Title: PSI 17/18 - Quantum Theory - Lecture 2

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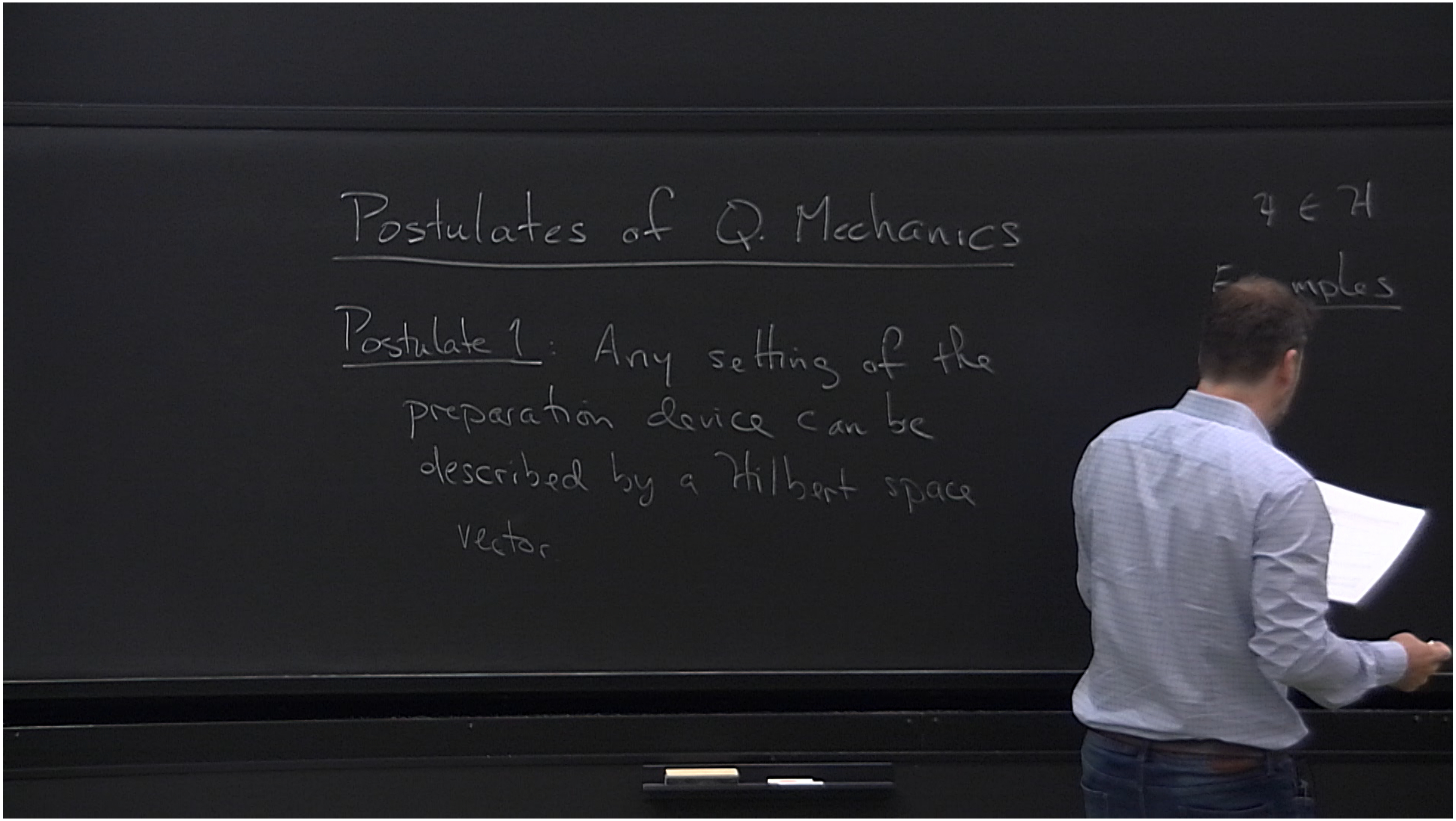
Abstract:


Postulates of Q. Mechanics
Postulate 1: Any setting of the preparation device can be described by a Hilbert space vector
$\psi \in \mathcal{H}$

$\psi \in \mathcal{H}$

Example 1:

The space ℓ^2 of square summable sequences

$$\phi = \{a_j\}, \quad a_j \in \mathbb{C}$$

$$\|\phi\|^2 = \sum_{j=1}^{\infty} \overline{a_j}\, a_j < \infty$$

Endowed with inner-product

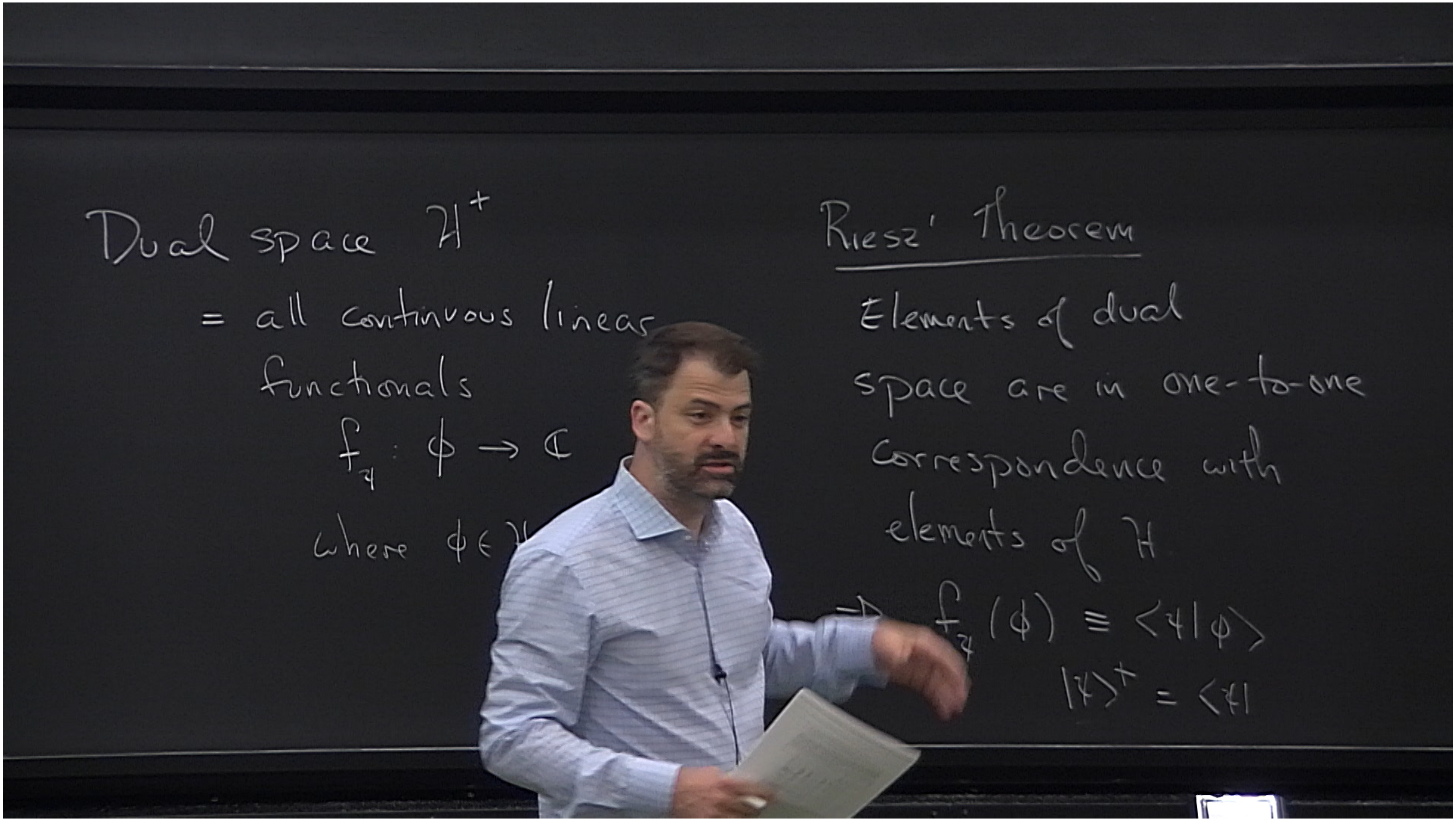
$$\langle \psi | \varphi \rangle \equiv (\psi, \varphi) = \sum_k \bar{b}_k a_k$$

$$< \infty$$

by Cauchy-Schwarz

where $\psi = \{ b_k \}$

$< \infty$


Dual space $\mathcal{H}^+$
= all continuous linear
functionals
$f_\psi : \phi \rightarrow \mathbb{C}$
where $\phi \in$
Riesz' Theorem
Elements of dual
space are in one-to-one
correspondence with
elements of $\mathcal{H}$
$f_\psi(\phi) \equiv \langle\psi|\phi\rangle$
$|\psi\rangle^+ = \langle\psi|$

Dual space $\mathcal{H}^+$

= all continuous linear functionals

$$f_\psi : \phi \to \mathbb{C}$$

where $\phi \in \mathcal{H}$.

Riesz' Theorem

Elements of dual space are in one-to-one correspondence with elements of $\mathcal{H}$.

$$\Rightarrow f_\psi(\phi) \equiv \langle \psi | \phi \rangle$$

$$|\psi\rangle^+ = \langle \psi |$$

Endowed with inner-product

$$\langle \psi | \varphi \rangle \equiv (\psi, \varphi) = \sum_k \bar{b}_k a_k$$

$$< \infty$$

by Cauchy-Schwarz

where $\psi = \{ b_k \}$

$\in \mathbb{C}$

$< \infty$

Remark: For $L^2(\mathbb{R})$

$$\int d\mu(x)\,(\cdots)$$

is a Lebesgue integral and functions that differ on sets of measure zero are equivalent.

$$\|\psi\| = \sum_{j=1} \bar{a}_j a_j < \infty$$

ntegral

differ

re zero

<u>Example 3</u> $\mathcal{H} = L^2([a,b])$

where $\int_{-\infty}^{\infty} d\mu(x) \to \int_a^b d\mu(x)$

- These are all separable Hilbert spaces.
- All ∞-dim$^{\underline{l}}$ Hilbert are "equivalent":

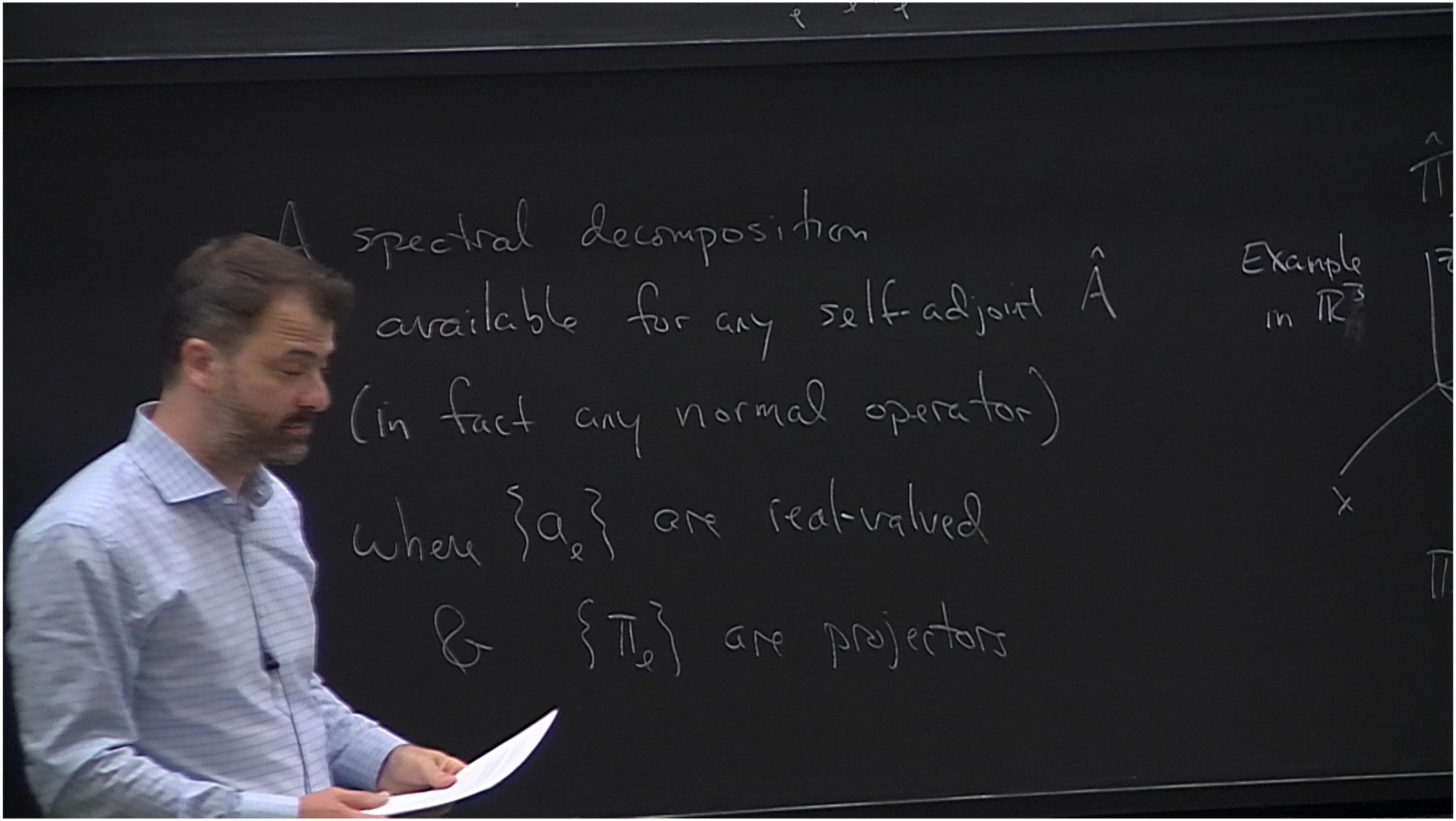
$\exists$ bijective, structure preserving mappings b/w them.

Postulate 2

Any setting of the measurement device can be described by a self-adjoint operator $\hat{A}$ ~~which admits a~~ with spectral decomposition $\hat{A} = \sum_{\ell} a_{\ell} \hat{\Pi}_{\ell}$

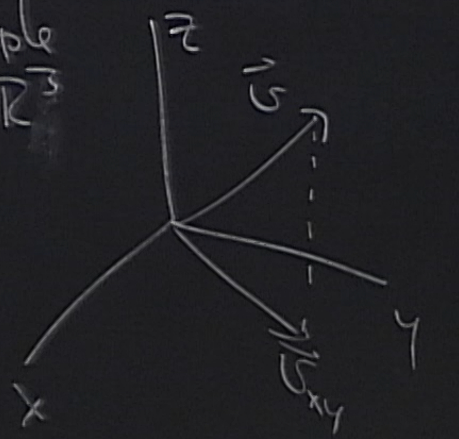
2.a) there is a discrete set of outcomes labelled by $\{a_\ell\}$.

2.b) the probability of observing outcome label a_ℓ, is $\Pr(a_\ell) = \mathrm{Tr}(\hat{\Pi}_\ell |\psi\rangle\langle\psi|) = \langle\psi|\hat{\Pi}_\ell|\psi\rangle$


A spectral decomposition
available for any self-adjoint Â
(in fact any normal operator)
where {a_ℓ} are real-valued
& {Π_ℓ} are projectors
Example
in ℝ³

$$\boxed{\hat{\Pi}_e \hat{\Pi}_n = \delta_{en} \Pi_e}$$

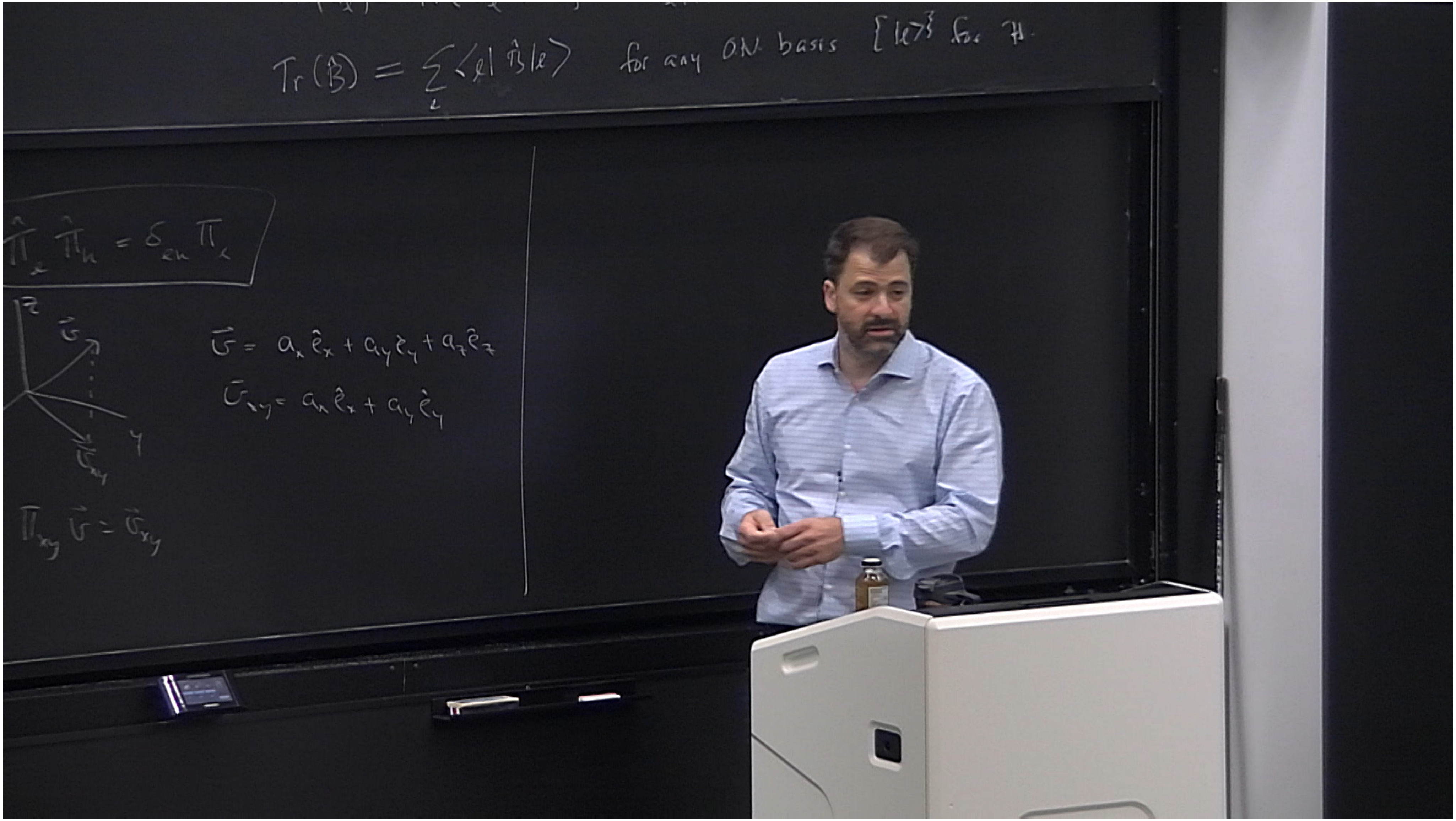
Example in $\mathbb{R}^3$

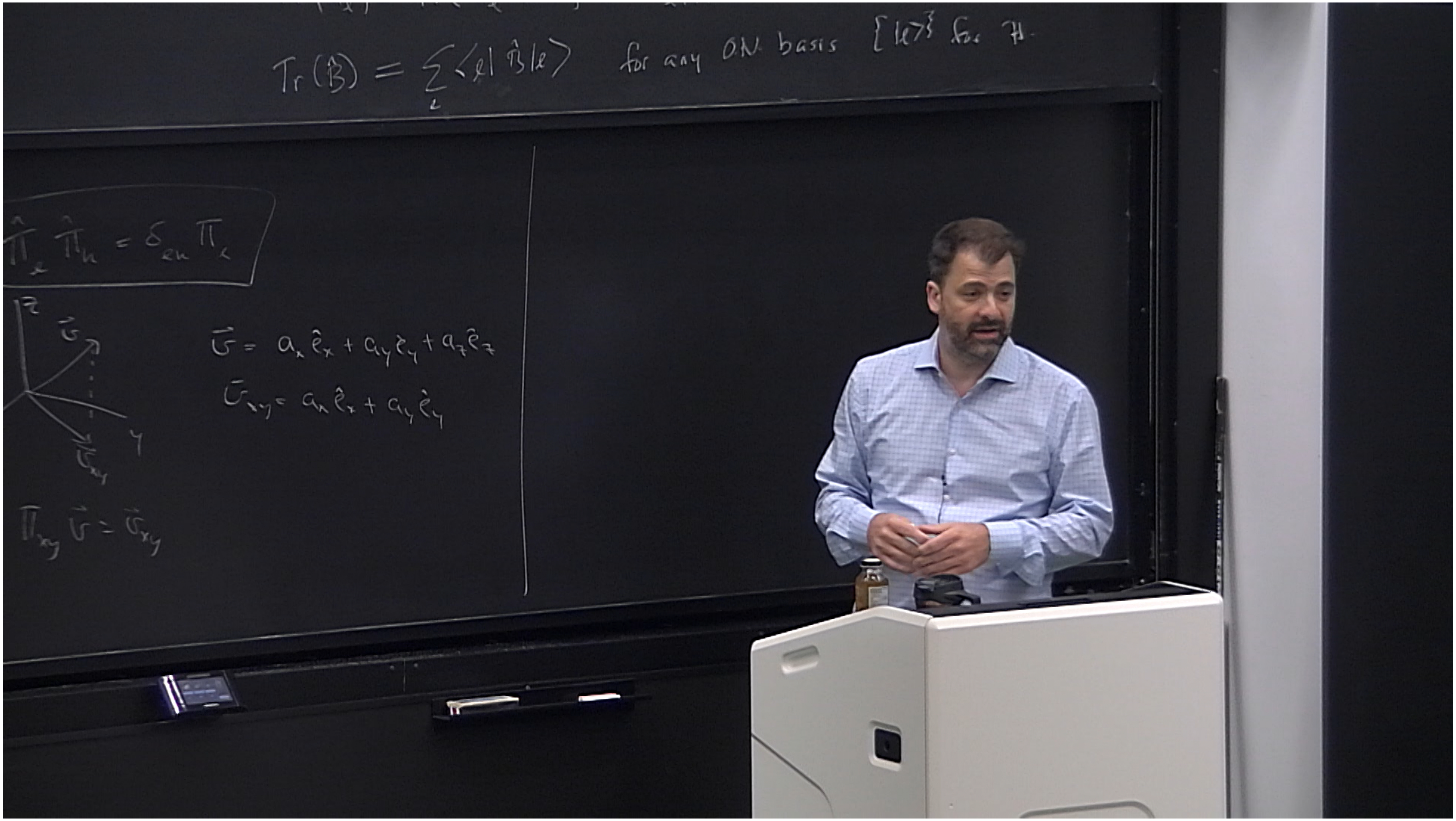


$$\vec{v} = a_x \hat{e}_x + a_y \hat{e}_y + a_z \hat{e}_z$$

$$\vec{v}_{xy} = a_x \hat{e}_x + a_y \hat{e}_y$$

$$\Pi_{xy} \vec{v} = \vec{v}_{xy}$$


Tr(B̂) = Σ_l ⟨l| B̂ |l⟩ for any O.N. basis {|l⟩} for H.
v⃗ = a_x ê_x + a_y ê_y + a_z ê_z
v⃗_xy = a_x ê_x + a_y ê_y
Π_xy v⃗ = v⃗_xy


Tr(B̂) = ∑ ⟨l| B̂ |l⟩ for any O.N. basis {|l⟩} for H.
Π̂ₗ Π̂ₖ = δₗₖ Πₗ
v⃗ = aₓ êₓ + a_y ê_y + a_z ê_z
v_xy = aₓ êₓ + a_y ê_y
Π_xy v⃗ = v⃗_xy

Born rule

Probability density

$$\rho(x) = \|\psi(x)\|^2$$

$$\mathrm{Prob}(x \in [a,b]) = \int_a^b \underline{d\mu(x)}\, |\psi(x)|^2$$

$$\mathrm{Prob}(\ell) = \underline{\mathrm{Tr}\left[\, |\ell\rangle\langle\ell|\; |\psi\rangle\langle\psi| \,\right]} = \underline{\underline{|\langle\psi|\ell\rangle|^2}}$$

Rank of projector

is the dim$^{\underline{n}}$ of the Hilbert

on which a projector has

eigenvalue 1.

For any projector, eigenvalues $(\Pi_e) \in \{0,1\}$

$$\text{Rank-}1 \Rightarrow \Pi_\ell = |\ell\rangle\langle\ell|$$

$$\text{Rank-}2 \Rightarrow \Pi = |\ell\rangle\langle\ell| + |\ell+1\rangle\langle\ell+1|$$

$\mathcal{H} = \mathbb{C}^3$ 3-dim$^{\ell}$ complex vector space.

An O.N. basis $\{|0\rangle, |1\rangle, |2\rangle\}$, $|\psi\rangle \in \mathbb{C}^3$

$$|\psi\rangle = \sum_\ell c_\ell |\ell\rangle$$

$$= c_0|0\rangle + c_1|1\rangle + c_2|2\rangle$$

$$\hat{\Pi}_{12}|\psi\rangle = c_1|1\rangle + c_2|2\rangle$$

$$= \{0,1\}$$

$$\Pr\left(\begin{matrix}\text{finding particle}\\ \text{in } |1\rangle \text{ or } |2\rangle\end{matrix}\right) = \mathrm{Tr}\left[\hat{\Pi}_{12}|\psi\rangle\langle\psi|\right]$$

Postulate 3: Any setting of the transformation device can be described by a unitary operator U satisfying $i\hbar \frac{\partial \hat{U}(t_2, t_1)}{\partial t_2} = \hat{H}(t_2)\,\hat{U}(t_2, t_1)$

where $\hat{H}$ is a self-adjoint operator representing the energy and $t \in \mathbb{R}$ is time parameter with initial condition

$\hat{U}(t_1, t_1) = \hat{\mathbb{1}}$ and

s.t. $|\psi(t_2)\rangle = \hat{U}(t_2, t_1)\, |\psi(t_1)\rangle$.

Remarks: this implies

$U^\dagger U = U U^\dagger = \mathbb{1}$

$\Rightarrow \hat{U}$ is unitary

$\hat{U} = \sum_\ell e^{i\phi_\ell} |\ell\rangle\langle\ell|$

Gen^l Sol^n: $U(t_2, t_1) = T \exp\left(-i \int_{t_1}^{t_2} H(\tau)\, d\tau\right)$

If H is time-independent then

$U(t) = \exp(-iHt/\hbar)$

$\mathcal{H} = L^2([a,b])$

where $\hat{H}$ is a self-adjoint operator representing the energy and $t \in \mathbb{R}$ is time parameter

From postulate 3, we can deduce $i\hbar \frac{d|\psi(t)\rangle}{dt} = \hat{H}(t)|\psi(t)\rangle$ ← Postulate 3 in Schrodinger picture.

Heisenberg picture: $\hat{A}_H(t_2) = \hat{U}^\dagger(t_2,t_1)\,\underline{\hat{A}(t)}\,\hat{U}(t_2,t_1)$

⇒ Postulate 3 in Heisenberg picture.

$$\frac{d\hat{A}_H(t)}{dt} = \frac{1}{i\hbar}\left[\hat{A}_H(t), \hat{H}_H(t)\right] + \left(\frac{\partial A}{\partial t}\right)_H$$

$U(t) = \exp(-iHt/\hbar)$

The von Neumann Eq^n

Gauge freedom in Q.Theory

$$|\psi\rangle \longrightarrow e^{i\phi}|\psi\rangle = |\psi'\rangle$$

$\Rightarrow$ not physically observable.

$$\hat{A} = \sum_e a_e \hat{\Pi}_e$$

$$\mathrm{Prob}(a_e) = \langle\psi|\Pi_e|\psi\rangle = \mathrm{Tr}\{\hat{\Pi}_e\, |\psi\rangle\langle\psi|\}$$

$$= \langle\psi'|\Pi_e|\psi'\rangle = e^{i\varphi}e^{-i\varphi}\langle\psi|\Pi_e|\psi\rangle$$

Notation

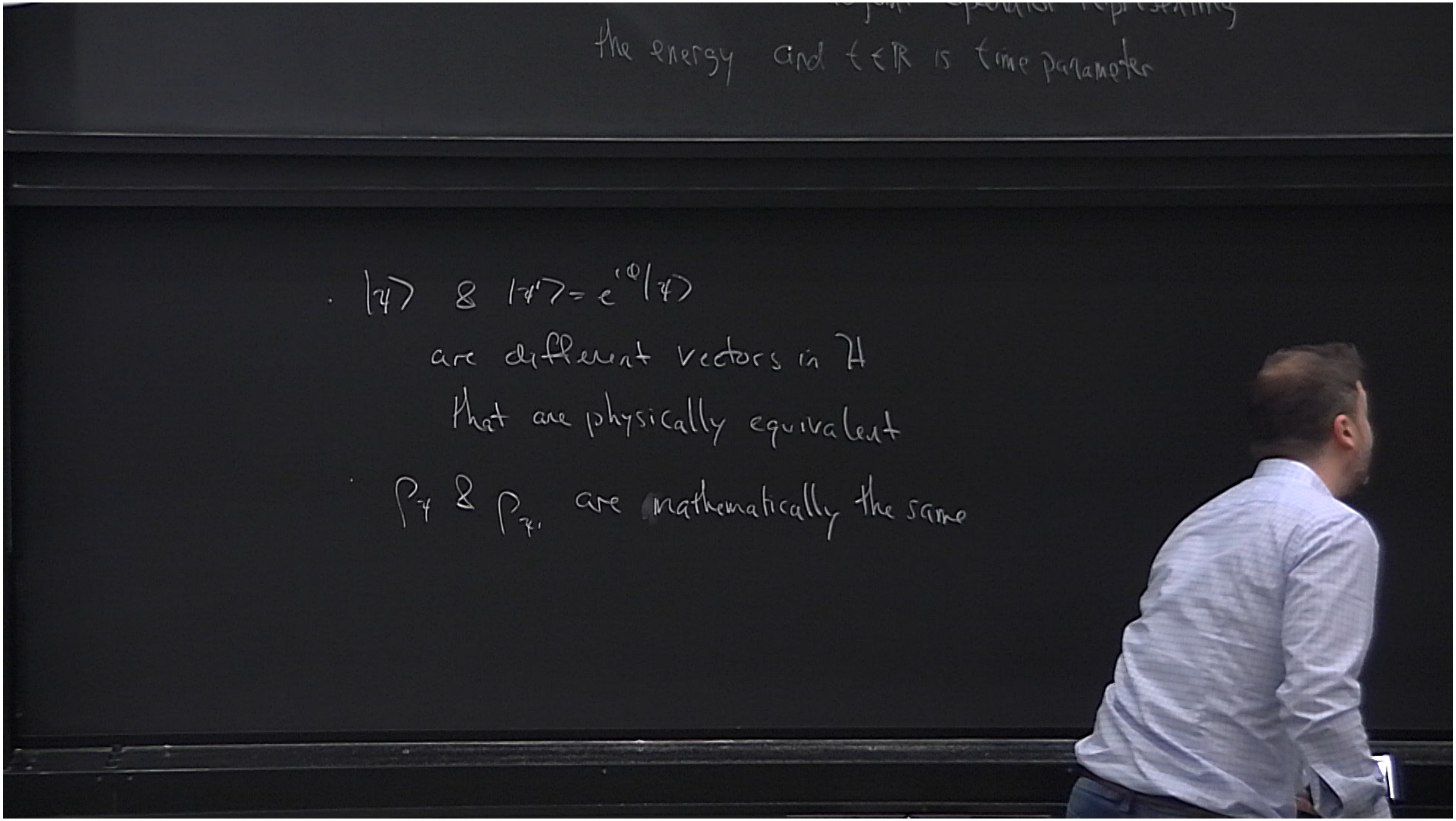
Density operator

$$\rho_\psi = |\psi\rangle\langle\psi|$$

"outer product"

formed vector $|\psi\rangle$

$|\psi\rangle\langle\psi|$


the energy and $t \in \mathbb{R}$ is time parameter
· $|\psi\rangle$ & $|\psi'\rangle = e^{i\varphi}|\psi\rangle$
are different vectors in $\mathcal{H}$
that are physically equivalent
· ρ_ψ & $\rho_{\psi'}$ are mathematically the same

or representing
parameter

if H is time-independent the
$U(t) = \exp$

Postulate 3 as the
Von Neumann ~~representation~~
equation

$$\boxed{\frac{d\hat{\rho}(t)}{dt} = \frac{1}{i\hbar}\left[\hat{H}(t), \hat{\rho}(t)\right]}$$

Q. State $\hat{\rho}$ is now a (linear) operator
acts on $\mathcal{H}$, rather than a vector $\in \mathcal{H}$.

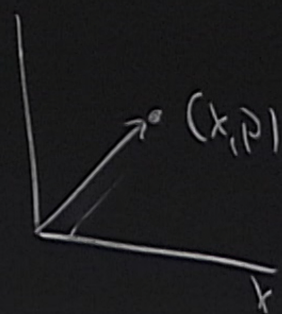
Classical dynamics of
a prob density
"Liouville dynamics"

Classical dynamics of
a prob. density
"Liouville dynamics"

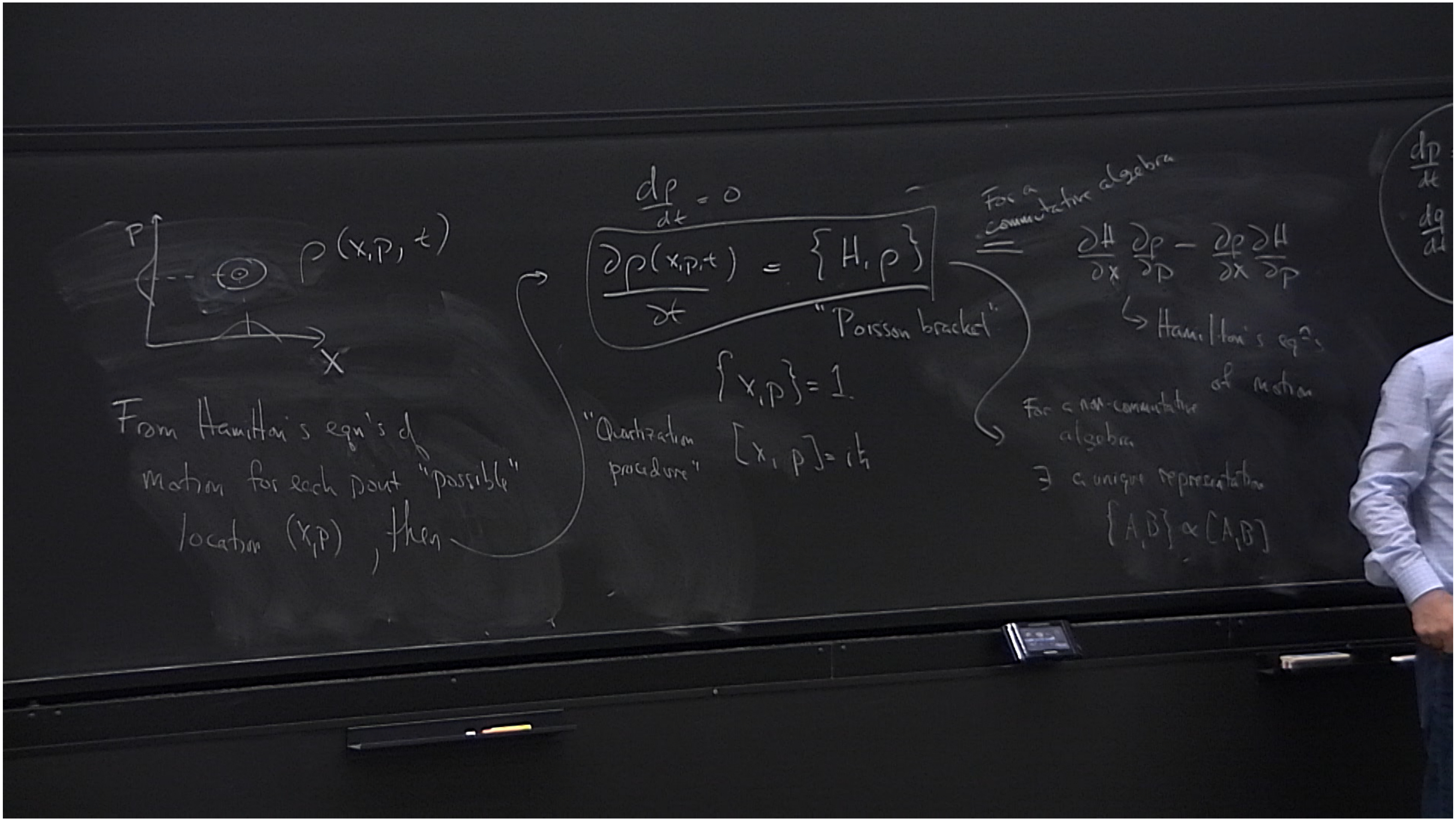
Phase space

Coordinates of
a particle
"planet" or
"cannon ball"

(x, p)



vector $\in \mathcal{H}$.


p
x
ρ(x,p,t)
From Hamilton's eqn's of motion for each point "possible" location (x,p), then
dρ/dt = 0
∂ρ(x,p,t)/∂t = {H, ρ}
"Poisson bracket"
{x,p} = 1
"Quantization procedure"
[x, p] = iħ
For a commutative algebra
∂H/∂x ∂ρ/∂p − ∂ρ/∂x ∂H/∂p
Hamilton's eqⁿs of motion
For a non-commutative algebra
∃ a unique representation
{A,B} ∝ [A,B]
dp/dt
dq/dt