Title: PSI 17/18 - Relativity - Lecture 10

Date: Sep 18, 2017 09:00 AM

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Abstract:

remember cov. diff^n along a curve $x^\mu(\tau)$

$$\frac{Du^\mu}{D\tau} \equiv \frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} u^\nu \frac{dx^\lambda}{d\tau} = 0$$ if u^μ is parallel transported along $x^\mu(\tau)$

geodesic eq^n states that 4-velocity $u^\mu = \frac{dx^\mu}{d\tau}$ is parallel transported along world line.

$= \left\{ {\mu \atop \nu\lambda} \right\}$, the connection

∂t

...ravitational field has $|T^{ij}| \ll T^{00}$

S, V, $\underline{J}$

$$\frac{d}{dt}\int_V d^3x\,\rho_e = -\int_S d\underline{S}\cdot\underline{J}$$

Cons^n of energy

energy

momentum

$$\partial_0 T^{00} = -\partial_i T^{i0}$$

$$\partial_0 T^{0i} = -\partial_j T^{ji}$$

T^{ij} = spatial stress tensor.

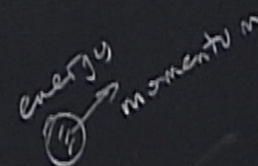
, where $|h_{\mu\nu}| \ll 1$, i.e. weak gravitational fields.

$|v| = \left|\frac{dx}{dt}\right|$ for particle is also $\ll 1$. (in units where $c = 1$)

the stress-energy $T^{\mu\nu}$ creating the gravitational field has $|T^{ij}|$

e.g. for perfect fluid in its rest frame, $T^{ij} = p\,\delta^{ij}$

pressure.



$\partial_0 T^{00} = -$

$\partial_0 T^{0i} = -$

T^{i}

Go to Newtonian limit: 1) $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$

2) Assume that $|\underline{v}| = \left|\frac{dx}{dt}\right|$ for particle is a

3) Assume that the stress-energy $T^{\mu\nu}$ crea

e.g. for perfect fluid in

our assumption is $\left|\frac{P}{\rho_e}\right| \ll 1$.

gas in a room

ideal gas law

$$\rho_e = n\left(mc^2 + \frac{1}{2} m \underline{v}^2\right)$$

$$P = \frac{1}{3} n m \underline{v}^2 \qquad \frac{P}{\rho} \sim \frac{1}{3}\left(\frac{v}{c}\right)^2 \ll 1.$$

4) Assume $\frac{\partial}{\partial x^0}(g_{\mu\nu}) = 0$.

Algebra:

if $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$.

check $(\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\beta} - \eta^{\nu\lambda}\eta^{\beta\delta}h_{\lambda\delta}) = \delta_\mu^\beta + h_\mu^\beta - h_\mu^\beta = O(h^2)$.

$$d\tau = \sqrt{dt^2 - d\underline{x}^2} \approx dt\left(1 - \frac{1}{2}\left(\frac{dx}{dt}\right)^2\right) \approx dt.$$

4 components of geodesic eqn are

$$\frac{d^2x^i}{d\tau^2} \approx -\Gamma^i_{00}\left(\frac{dt}{d\tau}\right)^2 \approx -\Gamma^i_{00}$$

$$\frac{d^2t}{d\tau^2} \approx -\Gamma^0_{00}\left(\frac{dt}{d\tau}\right)^2 \approx -\Gamma^0_{00}.$$

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}\underset{\substack{\Downarrow \\ \text{drop } h \\ \text{here}}}{g^{\mu\alpha}}\underbrace{\left(g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\lambda\nu,\alpha}\right)}_{O(h)} \approx \frac{1}{2}\eta^{\mu\alpha}\left(h_{\nu\alpha,\lambda} + h_{\lambda\alpha,\nu} - h_{\lambda\nu,\alpha}\right)$$

since $\frac{\partial}{\partial x^0} g_{\mu\nu} = 0$

$\Gamma^0_{00} \approx 0$

$\Gamma^i_{00} \approx -\frac{1}{2}\eta^{ij} h_{00,j} = -\frac{1}{2} h_{00,i} = -\frac{1}{2}\frac{\partial}{\partial x^i} h_{00}$

(raising and lowering spatial indices does not involve any minus sign)

$$\Rightarrow \frac{d^2t}{d\tau^2} \simeq 0 \Rightarrow \frac{dt}{d\tau} = \text{const.}$$

$$\Rightarrow \frac{d^2x^i}{dt^2} \approx -\Gamma^i_{00} \simeq \frac{1}{2}\frac{\partial}{\partial x^i} h_{00}$$

$$\text{or} \quad \frac{d^2\underline{x}}{dt^2} \approx \frac{1}{2}\underline{\nabla} h_{00}$$

cf. Newton, $m\frac{d^2\underline{x}}{dt^2} = \underline{F}_g = -m\underline{\nabla}\phi_g$ gravitational potential ravity

remember cov. diff^n along a curve $x^\mu(\tau)$

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where $\Gamma^\mu_{\nu\lambda} = \left\{ {\mu \atop \nu\lambda} \right\}$, the connection

geodesic eq^n states that 4-velocity $u^\mu = \frac{dx^\mu}{d\tau}$ is parallel transported along world line.

$= -1$

(one can check that $\frac{d}{d\tau}\left(g_{\mu\nu}(x(\tau)) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) = 0$ by the geodesic eq^n.)

$$\Rightarrow \frac{d^2 t}{d\tau^2} \simeq 0 \Rightarrow \frac{dt}{d\tau} = \text{const.}$$

$$\Rightarrow \frac{d^2 x^i}{dt^2} \simeq -\Gamma^i_{00} \simeq \frac{1}{2}\frac{\partial}{\partial x^i} h_{00}$$

$$\text{or} \quad \frac{d^2 \underline{x}}{dt^2} \simeq \frac{1}{2} \underline{\nabla} h_{00}$$

cf. Newton, $m\dfrac{d^2\underline{x}}{dt^2} = \underline{F}_g = -m\underline{\nabla}\phi_g$

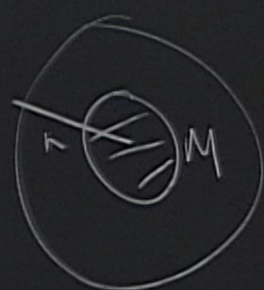
gravitational potential

Same mass, but unexplained

in Newtonian limit $\left(|h_{\mu\nu}| \ll 1,\ \left|\frac{v}{c}\right| \ll 1\right)$

$$h_{00} \simeq -2\phi_g$$

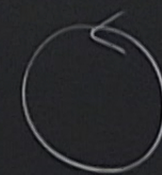
So for a mass M (like the earth or the sun)



$$h_{00} \simeq +2\frac{GM}{r}$$

in Newtonian limit $\left(|h_{\mu\nu}| \ll 1 \, , \, \left|\frac{v}{c}\right| \ll 1 \right)$

$h_{00} \simeq -2\phi_g$



$\underbrace{\frac{GM}{r}}_{\text{units of } v^2} M \quad \frac{1}{2}mv^2$

So for a mass M (like the earth or the sun)

$h_{00} \approx +\frac{2GM}{rc^2} \, , \quad g_{00} \approx -\left(1 - \frac{2GM}{rc^2}\right)$

$\parallel$

dim^{n}less

... mass, but
unexplained

Suggests something strange happens for $r \lesssim \frac{2GM}{c^2} \equiv$ Schwarz

		$\lvert h_{00} \rvert$
e.g.'s	Surface of Earth	10^{-9}
	" Sun	10^{-6}
	" white dwarf star	10^{-4}
	" neutron star	10^{-1}
	black hole	1.

$$= \frac{1}{3} n m \underline{v}^2 \qquad \frac{P}{\rho} \sim \frac{1}{3}\left(\frac{v}{c}\right)^2 \ll 1.$$

In Newton's theory, $\phi_g(\underline{x})$ is determined by solving Poisson's equation

$$\underline{\nabla}^2 \phi_g = 4\pi G \rho_m(\underline{x})$$

mass density.

$\phi_g(\underline{x})$ is determined by solving Poisson's equation

$$\boxed{\underline{\nabla}^2 \phi_g = 4\pi G \rho_m(\underline{x})} \quad \longleftrightarrow \quad \phi_g = \sum_i -\frac{G m_i}{|\underline{x} - x_i|}$$

mass density.

$$\rho_m(\underline{x}) = \sum_i m_i \delta(\underline{x} - x_i)$$

$$\underline{\nabla}^2 \left(- \frac{1}{4\pi |\underline{x} - x_i|} \right) = \delta$$

In Newton's theory, $\phi_g(\underline{x})$ is determined by solving

$$\boxed{\underline{\nabla}^2 \phi_g = 4\pi G \rho_m(\underline{x})}$$

mass den

$T^{00} \sim \rho_{energy}$

$\sim \rho_m \cdot c^2$

$$\Rightarrow -\underline{\nabla}^2 h_{00} = 2\underline{\nabla}^2 \phi_g = 8\pi G \rho_m \simeq 8\pi G T^{00}$$

In Newton's theory, $\phi_g(\underline{x})$ is determined by solving Poisson's eq

$$\boxed{\underline{\nabla}^2 \phi_g = 4\pi G \rho_m(\underline{x})} \longleftrightarrow$$

mass density.

$$\sum_i m_i \delta(\underline{x} -$$

$$\Longrightarrow -\underline{\nabla}^2 h_{00} = 2\underline{\nabla}^2 \phi_g = 8\pi G \rho_m \simeq 8\pi G T^{00}$$

Could be the 00 component of a tensor eq^n

$$X^{\mu\nu}(g, \partial g, \partial^2 g) = 8\pi G T^{\mu\nu}$$

to Newtonian limit: 1) $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$, i.e.

built from $g_{\mu\nu}$, and up to second derivatives. (obv. need second derivs) :-

tensor which reduces to $-\underline{\nabla}^2 h_{00}$ in the Newt built from $g_{\mu\nu}$, and up to second derivatives (obv. need sec

$X^{\mu\nu}$ must be a tensor which reduces to $-\underline{\nabla}^2 h_{00}$ in t

Riemann: $[\nabla_\mu, \nabla_\nu] V_\sigma = -R^\lambda{}_{\sigma\mu\nu} V_\lambda$

$X_{\mu\nu} \overset{?}{=} \nabla^\lambda \nabla_\lambda g_{\mu\nu}$

$= 0.$

would like to write the field eq^{ns} as

$X^{\mu\nu} = 8\pi G T^{\mu\nu}$

second derivs) : ...

in the Newtonian limit: $|h_{\mu\nu}| \ll 1$.

$$R^{\lambda}{}_{\sigma\lambda\nu}(g, \partial g, \partial^2 g) = R_{\sigma\nu}$$

Ricci tensor

Symm under $\sigma \leftrightarrow \nu$.

Einstein's first attempt: $X^{\mu\nu} \propto R^{\mu\nu} \Rightarrow$ wrong.

n:

$$[\nabla_\mu, \nabla_\nu] V_\sigma = -R^\lambda{}_{\sigma \underbrace{\mu\nu}_{\text{antisymm}}} V_\lambda$$

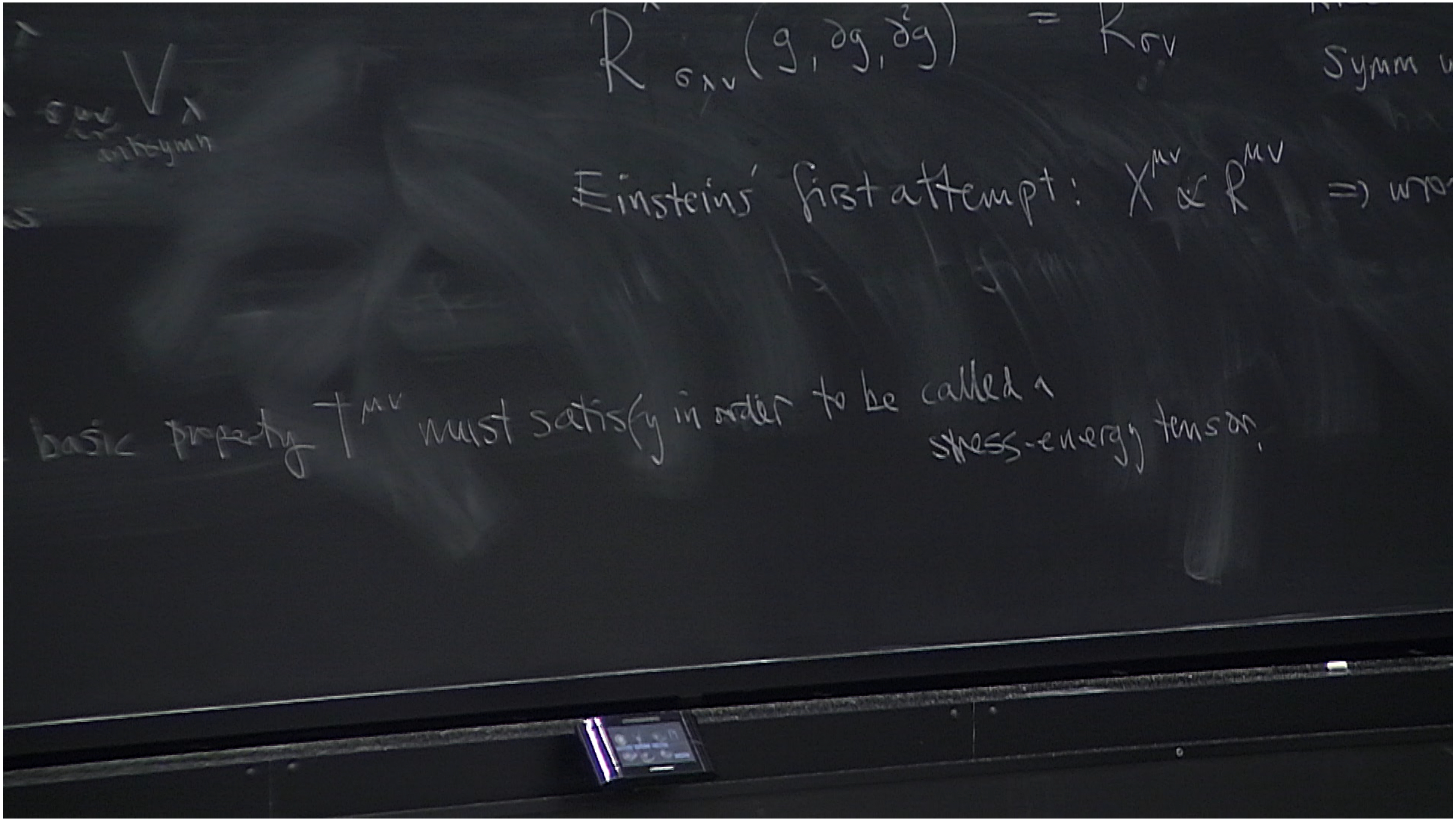
R

Einst

would like to write the field eq^ns as

$$X^{\mu\nu} = 8\pi G T^{\mu\nu}$$

but $\nabla_\mu T^{\mu\nu} = 0$ - this is the basic property $T^{\mu\nu}$ must satisfy i


antisymm
$R^{\lambda}{}_{\sigma\lambda\nu}(g, \partial g, \partial^2 g) = R_{\sigma\nu}$
Symm
Einstein's first attempt: $X^{\mu\nu} \propto R^{\mu\nu}$ ⇒ wro
basic property $T^{\mu\nu}$ must satisfy in order to be called a stress-energy tensor.

So, to obtain a consistent equation, we must have $\nabla_\mu X^{\mu\nu} =$

Start from Bianchi identity:

(Jacobi) $$\Big([\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] \Big) V_\sigma$$

$_{,\nu} = \partial_\nu$

$_{;\nu} = \nabla_\nu$

$$\Longrightarrow \quad R_{\sigma\rho\underbrace{\mu\nu;\kappa}_{\text{cycle}}} + R_{\sigma\rho\nu\kappa;\mu} + R_{\sigma\rho\kappa\mu;\nu} = 0$$

$$\Rightarrow \quad R_{\sigma\rho\underbrace{\mu\nu;\kappa}_{\text{cycle}}} + R_{\sigma\rho\nu\kappa;\mu} + R_{\sigma\rho\kappa\mu;\nu} = 0$$

$;\nu =$

BIANCHI

$\Downarrow$

$g^{\sigma\mu} \cdot$ BIANCHI

$$R_{\rho\nu;\kappa} + R^{\sigma}{}_{\rho\nu\kappa;\sigma} - R_{\rho\kappa;\nu} = 0$$

(use $\nabla_\lambda g^{\sigma\mu} = 0$)

$\Rightarrow$ move g through covariant derivs.

Einstein Field Equations for Gravity

Recall the geodesic equation :

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \quad (1) \quad \text{wher } \Gamma^\mu_{\nu\lambda} = \left\{ {\mu \atop \nu\lambda} \right\}, \text{th}$$

(eqn of motion)

For massive particles, we defined τ by $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$ (2) (one can check

(constraint)

$S = -m \cdot$ length of world line

(1) and (2) describe the motion of a test particle (of any mass)

$$\nabla_\mu X^{\mu\nu} = 0$$

$R_{\sigma\rho} \equiv R^{\lambda}{}_{\sigma\lambda\rho}$ Ricci tensor

$R = g^{\sigma\rho} R_{\sigma\rho}$ Ricci scalar.

$$R_{\sigma\rho\nu\kappa} = -R_{\rho\sigma\nu\kappa}$$

antisym

antisym

$$R_{\rho\nu;\kappa} + R^{\sigma}{}_{\rho\nu\kappa;\sigma} - R_{\rho\kappa;\nu} = 0$$

$$\downarrow$$

$$R_{;\kappa} - \underbrace{R^{\nu\sigma}{}_{\nu\kappa;\sigma}}_{R^{\sigma}{}_{\kappa;\sigma}} - R^{\rho}{}_{\kappa;\rho} = 0$$

$$\Rightarrow \left(-2R^{\sigma}{}_{\kappa} + Rg^{\sigma\kappa}\right)_{;\sigma} = 0$$

$$\Rightarrow \left(R^{\sigma\kappa} - \frac{1}{2}g^{\sigma\kappa}R\right)_{;\sigma} = 0 \Rightarrow X^{\sigma\kappa} \equiv R^{\sigma\kappa} - \frac{1}{2}g^{\sigma\kappa}R$$

$R_{\sigma\rho} \equiv R^{\lambda}{}_{\sigma\lambda\rho}$ Ricci tensor

$R = g^{\sigma\rho} R_{\sigma\rho}$ Ricci scalar.

$R_{\underbrace{\sigma\rho}_{\text{antisym}}\underbrace{\nu\kappa}_{\text{antisym}}} = - R_{\rho\sigma\nu\kappa}$

$\Rightarrow \; X^{\sigma\kappa} \equiv R^{\sigma\kappa} - \frac{1}{2} g^{\sigma\kappa} R$ obeys $\nabla_\sigma \left(X^{\sigma\kappa} \right) = 0$,

$$\nabla_\mu X^{\mu\nu} = 0$$

0

$R_{\sigma\rho} \equiv R^{\lambda}{}_{\sigma\lambda\rho}$ Ricci tensor

$= 0$

$R = g^{\sigma\rho} R_{\sigma\rho}$ Ricci scalar.

$R_{\sigma\rho\nu\kappa} = - R_{\rho\sigma\nu\kappa}$ (prove next time)

antisym antisym

$= 0$

$\equiv$ Einstein tensor $G^{\sigma\kappa}$

$$= 0 \Rightarrow \boxed{X^{\sigma\kappa} \equiv R^{\sigma\kappa} - \frac{1}{2} g^{\sigma\kappa} R} \text{ obeys } \nabla_\sigma (X^{\sigma\kappa}) = 0.$$

Same mass, but

$\Rightarrow$ Candidate equation is

$\nabla_\mu (A G^{\mu\nu}) = 0$ $\quad\quad$ $A\, G^{\mu\nu} = 8\pi G\, T^{\mu\nu}$ $\quad$ where $G^{\mu\nu} = R^{\mu\nu} -$

where A is a constant., fixed by checking correspondence w/ Newtonian

Same mass, but

$\Rightarrow$ candidate equation is

$\nabla_\mu (A G^{\mu\nu}) = 0$ $\quad\quad$ $\underline{\underline{A}}\, G^{\mu\nu} = 8\pi G T^{\mu\nu}$ $\quad\quad$ where $G^{\mu\nu} =$

where A is a constant., fixed by checking correspondence w/ N

tomorrow: lhs $= \; - \nabla^2 h_{00}$, if $A = 1$

dim^{n}less

where $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$, $R^{\mu\nu} = R^{\lambda\mu}{}_{\lambda}{}^{\nu}$, $R = R^{\lambda\sigma}{}_{\lambda\sigma}$

...king correspondence w/ Newtonian limit.

$g^{\sigma\mu} \cdot$ BIANCHI

(use $\nabla_\lambda g^{\sigma\mu} = 0$)

$\Rightarrow$ move g through covariant derivs.

$$\Downarrow$$

$$R_{\rho\nu;\kappa} + R^{\sigma}{}_{\rho\nu\kappa;\sigma} - R_{\rho\kappa;\nu}$$

$$\downarrow \quad \times g^{\nu\rho}$$

$$R_{;\kappa} - \underbrace{R^{\nu\sigma}{}_{\nu\kappa;\sigma}}_{R^{\sigma}{}_{\kappa;\sigma}} - R^{\rho}{}_{\kappa;\rho}$$

$$\nabla_\mu(\nabla_\nu\nabla_\lambda - \nabla_\lambda\nabla_\nu) - (\nabla_\nu\nabla_\lambda - \nabla_\lambda\nabla_\nu)\nabla_\mu$$

$$\Rightarrow \left(-2R^{\sigma}{}_{\kappa} + R g^{\sigma\kappa}\right)_{;\sigma}$$

$$\Rightarrow \left(R^{\sigma\kappa} - \tfrac{1}{2} g^{\sigma\kappa} R\right)_{;\sigma}$$

line ① and ② describe the motion of a test particle (of any mass) under

Go to Newtonian limit: 1) $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$, i.e. weak

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3) Assume that the stress-energy $T^{\mu\nu}$ creating the g

$g^{\sigma\mu} \cdot$ BIANCHI

(use $\nabla_\lambda g^{\sigma\mu} = 0$)

$\Rightarrow$ move g through covariant derivs.

$$\Downarrow$$

$$R_{\rho\nu;\kappa} + R^{\sigma}{}_{\rho\nu\kappa;\sigma} - R_{\rho\kappa;\nu}$$

$$\times g^{\nu\rho} \quad R_{;\kappa} - \underbrace{R^{\nu\sigma}{}_{\nu\kappa;\sigma}}_{R^{\sigma}{}_{\kappa;\sigma}} - R^{\rho}{}_{\kappa;\rho}$$

$$\Rightarrow \left(-2R^{\sigma}{}_{\kappa} + R g^{\sigma\kappa}\right)_{;\sigma}$$

$$\Rightarrow \left(R^{\sigma\kappa} - \tfrac{1}{2} g^{\sigma\kappa} R\right)_{;\sigma}$$

$$\nabla_\mu(\nabla_\nu\nabla_\lambda - \nabla_\lambda\nabla_\nu) - (\nabla_\nu\nabla_\lambda - \nabla_\lambda\nabla_\nu)\nabla_\mu$$

$$\nabla_\mu(\nabla_\nu\nabla_\lambda) = (\nabla_\mu\nabla_\nu)\nabla_\lambda$$

$$\Rightarrow \left(R^{\sigma\kappa} - \tfrac{1}{2} g^{\sigma\kappa} R\right)_{;\sigma} = 0 \quad \Rightarrow \quad \boxed{X^{\sigma\kappa} \equiv R^{\sigma\kappa} - \tfrac{1}{2} g^{\sigma\kappa} R}$$

Bianchi identity:

$$\left([\nabla_\mu,[\nabla_\nu,\nabla_\lambda]] + [\nabla_\nu,[\nabla_\lambda,\nabla_\mu]] + [\nabla_\lambda,[\nabla_\mu,\nabla_\nu]] \right) V_\sigma = 0$$

$${}_{,\nu} = \partial_\nu$$

$${}_{;\nu} = \nabla_\nu$$

$$\Rightarrow \quad R_{\sigma\rho\underbrace{\mu\nu;\kappa}_{\text{cycle}}} + R_{\sigma\rho\nu\kappa;\mu} + R_{\sigma\rho\kappa\mu;\nu} = 0$$

BIANCHI

Start from Bianchi identity:

(Jacobi) $\left([\nabla_\mu,[\nabla_\nu,\nabla_\lambda]] + [\nabla_\nu,[\nabla_\lambda,\nabla_\mu]] + [\nabla_\lambda,[\nabla_\mu,\nabla_\nu]]\right) V_\sigma \overset{}{=} 0$

$$\Rightarrow R_{\sigma\rho\underbrace{\mu\nu;\kappa}_{\text{cycle}}} + R_{\sigma\rho\nu\kappa;\mu} + R_{\sigma\rho\kappa\mu;\nu} = 0$$

BIANCHI

$$X^{\mu\nu} = 8\pi G T^{\mu\nu}$$

$_{,\nu} = \partial_\nu$

$_{;\nu} = \nabla_\nu$

$$(\text{Jacobi}) \qquad \Big([\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] + \ldots$$

$$\nabla_\mu X^{\mu\nu}(g) = 0 \qquad \Longrightarrow \qquad R_{\sigma\rho\underbrace{\mu\nu;\kappa}_{\text{cycle}}} + R_{\sigma\rho\nu\kappa;\mu} + R_{\sigma\ldots}$$

$$X^{\mu\nu} = 8\pi G T^{\mu\nu}$$

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\nabla_\mu X^{\mu\nu} = 8\pi G \underbrace{\cancel{\nabla_\mu T^{\mu\nu}}}_{=0} \qquad \boxed{\nabla_\mu T^{\mu\nu} = 0} \;\; \forall g.$$