

Title: PSI 17/18 - Relativity - Lecture 14

Date: Sep 22, 2017 09:00 AM

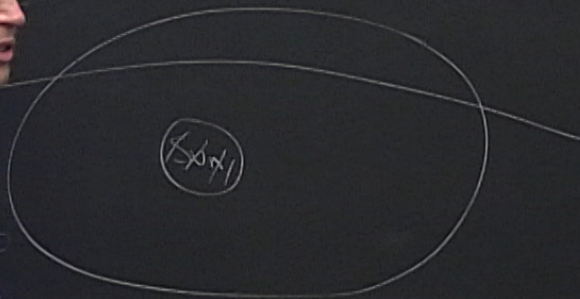
URL: <http://pirsa.org/17090033>

Abstract:

Schwarzschild $ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)} dr^2 + r^2 d\Omega^2$

$R_{\mu\nu} = 0$ $\uparrow$ sph. symm

trajectories of particles in this metric



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$R_{\mu\nu} = 0$ $\uparrow$ spherically-symm

trajectories of particles in this metric

Define $r_s = 2GM$: action for massive particle $= -m \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \Rightarrow$ geodesic eq.
conservation laws

turns out there is so much symmetry, we don't need to use the geodesic eq.

(x, y, z)

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↑
sph. symm

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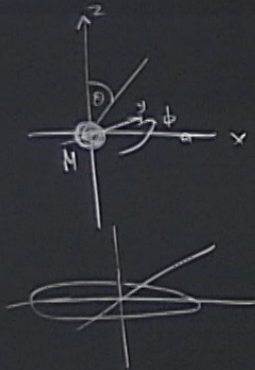
turns out there is so much symmetry, we don't need to use the geodesic eq.

lets study trajectory in proper time τ : $dt = \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda \Rightarrow \boxed{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1}$, where $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$.

(Ex check that geodesic eq. $\Rightarrow \frac{d}{d\tau}(g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu) = 0$)

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(Ex check that geodesic eqⁿ implies $\frac{d}{d\tau}(g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu) = 0$)



Convenient
to
set $\theta = \pi/2$

Conservation laws:

t, r, ϕ

Symm $t \rightarrow t + \text{const} \Rightarrow$ Energy
 $\phi \rightarrow \phi + \text{const} \Rightarrow$ angular momentum

$$S = \int -m \sqrt{-(1 - \frac{r_s}{r}) \left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{(1 - \frac{r_s}{r})} \left(\frac{dr}{d\lambda}\right)^2 - r^2 \left(\frac{d\phi}{d\lambda}\right)^2} d\lambda = - \int d\lambda L(\lambda)$$

$$\Rightarrow P_t \equiv \frac{\partial L}{\partial \left(\frac{dt}{d\lambda}\right)}, \quad P_\phi \equiv \frac{\partial L}{\partial \left(\frac{d\phi}{d\lambda}\right)} \quad \text{both conserved}$$

$$= -M \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\lambda}; \quad = \frac{m r^2 \frac{d\phi}{d\lambda}}{\left(1 - \frac{r_s}{r}\right)} = m r^2 \frac{d\phi}{d\tau}$$

$$= -M \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau}$$

$$= -M \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\lambda} ; \quad \left\{ \frac{dt}{d\lambda} \right\} = m \frac{dt}{d\lambda} \frac{1}{\left(1 - \frac{r_s}{r}\right)} = m^2 \frac{dt}{d\tau}$$

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1 \Rightarrow -\left(1 - \frac{r_s}{r}\right) \dot{t}^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 = -1$$

$$\frac{p_t}{m} \equiv E$$

$$\frac{p_\phi}{m} \equiv J$$

$$\Rightarrow -\frac{E^2}{\left(1 - \frac{r_s}{r}\right)} + \frac{\dot{r}^2}{\left(1 - \frac{r_s}{r}\right)} + \frac{J^2}{r^2} = -1$$

$$r^2 \dot{\phi} = J$$

$$\dot{t} = \frac{E}{1 - r_s/r}$$

$$\times \left(1 - \frac{r_s}{r}\right)$$

$$\underbrace{\dot{r}^2 - \frac{r_s}{r}}_{\text{usual Newtonian terms}} + \frac{J^2}{r^2} - \frac{r_s J^2}{r^3} = E^2 - 1$$

$$\downarrow \text{down by } \frac{r_s}{r} = \frac{2GM}{rc^2} \sim \frac{v^2}{c^2} \ll 1 \quad \text{for a circular Newtonian orbit } \frac{GM}{r} = \frac{v^2}{r}$$

$$= -M \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\lambda} ; \quad \left\{ \frac{dt}{d\lambda} \right\} = M \frac{\frac{dt}{d\tau}}{\left(1 - \frac{r_s}{r}\right)} \frac{1}{\sqrt{\dots}} = m^2 \frac{d\phi}{d\tau}$$

$$= -M \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau} \sqrt{\dots}$$

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1 \Rightarrow -\left(1 - \frac{r_s}{r}\right) \dot{t}^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 = -1$$

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$$r^2 \dot{\phi} = J$$

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$$\times \left(1 - \frac{r_s}{r}\right)$$

$$\dot{r}^2 - \frac{r_s}{r} + \frac{J^2}{r^2} - \frac{r_s J^2}{r^3} = E^2 - 1$$

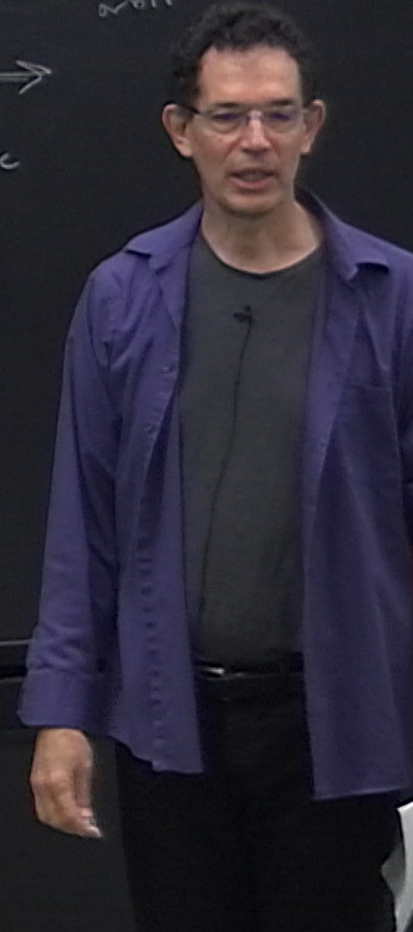
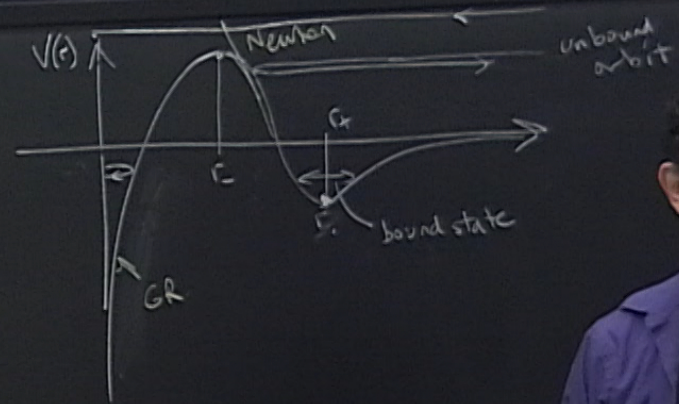
usual Newtonian terms

GR correction

down by $\frac{r_s}{r} = \frac{2GM}{rc^2} \sim \frac{v^2}{c^2} \ll 1$ for a circular Newtonian orbit $\frac{GM}{r} = \frac{v^2}{r}$

Define $V(r) = -\frac{r_s}{r} + \frac{J^2}{r^2} - \frac{r_s J^2}{r^3}$

2 critical radii, where $\frac{dV}{dr} = 0$



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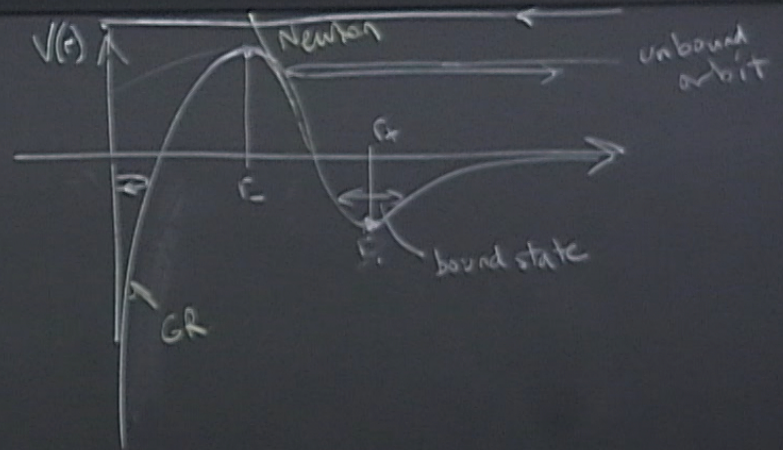
r_+ stable

r_- unstable

$$+\frac{r_s}{r^2} - 2\frac{J^2}{r^3} - 3\frac{r_s J^2}{r^4} = 0$$

$$\Rightarrow r_{\pm} = \frac{J^2}{r_s} \pm \sqrt{\left(\frac{J^2}{r_s}\right)^2 - 3J^2}$$

$$= \frac{J^2}{r_s} \left(1 \pm \sqrt{1 - \frac{3r_s^2}{J^2}} \right)$$



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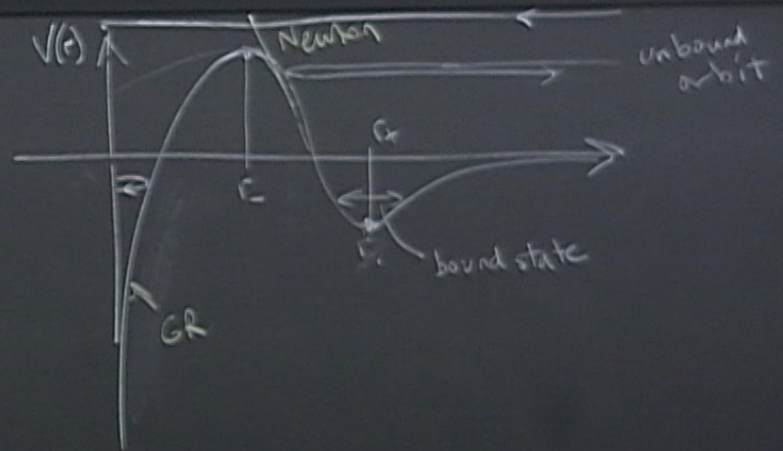
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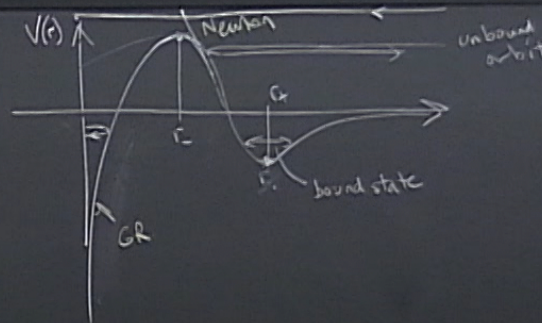
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$\Rightarrow$ as we decrease J to limit $\sqrt{3} r_s$
at the limit, $r_- = r_+ = 3r_s = \frac{6GM}{c^2}$
Innermost stable circular orbit (ISCO)



$\downarrow$ dec. J

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innermost stable circular orbit (ISCO).

$$\overbrace{r'^2 - \frac{r_s}{r} + \frac{J^2}{r^2} - \frac{r_s J^2}{r^3}}^{\text{Newton}} = \text{const}$$

$$\dot{\phi} = \frac{J}{r^2}$$

Perihelion shift

Consider $r(\phi)$ $r' \equiv \frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} \Rightarrow$

$$r'^2 = \frac{\dot{r}^2}{\dot{\phi}^2} = \frac{\dot{r}^2}{J^2} r^4 = \left(\frac{r_s}{r} - \frac{J^2}{r^2} \left(1 - \frac{r_s}{r}\right) + \text{const}\right) \frac{r^4}{J^2}$$

Define $u = \frac{1}{r}$ $u' = -\frac{r'}{r^2}$

$$\Rightarrow \underbrace{u'^2 - \frac{r_s u}{J^2} + u^2 - r_s u^3}_{\text{Newton}} = \text{const}$$

Define $u = \frac{1}{r}$ $u' = -\frac{r'}{r^2}$ $\Rightarrow$ $\underbrace{u^2 - \frac{r_s u}{j^2}}_{\text{SHO}} + \underbrace{u^2 - r_s u^3}_{\text{Newton}} = \text{const}$



SHO: eliminate linear term by a shift $u = \tilde{u} + a$ $a = +\frac{1}{2} \frac{r_s}{j^2}$ cancels linear term in $\tilde{u}$.

Soln. is $u = \frac{r_s}{2j^2} \left(1 + e \cos \varphi \right)$
Simple Harmonic motion, with amplitude $\propto e$.

$$\Rightarrow r = \frac{2j^2}{r_s} \frac{1}{(1 + e \cos \varphi)}$$

Define $u = \frac{1}{r}$ $u' = -\frac{r'}{r^2}$

$\Rightarrow \underbrace{u^2 - \frac{r_s u}{j^2}}_{\text{SHO}} + \underbrace{u^2 - r_s u^3}_{\text{small}} = \text{const}$

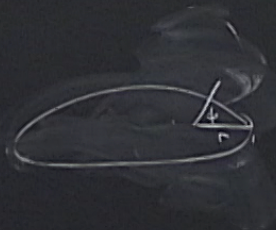
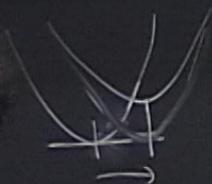
Newton

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Sol'n is $u = \frac{r_s}{2j^2} (1 + e \cos \phi)$

Simple Harmonic motion, with amplitude $\propto e$.

$\Rightarrow r = \frac{2j^2}{r_s} \frac{1}{(1 + e \cos \phi)}$ elliptical orbit, ellipticity $0 \leq e < 1$.



$$\Rightarrow P_t \equiv \frac{\partial L}{\partial \left(\frac{dt}{d\lambda}\right)}, \quad P_\phi \equiv \frac{\partial L}{\partial \frac{d\phi}{d\lambda}} \quad \text{both conserved}$$

$$= -M \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\lambda}; \quad = \frac{M r^2 \frac{d\phi}{d\lambda}}{\left(1 - \frac{r_s}{r}\right)} = m r^2 \frac{d\phi}{dt}$$

$$= -M \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau}$$

treat $r_s u^3$ term as a pert.

eq. of motion for u is $u'' = - \underbrace{\left(u - \frac{r_s}{2u^2}\right)}_{\text{shifted SHO}} + \underbrace{\frac{3}{2} r_s u^2}_{\text{GR pert}}$

Solve in pert. theory: $u = u^{(0)} + \delta u$, with δu small

$$\rightarrow \delta u'' = -\delta u + \frac{3}{2} r_s u^{(0)2} = -\delta u + \frac{3}{2} r_s \left(\frac{r_s}{2u^2}\right)^2 (1 + \epsilon \cos \phi)^2 \quad \text{driving force}$$

driven SHO

$$\Rightarrow P_t \equiv \frac{\partial L}{\partial \left(\frac{dt}{d\lambda} \right)}, \quad P_\phi \equiv \frac{\partial L}{\partial \frac{d\phi}{d\lambda}} \quad \text{both conserved}$$

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$$= -M \left(1 - \frac{r_s}{r} \right) \frac{dt}{d\tau}$$

treat $r_s u^3$ term as a pert.

e.g. of motion for u is $\left(\frac{d}{d\phi} \right)$

$$u'' = - \underbrace{\left(u - \frac{r_s}{2J^2} \right)}_{\text{shifted SHO}} + \underbrace{\frac{3}{2} r_s u^2}_{\text{GR pert.}}$$

Solve in pert. theory: $u = u^{(0)} + \delta u$, with δu small

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forget const, $\cos\varphi$ terms
(uninteresting)

$$\delta u' = -\delta u + \frac{3}{4} \frac{r_s^3}{J^4} \cos\varphi$$

$$1 + 2\cos\varphi + \cos^2\varphi$$

resonant !!! $\frac{\cos\varphi + 1}{2}$

forget const, $\cos^2 \phi$
terms
(uninteresting)

$$\delta u' = -\delta u + \frac{3}{4} \frac{\Gamma_s^3}{J^4} e \cos \phi$$

$$1 + 2e \cos \phi + e^2 \cos^2 \phi$$

resonant !!! $\frac{\omega_2 \phi + 1}{2}$

growing
soln. is

$$\delta u = \frac{3 \Gamma_s^3}{8 J^4} e \phi \sin \phi \Rightarrow \delta u'' \sim \frac{2 \phi' (\sin \phi)'}{2 \cos \phi} - \phi \sin \phi$$

small

$$\rightarrow u \approx \frac{\Gamma_s}{2 J^2} (1 + e \cos \phi) + \frac{3 \Gamma_s^3}{8 J^4} e \phi \sin \phi$$

secularly growing term ← pert. theory fails

$$\approx \frac{\Gamma_s}{2 J^2} \left(1 + e \cos((1-\epsilon)\phi) \right) + o(\epsilon^2)$$

where $\epsilon = \frac{3 \Gamma_s^2}{4 J^2}$
resummation of pert. th. to prevent its breakdown.

forget const, $\cos^2 \phi$
terms
(uninteresting)

$$\delta u' = -\delta u + \frac{3}{4} \frac{\Gamma_s^3}{J^4} e \cos \phi$$

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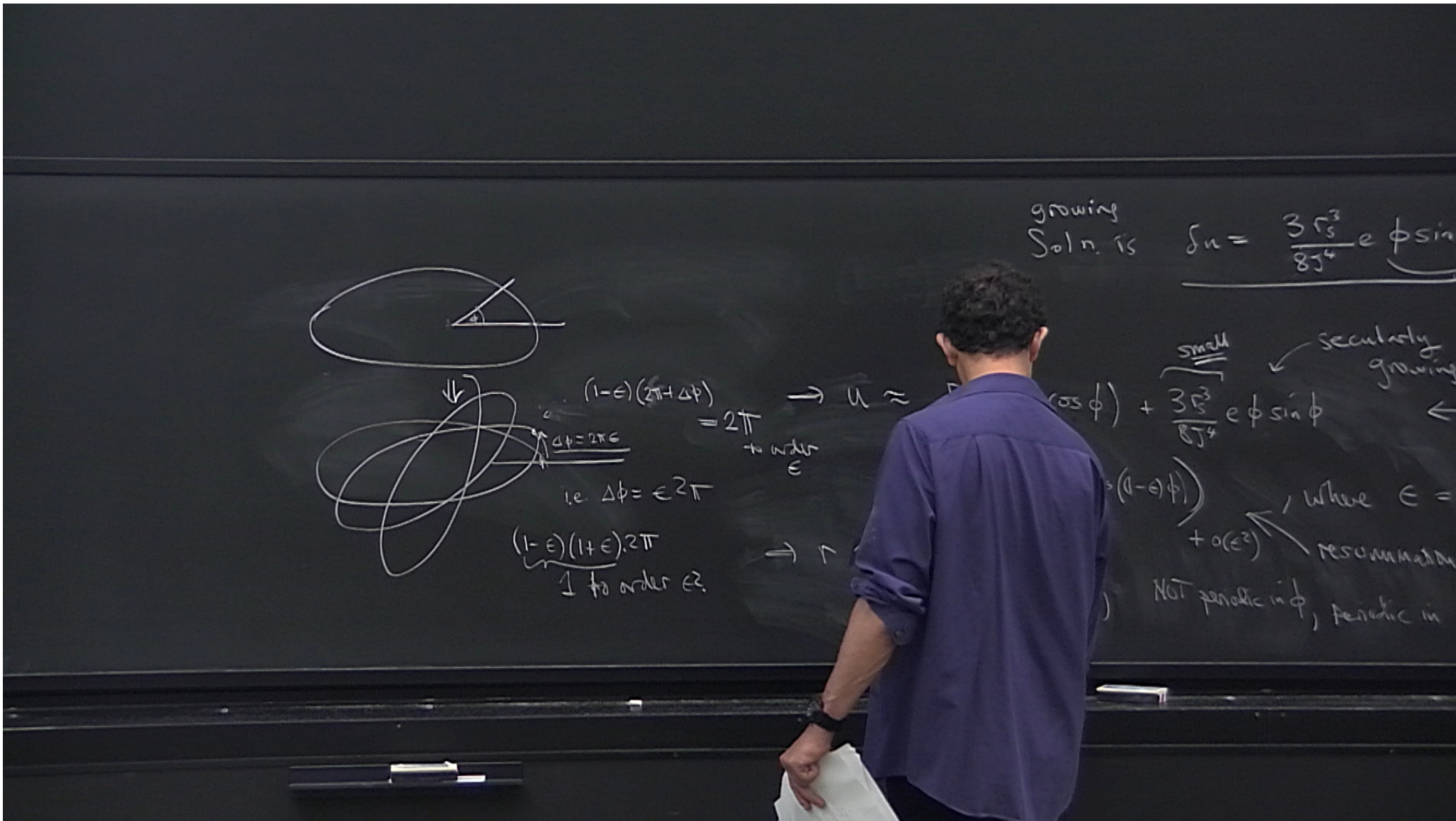
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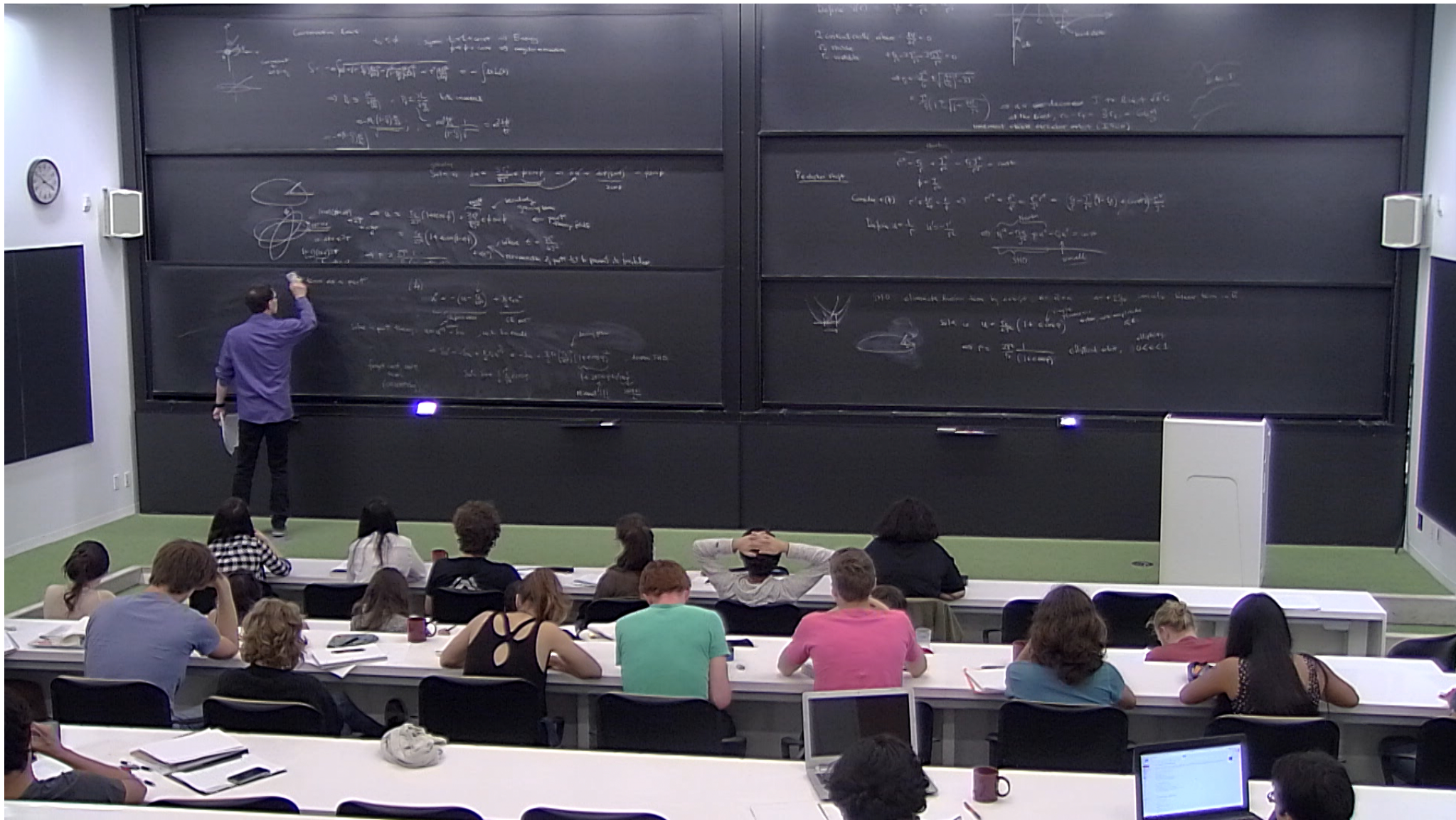
small $\leftarrow$ secularly growing term $\leftarrow$ pert. theory fails

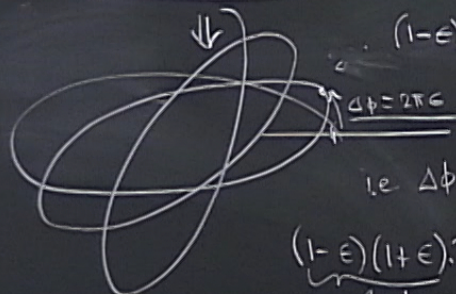
$\approx \frac{\Gamma_s}{2J^2} (1 + e \cos((1-\epsilon)\phi)) + o(\epsilon^2)$, where $\epsilon = \frac{3\Gamma_s^2}{4J^2}$.

$\rightarrow r \approx \frac{2J^2}{\Gamma_s} \frac{1}{(1 + e \cos((1-\epsilon)\phi))}$

resummation of pert. th. to prevent its breakdown.
NOT periodic in ϕ , periodic in $(1-\epsilon)\phi$.







$$(1-\epsilon)(2\pi + \Delta\phi) = 2\pi \quad \text{to order } \epsilon$$

$$\text{i.e. } \Delta\phi = \epsilon 2\pi$$

$$(1-\epsilon)(1+\epsilon)2\pi$$

$$\Rightarrow r \approx \frac{2J^2}{\Gamma_s} \frac{1}{(1-\epsilon \cos((1-\epsilon)\phi))}$$

growing
Soln. is
$$u = \frac{3\Gamma_s^3}{8J^4} e \phi \sin \phi \rightarrow$$

$$u \approx \frac{\Gamma_s}{2J^2} (1 + e \cos \phi) + \frac{3\Gamma_s^3}{8J^4} e \phi \sin \phi$$

← secularly growing term
← pert. theory

$$\approx \frac{\Gamma_s}{2J^2} (1 + e \cos((1-\epsilon)\phi)) + o(\epsilon^2)$$

where $\epsilon = \frac{3\Gamma_s^2}{4J^2}$
← resummation of pert. theory

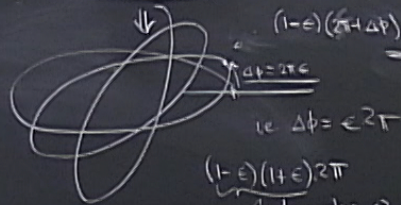
$$\left(\frac{d}{d\phi} \right)$$

$$- \left(u - \frac{\Gamma_s}{2J^2} \right) + \frac{3}{2} \Gamma_s u^2$$

STHO GR pert.

with du small

$$\frac{3}{2} \Gamma_s u^2 = -du + \frac{3}{2} \Gamma_s$$



growing
Soln. is
$$u = \frac{3r_s^2}{8J^2} e \phi \sin \phi \rightarrow u'' \sim \frac{2\phi'(g m)}{2 \cos \phi} - \phi \sin \phi$$

$$(1-\epsilon)(\pi + \Delta\phi) = 2\pi \rightarrow u \approx \frac{r_s}{2J^2} (1 + e \cos \phi) + \frac{3r_s^2}{8J^2} e \phi \sin \phi$$

+ order ϵ

$$\approx \frac{r_s}{2J^2} (1 + e \cos((1-\epsilon)\phi))$$

ie $\Delta\phi = \epsilon 2\pi$

(1-epsilon)(1+epsilon)2pi
1 + order ϵ^2

$\rightarrow r \approx \frac{2J^2}{r_s} \frac{1}{1 + e \cos((1-\epsilon)\phi)}$

secularly growing term
pert. theory fails
where $\epsilon = \frac{3r_s^2}{4J^2}$
resummation of parts that prevent its breakdown

$\Rightarrow$ Perihelion shift
(change in ϕ required to reach the max radius)

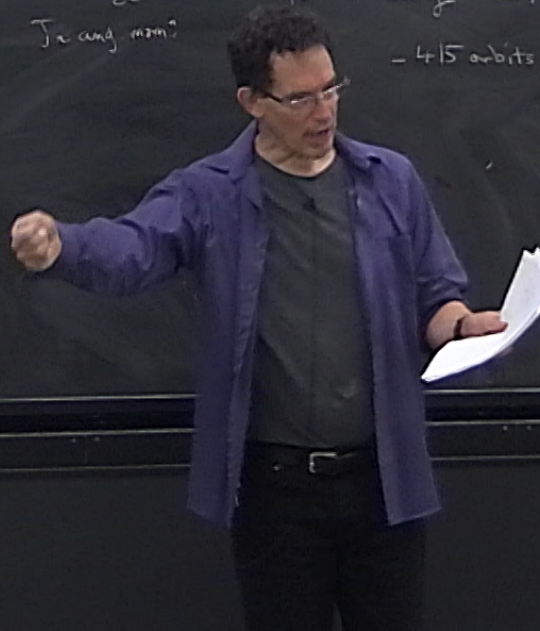
$$\Delta\phi = \frac{2\pi \cdot 3r_s^2}{4J^2}$$

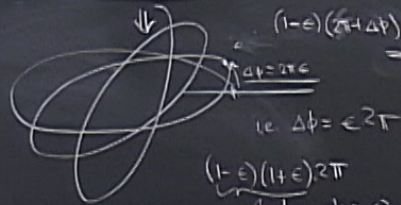
$$r_s = \frac{2GM_0}{c^2}$$

for any mass?

for Mercury $\Delta\phi = 5 \times 10^{-7}$ radians/orbit

- 415 orbits per century $\Rightarrow \Delta\phi = 43''/\text{century}$ in precise agreement with astronomical anomaly.





growing
Soln. is
$$u = \frac{3r_s^2}{8J^2} e \phi \sin \phi \Rightarrow u'' \sim \frac{2\phi'(g m)}{2 \cos \phi} - \phi \sin \phi$$

$$(1-\epsilon)(2\pi + \Delta\phi) = 2\pi \quad \rightarrow \quad u \approx \frac{r_s}{2J^2} (1 + e \cos \phi) + \frac{3r_s^2}{8J^2} e \phi \sin \phi$$

+ order ϵ

ie $\Delta\phi = \epsilon 2\pi$

$$\approx \frac{r_s}{2J^2} (1 + e \cos((1-\epsilon)\phi))$$

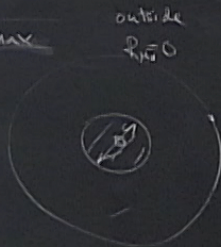
(1- ϵ)(1+ ϵ)2 π
1 under ϕ

$\rightarrow r \approx \frac{2J^2}{r_s} \frac{1}{1 + e \cos((1-\epsilon)\phi)}$

secularly growing term
pert. theory fails
where $\epsilon = \frac{3r_s^2}{4J^2}$
resummation of parts that prevent its breakdown.

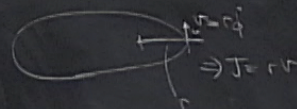
$\Rightarrow$ Perihelion shift $\Delta\phi = \frac{2\pi \cdot 3r_s^2}{4J^2}$

(change in ϕ required to reach the max radius)



outside
 $r_{\text{min}} = 0$

$$\dot{\phi} = \frac{J}{r^2} \Rightarrow J = \dot{\phi} r^2 = \text{const}$$



for Mercury $\Delta\phi = 5 \times 10^{-7}$ radians/orbit

- 415 orbits per century $\Rightarrow \Delta\phi = 43''/\text{century}$, in precise agreement with astronomical anomaly.