remember cov. diff. along a curve $X^\mu(t)$

$$\frac{D u^\mu}{D\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha \frac{dx^\beta}{d\tau} = 0 \quad \text{if} \quad U^\mu \text{ is parallel transported along} \quad X^\mu(t)$$

geodesic eqn. states that 4-velocity $U^\mu = \frac{dx^\mu}{d\tau}$ is parallel transported along world line.
The gravitational field has $|T_{ij}| \ll T_{00}$.

Energy-momentum tensor

$\partial_\nu T^{\mu\nu} = -\partial_\mu T^{\nu\nu}$

$\partial_\nu T^{\mu\nu} = -\partial_\mu T^{\nu\nu}$

$T_{ij} = \text{spatial stress tensor}$.
\[
\frac{\partial}{\partial t} \left| \frac{dx}{dt} \right| \leq 1, \quad \text{i.e. weak gravitational fields.}
\]

The stress-energy tensor \( T^{\mu \nu} \) creating the gravitational field has \( |T_{ij}| \)

- e.g. for perfect fluid in its rest frame, \( T_{ij} = \rho \delta_{ij} \)

- \( \rho \) is pressure.

\[\begin{align*}
\partial_0 T_{00} &= -
\partial_0 T_{0i} &= -
\partial_i T_{ij} &= -
\end{align*}\]
Go to Newtonian limit:

1) \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), where \( |h_{\mu\nu}| \ll 1 \)

2) Assume that \( U = \frac{dx}{dt} \) for particle is a scalar.

3) Assume that the stress-energy \( T_{\mu\nu} \) creates

e.g. for perfect fluid in

\[ P_e = n(m^2 + \frac{1}{2}mv^2) \]

\[ P = \frac{1}{3} nmv^2 \]

\[ \frac{p}{\rho} \sim \frac{1}{3} (\frac{v}{c})^2 \ll 1 \]

\[ \rho \sim \text{mass density} \]

\( P_e \) and \( P \) are pressure

Gas in a room

Ideal gas law

\( n \) = number of particles

\( m \) = mass

\( c \) = speed of light

Our assumption is \( \left| \frac{P}{P_e} \right| \ll 1 \).
4) Assume $\frac{\partial}{\partial x^0}(g_{\mu\nu}) = 0$.

Algebra:

If $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$.

Check

$$(\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\rho} - \eta^{\nu\rho}\eta^{\mu\sigma}h_{\mu\sigma}) = \delta^{\rho}_{\mu} + h^{\rho}_{\mu} - h^{\rho}_{\mu} = O(h^2).$$
\[ dt = \sqrt{dt^2 - dx^2} \approx dt \left(1 - \frac{1}{2} \left(\frac{dx}{dt}\right)^2\right) \approx dt. \]

4 components of geodesic eqn are

\[ \frac{d^2x^i}{dt^2} \approx - \Gamma^i_{oo} \left(\frac{dt}{dc}\right)^2 \approx - \Gamma^i_{oo}. \]

\[ \frac{d^2t}{dt^2} \approx - \Gamma^0_{0o} \left(\frac{dt}{dc}\right)^2 \approx - \Gamma^0_{0o}. \]

\[ \Gamma^i_{k\ell} = \frac{1}{2} g^{in} \left(g_{nk\ell} + g_{n\ell k} - g_{nk\ell} \right) \approx \frac{1}{2} g^{\mu\nu} \left( h_{\mu\lambda,\ell} + h_{\mu\ell,\lambda} - h_{\mu\lambda\ell} \right) \text{ o}(h) \]
since $\frac{\partial g_{\mu\nu}}{\partial x_{\tau}} = 0$

$\Gamma^\mu_{\nu\tau} = 0$

$\Gamma^\mu_{\nu\tau} \approx -\frac{1}{2} \epsilon^{\mu ij} h_{\alpha \beta j} = -\frac{1}{2} h_{\alpha \beta i} = -\frac{1}{2} \frac{\partial h_{\alpha \beta}}{\partial x_i}$ (raising and lowering spatial indices does not involve any minus sign)
\[ \Rightarrow \frac{d^2 t}{d x^2} \approx 0 \Rightarrow \frac{d t}{d t} = \text{const.} \]

\[ \Rightarrow \frac{d^2 x}{d t^2} \approx -\Gamma_{00} \approx \frac{1}{2} \frac{2}{\partial x} h_{00} \]

\[ \Rightarrow \frac{d^2 x}{d t^2} \approx \frac{1}{2} \nabla h_{00} \]

cf. Newton, \[ m \frac{d^2 x}{d t^2} = F_j = -m \nabla \phi_j \]
Remember covariant differentiation along a curve $\mathbf{x}^\mu(\tau)$

\[
\frac{Dx^m}{dt} = \frac{dx^m}{dt} + \Gamma^m_{\nu\lambda} u^\nu \frac{dx^\lambda}{dt} = 0 \text{ if } u^\nu \text{ is parallel transported along } \mathbf{x}^\mu(\tau)
\]

where $\Gamma^m_{\nu\lambda} = \frac{\partial g^{mn}}{\partial x^\lambda}$ is the connection.

Geodesic equation states that 4-velocity $u^\mu = \frac{dx^\mu}{dt}$ is parallel transported along world line.

(one can check that $\frac{d}{dt}(g_{\nu\lambda}(x^\mu, t) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}) = 0$ by the geodesic eqn. )
- \frac{dt}{d\tau^2} = 0 \Rightarrow \frac{dt}{d\tau} = \text{const.}

\Rightarrow \frac{d^2 x}{dt^2} \approx - \Gamma_{00} \approx \frac{1}{2} \frac{2}{0x^2} h_{00}

or \quad \frac{d^2 x}{dt^2} \approx \frac{1}{2} \frac{\nabla}{h_{00}}

\text{cf. Newton,}

m \frac{d^2 x}{dt^2} = F_j = - m \nabla \phi_j

\text{Same mass, but unexplained gravitational potential}
in Newtonian limit \( |h_{\mu \nu}| \ll 1, \left( \frac{v}{c} \right) \ll 1 \)

\[ h_{00} = -2\Phi \]

So for a mass \( M \) (like the earth or the sun) \( h_{00} \approx \frac{2GM}{r} \)
In Newtonian limit: \( |h_{\mu\nu}| \ll 1, \left| \frac{v}{c} \right| \ll 1 \)

\[ h_{00} = -2\Phi_g \]

So for a mass \( M \) (like the earth or the sun):

\[ h_{00} = \frac{2GM}{rc^2}, \quad g_{00} = 1 - \frac{2GM}{rc^2} \]

\[ \frac{GM}{r} \text{ units} \cdot \text{m/s}^2 \]

\[ \frac{1}{2} m^2 \times \text{cgs units} \]
Suggests something strange happens for \( r \leq \frac{2GM}{c^2} = \) Schwarzschild horizon.

<table>
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<tr>
<th>Examples</th>
<th>Surface of Earth</th>
<th>Sun</th>
<th>White Dwarf Star</th>
<th>Neutron Star</th>
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<td></td>
<td>( 10^{-9} )</td>
<td>( 10^{-6} )</td>
<td>( 10^{-4} )</td>
<td>( 10^{-1} )</td>
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</tbody>
</table>
In Newton's theory, $\phi_j(x)$ is determined by solving Poisson's equation:

$$\nabla^2 \phi_j = 4\pi G \rho_m(x)$$

mass density.
\[ \nabla^2 \phi_j = 4\pi G \rho_m (x) \]

Mass density:

\[ \sum_{i} m_i \delta (x - x_i) \]

\[ \nabla^2 \left( \frac{1}{4\pi |x-x_c|} \right) = \delta \]
In Newton's theory, $\phi_0(x)$ is determined by solving

$$\nabla^2 \phi_0 = 4\pi G \rho_m(x)$$

$$T^{00} \sim \rho_{\text{energy}}$$

$$\sim \rho_m c^2$$

$$\Rightarrow \nabla^2 h_{00} = 2 \nabla^2 \phi_0 = 8\pi G \rho_m = 8\pi G T^{00}$$
In Newton's theory, \( \phi_j(x) \) is determined by satisfying Poisson's equation:

\[
\nabla^2 \phi_j = 4\pi G \rho_m(x)
\]

\[
\Rightarrow \nabla^2 h_{00} = 2\nabla^2 \phi_j = 8\pi G \rho_m = 8\pi G T^{00}
\]

Could be the \( 00 \) component of \( \gamma^2 \) a tensor eqn.

\[
X^{\mu\nu}(y, a\gamma, b\gamma) = 8\pi G T^{\mu\nu}
\]

to Newtonian limit: 1) \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), where \( |h_{\mu\nu}| \ll 1 \), i.e.
built from \( g_{mn} \), and up to second derivatives. (Observe second derivative) tensor \( T_{mn} \) which reduces to \(-\Delta h_{00}\). In the Newton
$X_{\mu\nu}$ must be a tensor which reduces to $-\Delta^2 h_{00}$ in $\eta$.

Riemann:
$$\left[\nabla_{\mu}, \nabla_{\nu}\right] V_0 = -R_{\lambda \mu \nu} V_\lambda$$

$X_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} g_{\mu\nu}$

$= 0$.

Wanted to write the field eqs as
$$X_{\mu\nu} = 8\pi GT_{\mu\nu}$$
In the Newtonian limit, $|\mathbf{w}| \ll 1$.

\[ R^\sigma_{\mu \nu}(\mathbf{g}, \mathbf{d}g, \mathbf{d}^2g) = R_{\mu \nu} \quad \text{Ricci tensor} \]

Symm under \( \sigma \leftrightarrow \nu \).

Einstein's first attempt: \( X^\mu \propto R^\mu \nu \) = \( \gamma X^\mu \) = \( \gamma g^\mu \nu \) = \text{wrong}. \]
\[ \nabla_{\mu} V_{\nu} - \nabla_{\nu} V_{\mu} = -R_{\mu \nu}^{\phantom{\mu \nu} \alpha \beta} V_{\alpha}^{\phantom{\alpha} \beta} \]

would like to write the field eqns as

\[ K^{\mu} = 8\pi G T^{\mu \nu} \]

but \[ \nabla_{\mu} T^{\mu \nu} = 0 \] - this is the basic property \[ T^{\mu \nu} \] must satisfy in
Einstein's first attempt: \( X^{\mu} \propto R_{\mu}^{\nu} \) was not right.

Basic property \( \tau_{\mu}^{\nu} \) must satisfy in order to be called a stress-energy tensor.
So, to obtain a consistent equation, we must have $R_{\mu} X^{\mu} = 0$.

Start from Bianchi identity:

(Jacobi) \[(\nabla_\mu [\nabla_\nu, \nabla_\lambda] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] + [\nabla_\lambda, \nabla_\mu, \nabla_\nu]) V_\sigma = a_\nu \]

$\Rightarrow R^{\rho \mu \nu \kappa} + R_{\rho \mu \nu \kappa} + R^{\rho \mu \nu \lambda} \Gamma_{\lambda \kappa} = 0$

$\Rightarrow 3^\nu = \nabla_\nu$
\[ \Rightarrow R_{\mu \nu \rho \kappa} + R_{\nu \rho \kappa \mu} + R_{\rho \kappa \mu \nu} = 0 \]

BIANCHI

\[ \Rightarrow g^{\mu \nu} B \]

\[(\text{use } \nabla \cdot g^{\mu \nu} = 0)\]

\[ \Rightarrow \text{use } g \text{ through covariant dens.} \]
Einstein Field Equations for Gravity

Recall the geodesic equation:

\[ \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0 \]

(\text{eqn of motion})

For massive particles, we defined \( S \) by

\[ S = -m \sqrt{\frac{\text{length of world line}}{c^2}} \]

(\text{constraint})

1. and 2. describe the motion of a test particle (of any mass)
\[ R_{\mu \nu}^{\alpha \beta} + R_{\rho \mu \nu \sigma} - R_{\rho k}^{\nu} = 0 \]
\[ R_{\mu \nu}^{\alpha \beta} - R_{\nu \mu}^{\alpha \beta} = 0 \]
\[ R_{\mu \nu}^{\rho \sigma} - R_{\rho \nu}^{\mu \sigma} = 0 \]
\[ (2R_{\kappa}^{\rho} + Rg_{\rho \kappa}) = 0 \]
\[ (R_{\kappa}^{\rho} - \frac{1}{2}g_{\rho \kappa}R)_{\sigma} = 0 \]
\[ X = R_{\kappa}^{\rho} - \frac{1}{2}g_{\rho \kappa}R \]
\[ R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta} \]

\[ R = g^{\alpha\beta} R_{\alpha\beta} \]

\[ R_{\alpha\beta\gamma\delta} = - R_{\alpha\delta\beta\gamma} \]

\[ X^k = R^k_{\alpha} - \frac{1}{2} g^{|k} R \quad \text{obeys} \quad \nabla_k (X^k) = 0, \]
\[ \nabla_{\mu} X^{\mu} = 0 \]

Ricci tensor

\[ R_{\mu\nu} = R_{\mu\nu}^{\alpha\beta} R_{\alpha\beta} \]

Ricci scalar

\[ R_{\mu\nu} = -R_{\mu\nu\alpha\beta} \]

(prove next time)

\[ \nabla_{\mu} X^{\mu} = R_{\mu}^{\mu} - \frac{1}{2} g^{\mu\nu} R_{\nu} \]

Einstein tensor

\[ \nabla_{\mu} \nabla_{\nu} (X^{\mu}) = 0, \]
\[ \nabla_m (A_0^{\mu}) = 0 \]

\[ A_0^{\mu} = \frac{8\pi}{\kappa} G^{\mu}_0 \]

where \( G^{\mu}_0 = R^{\mu}_0 \)

where \( A \) is a constant, fixed by checking correspondence w/ Newtonian
\[ \nabla_m (A \delta^m_n) = 0 \]

The candidate equation is:

\[ A G^{\mu \nu} = 8 \pi G T^{\mu \nu} \]

where \( G^{\mu \nu} \) is the stress-energy tensor.

where \( A \) is a constant, fixed by checking correspondence w/ Newtonian GR.

Tomorrow: \[ \text{LHS} = - \mathcal{D}^2 \text{h}_{00}, \text{ if } \text{A} = 1 \]
where \( G^{\mu \nu} = R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R \), \( R^{\mu \nu} = R^{\lambda \mu \nu \lambda} \), \( R = R^{\mu \nu} \).

Any correspondence w/ Newtonian limit.
\[ g^m_{\text{EM}} \text{ BIANCHI} \]

\[ (\text{use } \nabla_k g^m = 0) \]

\[ \Rightarrow \text{more } g \text{ through \nabla_k g^m = 0} \text{ covariant dens.} \]

\[ \nabla_m (\nabla_k g^m - R_k^m \nabla g^m) - (\nabla_k \nabla g^m) \nabla_m \]

\[ = (2R^s_{\text{g}} + R g^s) \]

\[ = (R^s_{\text{g}} - \frac{1}{2} g^s R) \]
To Newtonian limit:
1) \( g_{\mu\nu} = 2 \text{ar} + h_{\mu\nu} \), where \(|h_{\mu\nu}| \ll 1\), i.e. weak

2) Assume that \(|\mathbf{\mathbf{\Sigma}}| = \left| \frac{dx}{dt} \right| \), for particle is also \( \ll 1 \).

3) Assume that the stress-energy \( T^{\mu\nu} \) creating the gravity
\[ g^m_{\cdot m} \textbf{BIANCHI} \]

\[
\n(\text{use } \nabla_1 g^m_{\cdot m} = 0) \]

\[ \Rightarrow \text{more } g \text{ through covariant dens.} \]

\[ \nabla_m (x^p - x^p \nabla_l) = (x^p - x^p \nabla_l) \nabla_m \]

\[ \nabla_m \nabla_l \nabla_k \nabla_j = (x^p \nabla_l) \nabla_m \]

\[ \nabla_m (x^p - x^p \nabla_l) - (x^p - x^p \nabla_l) \nabla_m \]

\[ \Rightarrow (2 R^s_{\cdot k} + R g^{s k}) \]

\[ \Rightarrow (R^s_{\cdot k} - \frac{1}{2} g^{s k} R) \]
Bianchi identity:

\[ \left( \nabla_3, \nabla_2 \right) + [\nabla_3, [\nabla_1, \nabla_2, \nabla_2]] \circ V_6 = 0 \]

\[ \Rightarrow R_{\mu\nu\rho\sigma} + R_{\nu\rho\sigma\mu} + R_{\rho\sigma\mu\nu} = 0 \]

\[ \Rightarrow \nabla \nu = \nabla \nu \]

\[ \Rightarrow \nabla \nu = \nabla \nu \]

BIANCHI
Start from Bianchi identity:

\[
\text{Jacobi) } \quad \left( [\nabla_m, \nabla_n] + [\nabla_n, \nabla_m] + [\nabla_k, [\nabla_m, \nabla_n]] \right) V^k = 0
\]

\[x^{\mu} = 8\pi G T^{\mu} \]

\[\Rightarrow R_{\mu
u}^{\text{app}} + R_{\nu\mu}^{\text{jac}} + R_{\nu\mu}^{\text{cyclic}} = 0\]

**Bianchi**
\[ \nabla^\mu \nabla_{\nu}^\mu (g) = 0 \]

\[ \nabla^\mu \nabla_{\nu}^\mu = 8 \nabla^\mu T_{\mu}^\nu \]

\[ \nabla^\mu T_{\mu}^\nu = 0 \quad \forall g \]

\[ X^\mu = 8 T_{\mu}^\nu \]

\[ \Rightarrow R_{\rho \mu \nu \kappa} + R_{\rho \mu \nu k\mu} + R_{\rho \mu k\mu} \]