

Title: PSI 17/18 - Quantum Theory - Lecture 4

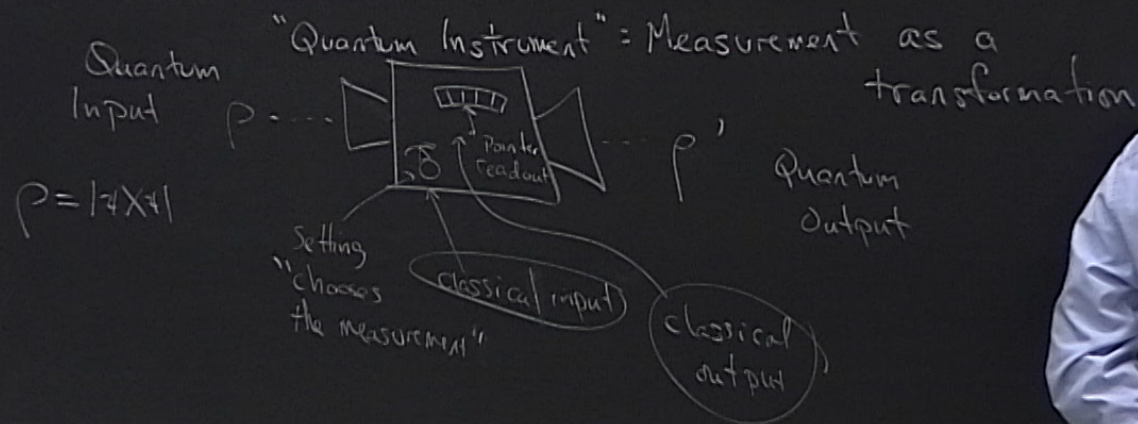
Date: Sep 11, 2017 09:00 AM

URL: <http://pirsa.org/17090037>

Abstract:

Sequential Measurements

& State-update Rule



Mathematical

Assume $\hat{A}$

Mathematical Rules

Assume $\hat{A} = \sum_m a_m \hat{\Pi}_m$

$a_m \in \text{real}$

$\hat{\Pi}_m$ are projectors

$$\hat{\Pi}_m \hat{\Pi}_{m'} = \delta_{mm'} \hat{\Pi}_m$$

& $\hat{\Pi}_m = \hat{\Pi}_m^\dagger$

Born rule (postulate 2)

$$P_r(m) = \text{Tr}[\hat{\Pi}_m \hat{\rho}]$$

In special case a_m is non-degenerate

then $\Pi_m = |\psi_m\rangle\langle\psi_m|$ a rank-one projector

$$\begin{aligned} \& \quad P_r(m) &= \text{Tr} [|\psi_m\rangle\langle\psi_m| \cdot |\psi\rangle\langle\psi|] \\ &= |\langle\psi_m|\psi\rangle|^2 \end{aligned}$$

Update

Conditional on outcome ' a_m '
being 'observed' ~ known

then Luders' Rule:

$$\rho \rightarrow \rho' = \frac{\hat{\pi}_m \hat{\rho} \hat{\pi}_m}{\text{Tr}(\hat{\pi}_m \hat{\rho})}$$

Born rule (postulate 2)

$$P_r(m) = \text{Tr} [\hat{\Pi}_m \hat{\rho}]$$

In special case a_m is non-degenerate

then $\Pi_m = |\psi_m\rangle\langle\psi_m|$ a rank-one projector

alternate notation

$$\hat{A}|\psi_m\rangle = a_m|\psi_m\rangle$$
$$\hat{A}|a_m\rangle = a_m|a_m\rangle$$

$$P_r(m) = \text{Tr} [|\psi_m\rangle\langle\psi_m| |\psi\rangle\langle\psi|]$$
$$= |\langle\psi_m|\psi\rangle|^2$$

Remark:

In special case

$$\text{that } \Pi_m = |\psi_m\rangle\langle\psi_m|$$

Recall, we can always
expand any state
wrt an ON basis

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle$$

$$\text{or } |\psi\rangle = \sum_n c_n |a_n\rangle$$

$$c_n = \langle\psi_n|\psi\rangle = \langle a_n|\psi\rangle$$

Remark:

In ^{the} special case

$$\text{that } \Pi_m = |\psi_m\rangle\langle\psi_m| \\ \equiv |a_m\rangle\langle a_m|$$

$$\rho \rightarrow \rho' = |a_m\rangle\langle a_m| = |\psi_m\rangle\langle\psi_m|$$

Von Neumann (1930's)

Recall, we can always
expand any state
wrt an O.N. basis

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle$$

$$\text{or } |\psi\rangle = \sum_n c_n |a_n\rangle$$

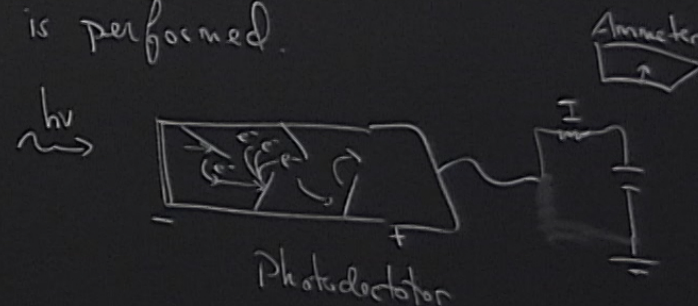
$$c_n = \langle\psi_n|\psi\rangle = \langle a_n|\psi\rangle$$

The usual textbook statement
is von Neumann rule

"after measurement the system
is left in the eigenstate of
the observable associated with
observed outcome"

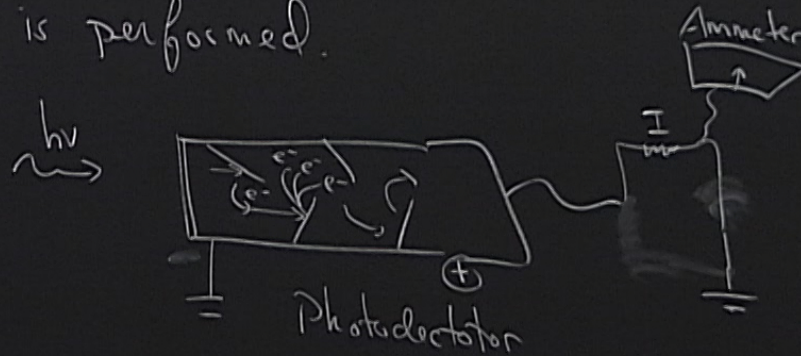
This is not always true,
it depends on how
measurement is performed.

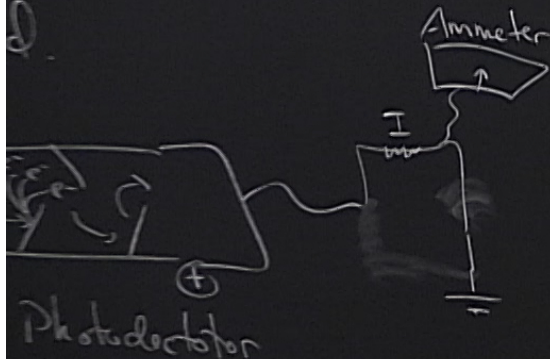
Example:



This is not always true,
it depends on how
measurement is performed.

Example:





First Non-rank 1 projectors

Example $H = \mathbb{C}^3$, $\pi_{\perp} = |a_1\rangle\langle a_1| + |a_2\rangle\langle a_2|$

$$\hat{A} = a_1 |a_1\rangle\langle a_1| + a_{\perp} (|a_2\rangle\langle a_2| + |a_3\rangle\langle a_3|)$$

$| \psi \rangle = c_1 |a_1\rangle + c_2 |a_2\rangle + c_3 |a_3\rangle$ 2-dim^e subspace of H

Suppose outcome $a_{\perp}$ is obtained

$$\rho = |\psi\rangle\langle\psi| \rightarrow \rho' = \frac{\hat{\pi}_{\perp} \rho \hat{\pi}_{\perp}}{\text{Tr}(\hat{\pi}_{\perp} \rho)}$$

Under outcome a_1

$$\rho' = |c_2|^2 |a_2\rangle\langle a_2| + |c_3|^2 |a_3\rangle\langle a_3| \\ + c_2 \bar{c}_3 |a_2\rangle\langle a_3| \\ + \bar{c}_2 c_3 |a_3\rangle\langle a_2|$$

Coherence in this subspace
is preserved.

Without post-select

Under outcome a_1

$$\tilde{\rho}' = |c_2|^2 |a_2\rangle\langle a_2| + |c_3|^2 |a_3\rangle\langle a_3| \\ + c_2 \bar{c}_3 |a_2\rangle\langle a_3| \\ + \bar{c}_2 c_3 |a_3\rangle\langle a_2|$$

↑
Add normalization

$$\rho' = \frac{\tilde{\rho}'}{\text{Tr}(\Pi_1 \rho)} \\ = \frac{\tilde{\rho}'}{|c_2|^2 + |c_3|^2}$$

Coherence in this subspace
is preserved.

Without post-select
i.e. w/o condition
on the observation
the "classical"
or "pointer"

$$\rho' = \sum_n$$

Without post-selection,
i.e. w/o conditioning
on the observation of
the "classical" output
viz "pointer position",

$$\rho \rightarrow \rho' = \frac{\sum_m \text{Pr}(m) \pi_m \rho \pi_m}{\text{Tr}(\hat{\pi}_m \hat{\rho})} = \sum_m \hat{\pi}_m \hat{\rho} \hat{\pi}_m$$
$$\text{Pr}(m) = \text{Tr}(\hat{\pi}_m \hat{\rho})$$

Without post-selection,
 i.e. w/o conditioning
 on the observation of
 the "classical" output
 i.e. "pointer position",

$$\rho \rightarrow \rho' = \sum_m \text{Pr}(m) \frac{\Pi_m \rho \Pi_m}{\text{Tr}(\hat{\Pi}_m \hat{\rho})} = \sum_m \hat{\Pi}_m \hat{\rho} \hat{\Pi}_m$$

$$\text{Pr}(m) = \text{Tr}(\hat{\Pi}_m \hat{\rho})$$

Then

$$\tilde{\rho}_{\text{sys}} = |c_1|^2 |a\rangle\langle a| + \tilde{\rho}'$$

Without post-selection,
 i.e. w/o conditioning
 on the observation of
 the "classical" output
 i.e. "pointer position",

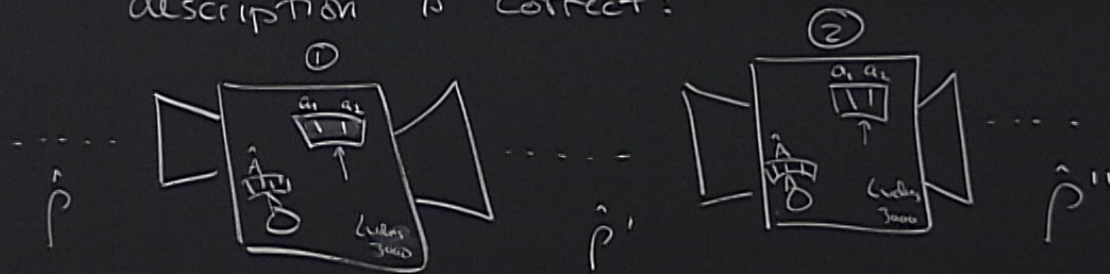
$$\rho \rightarrow \rho' = \sum_m \Pr(m) \left(\frac{\hat{\pi}_m \rho \hat{\pi}_m}{\text{Tr}(\hat{\pi}_m \rho)} \right) = \sum_m \hat{\pi}_m \rho \hat{\pi}_m$$

$$\Pr(m) = \text{Tr}(\hat{\pi}_m \rho)$$

Then

$$\rho'_{\text{NPS}} = |c_1|^2 |a_1\rangle\langle a_1| + \tilde{\rho}' \quad \leftarrow \text{automatically normalized}$$

How do we know this
description is correct?



Repeatability: If ① gives outcome a_1 (or a_2)
then ② must also give outcome a_1 (or a_2)

$$\rho = |\psi\rangle\langle\psi| \rightarrow \rho' = \frac{\Pi_{\perp} \rho \Pi_{\perp}}{\text{Tr}(\Pi_{\perp} \rho)}$$

$$P_r(a_{\perp} | a_{\parallel}) = \text{Tr} [|a_{\perp}\rangle\langle a_{\perp}| \rho']$$

$$\text{where } \rho' = |a_{\parallel}\rangle\langle a_{\parallel}|$$

$$\langle a_{\perp} | a_{\parallel} \rangle = 0$$

$$\rho = |4\rangle\langle 4| \rightarrow \rho' = \frac{\Pi_{\perp} \rho \Pi_{\perp}}{\text{Tr}(\Pi_{\perp} \rho)}$$

$$\text{Pr}(a_{\perp} | a_{\parallel}) = \text{Tr} [|a_{\perp}\rangle\langle a_{\perp}| \rho']$$

$$\text{where } \rho' = |a_{\parallel}\rangle\langle a_{\parallel}|$$

$$\langle a_{\perp} | a_{\parallel} \rangle = 0$$

How do we check that

$$A = a_{\parallel} \Pi_{\parallel} + a_{\perp} \Pi_{\perp}$$

$$\text{vs } \hat{A} = a_{\parallel} \Pi_{\parallel} + \underbrace{a_2 |q_2\rangle\langle q_2| + a_3 |q_3\rangle\langle q_3|}$$

$$\rho = |\psi\rangle\langle\psi| \rightarrow \rho' = \frac{\Pi_{\perp} \rho \Pi_{\perp}}{\text{Tr}(\Pi_{\perp} \rho)}$$

$$\text{Pr}(a_{\perp} | a_{\parallel}) = \text{Tr} [|a_{\perp}\rangle\langle a_{\perp}| \rho']$$

$$\text{where } \rho' = |q_{\parallel}\rangle\langle q_{\parallel}|$$

$$\langle a_{\perp} | a_{\parallel} \rangle = 0$$

How do we check that

$$A = a_{\parallel} \Pi_{\parallel} + a_{\perp} \Pi_{\perp}$$

$$\text{vs } \hat{A} = a_{\parallel} \Pi_{\parallel} + \underbrace{a_2 |q_2\rangle\langle q_2| + a_3 |q_3\rangle\langle q_3|}_{a_{\perp}}$$

What is $\hat{\rho}'$

under measurement $\hat{A}$

(w/o post-selection viz
w/o conditioning)?

$$\begin{aligned} \hat{\rho}' = & |c_1|^2 |q_1\rangle\langle q_1| \\ & + |c_2|^2 |q_2\rangle\langle q_2| \\ & + |c_3|^2 |q_3\rangle\langle q_3| \end{aligned}$$

Proof:

$$\rho \rightarrow \rho' = \sum_m \Pr(m) \frac{\Pi_m \rho \Pi_m}{\Pr(m)}$$

$$\text{For } \hat{A} = a_1 \Pi_1 + a_2 \Pi_2 + a_3 \Pi_3$$

→ "fully"
non-degenerate
measurement

$$\Pi_j = |a_j\rangle\langle a_j| \quad \forall j \in \{1, 2, 3\}$$

$$\frac{\langle \pi | \rho | \pi \rangle}{\langle \pi | \rho | \pi \rangle + \langle \sigma | \rho | \sigma \rangle}$$

$$P_r(m) = \text{Tr}(\Pi_m \hat{\rho})$$

Proof:

$$\rho = 14 \times 4$$

$$14 \times = \sum_j c_j |a_j\rangle$$

$$\rho \rightarrow \rho' = \sum_m P_r(m) \left(\frac{\Pi_m \rho \Pi_m}{P_r(m)} \right)$$

$$\text{For } \hat{A} = a_1 \Pi_1 + a_2 \Pi_2 + a_3 \Pi_3$$

"fully"
→ non-degenerate
measurement

$$\Pi_j = |a_j\rangle\langle a_j| \quad \forall j \in \{1, 2, 3\}$$

$$\rho' = \sum_m |e_m\rangle\langle e_m| \left(\frac{|a_m\rangle\langle a_m| \rho |a_m\rangle\langle a_m|}{|e_m\rangle\langle e_m|} \right)$$

$$= |a_1\rangle\langle a_1| \rho |a_1\rangle\langle a_1|$$

$$+ |a_2\rangle\langle a_2| \rho |a_2\rangle\langle a_2|$$

$$+ |a_3\rangle\langle a_3| \rho |a_3\rangle\langle a_3|$$

$$\text{Tr}(\Pi_i \rho) = \frac{\tilde{\rho}_i}{|\tilde{\rho}_i|^2}$$

$$Pr(m) = \text{Tr}(\Pi_m \rho)$$

Proof:

$$\rho = 14 \times 14$$

$$14 \times = \sum_j c_j |a_j\rangle$$

$$\langle a_m | a_m \rangle = \delta_{m,m'}$$

$$\rho \rightarrow \rho' = \sum_m Pr(m) \left(\frac{\Pi_m \rho \Pi_m}{Pr(m)} \right)$$

$$\text{For } \hat{A} = a_1 \Pi_1 + a_2 \Pi_2 + a_3 \Pi_3$$

"fully"
→ non-degenerate
measurement

$$\Pi_j = |a_j\rangle \langle a_j| \quad \forall j \in \{1, 2, 3\}$$

$$\rho' = \sum_m |c_m|^2 \left(\frac{|a_m\rangle \langle a_m| \rho |a_m\rangle \langle a_m|}{|c_m|^2} \right)$$

$$= |a_1\rangle \langle a_1| \rho |a_1\rangle \langle a_1|$$

$$+ |a_2\rangle \langle a_2| \rho |a_2\rangle \langle a_2|$$

$$+ |a_3\rangle \langle a_3| \rho |a_3\rangle \langle a_3|$$

$$\frac{c_1 \rho |a_1\rangle\langle a_1|}{\text{left}}$$

$$|a_2\rangle\langle a_2|$$

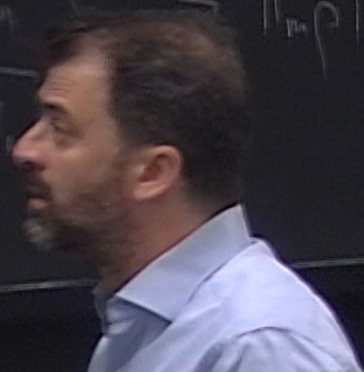
$$\rho |a_3\rangle\langle a_3|$$

$$= |c_1|^2 |a_1\rangle\langle a_1| + |c_2|^2 |a_2\rangle\langle a_2| + |c_3|^2 |a_3\rangle\langle a_3|.$$

$\Rightarrow$ No cross-terms b/c measurement
has destroyed all coherences (relative phases)
b/w eigenstates of $\hat{A}$.

Then $\rho'_{\text{sys}} = |c_1|^2 |a_1\rangle\langle a_1| + \tilde{\rho}' \leftarrow \text{automatically normalized}$

↑
coherences b/w
 $|a_2\rangle$ & $|a_3\rangle$ are
preserved



Sequential Measurements

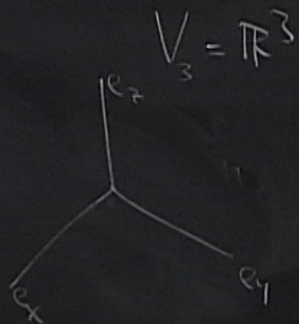
& State-update Rule

⇒ How to operationally/experimentally
verify if a physical device (q. instrument)
is described by $\hat{A}$ vs $\hat{A}'$?

Idea: Measure a non-commuting
observable in $H_{\perp}$ subspace.

$H = H_{\parallel} \oplus H_{\perp}$, $\oplus$ = adding orthogonal vector spaces

Simple example in $\mathbb{R}^3$ = Cartesian vector space



$$V_3 = \mathbb{R}^3$$

Plane x-y is a 2 dim^e

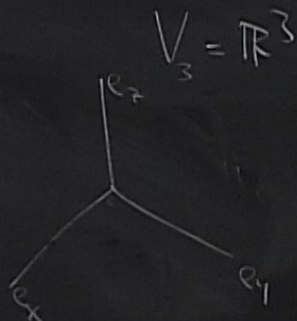
subspace V_{xy} of V_3

$$V_3 = V_z \oplus V_{xy}$$

Idea: Measure a non-commuting observable in $H_{\perp}$ subspace.

$H = H_{\parallel} \oplus H_{\perp}$, $\oplus$ = "tensor sum" = adding orthogonal vector spaces

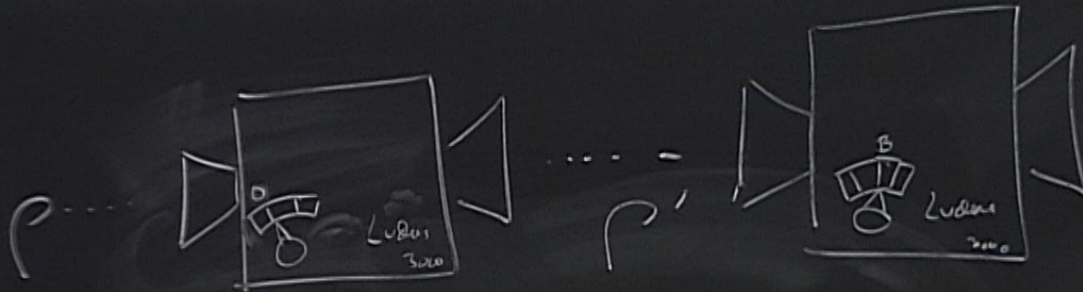
Simple example in $\mathbb{R}^3$ = Cartesian vector space



Plane $x-y$ is a 2 dim^e subspace V_{xy} of V_3

$$V_3 = V_z \oplus V_{xy}$$

$$= \frac{p}{|c_1|^2 + |c_2|^2}$$



$$[B, \tilde{A}] \neq 0$$

$$0 = A \text{ or } \tilde{A} ?$$

$$\text{Let } B = b_1 |a\rangle \langle a| + b_2 P_\perp$$

$$[P_\perp, \pi_\perp] \neq 0$$

$\tilde{A}:$

$$\Pi_{\perp} = |a_2\rangle\langle a_2| + |a_3\rangle\langle a_3|$$

$$|b_2\rangle = \frac{|a_2\rangle + |a_3\rangle}{\sqrt{2}}$$

$$|b_3\rangle = \frac{|a_2\rangle - |a_3\rangle}{\sqrt{2}}$$

A or $\tilde{A}$?

$$\langle b_2 | b_3 \rangle = 0$$

expand any
w.r.t an ON.

$$|\psi\rangle = \sum_n c_n$$

$$\text{or } |\psi\rangle = \sum_n c_n$$

$$c_n = \langle \psi_n | \psi \rangle =$$

Condition on outcome a_1

under measurement ①.

Either (i) $\rho' = \tilde{\rho}' = \rho'_1 + c_2 \bar{c}_3 |a_2\rangle\langle a_3| + \bar{c}_2 c_3 |a_3\rangle\langle a_2|$

(ii) $\rho' = \rho'_1 = |c_2|^2 |a_2\rangle\langle a_2| + |c_3|^2 |a_3\rangle\langle a_3|$

Born rule:

$$P_r(b_2 | \rho'_1) = \text{Tr} [|b_2\rangle\langle b_2| \rho'_1]$$

$\hookrightarrow$

$$P_r(b_2 | \tilde{\rho}'_1) = \text{Tr} [|b_2\rangle\langle b_2| \tilde{\rho}'_1]$$

Born rule:

$$\Pr(b_2 | \rho'_1) = \text{Tr} [|b_2\rangle\langle b_2| \rho'_1] \\ = |c_2|^2 |\langle b_2 | a_2 \rangle|^2 + |c_3|^2 |\langle b_2 | a_3 \rangle|^2$$

vs

$$\Pr(b_2 | \tilde{\rho}'_1) = \text{Tr} [|b_2\rangle\langle b_2| \tilde{\rho}'_1] = \Pr(b_2 | \rho'_1) \\ + c_2 \bar{c}_3 \langle b_2 | a_2 \rangle \langle a_3 | b_2 \rangle \\ + \bar{c}_2 c_3 \langle b_2 | a_3 \rangle \langle a_2 | b_2 \rangle \\ = \Pr(b_2 | \rho'_1) + \frac{2 \text{Re}(\bar{c}_2 c_3)}{2}$$

$\Rightarrow$ Different exp^t predictions
under subsequent measurement
if $\langle$ non-commuting observable