Title: PSI 17/18 - Quantum Theory - Lecture 3

Date: Sep 08, 2017 10:45 AM

URL: http://pirsa.org/17090036

Abstract:

Proof Sketch

For deriving Q vs Cl Mechanics

Some algebra X, P

Poisson algebra

with a Poisson bracket

$\{X, P\} = 1$

$A(x,p)$, $B(x,p)$

$\forall$ A & B

Poisson bracket

satisfies some 4 or 5

defining properties

- Leibniz Law
- Jacobi Identity
- Linearity
- Anti-symmetry

Physical Assumptions

$\rho(x, p, t)$, parameter "state"

X generates translations in momentum

P generates " " position

$H(x, p)$ generates translations in time.

$$\hookrightarrow \frac{\partial \rho}{\partial t} = \{H, \rho\}$$

I'm guessing

$$\rho(x, p, t) \to \rho(x-a, p, t)$$

$$\Rightarrow e^{ipa} \rho(x, p, t)$$

$$\to \rho(x+a, p, t)$$

I'm guessing

$$\rho(x,p,t) \to \rho(x+a,p,t)$$

$$\rightsquigarrow e^{ipa}\,\rho(x,p,t)$$

$$\to \rho(x+a,p,t)$$

Do I assume that x & p generate commutative transformations?

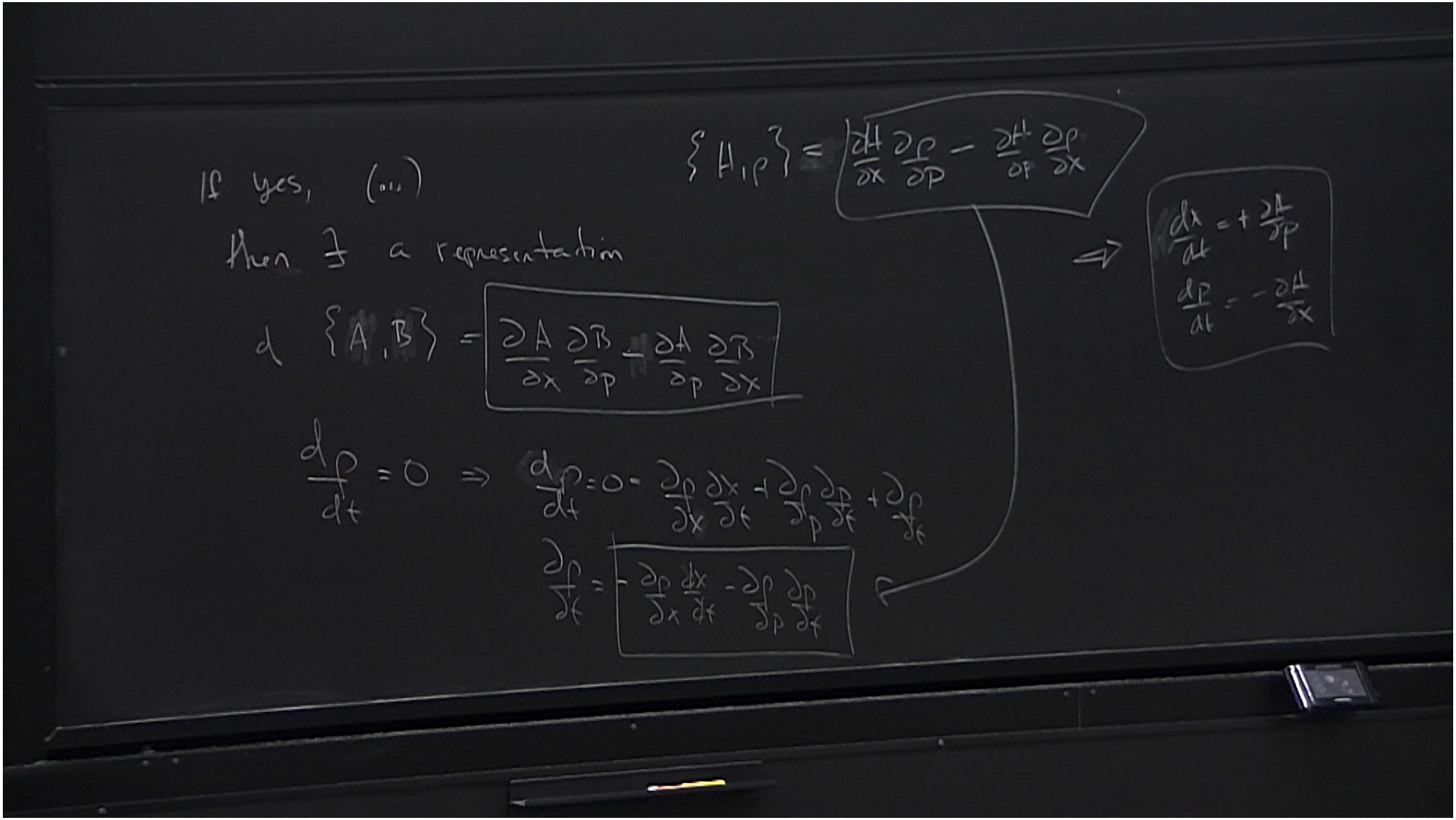
If yes, (...)

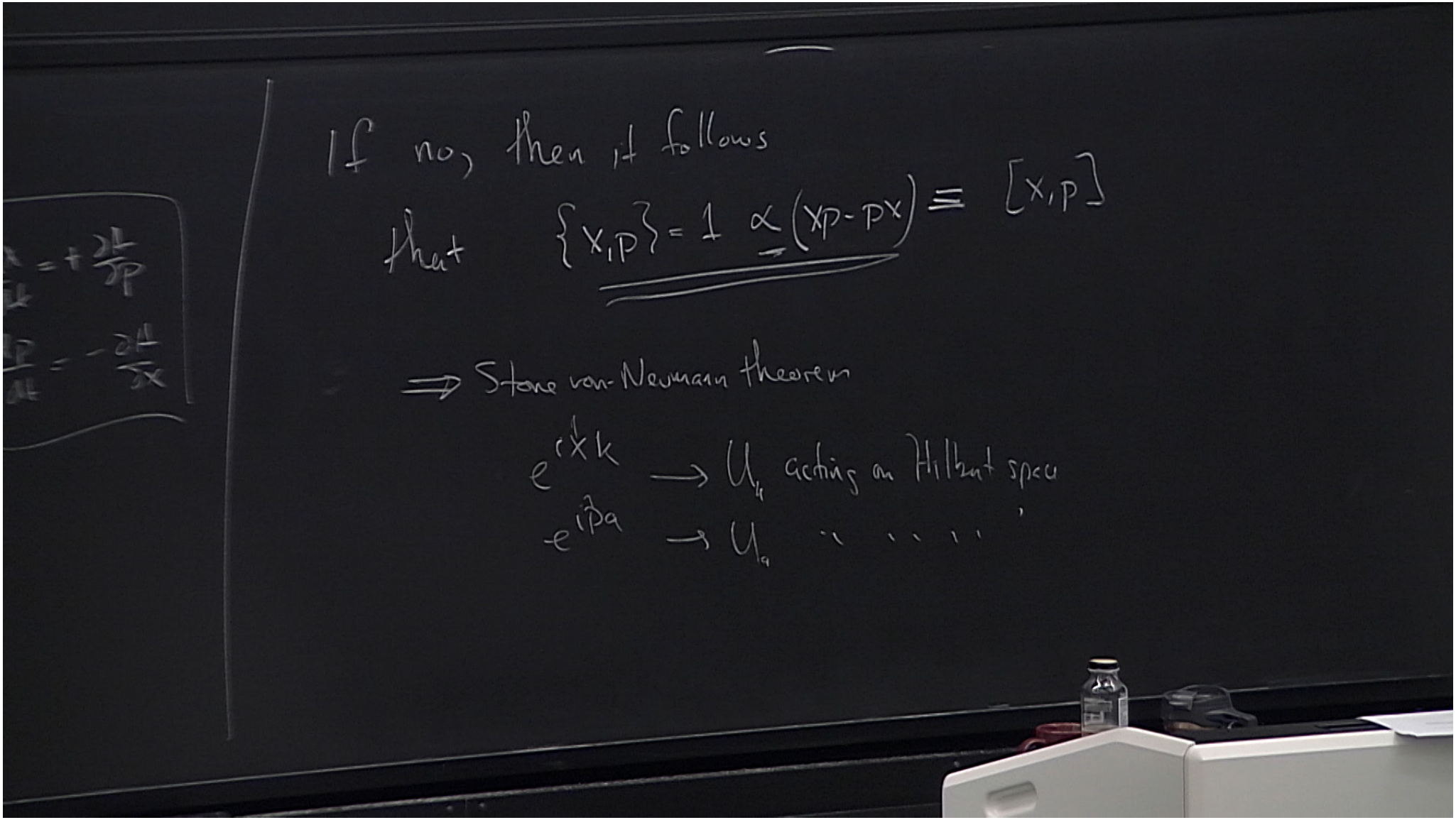
then $\exists$ a representation

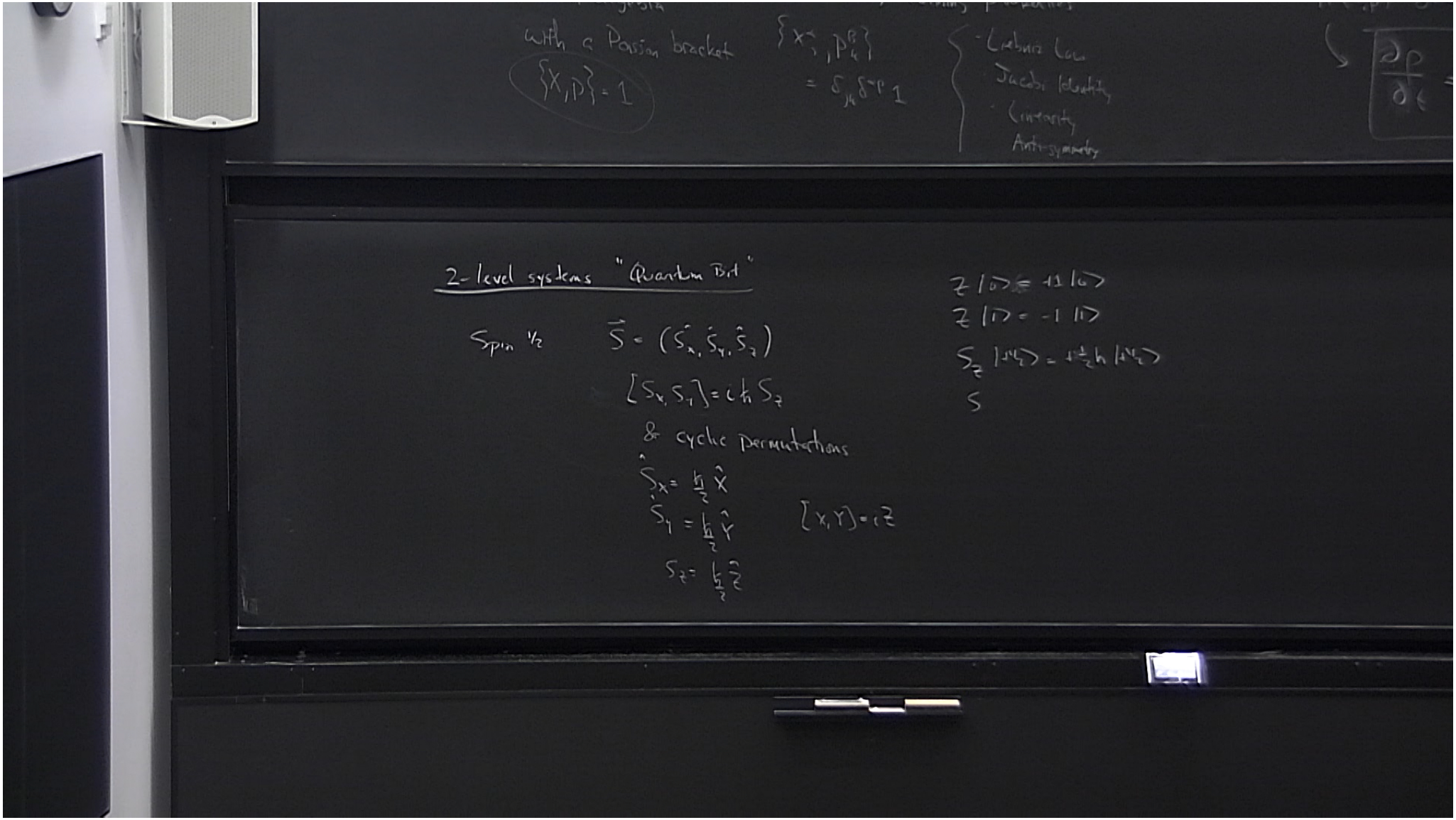
$$d\{A,B\} = \boxed{\frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}}$$

$$\frac{df}{dt} = 0 \Rightarrow \frac{df}{dt} = 0 = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial f}{\partial t}$$

$$\frac{\partial f}{\partial t} = \boxed{-\frac{\partial p}{\partial x}\frac{\partial x}{\partial t} - \frac{\partial f}{\partial p}\frac{\partial p}{\partial t}}$$


If yes, (...)
Then ∃ a representation
$\{H, p\} = \frac{\partial H}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial x}$
$\Leftrightarrow$
$\frac{dx}{dt} = +\frac{\partial H}{\partial p}$
$\frac{dp}{dt} = -\frac{\partial H}{\partial x}$
$d\{A, B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}$
$\frac{dp}{dt} = 0 \Rightarrow \frac{dp}{dt} = 0 = \frac{\partial p}{\partial x}\frac{dx}{dt} + \frac{\partial p}{\partial p}\frac{dp}{dt} + \frac{\partial p}{\partial t}$
$\frac{\partial f}{\partial t} = -\frac{\partial p}{\partial x}\frac{dx}{dt} - \frac{\partial p}{\partial p}\frac{dp}{dt}$


If no, then it follows
that $\{x,p\} = 1 \propto (xp - px) \equiv [x,p]$
$\Rightarrow$ Stone von-Neumann theorem
$e^{ixk} \rightarrow U_k$ acting on Hilbert space
$e^{ipa} \rightarrow U_a$ " " "


with a Poisson bracket
$\{x, p\} = 1$
$\{x^\alpha, p^\beta_\mu\} = \delta_\mu \delta^\alpha 1$
- Leibniz law
Jacobi identity
- Linearity
Anti-symmetry
$\frac{\partial f}{\partial t}$
2-level systems "Quantum Bit"
Spin 1/2
$\vec{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$
$[S_x, S_y] = i\hbar S_z$
& cyclic permutations
$\hat{S}_x = \frac{\hbar}{2}\hat{X}$
$S_y = \frac{\hbar}{2}\hat{Y}$
$S_z = \frac{\hbar}{2}\hat{Z}$
$[X, Y] = iZ$
$Z|\uparrow\rangle = +1|\uparrow\rangle$
$Z|\downarrow\rangle = -1|\downarrow\rangle$
$S_z|\uparrow\downarrow\rangle = \pm\frac{1}{2}\hbar|\uparrow\downarrow\rangle$
S

-level systems "Quantum Bit"

Spin 1/2 $\quad \vec{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$

$[S_x, S_y] = i\hbar S_z$

& cyclic permutations

$\hat{S}_x = \frac{\hbar}{2}\hat{X}$

$\hat{S}_y = \frac{\hbar}{2}\hat{Y}$ $\qquad [X, Y] = iZ$

$\hat{S}_z = \frac{\hbar}{2}\hat{Z}$

$Z|0\rangle = +1|0\rangle$

$Z|1\rangle = -1|1\rangle$

$S_z|+\tfrac{1}{2}\rangle = +\tfrac{1}{2}\hbar|+\tfrac{1}{2}\rangle$

$S_z|-\tfrac{1}{2}\rangle = -\tfrac{1}{2}\hbar|-\tfrac{1}{2}\rangle$

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in basis of S_z eigenstates

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

-level systems "Quantum Bit"

Spin 1/2 $\quad \vec{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$

$$[S_x, S_y] = i\hbar S_z$$

& cyclic permutations

$$\hat{S}_x = \frac{\hbar}{2}\hat{X} \qquad [X, Y] = iZ$$

$$\hat{S}_y = \frac{\hbar}{2}\hat{Y}$$

$$S_z = \frac{\hbar}{2}\hat{Z}$$

$$Z|0\rangle = +1|0\rangle$$

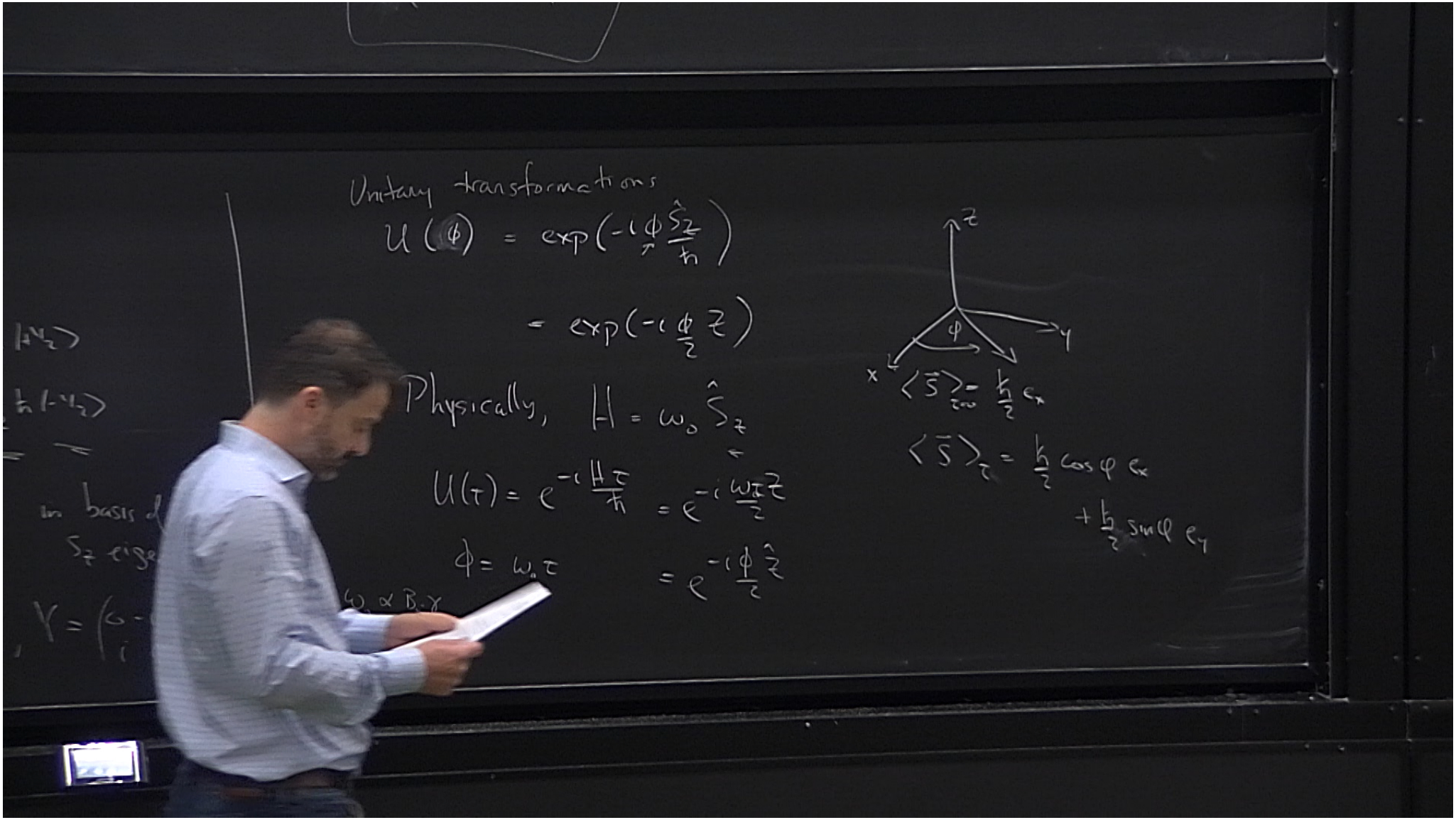
$$Z|1\rangle = -1|1\rangle$$

$$S_z|+\tfrac{1}{2}\rangle = +\tfrac{1}{2}\hbar|+\tfrac{1}{2}\rangle$$

$$S_z|-\tfrac{1}{2}\rangle = -\tfrac{1}{2}\hbar|-\tfrac{1}{2}\rangle$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ in basis of S_z eigenstates

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$


Unitary transformations
$U(\phi) = \exp\left(-i\frac{\phi}{\hbar}\hat{S}_z\right)$
$= \exp\left(-i\frac{\phi}{2}\hat{z}\right)$
Physically, $H = \omega_0 \hat{S}_z$
$U(\tau) = e^{-i\frac{H\tau}{\hbar}} = e^{-i\frac{\omega\tau}{2}\hat{z}}$
$\phi = \omega_0 \tau$ $= e^{-i\frac{\phi}{2}\hat{z}}$
$\omega_0 \propto B_0$
z
y
x
φ
$\langle \vec{S} \rangle_{z=0} = \frac{\hbar}{2} e_x$
$\langle \vec{S} \rangle_z = \frac{\hbar}{2}\cos\varphi\, e_x + \frac{\hbar}{2}\sin\varphi\, e_y$

$|+\rangle_z$

$\hbar|-\rangle_z$

in basis of S_z eigenstates

$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Unitary transformations

$$U(\phi) = \exp\left(-i\frac{\phi}{\hbar}\hat{S}_z\right)$$

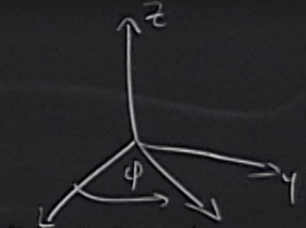
$$= \exp\left(-i\frac{\phi}{2}\hat{z}\right)$$

Physically, $H = \omega_0 \hat{S}_z$

$$U(\tau) = e^{-i\frac{H\tau}{\hbar}} = e^{-i\frac{\omega\tau}{2}\hat{z}}$$

$$\phi = \omega_0 \tau \qquad = e^{-i\frac{\phi}{2}\hat{z}}$$

$\omega_0 \propto B_0 \gamma$



$\langle \vec{S} \rangle_{z=0} = \frac{\hbar}{2} e_x$

$\langle \vec{S} \rangle_\tau = \frac{\hbar}{2}\cos\varphi\, e_x + \frac{\hbar}{2}\sin\varphi\, e_y$

Useful Identity

Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$

Quantum vs Classical Rotations

Any classical rotation $R \in O(3)$

Can be decomposed

$$R = R_z(\alpha) R_y(\beta) R_z(\gamma)$$

"Euler angles"

$SU(2) \sim O(3)$

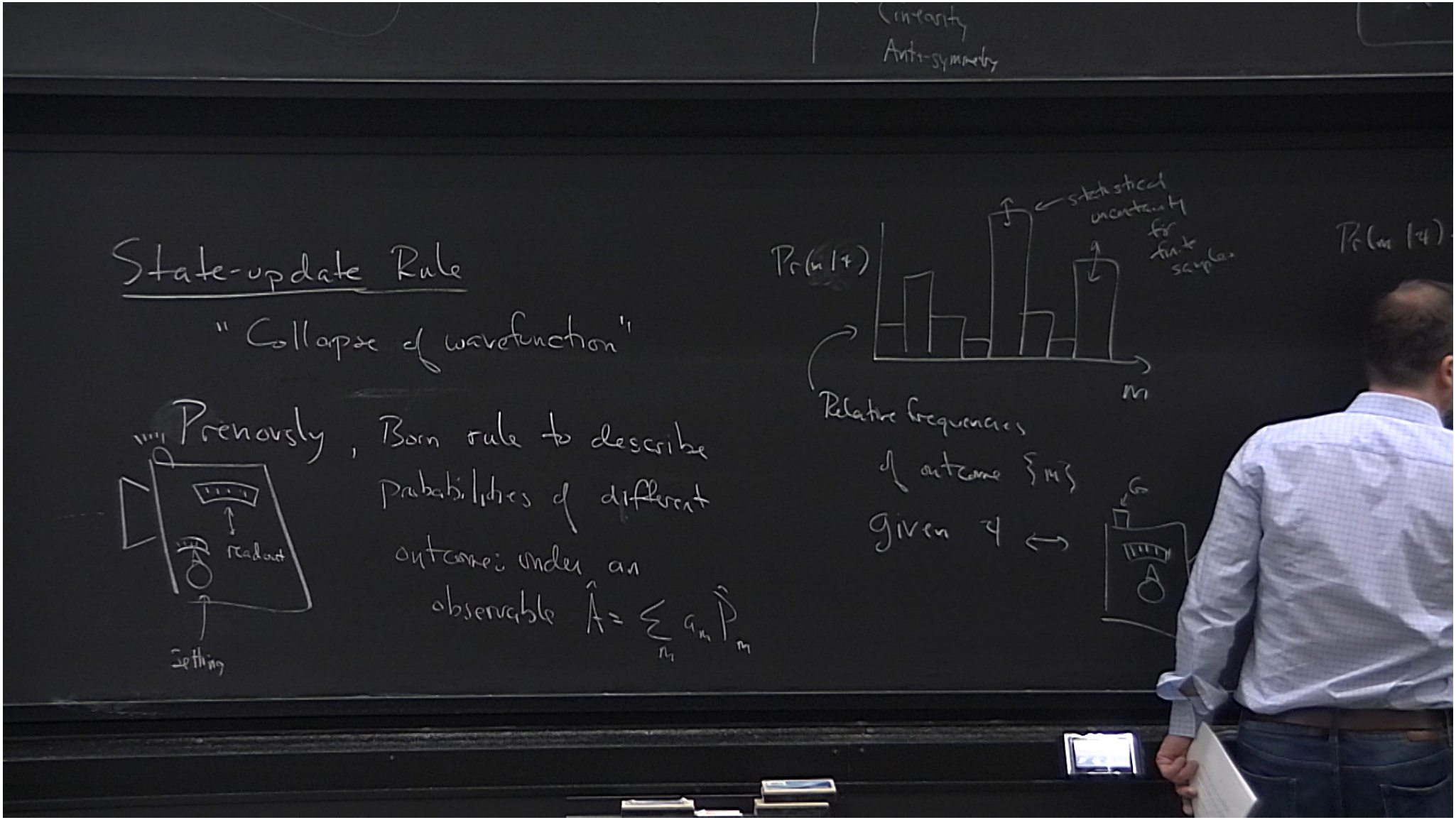
$U \in SU(2)$

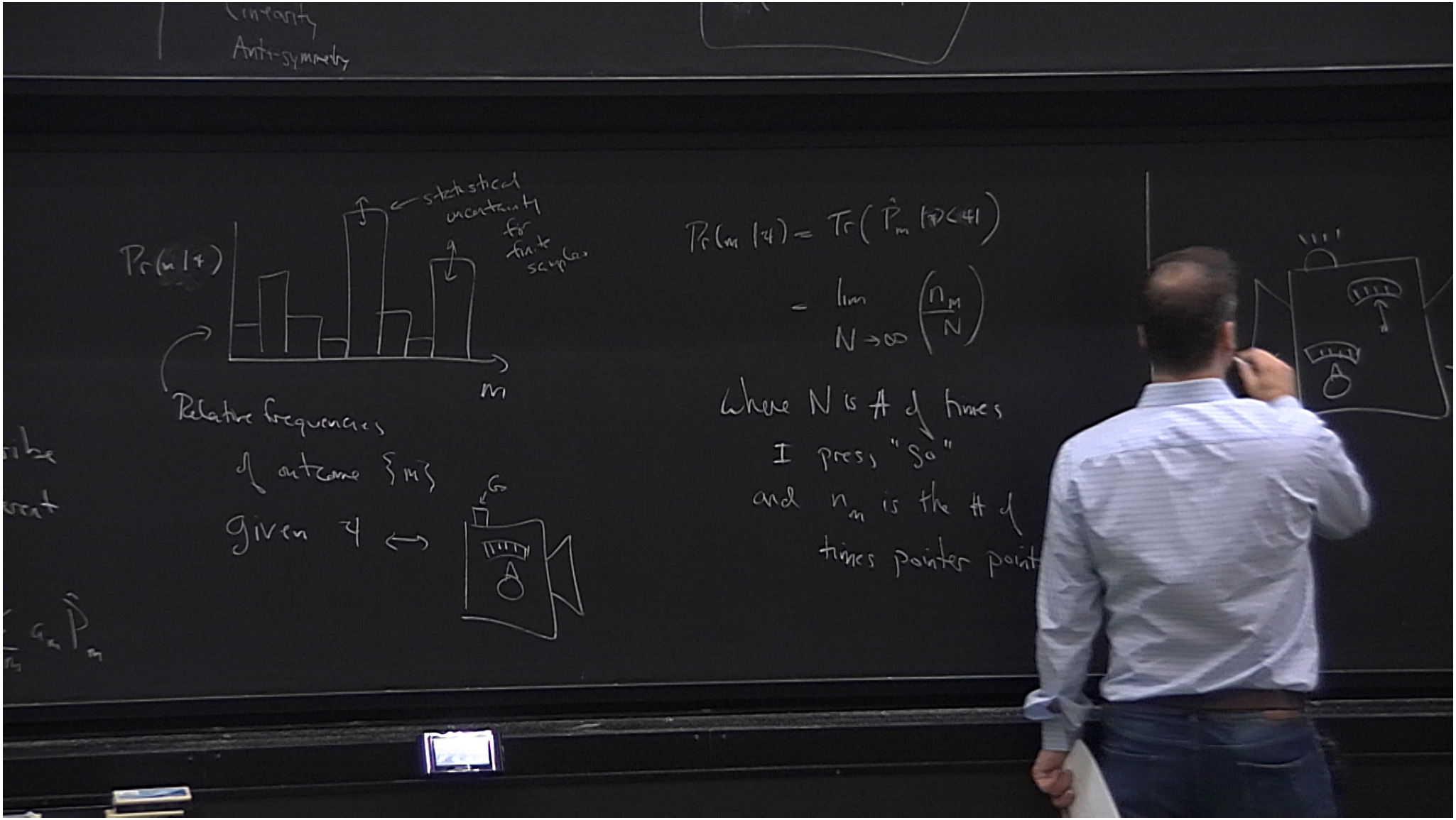
$$U = U_z(\alpha) U_y(\beta) U_z(\gamma)$$

$$U_z(\alpha) = \exp\left(-i\alpha \frac{S_z}{\hbar}\right)$$

$$U_y(\beta) = \exp\left(-i\beta \frac{S_y}{\hbar}\right)$$

$$U(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i\left(\frac{\alpha+\gamma}{2}\right)}\cos(\beta/2) & -e^{-i\left(\frac{\alpha-\gamma}{2}\right)}\sin(\beta/2) \\ e^{i\left(\frac{\alpha-\gamma}{2}\right)}\sin(\beta/2) & e^{+i\left(\frac{\alpha+\gamma}{2}\right)}\cos(\beta/2) \end{pmatrix}$$

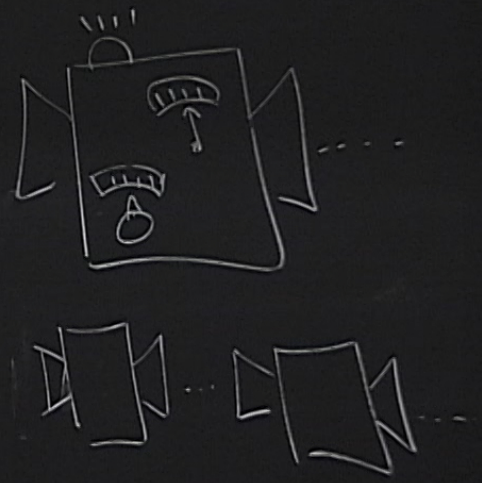

Unitarity
Anti-symmetry
State-update Rule
"Collapse of wavefunction"
Previously, Born rule to describe probabilities of different outcomes under an observable $\hat{A} = \sum_m a_m \hat{P}_m$
readout
Something
$\Pr(m|\psi)$
statistical uncertainty for finite samples
m
Relative frequencies of outcome {m} given ψ
G
$\Pr(m|\psi)$


Linearity
Anti-symmetry
$\Pr(m|\psi)$
statistical uncertainty for finite samples
m
Relative frequencies of outcome $\{m\}$ given ψ
$\Pr(m|\psi) = \mathrm{Tr}(\hat{P}_m |\psi\rangle\langle\psi|)$
$= \lim_{N\to\infty} \left(\frac{n_m}{N}\right)$
where N is # of times I press "Go" and n_m is the # of times pointer point

$P_\Gamma(m|\psi) = \mathrm{Tr}\left(\hat{P}_m |\psi\rangle\langle\psi|\right)$

$= \lim_{N\to\infty} \left(\frac{n_m}{N}\right)$

where N is # of times I press "Go" and n_m is the # of times pointer points to m.



Quant

$= \delta_{\mu} \delta^{-1} 1$

Jacobi Identity
Linearity
Anti-symmetry

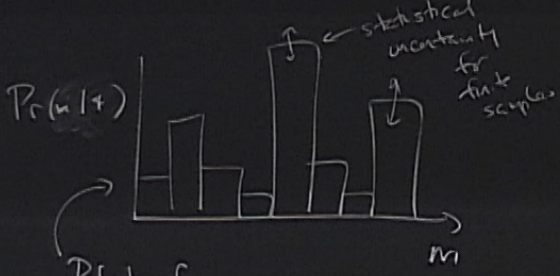
$$\frac{\partial f}{\partial t} = \{H, \rho\}$$

generate commutative transformat

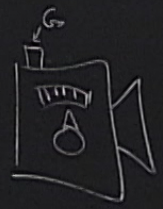
ction"

to describe
of different

der an
$\hat{A} = \sum_{m} a_m \hat{P}_m$



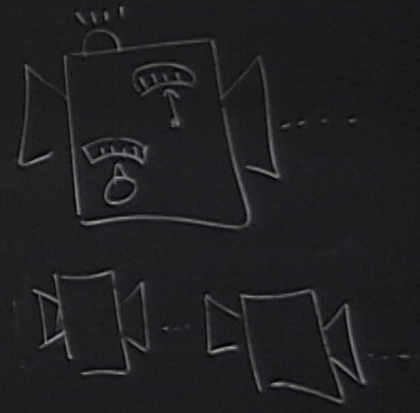
Relative frequencies of outcome $\{m\}$ given $\psi \leftrightarrow$



$$Pr(m|\psi) = Tr\left(\hat{P}_m |\psi\rangle\langle\psi|\right)$$

$$= \lim_{N \to \infty} \left(\frac{n_m}{N}\right)$$

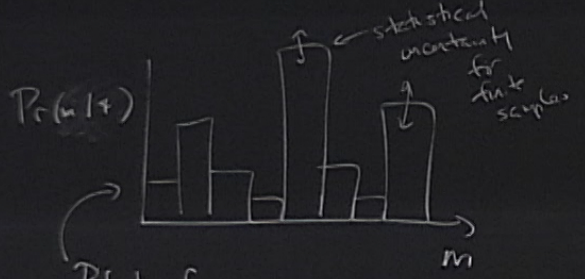
Where N is # of times I press "Go" and n_m is the # of times pointer points to m.



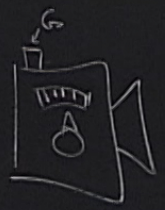
Jacobi Identity
Linearity
Anti-symmetry

$$\frac{\partial \rho}{\partial \epsilon} = \{H, \rho\}$$

generate commutative transformations?



Relative frequencies of outcome $\{m\}$ given ψ $\Leftrightarrow$

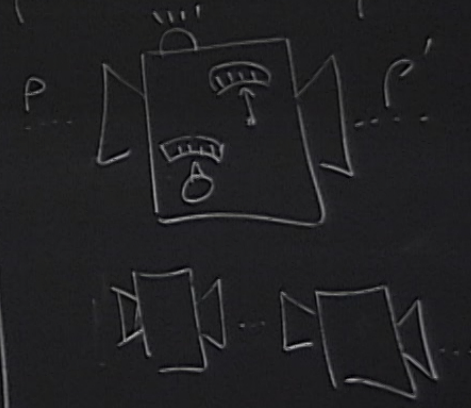


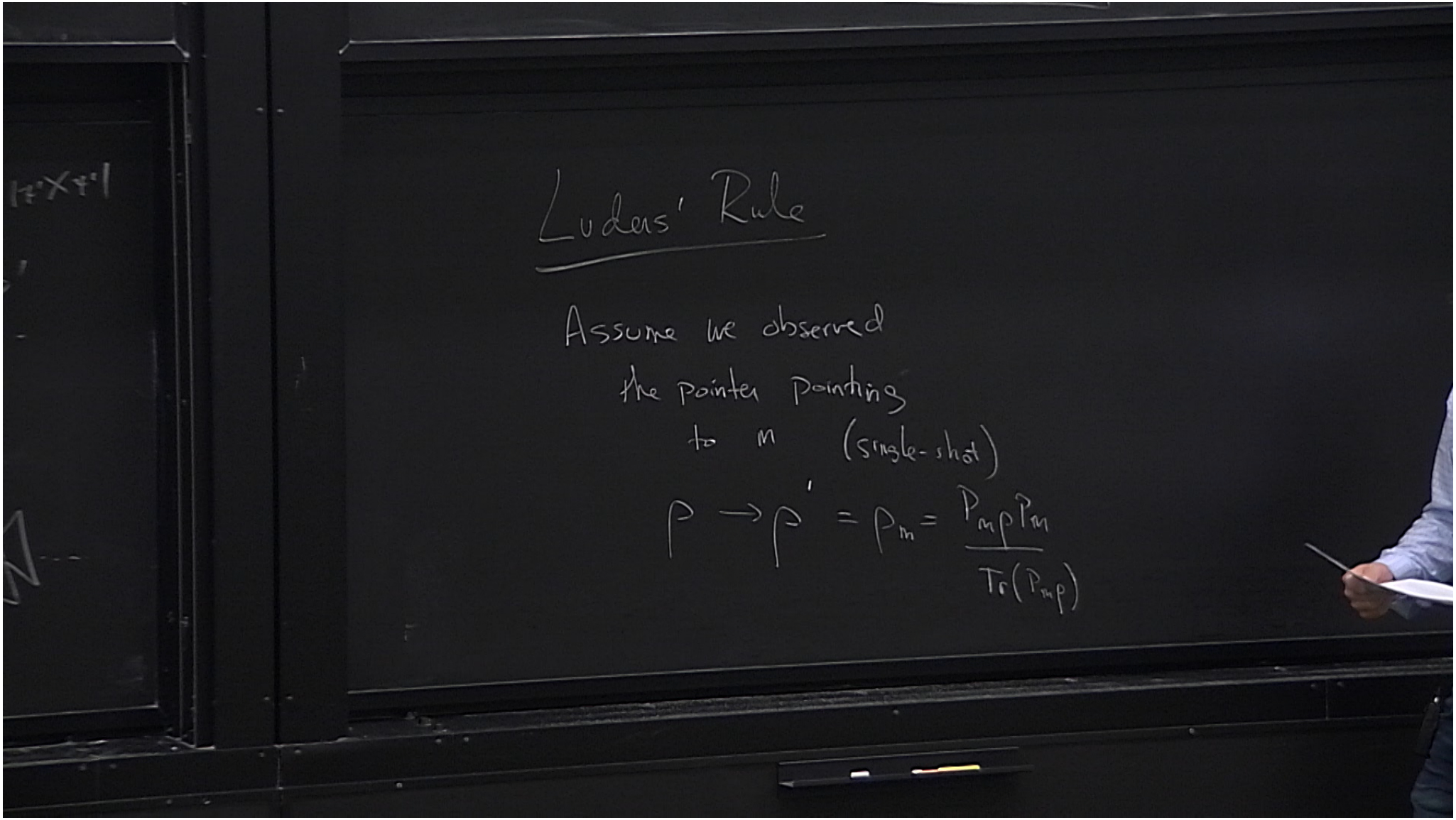
$$Pr(m|\psi) = Tr\left(\hat{P}_m |\psi\rangle\langle\psi|\right)$$

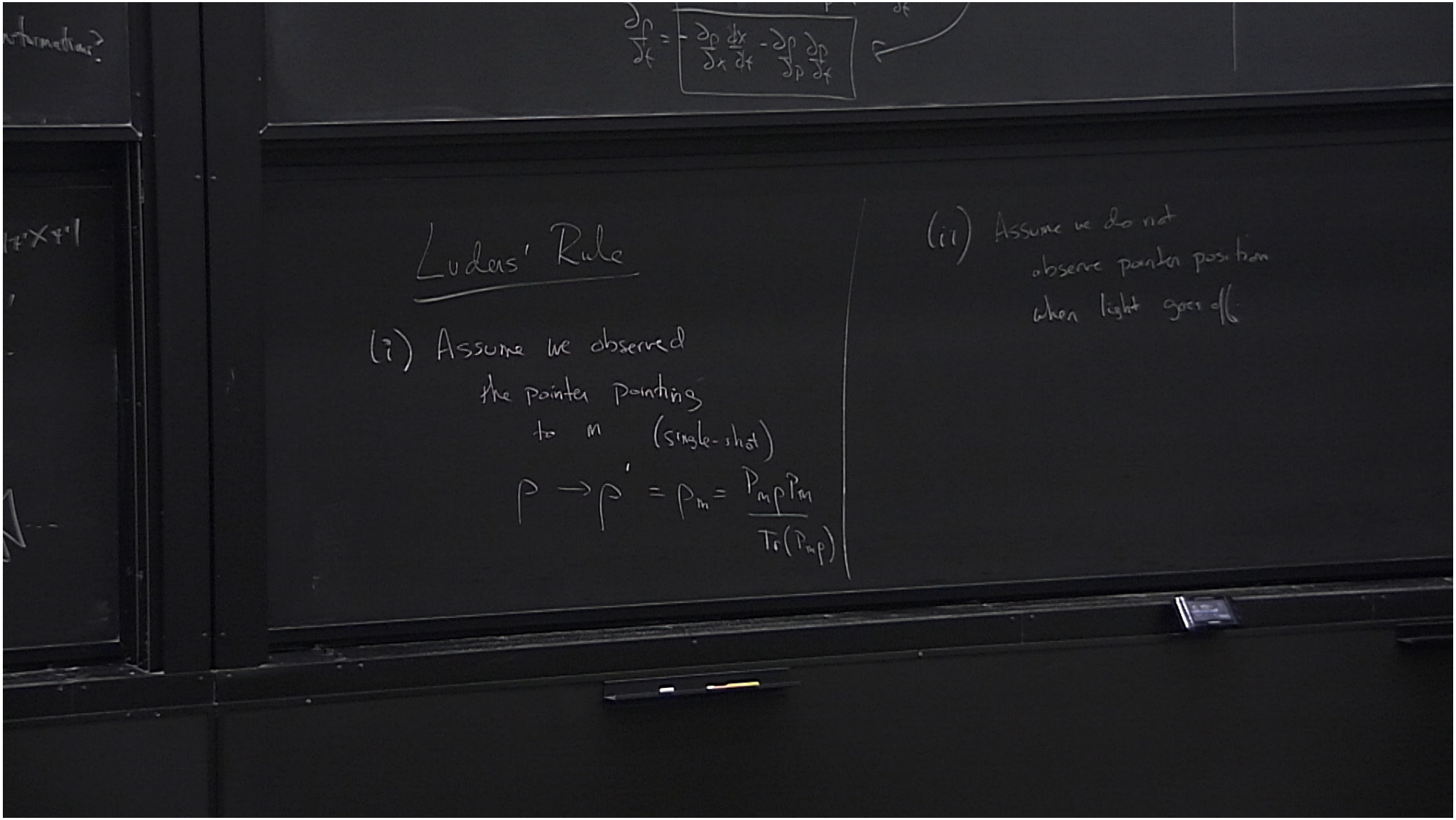
$$= \lim_{N\to\infty} \left(\frac{n_m}{N}\right)$$

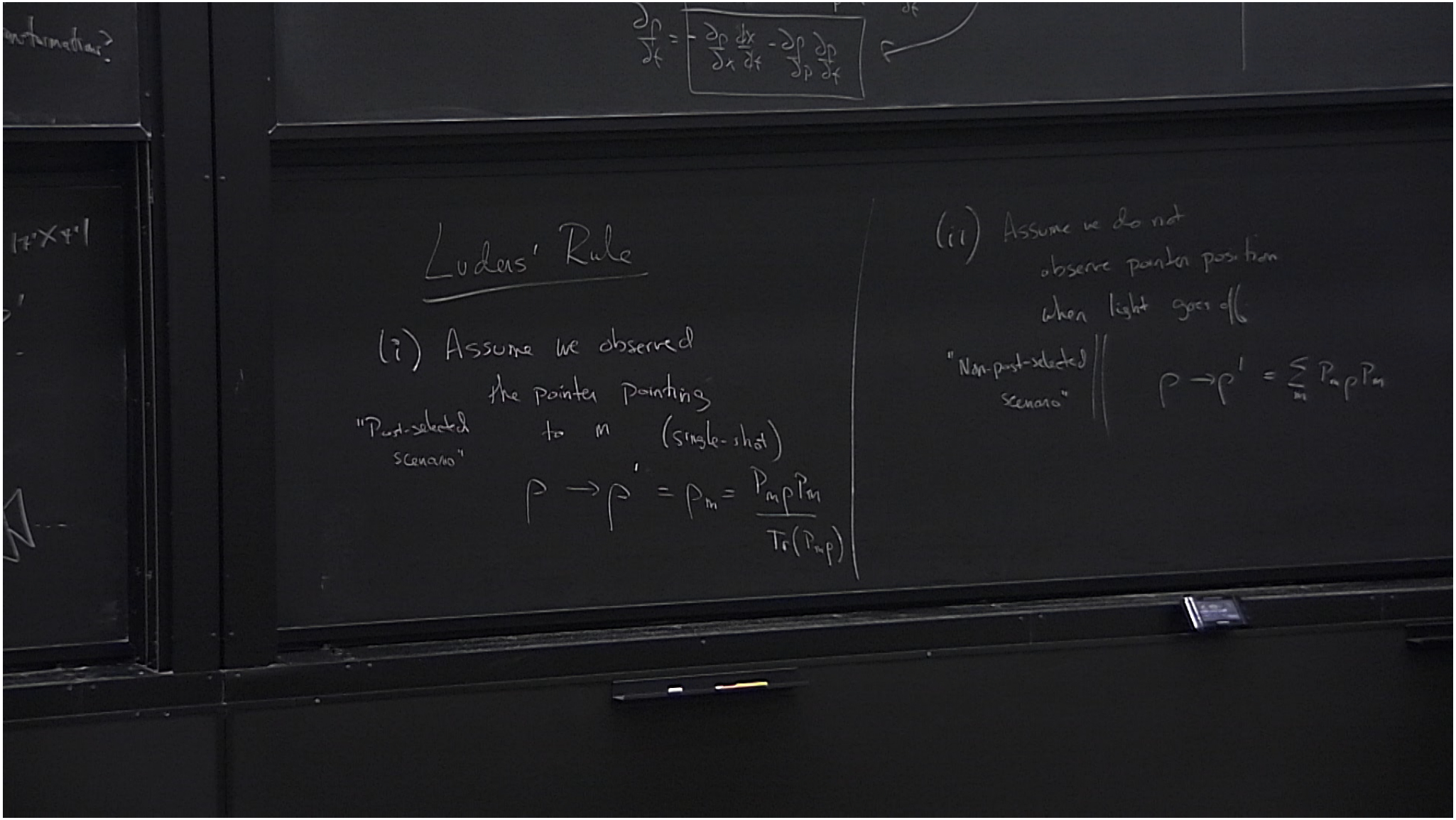
where N is # of times I press "Go" and n_m is the # of times pointer points to m.

$\rho = |\psi\rangle\langle\psi|$ $\rho' = |\psi'\rangle\langle\psi'|$




Ludens' Rule
Assume we observed
the pointer pointing
to m (single-shot)
$\rho \to \rho' = \rho_m = \frac{P_m \rho P_m}{\mathrm{Tr}(P_m \rho)}$


Ludens' Rule
(i) Assume we observed the pointer pointing to m (single-shot)
$\rho \to \rho' = \rho_m = \frac{P_m \rho P_m}{\mathrm{Tr}(P_m \rho)}$
(ii) Assume we do not observe pointer position when light goes off


Lüders' Rule
(i) Assume we observed the pointer pointing to m (single-shot)
"Post-selected scenario"
$\rho \to \rho' = \rho_m = \frac{P_m \rho P_m}{\mathrm{Tr}(P_m \rho)}$
(ii) Assume we do not observe pointer position when light goes off
"Non-post-selected scenario"
$\rho \to \rho' = \sum_m P_m \rho P_m$