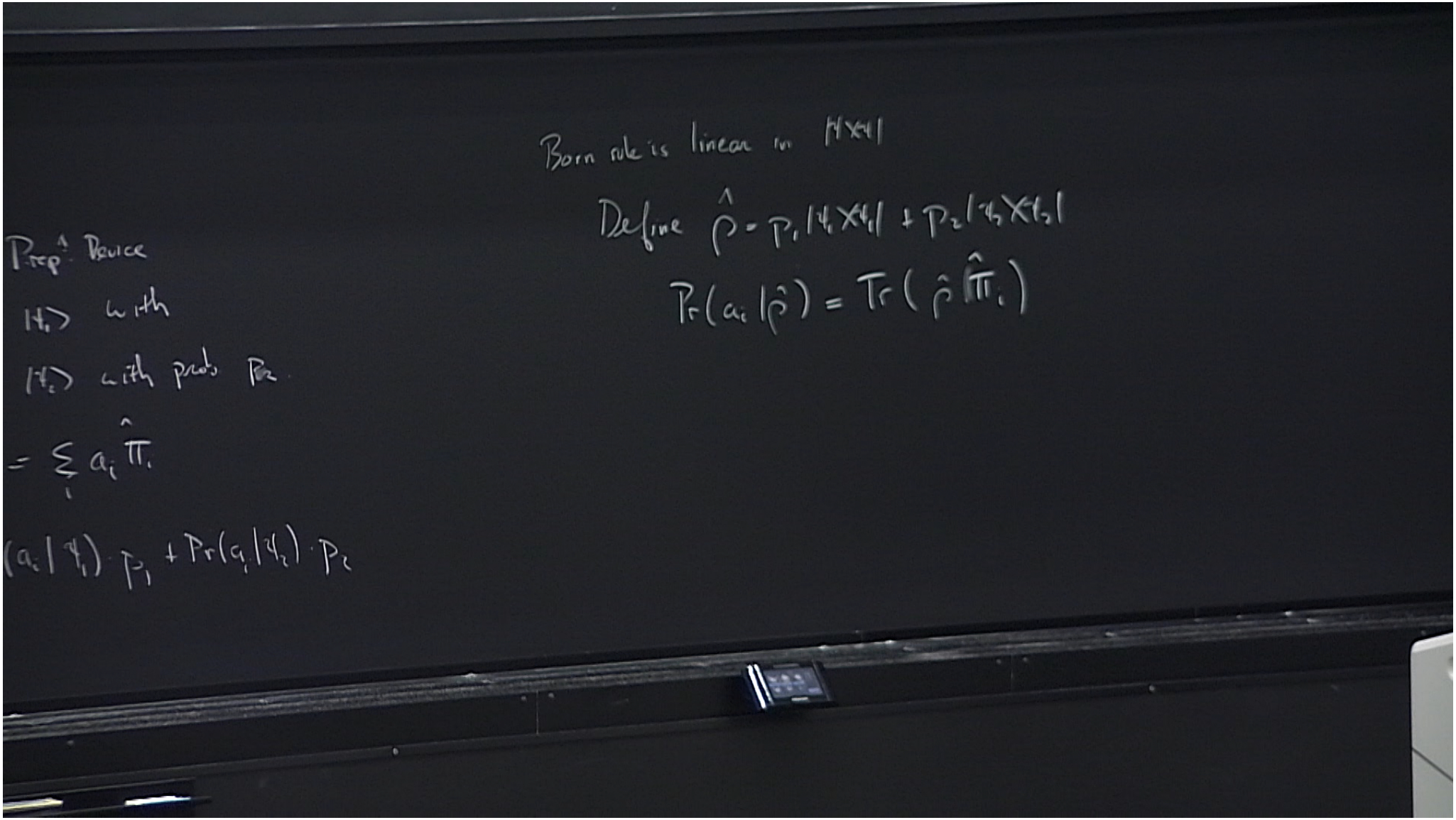
Completely Positive Maps "CP maps"

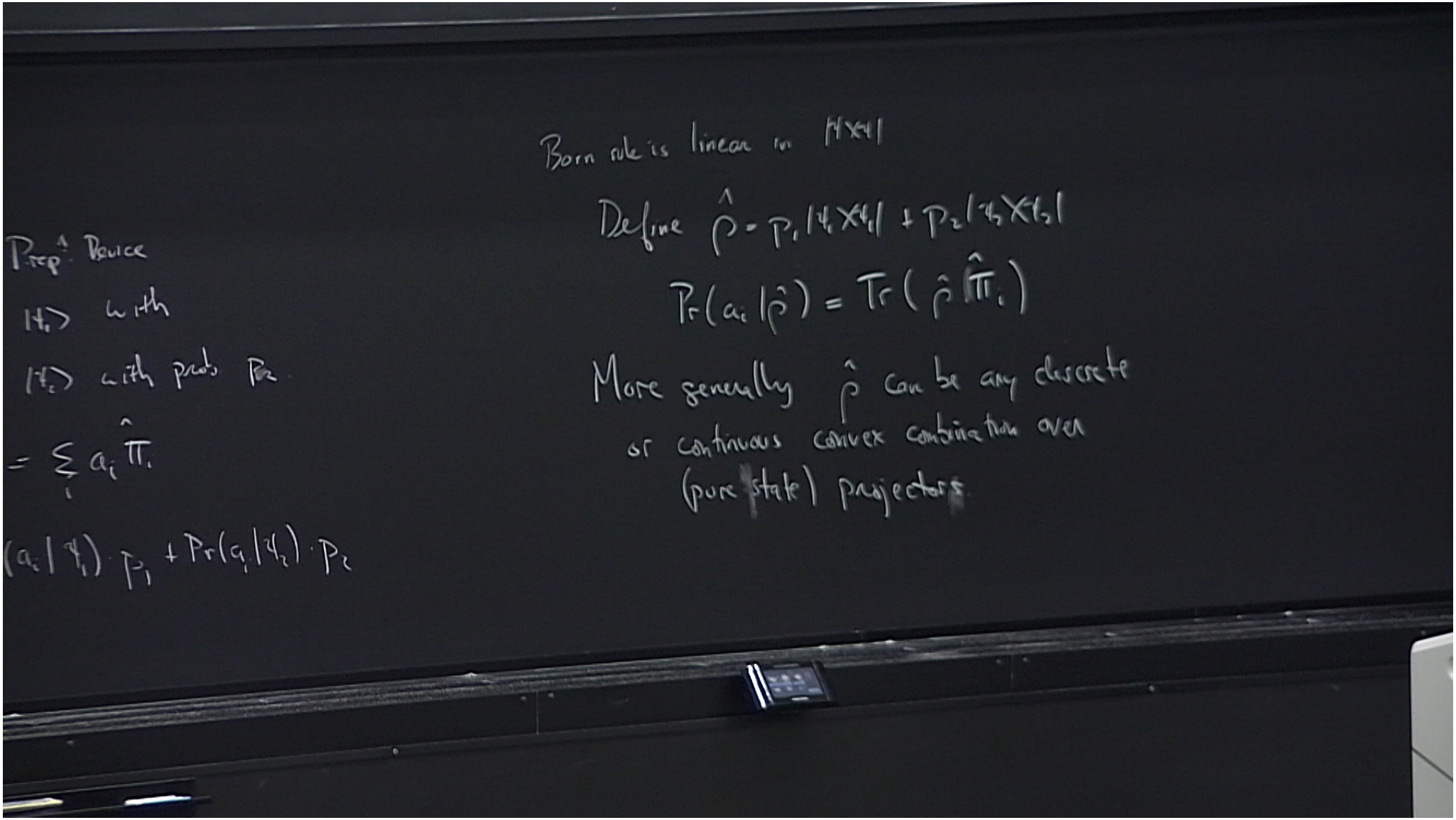


POVMs Positive-Operator Valued Measures

- Questionable speculation by most current researchers that these open system structures give a fundamental set of postulates for Q. theory.


Preparation procedures
Case (i): Imagine a Prep Device
that produces |ψ1⟩ with
prob p1 & |ψ2⟩ with prob p2
- Measure Â = Σ_i a_i Π̂_i
Pr(a_i) = Pr(a_i|ψ1) · p1 + Pr(a_i|ψ2) · p2


Born rule is linear in |ψ><ψ|
Define ρ̂ = p1 |ψ1><ψ1| + p2 |ψ2><ψ2|
Pr(ai|ρ̂) = Tr(ρ̂ Π̂i)
Prep: Source
|ψ1> with
|ψ2> with prob p2
= Σi ai Π̂i
(ai|ψ1) · p1 + Pr(ai|ψ2) · p2


Born rule is linear in |ψ><ψ|
Define ρ̂ = P1 |ψ1><ψ1| + P2 |ψ2><ψ2|
Pr(ai | ρ̂) = Tr(ρ̂ Π̂i)
More generally ρ̂ can be any discrete or continuous convex combination over (pure state) projectors
Prep: Source
|ψ1> with
|ψ2> with prob P2
= Σi ai Π̂i
(ai | ψ1) · P1 + Pr(ai | ψ2) · P2

Preparation procedures

Case (i): Imagine a $\text{Prep}^{\hat{}}$ device that produces $|\psi_i\rangle$ with prob p_i & $|\psi_2\rangle$ with prob p_2

- Measure $\hat{A} = \sum_i a_i \hat{\pi}_i$

$$\Pr(a_i) = \Pr(a_i|\psi_1)\cdot p_1 + \Pr(a_i|\psi_2)\cdot p_2$$

Born rule is linear in $|\psi\rangle\langle\psi|$

Define $\hat{\rho} = p_1 |\psi_1\rangle\langle\psi_1| \ldots$

$\Pr(a_i|\hat{\rho}) = T$

More generally $\hat{\rho}$ or continuous convex (pure state) pr

Born rule is linear in $|\psi\rangle\langle\psi|$

Define $\hat{\rho} = p_1 |\psi_1\rangle\langle\psi_1| + p_2 |\psi_2\rangle\langle\psi_2|$

$$\Pr(a_i|\hat{\rho}) = \mathrm{Tr}(\hat{\rho}\hat{\Pi}_i)$$

More generally $\hat{\rho}$ can be any discrete or continuous convex combination over (pure state) projectors.

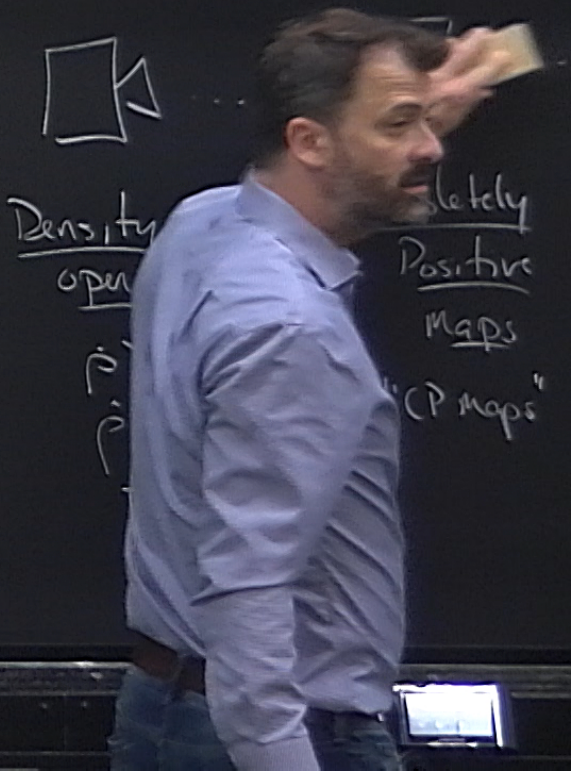
$[A, A']$ is defined by:

Case (i) $\mathcal{H}_{system} = \mathcal{H}_A$

$\mathcal{H}_{env.} = \mathcal{H}_B$

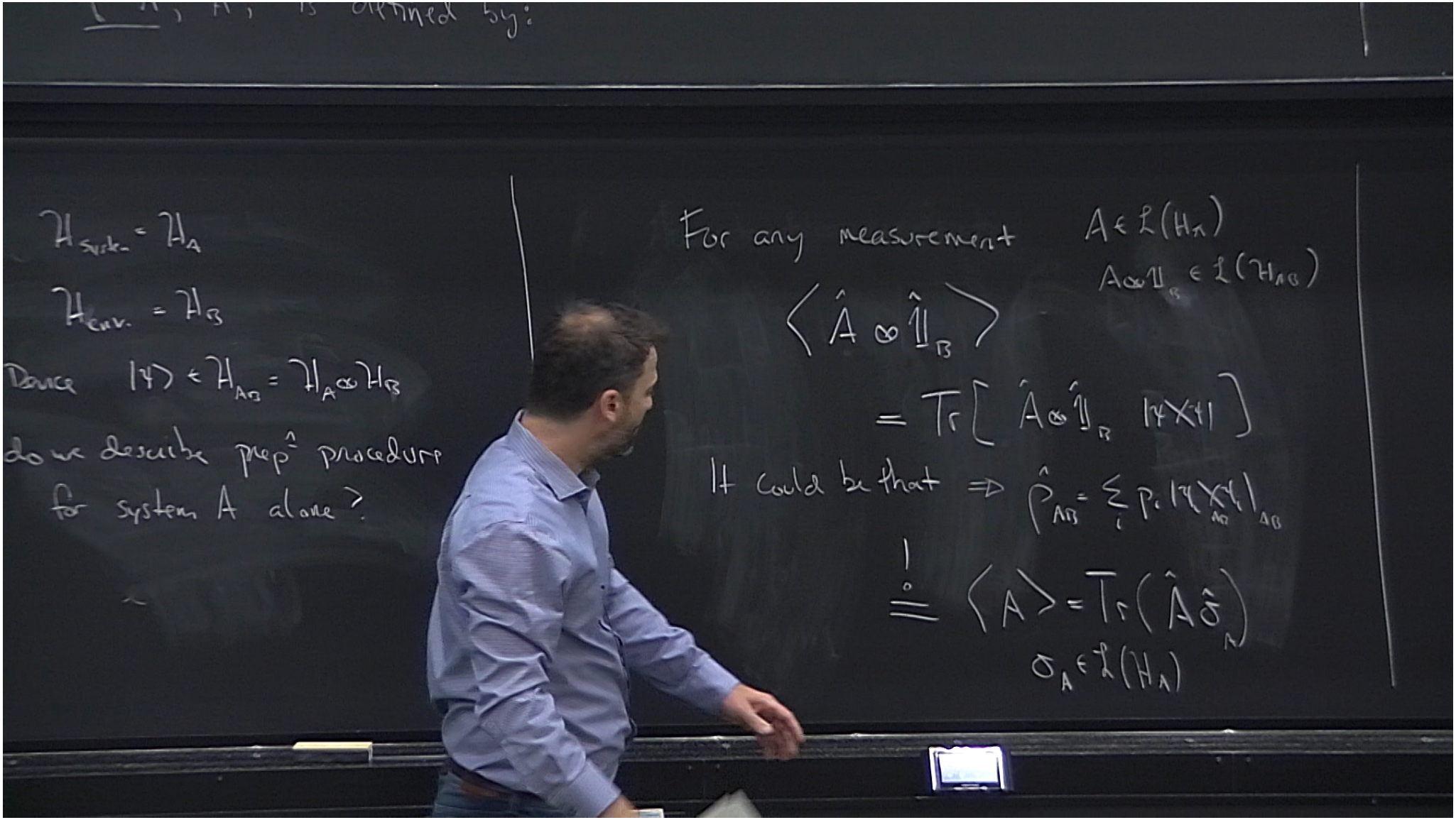
Prep $\hat{=}$ Device $|\psi\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$

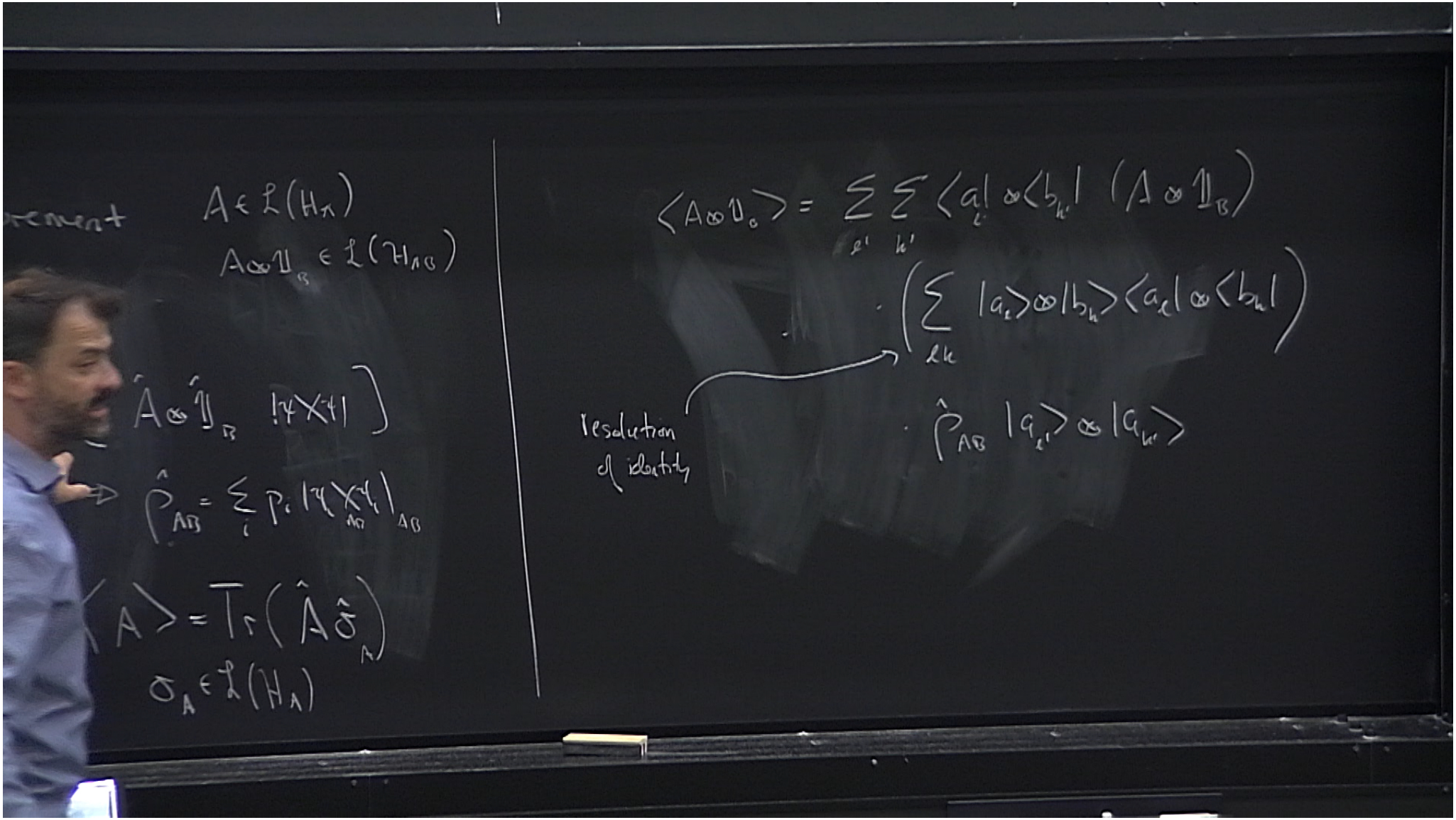
How do we describe prep $\hat{=}$ procedure for system A alone?

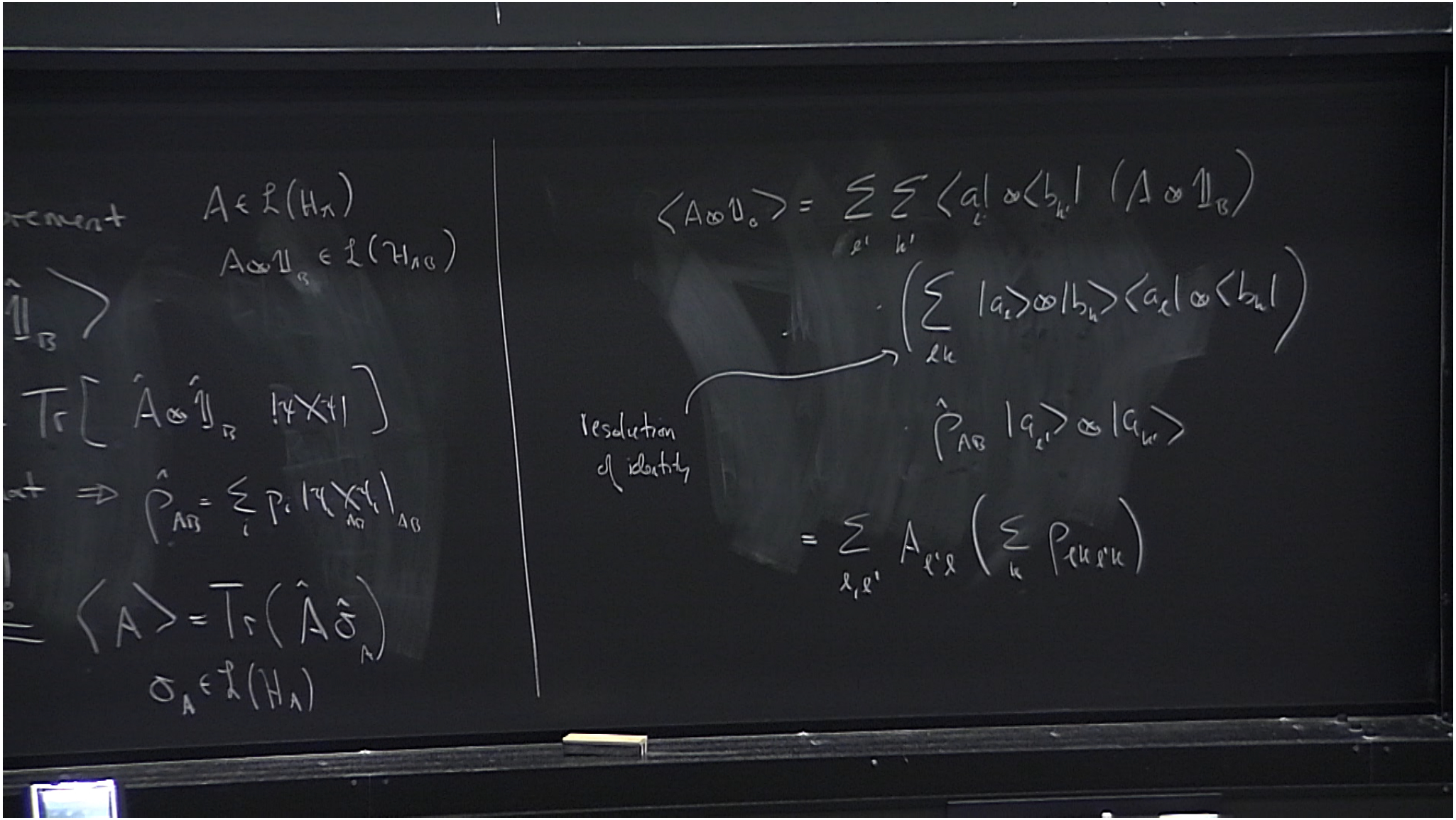


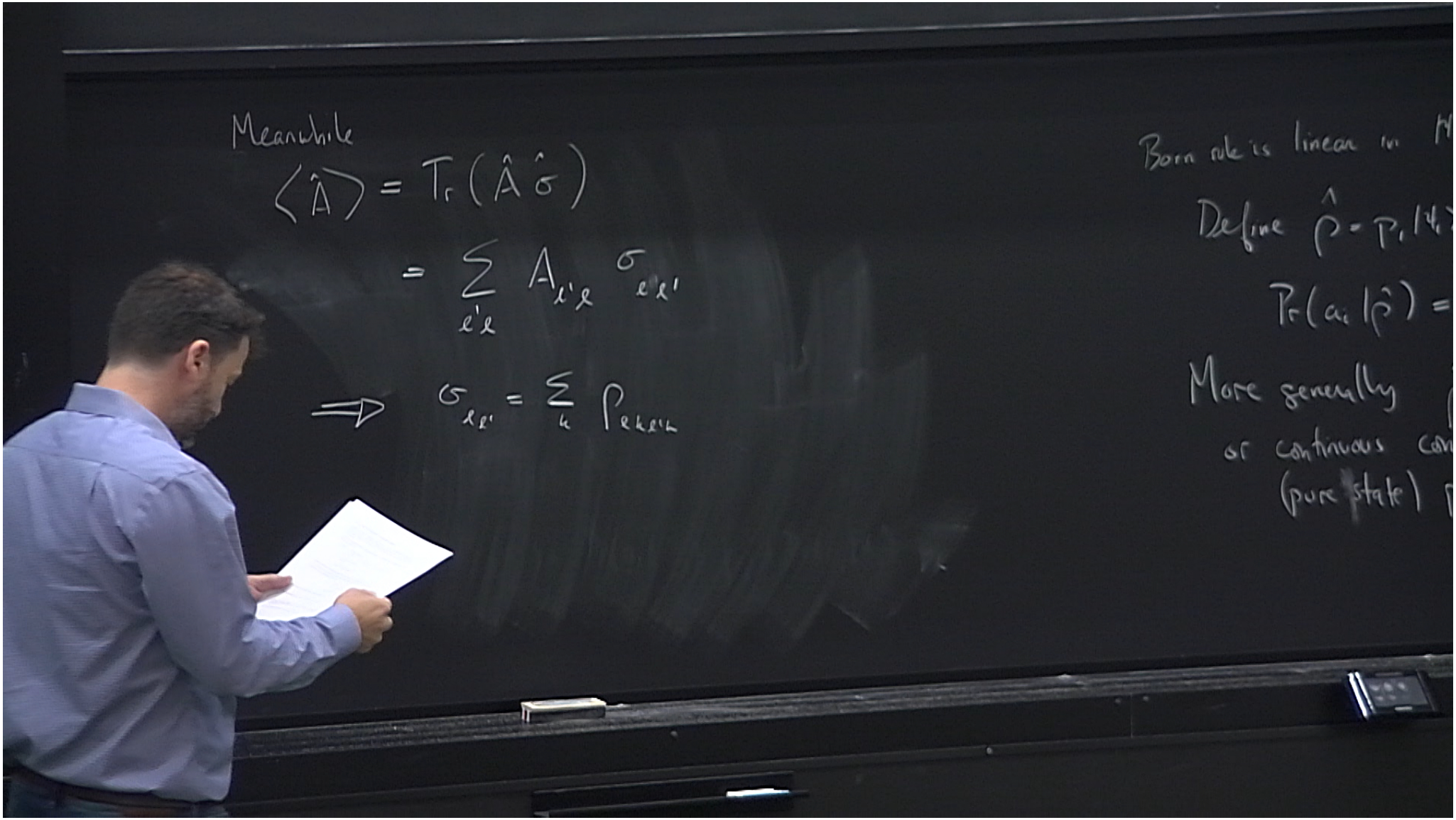
Density
op

letely
Positive
maps
"CP maps"


$\mathcal{H}_{Sys.} = \mathcal{H}_A$
$\mathcal{H}_{Env.} = \mathcal{H}_B$
Bonus $|\psi\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$
do we describe prep procedure for system A alone?
For any measurement $A \in \mathcal{L}(H_A)$
$A \otimes \mathbb{1}_B \in \mathcal{L}(\mathcal{H}_{AB})$
$\langle \hat{A} \otimes \hat{\mathbb{1}}_B \rangle$
$= Tr[\hat{A} \otimes \hat{\mathbb{1}}_B |\psi\rangle\langle\psi|]$
It could be that $\Rightarrow \hat{\rho}_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|_{AB}$
$\stackrel{!}{=} \langle A \rangle = Tr(\hat{A}\hat{\sigma}_A)$
$\sigma_A \in \mathcal{L}(H_A)$


$A \in \mathcal{L}(\mathcal{H}_A)$
$A \otimes \mathbb{1}_B \in \mathcal{L}(\mathcal{H}_{AB})$
$\langle A \otimes \mathbb{1}_B \rangle = \sum_{k'} \sum_{k'} \langle a_k | \otimes \langle b_{k'} | \left(A \otimes \mathbb{1}_B \right)$
$\left(\sum_{kk} |a_k\rangle \otimes |b_k\rangle \langle a_k| \otimes \langle b_k| \right)$
Resolution of identity
$\hat{\rho}_{AB} |a_{k'}\rangle \otimes |a_{k'}\rangle$
$\hat{A} \otimes \hat{\mathbb{1}}_B$
$\hat{\rho}_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|_{AB}$
$\langle A \rangle = \mathrm{Tr}\left(\hat{A} \hat{\sigma}_A \right)$
$\sigma_A \in \mathcal{L}(\mathcal{H}_A)$


$A \in \mathcal{L}(\mathcal{H}_A)$
$A \otimes \mathbb{1}_B \in \mathcal{L}(\mathcal{H}_{AB})$
$\langle A \otimes \mathbb{1}_B \rangle = \sum_{k,k'} \langle a_k| \otimes \langle b_{k'}| \, (A \otimes \mathbb{1}_B)$
$\cdot \left(\sum_{k,k} |a_k\rangle \otimes |b_k\rangle \langle a_k| \otimes \langle b_k| \right)$
resolution of identity
$\hat{\rho}_{AB} \, |a_{k'}\rangle \otimes |a_{k''}\rangle$
$= \sum_{k,k'} A_{k'k} \left(\sum_k \rho_{kk'k''} \right)$
$\langle A \rangle = \mathrm{Tr}\left(\hat{A} \hat{\sigma}_A \right)$
$\sigma_A \in \mathcal{L}(\mathcal{H}_A)$
$\mathrm{Tr}\left[\hat{A} \otimes \mathbb{1}_B \, |\psi\rangle\langle\psi| \right]$
$\Rightarrow \hat{\rho}_{AB} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|_{AB}$


Meanwhile
$\langle \hat{A} \rangle = \mathrm{Tr}(\hat{A}\hat{\sigma})$
$= \sum_{k'k} A_{k'k}\, \sigma_{kk'}$
$\Rightarrow \quad \sigma_{kk'} = \sum_{\mu} \rho_{\mu k k' \mu}$
Born rule is linear in
Define $\hat{\rho}$
$\mathrm{Tr}(a_i|\hat{\rho}) =$
More generally
or continuous
(pure state)

Meanwhile

$$\langle \hat{A} \rangle = \mathrm{Tr}(\hat{A}\hat{\sigma})$$

$$= \sum_{\ell' \ell} A_{\ell' \ell} \, \sigma_{\ell \ell'}$$

$$\Rightarrow \quad \sigma_{\ell \ell'} = \sum_{k} \rho_{\ell k \ell' k}$$

$\Rightarrow$ Matrix dimension: $\dim(\mathcal{H}_A) \times \dim(\mathcal{H}_A)$

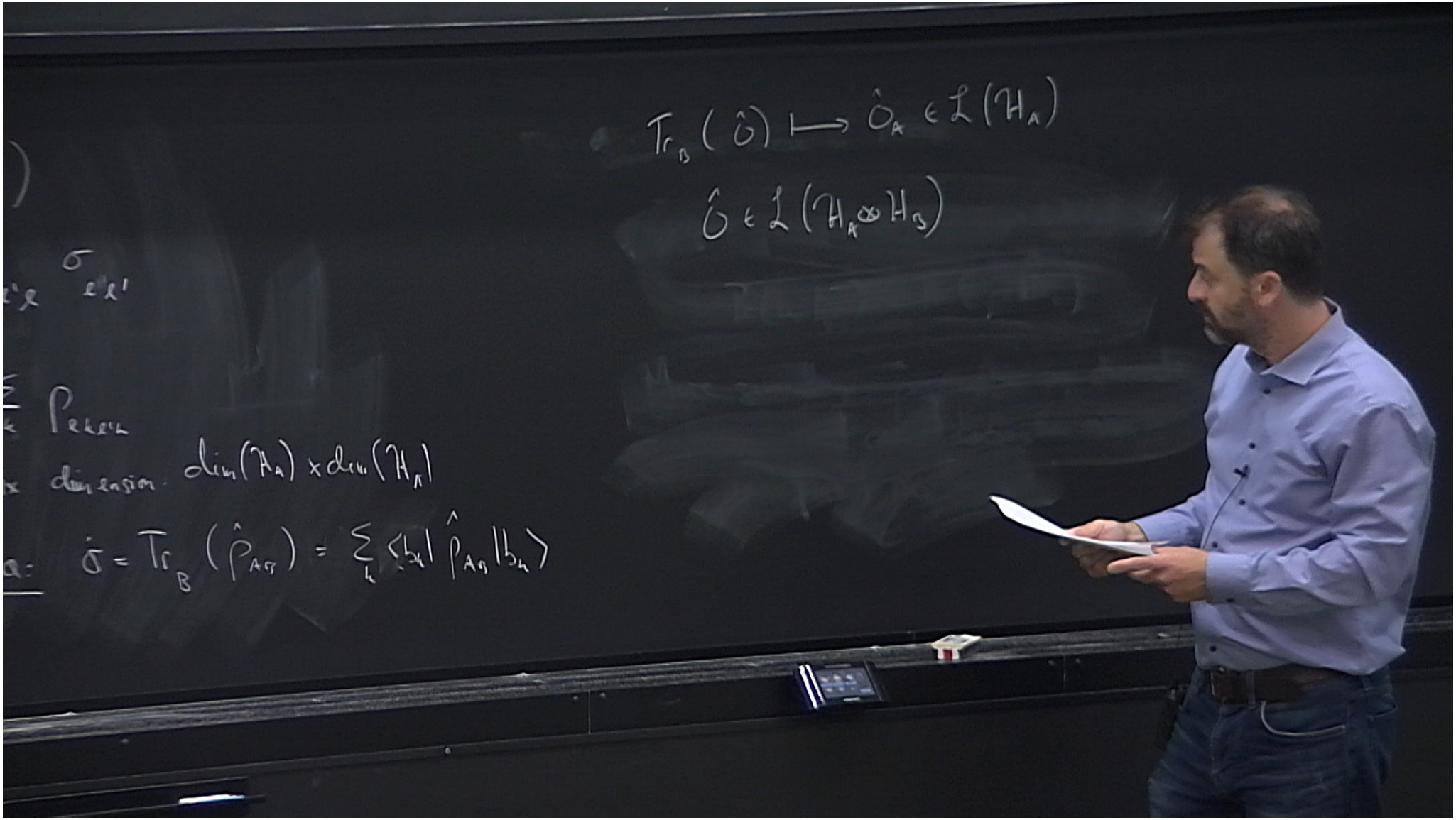
Partial trace: $\hat{\sigma} = \mathrm{Tr}_B(\hat{\rho}_{AB}) = \sum_k \langle b_k | \hat{\rho}_{AB} | b_k \rangle$

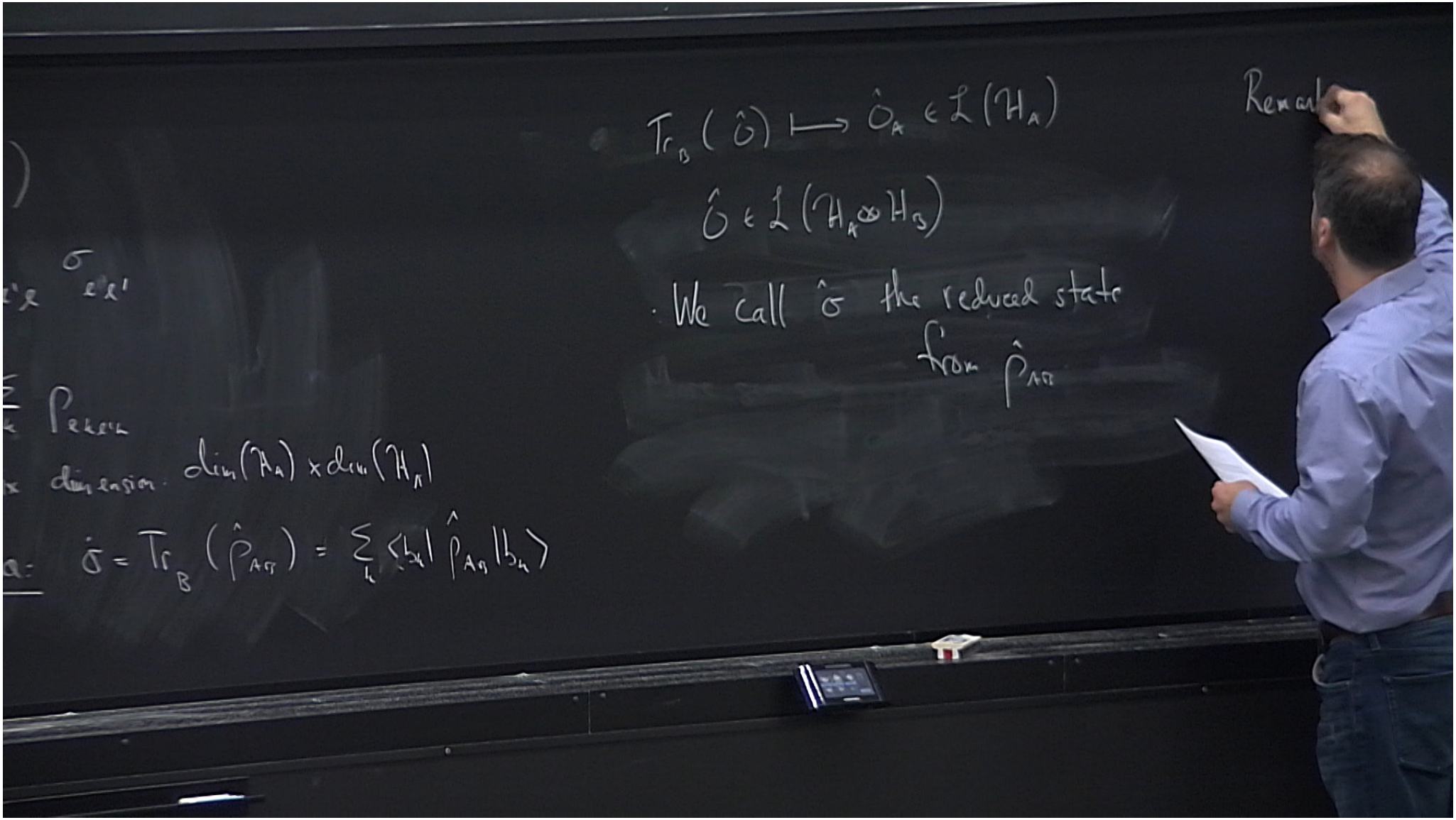
$\mathrm{Tr}(a_i | \hat{\rho}) =$

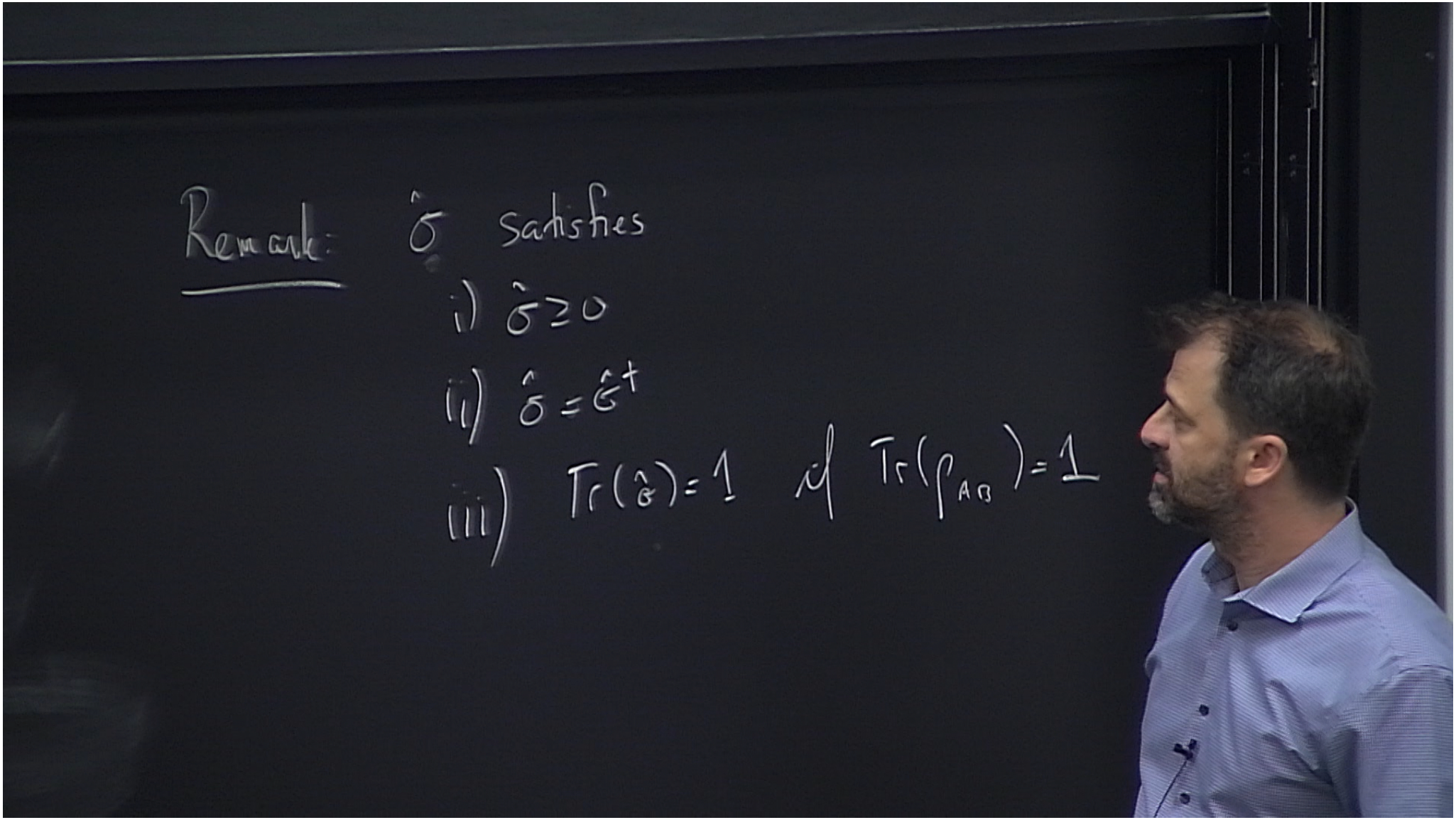
More generally

or continuous

(pure state)


$\mathrm{Tr}_B(\hat{O}) \longmapsto \hat{O}_A \in \mathcal{L}(\mathcal{H}_A)$
$\hat{O} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$
σ
ρ
dimension $\dim(\mathcal{H}_A) \times \dim(\mathcal{H}_B)$
$\hat{\sigma} = \mathrm{Tr}_B(\hat{\rho}_{AB}) = \sum_k \langle b_k | \hat{\rho}_{AB} | b_k \rangle$


$\mathrm{Tr}_B(\hat{\sigma}) \longmapsto \hat{\sigma}_A \in \mathcal{L}(\mathcal{H}_A)$
$\hat{\sigma} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$
· We call $\hat{\sigma}$ the reduced state
from $\hat{\rho}_{AB}$
Remark
$\hat{\rho}_{\text{reduit}}$
dimension $\dim(\mathcal{H}_A) \times \dim(\mathcal{H}_B)$
$\hat{\sigma} = \mathrm{Tr}_B(\hat{\rho}_{AB}) = \sum_k \langle b_k | \hat{\rho}_{AB} | b_k \rangle$


Remark: $\hat{\sigma}$ satisfies
i) $\hat{\sigma} \geq 0$
ii) $\hat{\sigma} = \hat{G}^{\dagger}$
iii) $Tr(\hat{\sigma}) = 1$ if $Tr(\rho_{AB}) = 1$

Remark: $\hat{\sigma}$ satisfies

i) $\hat{\sigma} \geq 0$

ii) $\hat{\sigma} = \hat{\sigma}^{\dagger}$

iii) $Tr(\hat{\sigma}) = 1$ if $Tr(\rho_{AB}) = 1$

Def^{n} of Density Operator

Operational Postulate 1

Any preparation procedure on a Hilbert space $\mathcal{H}$ can be described by a density operator $\hat{\rho} \in \mathcal{L}(\mathcal{H})$.

For any measurement $A \in \mathcal{L}(\mathcal{H}_A)$

$A \otimes \mathbb{1}_B \in \mathcal{L}(\mathcal{H}_{AB})$

$$\langle \hat{A} \otimes \hat{\mathbb{1}}_B \rangle$$

$$= \mathrm{Tr}\left[\hat{A} \otimes \hat{\mathbb{1}}_B \, |\psi\rangle\langle\psi| \right]$$

It could be that $\Rightarrow$ $\hat{\rho}_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|_{AB}$

$$\overset{!}{=} \langle A \rangle = \mathrm{Tr}\left(\hat{A} \hat{\sigma}_A \right)$$

$\sigma_A \in \mathcal{L}(\mathcal{H}_A)$

$$\langle A \otimes \mathbb{1}_B \rangle = \sum_{k'} \sum_{k} \langle a_{k'}| \otimes \langle b_{k'}|$$

$$\left(\sum_{kk} |$$

Resolution of identity

$$=$$