recap

\[ R^\sigma_{\rho \mu \nu} = - \Gamma^\sigma_{\rho \mu \nu} + \Gamma^\sigma_{\rho \mu} \Gamma^\rho_{\mu \nu} - (\mu \leftrightarrow \nu) \]

\[ \approx \frac{1}{2} \nabla^\alpha (g_{\nu \mu, \rho} - g_{\rho \mu, \nu} - g_{\mu \nu, \rho} + g_{\rho \nu, \mu}) \]
\[ \mathcal{R}_{\rho\mu\nu} = -\Gamma_{\rho\mu\nu}^\sigma + \Gamma_{\rho\mu\lambda}^\sigma \Gamma_{\lambda\nu}^\rho - (\mu\nu) \]

\[ g_{\mu\nu} \approx \eta_{\mu\nu} + \frac{1}{2} \eta_{\lambda\lambda} \frac{\partial}{\partial x^\lambda} \]

\[ \approx \frac{1}{2} \eta^\lambda \left( 9 \eta_{\lambda\rho\mu} - 9 \eta_{\lambda\mu\rho} - 9 \eta_{\rho\lambda\mu} + 9 \eta_{\rho\mu\lambda} \right) \]
Recap

\[ R^\sigma_{\rho\mu\nu} = -\Gamma^\sigma_{\rho\mu\nu} + \Gamma^\sigma_{\rho\nu\sigma} \Gamma^\rho_{\mu\sigma} - (\text{mass}) \]

\[ R^\sigma_{\rho\mu\nu}(0) = \frac{1}{2} \nabla^\sigma \left( g_{\nu\xi,\rho\mu} - g_{\rho,\mu,\nu} - g_{\rho,\sigma,\mu} + g_{\rho,\mu,\sigma} \right) (0) \]
Recap

\[ R^\sigma_{\mu\nu} = -\Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\rho\lambda} \Gamma^{\rho\lambda}_{\mu\nu} - (\text{tr}R) \]

\[ R^\sigma_{\mu\nu}(0) = \frac{1}{2} \nabla^\lambda \left( g_{\lambda\mu,\rho\nu} - g_{\rho\nu,\lambda\mu} - g_{\rho\lambda,\mu\nu} + g_{\mu\nu,\rho\lambda} \right) \]

\[ \text{Other equations:} \]

\[ \Gamma^\sigma_{\mu\nu} \]
\[ g_{\mu\nu} = 0 \]
\[ \nabla^\lambda = 0 \]
\[ \nabla^\sigma = 0 \]
Recap

\[ R_{\mu
u}^\sigma = -\Gamma^\sigma_{\mu\rho\nu} + \Gamma^\sigma_{\mu\nu\rho} - \Gamma^\sigma_{\rho\mu\nu} \]

\[ R_{\mu
u}^\sigma (x) = \frac{1}{2} \varepsilon_{\sigma}^{\alpha \beta} (g_{\alpha\mu} - \eta_{\alpha\mu} - g_{\beta\nu} + \eta_{\beta\nu}) \]

\[ R_{\mu
u} = \begin{cases} 1) \text{Symm under exchange of 1st pair and 2nd pair of indices} \quad (= R_{\nu\mu\rho} ) \\ 2) \text{Antisymm on 1st pair and 2nd pair of indices} \quad (= - R_{\nu\mu\rho} , = - R_{\mu\nu\rho} ) \end{cases} \]
Recap

\[ R^\alpha_{\mu
u} = - \Gamma^\alpha_{\rho
u \mu} + \Gamma^\alpha_{\rho \mu} \Gamma^\rho_{\nu} = (u_{\alpha v}) \]

\[ R^\alpha_{\mu
u} = \frac{1}{2} \xi^\lambda \left( 9 g_{\lambda \mu} g_{\nu \alpha} - 9 g_{\lambda \nu} g_{\mu \alpha} - 9 g_{\lambda \alpha} g_{\mu \nu} + 9 g_{\lambda \nu} g_{\alpha \mu} \right) (0) \]

\[ R_{\xi_{\mu \nu}} = 0 \]

\[ g_{\mu \nu} = 0 \]

\[ \xi_{\lambda \nu} = 0 \]

\[ \xi_{\mu \nu} = 0 \]

\[ \xi_{\alpha \nu} = 0 \]

1) Symmetric under exchange of 1st pair and 2nd pair of indices

\[ R_{\xi_{\mu \nu}} = R_{\xi_{\nu \mu}} \]

2) Anti-symmetric on 1st pair and 2nd pair of indices

\[ (- R_{\xi_{\mu \nu}}) = - R_{\xi_{\nu \mu}} \]

3) Cyclic in last 3 indices

\[ R_{\xi_{\mu \nu}} + R_{\xi_{\nu \rho}} + R_{\xi_{\rho \mu}} = 0. \]
Ex: Can prove that $R^\mu{}_{\nu\rho\sigma}$ has $\frac{5^2(5^2-1)}{12}$ independent components.

A pair of indices:

\(-R^\mu{}_{\nu\rho\sigma}, -R^\mu{}_{\nu\rho\sigma}\)

$R^\mu{}_{\nu\rho\sigma} = 0$. 

A pair of indices
Ex. can prove that $R_{\mu
u}$ has $\frac{D(D^2-1)}{12}$ independent components in $D$ spacetime dims.
Recap:

\[ A X^\mu^\nu = 8 \eta T^\mu^\nu \]

where \( \nabla^\mu X^\nu = 0 \neq g^\mu\nu \).
Recap:

\[ A X^{\mu \nu} = 8 \pi G T^{\mu \nu} \]

where \[ \nabla_\mu X^{\mu \nu} = 0 \neq g_{\mu \nu} \text{ as an identity.} \]

candidate for \[ X^{\mu \nu} = G^{\mu \nu} = R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R \], since this identically obeys \[ \nabla_\mu G^{\mu \nu} = 0. \]
This identically obeys $\nabla \! g^{\mu} = 0$. (Consequence of Bianchi identity)
Recap:

\[ AX^{\mu \nu} = 8 \pi G T^{\mu \nu} \]

where \( \nabla_{\alpha} X^{\mu \nu} = 0 \neq g_{\mu \nu} \) as an identity.

Candidate for \( X^{\mu \nu} = G^{\mu \nu} = R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R \), since this identically obeys...
Recap:

\[ A X^{\mu \nu} = 8 \pi G T^{\mu \nu} \]

where \( \nabla_{\mu} X^{\mu \nu} = 0 \neq g_{\mu \nu} \) as an identity.

Candidate for \( X^{\mu \nu} = G^{\mu \nu} = R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R \), since this identically obeys

Would like to fix constant \( A \) by comparing with the Newtonian limit.

First, consider spatial components of \( \mathbf{0} \):

\[ T^{ij} \ll T^{oo} \]
\[ A X_{\mu \nu} = 8 \pi G T_{\mu \nu} \]

where \( \nabla_{\mu} X_{\nu} = 0 \neq g_{\mu \nu} \) as an identity.

Consider spatial components at \( r \): \( T_00 \leq T^{00} \Rightarrow G^0 \) is small re. \( R^0 \approx 3/3 \delta_{\mu \nu} R \).
but \[ R = g^\mu{}_{\nu} R_{\mu\nu} \approx \eta^{\mu\nu} R_{\mu\nu} = -R_{\sigma\sigma} + R_{ii} \]

\[ g \approx \eta + h \]

\[ \eta^{\mu\nu} \]
Consider spatial components of $T_j^i$ \(\ll T^{00} \Rightarrow G_j^i \) is small, i.e., \( R^{ij} \approx \frac{1}{2} \Delta \, R \) $(\dagger)$. 

\[ R = g^{\mu\nu} R_{\mu\nu} \approx \eta^{\mu\nu} R_{\mu\nu} = -R_{00} + R_{ii} = -R_{00} + \frac{3}{2} R \quad \text{from } (\dagger). \]
but \[ R = g^{\mu \nu} R_{\mu \nu} \approx \eta^{\mu \nu} R_{\mu \nu} = -R_{00} + R_{ii} = -R_{00} + \frac{2}{3} R \] from (1).

\[ g - \eta + h \]

\[ g^{\mu \nu} h_{\mu \nu} \Rightarrow R_{00} \approx \frac{1}{2} R. \quad \Rightarrow G_{00} = R_{00} + \frac{1}{2} R \approx 2R_{00}. \]
\[ R^\gamma_{\mu\nu} = \frac{1}{2} \partial^\gamma (\partial_{\nu}\partial_{\mu} - \partial_{\nu}\partial_{\mu} - \partial_{\mu}\partial_{\nu} + \partial_{\mu}\partial_{\nu}) \]  

\[ \star \]

1) Symmetric under exchange of 1st pair and 2nd pair of indices  
\[ R^\gamma_{\mu\nu} = R_{\nu\mu\gamma} \]  

2) Anti-symmetric on 1st pair and 2nd pair of indices  
\[ R^\gamma_{\mu\nu} = -R_{\nu\mu\gamma} \]  

3) Cyclic in last 3 indices  
\[ R^\gamma_{\mu\nu} + R^\gamma_{\nu\rho} + R^\gamma_{\rho\mu} = 0 \]

\[ \Rightarrow R_{\gamma\gamma} = \frac{1}{2} R. \Rightarrow G_{\gamma\gamma} = R_{\gamma\gamma} + \frac{1}{2} R = 2 R_{\gamma\gamma} \]
2) antisymmetric on 1st pair and 2nd pair of indices \(-R_{\rho m \nu}, -R_{\rho m \nu}\)

3) cyclic in last 3 indices \(R_{\rho m \nu} + R_{\rho m \mu} + R_{\rho m \nu} = 0\).

\(\omega h_{\mu \nu} \Rightarrow R_{\alpha \beta} = \frac{1}{2} R. \Rightarrow G_{\alpha \beta} = R_{\alpha \beta} + \frac{1}{2} R = 2 R_{\alpha \beta}\).

But from (\(\ast\)), \(R_{\rho m \nu} = \frac{1}{2}(\omega_{\mu \nu} h_{\rho \alpha} - \omega_{\rho \nu} h_{\mu \alpha} - \omega_{\rho \alpha} h_{\mu \nu} + h_{\mu \nu} \omega_{\rho \alpha}) + o(\hbar^2)\).

\(\Rightarrow R_{\alpha \beta} = \gamma^\alpha_{\mu \nu} R_{\rho m \nu} \alpha \beta \).
2) antisymmetric on 1st pair and 2nd pair of indices \(-R_{\mu
u}\), \(-R_{\nu\mu}\)

3) cyclic in last 3 indices \(R_{\rho\mu
u} + R_{\nu\rho\mu} + R_{\mu\nu\rho} = 0\).

\[\nabla^2 h_{\mu\nu} \Rightarrow R_{\mu\nu} = \frac{1}{2} R. \Rightarrow G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R = 2 R_{\mu\nu}.\]

But from (\(*\)), \(R_{\rho\mu\nu} = \frac{1}{2} (\nabla^2 h_{\rho\mu\nu} - h_{\rho\mu\nu,\rho} - h_{\rho\mu,\nu} + h_{\rho,\mu\nu}) + o(h^2)\)

\[\Rightarrow R_{\mu\nu} = \nabla^2 R_{\mu\nu} + o(h^2)\]
2) antisymmetric on 1st pair and 2nd pair of indices \( (- = R_{\mu\nu\rho}, \quad = - R_{\nu\rho\mu}) \)

3) cyclic in last 3 indices \( R_{\mu\nu\rho} + R_{\rho\mu\nu} + R_{\nu\rho\mu} = 0 \).

\[ \nabla^2 h_{\mu\nu} \Rightarrow R_{\mu\nu} = \frac{1}{2} R. \Rightarrow G_{\mu\nu} = R_{\mu\nu} + \frac{1}{4} R = 2 R_{\mu\nu} \]

But from \((*)\), \( R_{\mu\nu\rho} = \frac{1}{2} (h_{\nu,\rho} - h_{\rho,\nu} - h_{\rho,\nu} + h_{\nu,\rho}) + o(h^2) \)

\[ \Rightarrow R_{\mu\nu} = h_{\mu,\rho} R_{\rho,\nu} = - R_{\nu,\rho,\mu} + R_{\rho,\nu,\mu} \]
\[ R^m_{n} \approx R^{\mu}_{\nu} = -R_{\mu\nu} + \Gamma_{\mu\nu} = -R_{\mu\nu} + \frac{3}{2} R^{(4)} \]

\[ \Rightarrow R_{\mu\nu} \approx \frac{1}{2} R. \Rightarrow G_{\mu\nu} \approx R_{\mu\nu} + \frac{1}{2} R \approx 2 \]

\[ R_{\mu\nu} = \frac{1}{2} \left( \nabla_{\mu} \nabla_{\nu} - \nabla_{\mu} \nabla_{\nu} + \nabla_{\mu} \nabla_{\nu} \right) + o(\hbar^2) \]

\[ R_{\mu\nu} \approx \nabla_{\mu} \nabla_{\nu} \approx -\frac{1}{2} \nabla^2 h_{\mu\nu} \]

\[ \Rightarrow R_{\mu\nu} \approx \nabla_{\mu} \nabla_{\nu} \approx \frac{3}{2} R_{\mu\nu} \approx -\frac{3}{2} \nabla^2 h_{\mu\nu} \]
\[ R_{\mu \nu} = \eta^{\mu \nu} R_{\rho \sigma} - R_{\mu \rho} + R_{\nu \sigma} = -R_{\mu \rho} + \frac{3}{2} R \text{ from (1).} \]

\[ \Rightarrow G_{\mu \nu} = R_{\mu \rho} + \frac{1}{2} R \approx 2 R_{\mu \rho}. \]

\[ R_{\rho \sigma \nu} = \frac{1}{2} \left( \eta_{\rho \nu} \partial_\sigma h_{\alpha \beta} - \eta_{\rho \beta} \partial_\sigma h_{\alpha \nu} + \eta_{\sigma \nu} \partial_\rho h_{\alpha \beta} - \eta_{\sigma \beta} \partial_\rho h_{\alpha \nu} \right) + o(h^{2}) \]

\[ R_{\rho 0 0} = -\frac{1}{2} \partial_\rho h_{00}, \quad R_{\rho 0 0} = -\frac{1}{2} \partial_\rho^2 h_{00}. \]
\[ R_{\mu
u} \approx \eta_{\mu
u} R - R_{\mu\nu} + R_{\mu\nu} = -R_{\mu\nu} + \frac{3}{2} R \text{ from (\#).} \]

\[ \Rightarrow R_{\mu\nu} \approx \frac{1}{2} R. \Rightarrow G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R \approx 2R_{\mu\nu}. (\#) \]

\[ R_{\mu\nu} = \frac{1}{2} \left( h_{\mu\nu} - h_{\mu\nu}h_{\rho\sigma}h_{\rho\sigma}^{\nu} - h_{\mu\nu}^{\rho} + h_{\mu\rho}h_{\rho\nu}^{\nu} \right) + O(h^3) \]

\[ R_{\mu\nu} = -\frac{1}{2} h_{\mu\nu}^{\rho}c_i = -\frac{1}{2} \Delta^2 h_{\mu\nu}. \]

\[ R_{\mu\nu} = \eta_{\mu\nu} R_{\mu\nu} = -R_{\mu\nu} + R_{\mu\nu} = -\frac{1}{2} \Delta^2 h_{\mu\nu}. (\#) \]

\[ \text{Back to 0: } AG^{\circ} \approx 2AR^{\circ} \approx -AV^{2}h_{\mu\nu}^{c} = 8\pi G T_{\mu\nu} = 8\pi G \rho_m. \]
\[ R_{\mu \nu} = R_{\mu \nu}^{\text{ind}} = -R_{\mu \sigma} + R_{\sigma \mu} = -R_{\sigma \mu}^{\text{ind}} + \frac{3}{2} R \quad \text{from (1)}. \]

\[ \Rightarrow R_{\mu \mu} \approx \frac{1}{2} R. \quad \Rightarrow G_{\mu \mu} = R_{\mu \mu} + \frac{1}{2} R \approx 2 R_{\mu \mu}. \quad (\exists) \]

\[ R_{\sigma \rho \eta \nu} = \frac{1}{2} (h_{\mu \sigma, \rho} + h_{\rho \sigma, \mu} - h_{\mu \rho, \sigma} - h_{\sigma \rho, \mu}) + o(h^2) \quad R_{\rho \sigma \mu \nu} = \frac{1}{2} h_{\rho \sigma, \mu} = -\frac{1}{2} \nabla^2 h_{\rho \sigma}. \]

\[ h_{\sigma \sigma} = -2 \phi \]

\[ R_{\sigma \rho \eta \nu} = \frac{1}{2} R_{\rho \sigma \mu \nu} + R_{\rho \sigma \nu \mu} = -\frac{1}{2} \nabla^2 h_{\rho \sigma}. \quad (\exists) \]

Back to 0: \[ AG^{\text{out}} \approx 2 A R_{\rho \sigma} \approx -A \nabla^2 h_{\rho \sigma} = 8 \pi G T_{\rho \sigma} = 8 \pi G \rho_m \]
\[ R_{ii} = -R + \frac{3}{2} R \] 

\[ G_{oo} = R_{oo} \approx 2 R_{oo} \] 

\[ = -\frac{1}{2} h_{oo,ii} = -\frac{1}{2} \nabla^2 h_{oo} \]

\[ h_{oo} = -2 \phi \]

\[ 3 \pi G T_{oo} = 8 \pi G p_m \implies A \nabla^2 \phi = 4 \pi G p_m \text{ if } A = 1 \]
\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \Rightarrow \text{Poisson eqn in Newtonian limit.} \]
So, we have shown that \[ G_{\mu\nu} = 8\pi G T_{\mu\nu} \]

\[ \Rightarrow \text{Poisson eqn} \]

Einstein's field equation.
So, we have shown that \[ G_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^4} \] is Einstein’s field equation. By dimensional analysis, this equation implies the Poisson equation.
So, we have shown that

\[ G_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^4} \]

\[ \Rightarrow \text{Poisson equation} \]

Einstein's field equation.
Poisson eqn in Newtonian limit.
Poisson eq. in Newtonian limit.

Field equation.
Poisson eqn in Newtonian limit.

equation
Poisson eq in Newtonian limit.
Poisson eq. in Newtonian limit.

Confirmation # 1

Mercury
43" per century!
almost exactly predicted by G.R.
Poisson eqn. in Newtonian limit.

\[ \text{Poisson eqn. in Newtonian limit.} \]

\[ \text{Conformation #1} \]

\[ \text{Mercury} \]

\[ 43'' \text{ per century!} \]

\[ \text{almost exactly predicted by GR.} \]

\[ \text{Sun} \]

\[ \text{Eddington} \]

\[ \text{Sun} \]

\[ \text{Sun} \]
So, we have shown that \( G_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^4} \) is Einstein's field equation. By dimensional analysis, this equation leads to the Poisson equation:

- now the foundation for cosmology.
Gauge theories and gravitational waves

Analogy with Maxwell:
\[ \text{set } J^\nu = 0 \]
\[ \partial_\mu F^{\mu\nu} = 0 \]
(showed before that $E$ and $B$ propagate as waves, at speed of light).
Gauge theories and gravitational waves.

Analogy with Maxwell:

\( \partial \nu F^{\mu\nu} = 0 \)

(shown before that \( \mathbf{E} \) and \( \mathbf{B} \) propagate)

Recall \( F^{\mu\nu} = \partial^\mu A_\nu - \partial^\nu A_\mu \)
Gauge theories and gravitational waves.

With Maxwell:
\[(\text{set } J^\mu = 0)\]

In Flat Space,

\[\partial^\mu J_\mu = 0\]

Recall

\[\varepsilon_{\mu\nu\alpha\beta} \partial_\mu A_\nu = 0\]

(shown before that \( E \) and \( B \) propagate as waves, at)

\[A_\mu \leftrightarrow g_{\mu

\text{Gauge transformation:}

\[A_\mu \rightarrow A_\mu - \partial_\mu \Lambda\]

\[J_{\mu

\frac{\partial x_\nu}{\partial x_\nu} \frac{\partial x_{\mu'}}{\partial x_{\mu'}} J_{\mu' \nu'}(x') = \frac{\partial x_\mu}{\partial x_{\mu'}} \frac{\partial x_{\nu}}{\partial x_{\nu'}} J_{\mu \nu}(x)\]
Gauge theories and gravitational waves.

Maxwell:
\[ \partial_\mu F^{\mu \nu} = 0 \]  
(set \( J^\alpha = 0 \))

Recall:
\[ F^{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

(Shown before that \( E \) and \( B \) propagate as waves, at)

\[ A_\mu \rightarrow g^{\mu \nu} \]

\[ A_\mu \rightarrow g_{\mu \nu} \rightarrow g_{\mu \nu} (x) = \frac{2}{\sqrt{2}} \frac{\partial x^\mu}{\partial x'^{\nu}} g_{\mu \nu} \]
Gauge theories and gravitational waves.

Analogy with Maxwell:

\( \nabla \cdot F = 0 \)  
\( \nabla \times F = 0 \)  
\( \text{shown before that } E \text{ and } B \text{ propagate} \)

Recall:  
\( F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \)

(\star) \Rightarrow  
\( \partial_{\mu} \partial_{\nu} A_{\nu} - \partial_{\nu} \partial_{\mu} A_{\nu} = 0 \)

A \_\mu \rightarrow j_{\mu} \text{ (gauge transformation)}
Gauge theories and gravitational waves.

Analogy with Maxwell:
- \( \nabla F^\mu = 0 \) (set \( J^\mu = 0 \))
- Recall \( F^\mu_\nu = \partial_\mu A_\nu - \partial_\nu A_\mu \)
- \( (\star) \Rightarrow \nabla_\mu (\partial^\mu A_\nu - \partial_\nu A^\mu) = 0 \)

(not quite wave eq.)
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

\[ \exists \, \partial_A x^\mu A_\nu - \partial_\nu (\partial_A x^\mu) = 0 \]

\[ A_\mu \leftrightarrow g_{\mu\nu} \]

\[ \text{gauge transform: } A_\mu \rightarrow A_\mu' = g^{\mu\nu} \partial_\nu (\partial_\mu x^\nu) \]

\[ k^\nu x^\nu = -k^0 x^0 + k^i x^i \]

Fourier Transform \[ A_\mu = \sum E_\mu e^{ikx} \]
Recall $F_{uv} = \partial_u A_v - \partial_v A_u$

$\Rightarrow \partial_u x^\nu A_v - \partial_v x^\nu A_u = 0$  (**) 

Fourier Transform $A^\nu = \sum_k \tilde{A}_{k}^\nu e^{2\pi ik \cdot x}$

\[ (k^2 \delta^\nu_v - k^m k_v) A^v = 0 \]
Recall $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

\[ (*) \quad \partial_\mu A^\mu - 2\omega(a) = 0 \quad (**) \]

**Formal Theorem**

$A_{\mu} = \sum_k \tilde{A}_{\mu}^k e^{-ik\cdot x}$

\[ \Rightarrow \quad (k^2 \delta_{\mu}^\nu - k_\mu k_\nu) \tilde{A}^\nu_k = 0 \]
\[ \tilde{\mathcal{A}}(k) = 0 \]
\[ a_{\mu}(x) \Rightarrow a'_{\nu}(x') = \frac{2 x^n x^a}{dx^n dx^a} \frac{\partial}{\partial t} \frac{\partial}{\partial x'^n} f_{\nu} \]

\[ = k^0 x^0 + k \cdot x. \]

\[ \Rightarrow (k^2 \delta^M_{\nu} - k^M k_{\nu}) \tilde{A}^{\nu}(k) = 0 \]

algebraic eq

Matrix has zero eigenvalues.

Matrix operation
\[
\n\frac{\partial^2}{\partial x^2} \frac{1}{x} \frac{\partial}{\partial x} = -k^2 \frac{1}{x^2} \frac{\partial}{\partial x}
\]

\[
\frac{\partial^2}{\partial x^2} \frac{1}{x} \frac{\partial}{\partial x} = -k^2 \frac{1}{x^2} \frac{\partial}{\partial x}
\]

Matrix has zero eigenvalues.

\[
(k^2 \delta^m_v - k^m_k v)
\]

algebraic zero
\[ A^{\mu} G_{\mu} (x) \rightarrow g^{\mu\nu} (x') = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial x^\nu} g_{\mu\nu} \]

\[ \Rightarrow k^2 \delta^{\mu\nu} - k^\mu k^\nu \tilde{A}^\nu (k) = 0 \]

Matrix has zero eigenvalues.

\[ (k^2 \delta^{\mu\nu} - k^\mu k^\nu) k^\nu = 0. \]
\[ F_{\mu} = \partial_{\nu} A_{\nu} - \partial_{\nu} A_{\mu} \]

\[ A_{\mu} - \alpha(a_{\mu}^m) = 0 \quad (**) \]

\[ A_{\mu} \rightarrow A_{\mu}'(k') = g_{\mu\nu}(g_{\nu} = \frac{\partial x}{\partial x'} g) \]

\[ k' x' = -k x + k x' \]

\[ A_{\mu}' = \sum_{k} \tilde{A}_{\mu}'(k) e^{i k \cdot x} \quad (***) \]

\[ \Rightarrow \quad (k^2 \delta_{\mu}^{\nu} - k_{\mu} k_{\nu}) \tilde{A}'(k) = 0 \]

Matrix equation

\[ \tilde{A}'(k) \rightarrow \tilde{A}'(k) + \epsilon(k k' k) \quad \text{is still a solution, for all } \epsilon. \]
Recollect \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \)

\[ \partial_\mu A_\nu - \partial_\nu A_\mu = 0 \quad (**) \]

Gauge Transform: \( A_\mu \rightarrow A_\mu + \partial_\mu \theta \)

\[ k^\nu x_\nu = -k^0 x^0 + k^x \quad \text{matrix operation} \]

\[ A_\mu \rightarrow g_{\mu \nu} A_\nu \quad (**) \]

Fourier Transform: \( A_\mu = \sum_k \tilde{A}^{(k)} e^{i k \cdot x} \)

\[ \tilde{A}^{(k)} \rightarrow \tilde{A}^{(k)} + ie \rho (k) \]

\[ A_\nu \rightarrow A_\nu + \delta_\nu \theta \]

\( A_\nu \) is still a solution, for all \( \theta \).
Recall \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \)

\[ \partial_\mu A_\nu - \partial_\nu A_\mu = 0 \]  \hspace{1cm} (\star)

Gauge transformation: \( A_{\mu} \rightarrow A_{\mu}(x') = A_{\mu}(x) + \partial_\mu \Theta(x) \)

\[ k^\nu x_\nu = -k^0 x^0 + k \cdot x. \]

Fourier transform: \( A_{\mu}(x) \rightarrow \tilde{A}_{\mu}(k) \)

\[ (k^2 \delta_{\mu}^\nu - k^\nu k_\mu) \tilde{A}^\nu = 0 \]

Matrix operator

\( \tilde{A}''(k) \rightarrow \tilde{A}'(k) + i\epsilon(k)k^\nu \)

is still a solution for all \( \epsilon \).

\( A_\nu \rightarrow A_\nu + \delta_\nu \Theta \)

To solve Maxwell's equations, choose \( \Theta(x, \epsilon) \) to set \( \partial_\mu A_\mu = 0 \) \( \Leftrightarrow k_\mu \tilde{A}_\mu = 0 \)
\[ (k^2 \delta^m_n - k^m k^n) \tilde{A}^n(k) = 0 \quad \text{algebraic eqn} \]

Matrix has zero eigenvalues.

\[ (k^2 \delta^m_n - k^m k^n) k^v k_v = 0. \]

Still a solution, for all \( \epsilon \).

\[ \epsilon \text{ to set } \tilde{A}^m = 0 \iff k^m \tilde{A}^m = 0, \text{ then } \]

\[ \begin{align*}
(\text{\textit{ unknown})} & \quad \text{solved if } k^2 = 0 \\
(\text{\textit{ known})}
\end{align*} \]

\[ k^2 - k^2 = 0 \]

\[ k^0 = \pm |k| \]
Gauge fixing:

1) Set $\partial_{\mu} A^\mu = 0$ Lorentz gauge

2) residual gauge freedom

$A^\mu \to A^\mu + \partial_{\mu} \Theta$

$\tilde{A}^\mu \to \tilde{A}^\mu + i \epsilon(x) k^\mu$ with $k^2 = 0$. 

To solve Maxwell, choose $\Theta(x, t) = -\tilde{A}^\mu k_{\mu}$ is still a solution.
Gauge fixing:

1) Set $\partial_\mu A^\mu = 0$ Lorentz gauge

2) residual gauge freedom

$A^\mu \rightarrow A^\mu + \delta A^\mu$, with $\delta \Theta = 0$

$\tilde{A}_\mu^\prime \rightarrow \tilde{A}_\mu^\prime + i\epsilon(k)k^\mu$ with $k^2 = 0$

$\tilde{A}_\mu^\prime \rightarrow \tilde{A}_\mu^\prime + i\epsilon(k)k^\mu$ is still a solution

To solve Maxwell, choose $\Theta (\vec{x} \in \mathbb{R})$ to
So far EM, $A_m$ (4 components): $4 = 2 + 1 + 1$

↑

physical modes

↑

residual gauge freedom

↑

fixed by Lorentz gauge
So for EM, $A_{\mu}$ (4 components):

4 = 2 +

GR: first problem is to fix a gauge.
\[(k^2 \delta_{vv} - k^m k_v) \hat{A}(k) = 0\]

Matrix has zero eigenvalues.

\[(k^2 \delta_{vv} - k^m k_v) k^v = 0.\]

Still a solution, for all \( \epsilon \).

\[\text{to set} \quad \epsilon m \hat{A}^m = 0 \iff k_m \hat{A}^m = 0, \quad \text{then} \]

\[
\begin{align*}
(k^2) & \quad \text{solved if} \quad k^2 = 0 \\
(k k^v) & \quad k^2 - |k|^2 = 0 \\
& \quad k^0 = \pm |k|
\end{align*}
\]
So for EM, $A_\mu$ (4 components):

**GR:** first problem is to fix a gauge.
So for EM, \( A_\mu \) (4 components): \( 4 = 2 + 2 \)

**GR:** first problem is to fix a gauge.

\[ g'_\mu^\nu(x) = g_{\kappa\lambda}(x) \frac{dx^\kappa}{dx^\mu} \frac{dx^\lambda}{dx^\nu} \]

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]
So for EM, $A_\mu$ (4 components): $4 = 2 + 1$

**GR:** first problem is to fix a gauge.

$$g'_{\mu\nu}(x) = g_{\alpha\beta}(x) \frac{dx^\mu}{dx^\alpha} \frac{dx^\nu}{dx^\beta}$$

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$$

$$g'_{\mu\nu} \approx \eta_{\mu\nu} + h'_{\mu\nu}$$

$$x'^\mu \approx x^\mu + E^\mu(x)$$
So for EM, $A_\mu$ (4 components): $4 = 2 + \uparrow$

GR: first problem is to fix a gauge.

\[ g'_{\mu\nu}(x) = g_{\mu\nu}(x) \frac{dx^\alpha}{dx'} \frac{dx'}{dx^\mu} \]

\[ h'_{\mu\nu} = h_{\mu\nu} - \epsilon_{\mu\nu\alpha\beta} \nabla^\alpha \nabla^\beta \]

\[ g^\mu_\nu = g_{\mu\nu} + h_{\mu\nu} \]

\[ g'^\mu_\nu = g_{\mu\nu} + h'_{\mu\nu} \]

\[ x'^\mu = x^\mu + \delta^\mu_\nu (x) \]
So for EM, $A_\mu$ (4 components): $\nabla^2 A_\mu = 4 = 2 + \Gamma^\mu_{\nu \sigma} \Gamma^\nu_{\mu \sigma}$

GR: first problem is to fix a gauge.

\[
\begin{align*}
g'_{\mu \nu}(x) &= g_{\mu \nu}(x) \frac{\partial x^\lambda}{\partial x'_{\lambda}} \frac{\partial x'_{\sigma}}{\partial x^\sigma} \\
g_{\mu \nu} &= \delta_{\mu \nu} + h_{\mu \nu} \\
g'_{\mu \nu} &= \delta_{\mu \nu} + h'_{\mu \nu} \\
\eta_{\mu \nu} &= g_{\mu \nu}^{-1} \\
x'_{\mu} &= x^\nu + \epsilon_{\mu}^\nu(x)
\end{align*}
\]
So for EM, $A_\mu$ (4 components): $4 = 2 + \text{physical modes}$.

GR: first problem is to fix a gauge.

$$g'_{\mu\nu}(x') = g^-_{\alpha\beta}(x) \frac{dx^\alpha}{dx'^\alpha} \frac{dx^\beta}{dx'^\beta}$$

$$g_{\mu\nu} = h_{\mu\nu} + h_{\mu\nu}^\text{ex}$$

$$g'_{\mu\nu} = \eta_{\mu\nu} + \eta_{\mu\nu}^\text{ex}$$

$$x'^\mu = x^\mu + \text{EM}(x)$$

$$h_{\mu\nu} = h_{\mu\nu}^\text{em} - E_{\mu\nu} - E_{\nu|\mu}$$

$$E_\mu \rightarrow \Theta \text{ in EM, }$$

$\rightarrow$ 4 dof's in gauge trans.
So for EM, $A_\mu$ (4 components): $4 = 2 + 2.$

GR: first problem is to fix a gauge.

$$g'_\mu\nu(x') = g_{\mu\nu}(x) \frac{dx^\mu}{dx'^\mu} \frac{dx'^\nu}{dx^\nu}$$

$$g'_{\mu\nu} \approx h_{\mu\nu} + \dot{h}_{\mu\nu}$$

$$\dot{g}'_{\mu\nu} \approx \ddot{h}_{\mu\nu} + h'_{\mu\nu}$$

$$x'^\mu \approx x^\mu + \epsilon^\mu(x)$$

$$\epsilon^\mu \rightarrow \Theta \text{ in EM, } \Box = 0$$

$$\Theta_{\mu\nu} = -\epsilon_{\mu\nu} - \epsilon_{\mu\nu}$$

$$\epsilon_{\mu\nu} \rightarrow \Theta \text{ in EM, } \Box = 0$$

$\Theta A_\mu = 0$
So for EM, $A_\mu$ (4 components): $A_\mu = 0$

GR: first problem is to fix a gauge.

\[
g'_{\mu\nu}(\mathbf{x}) = g_{\mu\nu}(\mathbf{x}) \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\nu}{\partial x'^\nu}
\]

$g'_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$

$E^\alpha \rightarrow \Theta$ in EM.

$4$ dof's in gauge trans.
\[ g^\mu_\nu \Gamma^\tau_{\mu\nu} = 0 \]

\[ 4 \text{ eqns, which are not coord invariant, and therefore fix the gauge.} \]
\[ \sum_{\text{modes}} = 4 = 2 + 1 + 1 \]

4 eqns, which are not completely fixed by Lorentz gauge.

And therefore fix the gauge.

"Harmonic" gauge
\[ g_{\mu}^{\nu} \Gamma_{\mu}^{\nu} = 0 \]

4 eqns, which are not coord invariant, and therefore fix the gauge.

```
Harmonic gauge
\[ \Box \varphi = g^{\mu\nu} \nabla^\mu \nabla^\nu \varphi = g^{\mu\nu} \nabla^\mu \nabla_\mu \varphi = g^{\mu\nu} \partial_\nu \partial^\mu \varphi \]
```

\[ \partial_\mu A^\mu = 0 \]
physical modes
residual gauge freedom

are not coord invariant, and therefore fix the gauge.

in harmonic gauge

\[ \phi = g^{\mu\nu} \nabla^\nu \phi = g^{\mu\nu} \nabla_x \delta^0 \phi = g^{\mu\nu} (\partial_x \phi - \nabla_x \partial^\nu \phi) = g^{\mu\nu} \partial_x \phi. \]
\[ \square \phi = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \phi = g^{\mu\nu} \partial_{\mu} \partial_{\nu} \phi = g^{\mu\nu} (\partial_{\mu} \partial_{\nu} - \Gamma_{\mu\nu}^{\lambda} \partial_{\lambda}) \phi = g^{\mu\nu} \partial_{\mu} \partial_{\nu} \phi. \]

In particular, if we regard the coordinates \( x^\mu \) as four scalars.
Harmonic gauge \( \Delta \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = g^{\mu\nu} \partial_\mu \partial_\nu \phi = g^{\mu\nu} (\partial_\mu \phi - \Gamma^\rho_{\mu\nu} \partial_\rho \phi) = g^{\mu\nu} \partial_\mu \phi \).

In particular, if we regard the coordinates \( x^\mu \) as four scalars,

\[ \square x = g^{\mu\nu} \partial_\mu \partial_\nu x^\rho = 0 \]

acting on scalars.
Harmonic gauge \[ \Box \phi = g^{\mu \nu} \nabla_\mu \nabla_\nu \phi = g^{\mu \nu} \nabla_\mu \nabla_\nu \phi = g^{\mu \nu} (\epsilon_{\mu \nu \sigma \rho} - \Gamma^\lambda_{\mu \nu} \epsilon_{\lambda \sigma \rho}) = g^{\mu \nu} \partial_\mu \partial_\nu \phi. \]

In particular, if we regard the coordinates \( x^\mu \) as four scalars, \( \Box x^\mu = g^{\mu \nu} \partial_\mu \partial_\nu x^\mu = 0 \) in harmonic gauge.
\[ \nabla^2 \phi = g^{\mu \nu} \nabla_\mu \nabla_\nu \phi = g^{\mu \nu} \nabla_\mu \partial_\nu \phi = g^{\mu \nu} (\partial_\mu \phi - \Gamma^\mu_{\nu \rho} \partial_\rho \phi) = g^{\mu \nu} \partial_\mu \partial_\nu \phi . \]

Moreover, if we regard the coordinates \( x^\mu \) as four scalars, we get

\[ \Box x^\mu = g^{\mu \nu} \partial_\mu \partial_\nu x^\rho = 0 \quad \text{in harmonic gauge.} \]
\[ \Gamma^\mu_{\rho \nu} = 0 \]

4 eqns, which are not coord invar, and therefore fix the gauge.

"Harmonic" gauge \( \Box \phi = g^{\mu \nu} \nabla_\mu \nabla_\nu \phi = g^{\mu \nu} (\partial_\mu \phi - \Gamma_\mu^{\rho \nu} \partial_\nu \phi) \)

in particular, if we regard the coordinates \( x^\mu \) as four

\[ \Box x^\mu = g^{\mu \nu} \partial_\nu x^\mu = 0 \quad \text{in harmonic} \]

acting on scalars.
Linearized Einstein: \[ G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \]
\[ G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \quad x g^{\mu} \Rightarrow R = 2R \Rightarrow R = 0 \]

\[ g^{\mu\nu} g_{\mu\nu} = 5_{\mu} = 4 \]
\[ G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \quad \Rightarrow g^{\mu\nu} R = 2R \Rightarrow R = 0. \]

\[ \Rightarrow R_{\mu\nu} = 0 \]

Einstein equations for \( T^{\mu\nu} = 0. \)
Linearised Einstein: \[ G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \quad \times g^{\mu\nu} \Rightarrow \]

\[ g^{\mu\nu} R_{\mu\nu} = \delta_{\mu\nu} = 4 \]

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

\[ R_{\mu\nu} = \frac{1}{2} \left( h^\lambda_{\mu\nu\lambda} - h^\lambda_{\mu\lambda\nu} + h^\lambda_{\nu\mu\lambda} - h^\lambda_{\nu\lambda\mu} + h_{\mu\nu}^\lambda + h_{\nu\mu}^\lambda \right) \]
Linearised Einstein: $G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$, $x^m \Rightarrow \Rightarrow R_{\mu\nu} = 0$}

$g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$

$R_{\mu\nu} = \frac{1}{2} \left( h^{\lambda}_{\mu\lambda,\nu} - h^{\lambda}_{\nu,\mu\lambda} - h^{\lambda}_{\lambda,\mu\nu} + h^{\lambda}_{\lambda,\nu,\mu} \right)$

Einstein Eqns for $T^{\mu\nu} = 0$. 

$\delta_{\mu} = 4$
Linearised Einstein: \( G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \quad \times g^{\mu\nu} \)

\[ \Rightarrow R_{\mu\nu} = 0 \]  
**Einstein eqs for** \( T_{\mu\nu} = 0 \)

\[ g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \]

\[ R_{\mu\nu} = \frac{1}{2} \left( h_{\nu,\mu} - h_{\mu,\nu} + h_{\mu,\nu} \right) \]

\[ = -\frac{1}{2} \left( \partial h_{\mu\nu} - h_{\nu,\mu} - h_{\mu,\nu} + h_{\mu,\nu} \right) \]
$g_{\mu \nu} = 0 \Rightarrow R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 0 \Rightarrow R_{\mu \nu} = \frac{1}{2} g_{\mu \nu} R \Rightarrow g^{\mu \nu} \Rightarrow R = 2R \Rightarrow R = 0$

$\nabla^2 \phi = 0$

Einstein equations for $T^{\mu \nu} = 0$:

$g^{\mu \nu} \varepsilon_{\mu \nu} = 4$

Harmonic gauge is just:

$h^+_{\mu \nu} - \frac{1}{2} h_{\mu \nu} = 0$
\[ R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \quad \times g^\mu_\nu \Rightarrow R = 2R \Rightarrow R = 0. \]

\[ g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\nu = 4 \]

\[ \nabla^\alpha \Gamma^\beta_\alpha \]

Einstein Eqns for \( T^\mu_\nu = 0 \),

\[ \frac{1}{2} \left( \partial_{\alpha} h_{\beta\gamma}^\lambda - \partial_{\beta} h_{\gamma\alpha}^\lambda - \partial_{\gamma} h_{\alpha\beta}^\lambda + h_{\alpha\beta}^\lambda \right) \]

harmonic gauge is just

\[ h^\lambda_{\nu,\lambda} - \frac{1}{2} h^\lambda_{\nu} = 0 \]
\[ R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \quad \Rightarrow \quad R = 2R \Rightarrow R = 0. \]

\[ g^{\mu\nu} g_{\mu\nu} = \eta^{\alpha\beta} \eta_{\alpha\beta} \hat{T}^\mu_\mu = 0. \]

\[ h^\mu_{\nu\rho} - \frac{1}{2} h^\mu_{\nu} = 0 \quad \text{harmonic gauge is just} \]

\[ h^\mu_{\nu\rho} - \frac{1}{2} h^\mu_{\nu} = 0 \quad \text{in harmonic} \]
\[ R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \quad \Rightarrow R = 2 R \Rightarrow R = 0. \]

\[ g^{\mu\nu} g_{\mu\nu} = \delta^\mu_{\mu} \]

\[ R^\mu_{\nu\rho\sigma} = 0. \]

\[ \text{gauge is just} \]

\[ h_{\mu\nu} - \frac{i}{2} \eta_{\mu\nu} = 0 \]

\[ h_{\mu\nu} = 0 \text{ in harmonic gauge} \]
$$R \Rightarrow R_{\mu \nu} = \frac{1}{2} g_{\mu \nu} R, \quad x g_{\mu \nu} \Rightarrow R = 2 R \Rightarrow R = 0$$

$$g^{\mu \nu} g_{\mu \nu} = \delta_{\mu}^{\mu} = 4$$

Einstein equations for $T_{\mu \nu} = 0$.

$$= -\frac{1}{2} (h_{\mu \nu} - h^{\lambda}_{\mu \lambda} - h^{\lambda}_{\nu \lambda} + h^{\lambda}_{\mu \nu})$$

$$= 0 \text{ in harmonic gauge}$$

harmonic gauge is

$$h^{\lambda}_{\mu \lambda} = \frac{1}{2} h$$
\[ R \Rightarrow R_{\mu\nu} = \tfrac{1}{2} g_{\mu\nu} R, \quad x g^{\mu\nu} \Rightarrow R = 2R \Rightarrow R = 0. \]

\[ g^{\mu\nu} g_{\mu\nu} = \delta_{\mu}^{\mu} = 4 \]

Einstein \ EQS. \ FOR \ T^{\mu\nu} = 0:

\[ -\frac{1}{2} \left( h_{\mu\nu} - h_{\mu\nu}^{\perp} + h_{\mu\nu}^{\parallel} \right) - \frac{1}{4} h_{\mu\nu}^{\perp} = 0 \quad \text{in harmonic gauge.} \]

harmonic gauge is just

\[ h_{\mu\nu}^{\pi} - \frac{1}{2} h_{\mu\nu}^{\parallel} = 0 \]

"Transverse Trace Reversed" gauge
\[ R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \quad x g^{\mu\nu} \Rightarrow R = 2R \Rightarrow R = 0. \]

\[ g_{\mu\nu} g^{\mu\nu} = \delta_{\mu\nu} = 4 \]

\[ h_{\mu\nu} = \frac{1}{2} h_{\mu\nu}^T \]

"Transverse Trace Reversed" gauge

harmonic gauge is just

\[ h_{\mu\nu} = 0 \]

\( T^m \)

\( T^T \)

Einstein Eqns for \( T^{\mu\nu} = 0 \),

\[ \frac{1}{2}(h_{\mu\nu} - h_{\mu\times\nu} - h_{\times\mu\nu} + h_{\times\times\mu\nu}) \]

= 0 in harmonic gauge.
\[ R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \quad x g_{\mu\nu} \Rightarrow R = 2R \Rightarrow R = 0. \]

\[ g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\mu = 4 \]

harmonic gauge is just
\[ h^\mu_{\mu} = \frac{1}{2} h^\mu_{\nu} = 0 \]

"Transverse Trace Reversed" gauge

"Trace reversed"
\[ R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \quad \times g^{\mu\nu} \Rightarrow R = 2R \Rightarrow R = 0. \]

\[ g^{\mu\nu} g_{\mu\nu} = \delta_{\alpha}^{\alpha} = 4 \]

Consider

\[ h^\alpha_{\nu} - \frac{1}{2} h^\alpha_{\alpha} \delta^\alpha_{\nu} \]

"Trace reversed"

\[ h^\alpha_{\beta} - 2 h^\alpha_{\alpha} = -h^\alpha_{\beta} \]

"Transverse Trace Reversed" gauge

\[ h^\alpha_{\alpha} \frac{1}{2} h^\alpha_{\alpha} = 0 \]

harmomic gauge is just

\[ \Rightarrow 0 \text{ in harmonic gauge} \]
\[ R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \quad x g_{\mu\nu} \Rightarrow R = 2 R \Rightarrow R = 0 \]

\[ g^{\mu\nu} g_{\mu\nu} = \delta_\nu^\mu = 4 \]

Consider

\[ h^{\mu\nu} - \frac{1}{2} h^{\alpha\beta} \delta_\alpha^\mu \delta_\beta^\nu \]

"Trace reversed"

\[ h^{\mu\nu} - 2 h^{\alpha\beta} \delta_\alpha^\mu \delta_\beta^\nu = 0 \]

harmonic gauge is just

\[ h^{\mu\nu} = \frac{1}{2} h^{\alpha\beta} \delta_\alpha^\mu \delta_\beta^\nu = 0 \]

"Transverse Trace Reversal" gauge

\[ \alpha x (h^{\mu\nu} - \frac{1}{2} h^{\alpha\beta} \delta_\alpha^\mu \delta_\beta^\nu) = 0 \]

trace reversed.
Linearised Einstein: $G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$R_{\mu\nu} = \frac{1}{2}(h_{\nu,\mu}^{\lambda} - h_{\nu,\mu}^{\lambda} - h_{\lambda,\mu}^{\nu} + h_{\lambda,\mu}^{\nu})$

$\Rightarrow \Box_h (h_{\nu}^{\lambda} - \frac{1}{2} \delta_{\nu}^{\lambda} h_{,\alpha}^{\lambda}) = 0$
Linearsed Einstein: \( G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \)

\( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \)

\( R_{\mu\nu} = \frac{1}{2} \left( h_{\nu,\mu} - h_{\mu,\nu} - h^{,\lambda}_{\nu,\mu} + h^{,\lambda}_{\mu,\nu} \right) = -\frac{1}{2} \left( \partial h_{\mu\nu} - h^{,\lambda}_{\nu,\mu} - h^{,\lambda}_{\mu,\nu} \right) = 0 \) in ham

\( \partial_x \left( h^x - \frac{1}{2} \delta^x_{\Lambda} h^\Lambda \right) = 0 \)

\( R_{\mu\nu} = -\frac{1}{2} \partial h_{\mu\nu} \)
Linearised Einstein: \( G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \)

\( \Rightarrow R_{\mu\nu} = 0 \)

\( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \)

\( R_{\mu\nu} = \frac{1}{2} (h^\lambda_{\mu\nu} - \frac{1}{2} \delta^\lambda_{\mu\nu} h_{\lambda\nu}) - \frac{1}{2} \) \( \frac{\partial h_{\mu\nu}}{\partial \lambda} \)

\( \partial \lambda (h^\lambda_{\mu\nu} - \frac{1}{2} \delta^\lambda_{\mu\nu} h_{\lambda\nu}) = 0 \)

\( R_{\mu\nu} = -\frac{1}{2} \frac{\partial h_{\mu\nu}}{\partial \lambda} \)
Linearized Einstein: \( G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \)

\( \Rightarrow R_{\mu\nu} = 0 \) \text{ EINSTEIN EQNS FOR } T_{\mu\nu} = 0.

\( \partial x^a = 0 \)

\( x^a = x^a + \epsilon^a (x) \)

\( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \)

\( R_{\mu\nu} = \frac{1}{2} \left( \frac{\partial h_{\nu\mu a}}{\partial x^b} - h_{\nu\mu a}^4 - h_{\nu\mu a}^4 + h_{\nu\mu a}^4 \right) = -\frac{1}{2} \left( \partial h_{\mu\nu} - h_{\nu\mu a}^4 - h_{\nu\mu a}^4 \right) = 0 \) \text{ in ham}

\( \partial \left( h_{\nu}^a - \frac{1}{2} S_{\nu}^a h_{\mu}^4 \right) = 0 \)

\( R_{\mu\nu} = -\frac{1}{2} \partial h_{\mu\nu} \)
\[\begin{align*}
x^\mu &= x^\mu + \epsilon^\mu(x) \\
J &\rightarrow \text{4 dof in gauge trans.}
\end{align*}\]

**Linearised Einstein**

\[G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R\]

**Einstein Eqns for \( T^{\mu\nu} = 0 \)**

\[R_{\mu\nu} = 0\]

\[\begin{align*}
\Omega x^\mu &= 0 \\
\epsilon_{x^\mu} &= x^\mu + \epsilon(x)
\end{align*}\]

Still harmonic if \( \Omega x^\mu = 0 \)

\[\begin{align*}
\partial_x (h^\mu - h^\nu x_{\mu\nu}) &= 0 \\
R_{\mu\nu} &= -\frac{1}{2} \Omega h_{\mu\nu}
\end{align*}\]
Closed Einstein: \( G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \)

\[ \Rightarrow R_{\mu\nu} = 0 \]

Einstein Equations for \( T_{\mu\nu} = 0 \).

\[ \begin{aligned} \Delta x^x &= 0 \\ \Delta x^y &= \Delta x^x + h_{xy} \\ \Delta x^z &= \Delta x^x + h_{xz} + h_{yz} \\ \Delta x^t &= \Delta x^x + h_{xt} + h_{yt} \\ \Delta x^\mu &= \Delta x^x + h_{\mu x} + h_{\mu y} \\ \Delta x^\nu &= \Delta x^x + h_{\nu x} + h_{\nu y} \\ \Delta x^\tau &= \Delta x^x + h_{\tau x} + h_{\tau y} \\ \end{aligned} \]

\[ \frac{1}{2} (h_{\alpha \mu \nu} - h_{\mu \alpha \nu} + h_{\mu \nu \alpha} - h_{\nu \mu \alpha}) = \frac{1}{2} \left( \frac{\partial h_{\mu \nu}}{\partial \mu} - h_{\alpha \mu \nu} - h_{\mu \alpha \nu} \right) \]

\[ |h_{\mu \nu}| = 10^{-2} \]

\[ h_{\mu \nu} = 0 \text{ in h.a.} \]

\[ R_{\mu \nu} = -\frac{1}{2} \frac{\partial h_{\mu \nu}}{\partial t} \]