Title: PSI 17/18 - Relativity - Lecture 9

Date: Sep 15, 2017 09:00 AM

URL: http://pirsa.org/17090028

Abstract:

$$\partial_\mu V^\alpha + \Gamma^\alpha_{\lambda\mu} V^\lambda$$

$$\partial_\mu V_\alpha - \Gamma^\lambda_{\alpha\mu} V_\lambda$$

Lorentz-invariant length of world line $X^\mu(\lambda)$

...ski spacetime $= -m_0 \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$

$$\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\lambda\mu} V^\lambda$$

$$\nabla_\mu V_\alpha = \partial_\mu V_\alpha - \Gamma^\lambda_{\alpha\mu} V_\lambda$$

Lorentz-invariant length of world line $x^\mu(\lambda)$

lesics

action for particle in Minkowski spacetime $= -m_0 \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$

" in general spacetime $= -m_0 \int \sqrt{-g_{\mu\nu}(x(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$

Lorentz-invariant
length of world line
$x^\mu(\lambda)$

$$-m_0 \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

$$-m_0 \int \sqrt{-g_{\mu\nu}(x(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

"proper length" of world line
⇕
indept of coordinates

$$\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\lambda\mu} V^\lambda$$

$$\nabla_\mu V_\alpha = \partial_\mu V_\alpha - \Gamma^\lambda_{\alpha\mu} V_\lambda$$

Geometry: Geodesics

action for particle in Minkowski spacetime =

" in general spacetime =

$$\frac{\lambda}{\int g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\left(\overbrace{-g_{\mu\nu,\alpha}\delta x^\alpha}^{\delta g_{\mu\nu}}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - 2\,g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}\right)$$

$$\delta S = -m_0 \int \frac{d\lambda}{\frac{1}{2}\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}} \left(-\overbrace{g_{\mu\nu,\alpha}\delta x^\alpha}^{\delta g_{\mu\nu}} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - 2g \right.$$

re-express in terms of $d\tau = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$, define $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$ etc.

$$\delta S = m_0 \int d\tau \, \frac{1}{2}\left(g_{\mu\nu,\alpha}\delta x^\alpha \dot{x}^\mu \dot{x}^\nu + 2 g_{\mu\nu}\dot{x}^\mu \delta\dot{x}^\nu \right)$$

$$\delta S' = -m_0 \int \frac{d\lambda}{\frac{1}{2}\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}} \left(-\overbrace{g_{\mu\nu,\alpha}\delta x^\alpha}^{\delta g_{\mu\nu}} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - 2 g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda} \right)$$

re-express in terms of $d\tau = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$, define $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$ etc.

$$\delta S' = m_0 \int d\tau \, \frac{1}{2}\left(g_{\mu\nu,\alpha}\delta x^\alpha \dot{x}^\mu \dot{x}^\nu + 2 g_{\mu\nu}\dot{x}^\mu \delta\dot{x}^\nu \right)$$

$$-2 g_{\mu\nu}\ddot{x}^\mu - 2 g_{\mu\nu,\lambda}\dot{x}^\lambda \delta x^\nu$$

$$-m_0 \int \frac{1}{2\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}} \left(-g_{\mu\nu,\alpha}\delta x^\alpha \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - 2 g_{\mu\nu}\frac{dx^\mu}{d\lambda} \right.$$

$$d\tau = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda \Big), \quad \text{define} \quad \dot{x}^\mu = \frac{dx^\mu}{d\tau} \text{ etc.}$$

$$= m_0 \int d\tau \frac{1}{2}\Big(g_{\mu\nu,\alpha}\delta x^\alpha \dot{x}^\mu \dot{x}^\nu + \underbrace{2 g_{\mu\nu}\dot{x}^\mu \delta\dot{x}^\nu}$$

$$\Big(-2 g_{\mu\nu}\ddot{x}^\mu - 2\underbrace{g_{\mu\nu,\lambda}\dot{x}^\lambda}$$

$$-2 g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{d\delta x^\nu}{d\lambda}$$

$$-2 g_{\mu\nu,\lambda} \dot{x}^\lambda \dot{x}^\mu \delta x^\nu = 0 \quad \forall \delta x^\nu \Rightarrow \ddot{x}^\mu$$

$= \frac{dx^\mu}{d\tau}$ etc.

$$+ \underbrace{2 g_{\mu\nu} \dot{x}^\mu \delta\dot{x}^\nu}$$

$$\Big(-2 g_{\mu\nu} \ddot{x}^\mu \underbrace{- 2 g_{\mu\nu,\lambda} \dot{x}^\lambda \dot{x}^\mu}_{-(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu}) \dot{x}^\lambda \dot{x}^\mu} \Big) \delta x^\nu = 0 \quad \forall \delta x^\nu \Rightarrow \ddot{x}^\mu$$

$$\Rightarrow \quad g_{\mu\nu}\ddot{x}^{\mu} + \frac{1}{2}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)\dot{x}^{\mu}\dot{x}^{\lambda} = 0$$

$$\Rightarrow \ddot{x}^{\mu} + \frac{1}{2} g^{\mu\nu}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)\dot{x}^{\mu}\dot{x}^{\lambda} = 0$$

$\partial \forall \delta x^{\nu}$

$\cdot \forall \delta x^\nu$

$\Rightarrow$

$g^{\nu\alpha}$

$$g_{\mu\nu}\ddot{x}^\mu + \frac{1}{2}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)\dot{x}^\mu \dot{x}^\lambda = 0$$

$$\Rightarrow \ddot{x}^\alpha + \frac{1}{2} g^{\alpha\nu}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)\dot{x}^\mu \dot{x}^\lambda = 0$$

$$\Gamma^\alpha_{\mu\lambda}$$

$\Gamma^{\alpha}_{\mu\lambda}$

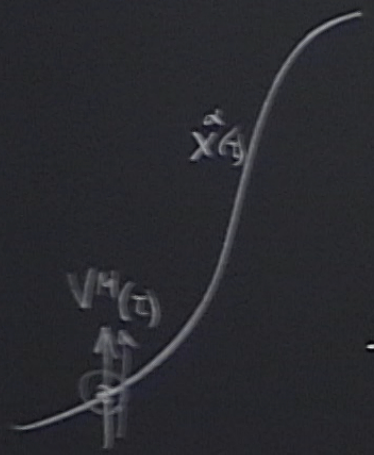
the proper length satisfies $\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\mu\lambda}\dot{x}^{\mu}\dot{x}^{\lambda} = 0$ GEODES EQUA

since it follows from an action principle respecting general

this is a tensor (Ex.) $\frac{dx'^{\mu}}{d\tau} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{dx^{\nu}}{d\tau}$

$= \frac{DV^{\mu}}{D\tau}(x(\tau))$ "covariant derivative of V along curve $x^{\alpha}(\tau)$"

useful to consider "parallel transport along a curve".



We say $V^\mu(\tau)$ is parallel transported along $x^\mu(\tau)$

likewise $U_\mu(\tau)$ " " "

If $\frac{DV^\mu}{D\tau} = 0$ and $\frac{DU_\mu}{D\tau} = 0$, then $\frac{d}{d\tau}(V^\mu U_\mu)$

t along a curve".

transported along $x^\alpha(\tau)$ if $\frac{DV^\mu}{D\tau} = 0$.

" " if $\frac{DU_\mu}{D\tau} = 0$; i.e. $\frac{d}{d\tau}U_\mu - \Gamma^\lambda_{\mu\alpha} U_\lambda \frac{dx^\alpha}{d\tau} = 0$.

then $\frac{d}{d\tau}(V^\mu U_\mu) = 0 \quad \forall\, U, V \Rightarrow \Gamma^\lambda_{\mu\alpha} = \Gamma^\lambda_{\alpha\mu}$ i.e. no torsion.

$$= \frac{D}{D\tau}(V^\mu U_\mu) = \underbrace{\frac{DV^\mu}{D\tau}}_{\left(\frac{d}{d\tau}V + \Gamma V\right)} U_\mu + V^\mu \underbrace{\frac{DU_\mu}{D\tau}}_{\left(\frac{d}{d\tau}U - \Gamma U\right)}$$

How do we build a tensor from the second

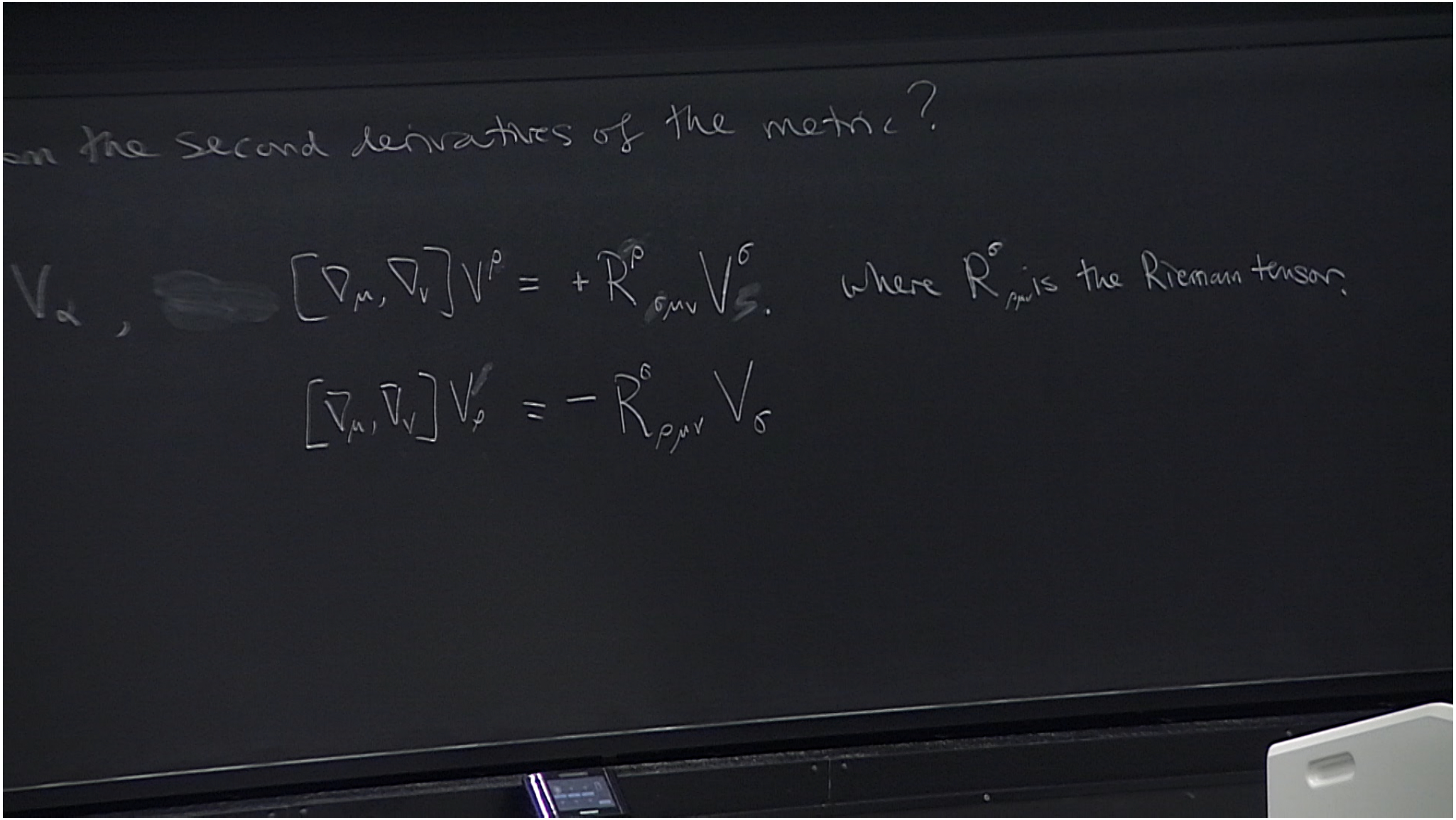
$$[\nabla_\mu, \nabla_\nu] V_\alpha = \text{tensor} \propto V_\alpha ,$$

$$[\partial_\mu + iA_\mu, \partial_\nu + iA_\nu]\phi = -i(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= iF_{\mu\nu}\phi$$

$\downarrow$

$\partial_\mu A_\nu - \partial_\nu A_\mu$


on the second derivatives of the metric?
V_α ,
$[\nabla_\mu, \nabla_\nu] V^\rho = + R^\rho_{\sigma\mu\nu} V^\sigma$.
where $R^\sigma_{\rho\mu\nu}$ is the Riemann tensor.
$[\nabla_\mu, \nabla_\nu] V_\rho = - R^\sigma_{\rho\mu\nu} V_\sigma$

econd derivatives of the metric?

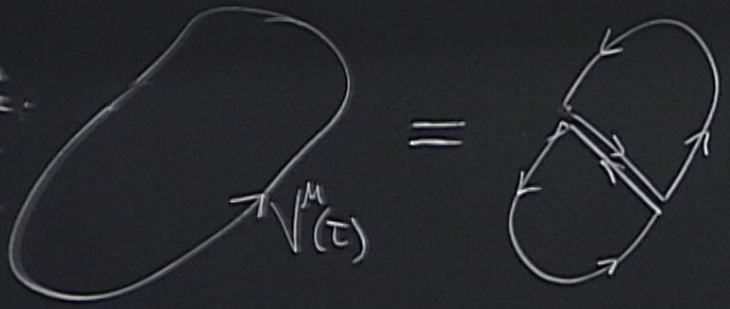
$[\nabla_\mu, \nabla_\nu] V^\rho = + R^\rho_{\sigma\mu\nu} V^\sigma$. where $R^\sigma_{\rho\mu\nu}$ is the Riemann tensor.

$[\nabla_\mu, \nabla_\nu] V_\rho = - R^\sigma_{\rho\mu\nu} V_\sigma$

$(V_{\rho\lambda}{}^{\sigma\alpha}{}_{\cdots}) \rightarrow$ + RV term for each upstairs index
− RV " " " downstairs index

More geometrically, Riemann tensor ca

closed curve: C



$\Rightarrow$ consider parallel transporting



geometrically, Riemann tensor can be derived by cons



$\Rightarrow$ consider parallel transporting a vector around an infinitesima

$x^\mu(\tau)$

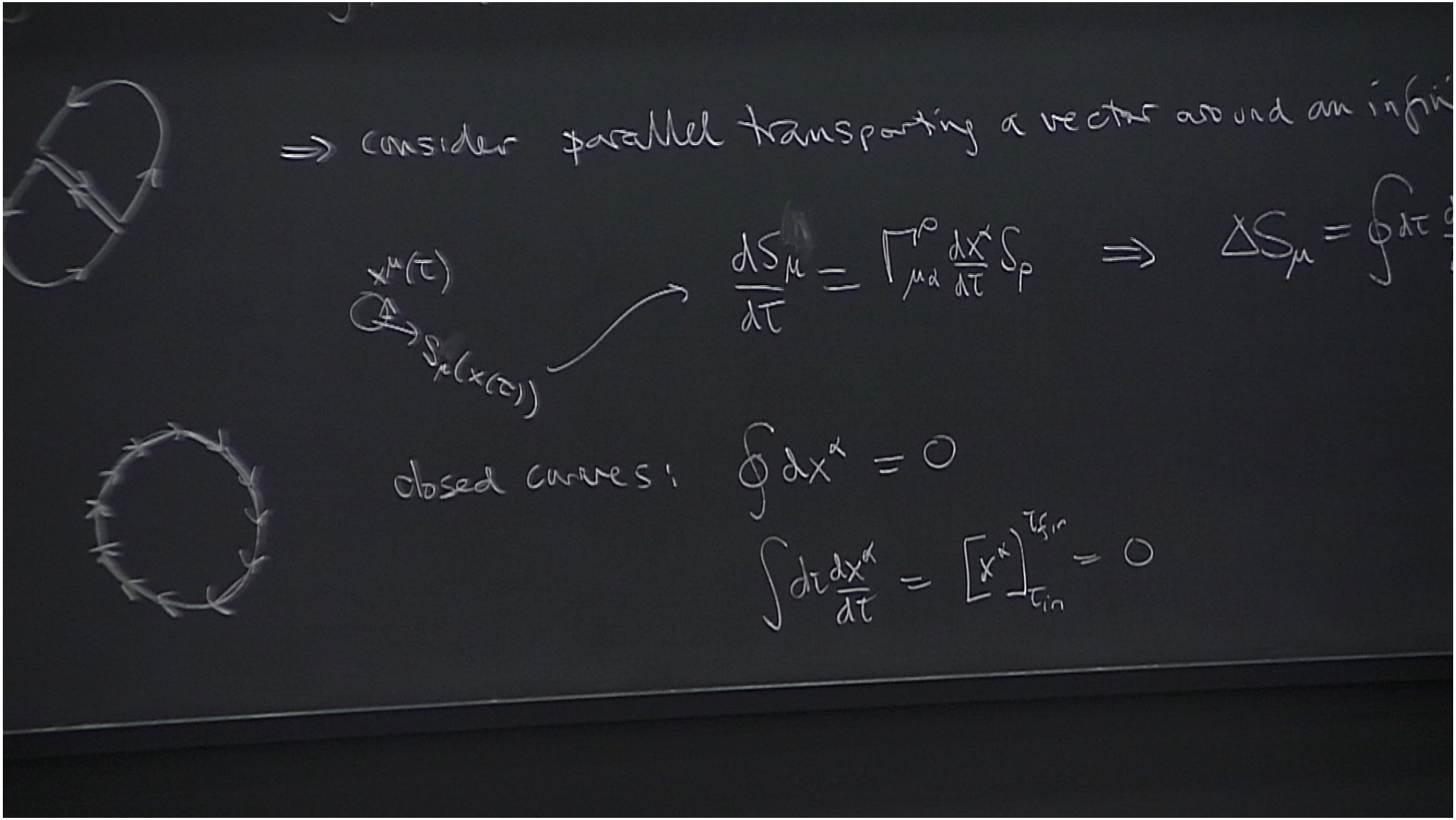
$S_\nu(x(\tau))$

$$\frac{dS_\mu}{d\tau} = \Gamma^\rho_{\mu\alpha} \frac{dx^\alpha}{d\tau} S_\rho$$

e derived by considering parallel transport.

around an infinitesimal closed curve.

$$\Rightarrow \Delta S_\mu = \oint d\tau \frac{dS_\mu}{d\tau} = \oint \Gamma^\rho_{\mu\alpha} \frac{dx^\alpha}{d\tau} S_\rho \, d\tau = \oint \Gamma^\rho_{\mu\alpha} S_\rho \, dx^\alpha$$


⇒ consider parallel transporting a vector around an infi
$x^\mu(\tau)$
$S_\mu(x(\tau))$
$\frac{dS_\mu}{d\tau} = \Gamma^\rho_{\mu\alpha}\frac{dx^\alpha}{d\tau}S_\rho$
⇒
$\Delta S_\mu = \oint d\tau$
closed curves: $\oint dx^\alpha = 0$
$\int d\tau \frac{dx^\alpha}{d\tau} = \left[x^\alpha\right]_{\tau_{in}}^{\tau_{fin}} = 0$

transporting a vector around an infinitesimal closed curve.

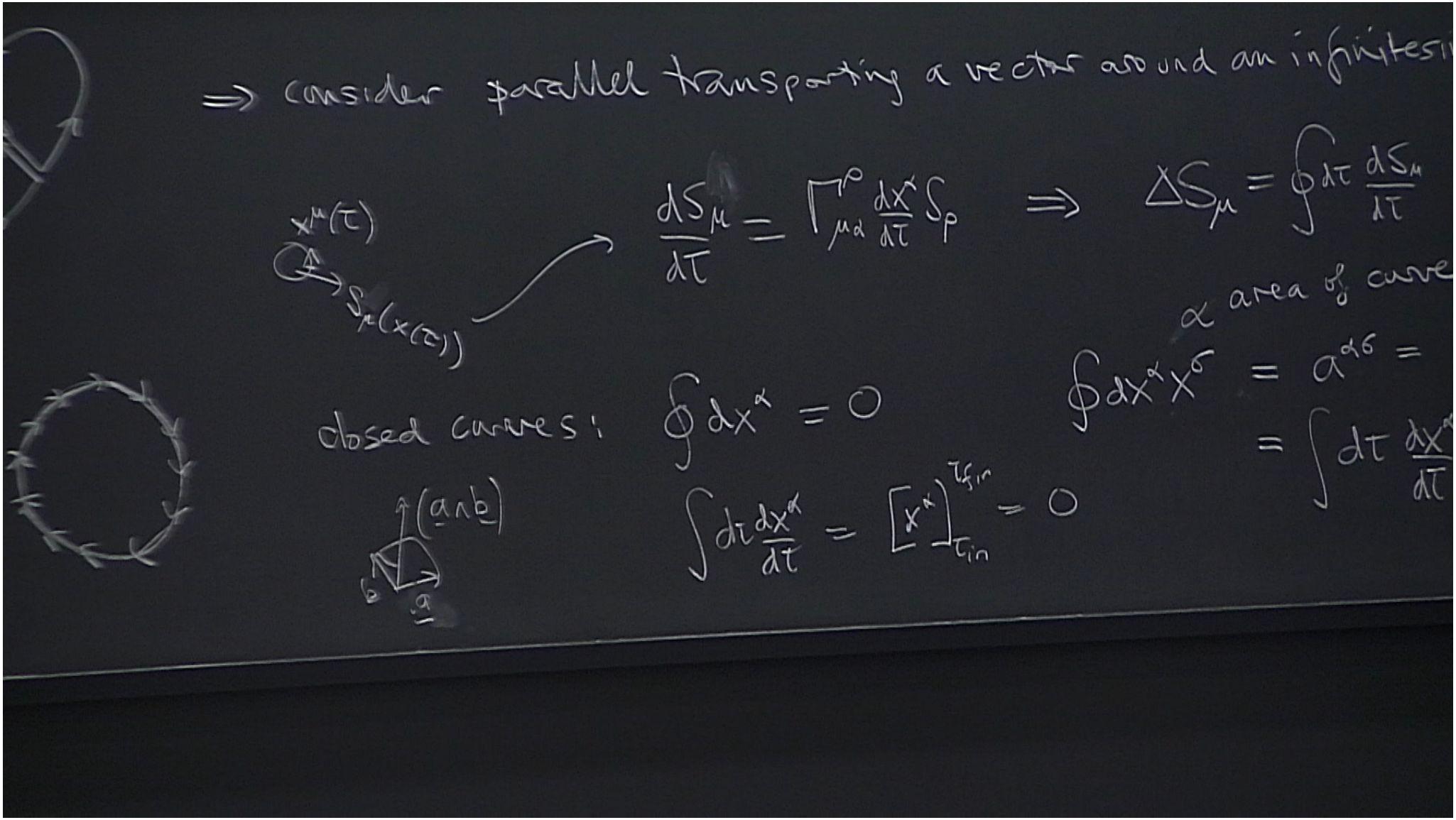
$$\delta^\mu = \Gamma^\mu_{\mu\alpha} \frac{dx^\alpha}{d\tau} \delta_\rho \quad \Rightarrow \quad \Delta\delta_\mu = \oint d\tau \frac{d\delta_\mu}{d\tau} = \oint \Gamma^\rho_{\mu\alpha} \frac{dx^\alpha}{d\tau} \delta_\rho \, d\tau = \oint \Gamma^\rho_{\mu\alpha} \delta_\rho \, dx^\alpha$$

α area of curve.

$$\oint dx^\alpha x^\sigma = a^{\alpha\sigma}$$

$$dx^\alpha = 0$$

$$\oint d\tau \frac{dx^\alpha}{d\tau} = \left[x^\alpha\right]_{\tau_{in}}^{\tau_{in}} = 0$$


⇒ consider parallel transporting a vector around an infinitesi
$x^\mu(\tau)$
$\delta_\mu(x(\tau))$
$\frac{d\delta_\mu}{d\tau} = \Gamma^\rho_{\mu\alpha}\frac{dx^\alpha}{d\tau}\delta_\rho \Rightarrow \Delta\delta_\mu = \oint d\tau \frac{d\delta_\mu}{d\tau}$
$\propto$ area of curve
closed curves: $\oint dx^\alpha = 0$
$\oint dx^\alpha x^\sigma = a^{\alpha\sigma} =$
$= \int d\tau \frac{dx^\alpha}{d\tau}$
$\int d\tau \frac{dx^\alpha}{d\tau} = \left[x^\alpha\right]_{\tau_{in}}^{\tau_{in}} = 0$
$(\underline{a}\wedge\underline{b})$
$\underline{b}$
$\underline{a}$

...sporting a vector around an infinitesimal closed curve.

$$= \Gamma^{\rho}_{\mu\alpha}\frac{dx^{\alpha}}{d\tau}\zeta_{\rho} \quad \Rightarrow \quad \Delta\zeta_{\mu} = \oint d\tau \frac{d\zeta_{\mu}}{d\tau} = \oint \Gamma^{\rho}_{\mu\alpha}\frac{dx^{\alpha}}{d\tau}\zeta_{\rho}\, d\tau = \oint \Gamma^{\rho}_{\mu\alpha}\zeta_{\rho}\, dx^{\alpha}$$

α area of curve.

$$\oint dx^{\alpha}x^{\sigma} = a^{\alpha\sigma} = -a^{\sigma\alpha}$$

$$= \int d\tau \frac{dx^{\alpha}}{d\tau}x^{\sigma} = -\int d\tau\, x^{\alpha}\frac{dx^{\sigma}}{d\tau} \qquad \text{(int}^{\text{n}}\text{ by parts)}$$

$$x^{\alpha} = 0$$

$$\frac{dx^{\alpha}}{d\tau} = \left[x^{\alpha}\right]_{\tau_{in}}^{\tau_{fin}} = 0$$

r can be derived by considering parallel transport.

ting a vector around an infinitesimal closed curve.

$$\Gamma^{\rho}_{\mu\alpha}\frac{dx^{\alpha}}{d\tau}S_{\rho} \Rightarrow \Delta S_{\mu} = \oint d\tau \frac{dS_{\mu}}{d\tau} = \oint \Gamma^{\rho}_{\mu\alpha}\frac{dx^{\alpha}}{d\tau}S_{\rho}\,d\tau = \oint \Gamma^{\rho}_{\mu\alpha}S_{\rho}\,dx^{\alpha}$$

$\propto$ area of curve.

$$\oint dx^{\alpha}x^{\sigma} = a^{\alpha\sigma} = -a^{\sigma\alpha} = \text{"area element"}$$

$$= \int dt \frac{dx^{\alpha}}{d\tau}x^{\sigma} = -\int d\tau\, x^{\alpha}\frac{dx^{\sigma}}{d\tau} \quad \text{(int'n by parts)}$$

$= 0$

$= \left[x^{\alpha}\right]_{\tau_{in}}^{\tau_{fin}} = 0$

infinitesimal curve $\Longrightarrow$

centred on $x^\mu = 0$.

Taylor series around $x^\mu = 0$:

$$\Gamma^\rho_{\mu\alpha} = \Gamma^\rho_{\mu\alpha}(0) + \partial_\lambda \Gamma^\rho_{\mu\alpha}\Big|_0 x^\lambda + \cdots$$

$$S_\rho = S_\rho(0) + \partial_\nu S(0) x^\nu + \cdots$$

Const term vanishes: $\oint dx^\alpha = 0$

$$\Delta S_\mu = \oint \Gamma^\rho_{\mu\alpha} S_\rho dx^\alpha \approx \oint \left(\partial_\lambda \Gamma^\rho_{\mu\alpha} S_\rho + \Gamma^\rho_{\mu\alpha} \partial_\lambda S_\rho\right) x$$

lines around $x^\mu = 0$:

$$\Gamma^\rho_{\mu\alpha}(0) + \partial_\lambda \Gamma^\rho_{\mu\alpha}(0) x^\lambda + \cdots$$

$$S_\rho(0) + \partial_\nu S(0) x^\nu + \cdots$$

Const term vanishes: $\oint dx^\alpha = 0$

"area" 2nd order

$$\oint S_\rho dx^\rho \approx \oint \left(\partial_\lambda \Gamma^\rho_{\mu\alpha} S_\rho + \Gamma^\rho_{\mu\alpha} \partial_\lambda S_\rho\right) x^\lambda dx^\mu$$

(dropping $x^\lambda x^\nu$ in Integrand

es around $x^\mu = 0$:

$(0) + \partial_\lambda \Gamma^{\rho}_{\mu\alpha}(0) x^\lambda + \cdots$

$(0) + \partial_\nu S(0) x^\nu + \cdots$

Const term vanishes : $\oint dx^\alpha = 0$

"area" 2nd order

$dx^2 \approx \oint \left(\partial_\lambda \Gamma^{\rho}_{\mu\alpha}(0) S^{(0)}_\rho + \Gamma^{\rho}_{\mu\alpha}(0) \partial_\lambda S^{(0)}_\rho \right) x^\lambda dx^\mu$

(dropping $x^\lambda x^\nu$ in Integrand

$\Gamma^{\alpha}_{\rho\lambda} S_\alpha$ because S_ρ is parallel transported around the curve

$= \frac{1}{2} R^{\rho}_{\mu\lambda\alpha} S_\rho a^{\lambda\alpha}$

$$\Gamma^{\rho}_{\mu\alpha} = \Gamma^{\rho}_{\mu\alpha}(0) + \partial_\lambda \Gamma^{\rho}_{\mu\alpha}(0) x^\lambda + \cdots$$

$$S_\rho = S_\rho(0) + \partial_\nu S(0) x^\nu + \cdots$$

$a^{\lambda\mu} = -a$

Const term vanishes: $\oint dx^\alpha = 0$

"area" 2nd order

$$\Delta S_\mu = \oint \Gamma^{\rho}_{\mu\alpha} S_\rho \, dx^\alpha \approx \oint \left(\partial_\lambda \Gamma^{\rho}_{\mu\alpha}(0) S^{(0)}_\rho + \Gamma^{\rho}_{\mu\alpha}(0) \partial_\lambda S^{(0)}_\rho \right) x^\lambda \, dx^\mu$$

$\Gamma^{\kappa}_{\rho\lambda} S_\kappa$ because S_ρ

$$= \frac{1}{2} R^{\rho}_{\mu\lambda\alpha} S_\rho a^{\lambda\alpha}$$

where $\boxed{R^{\rho}_{\mu\lambda\alpha} = \partial_\lambda \Gamma^{\rho}_{\mu\alpha} + \Gamma^{\delta}_{\mu\alpha} \Gamma^{\rho}_{\delta\lambda} - (\lambda \leftrightarrow \alpha)}$

Taylor series around $x^\mu = 0$:

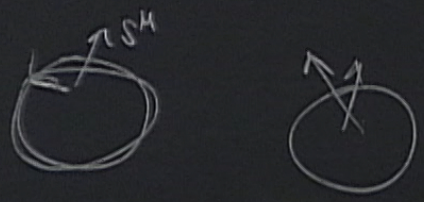
infinitesimal curve $\Longrightarrow$ centred on $x^\mu = 0$.

$$\Gamma^\rho_{\mu\alpha} = \Gamma^\rho_{\mu\alpha}(0) + \partial_\lambda \Gamma^\rho_{\mu\alpha}\Big|_0 x^\lambda + \cdots$$

$$S_\rho = S_\rho(0) + \partial_\nu S(0) x^\nu + \cdots$$

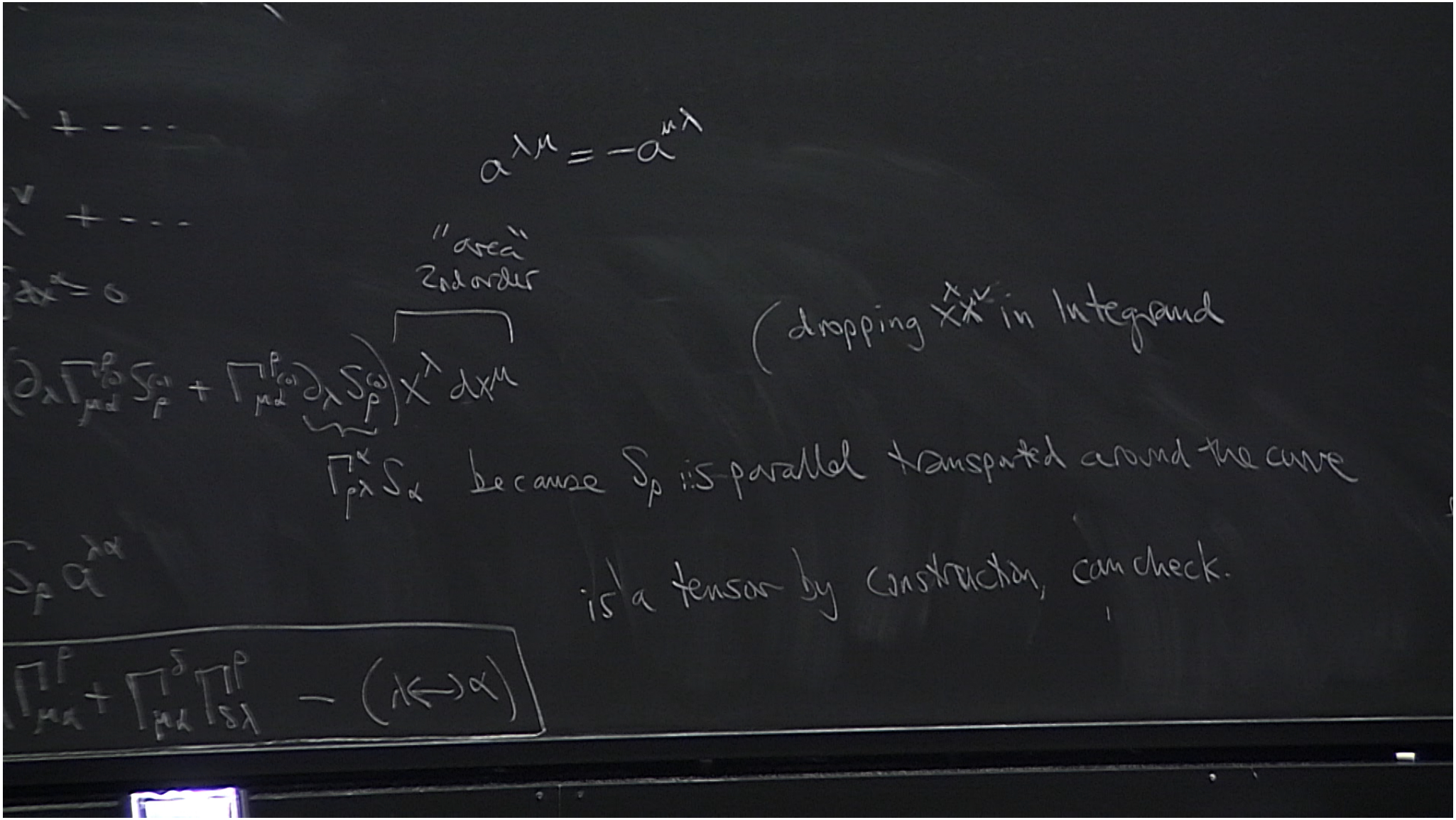
Const term vanishes: $\oint dx^\alpha = 0$

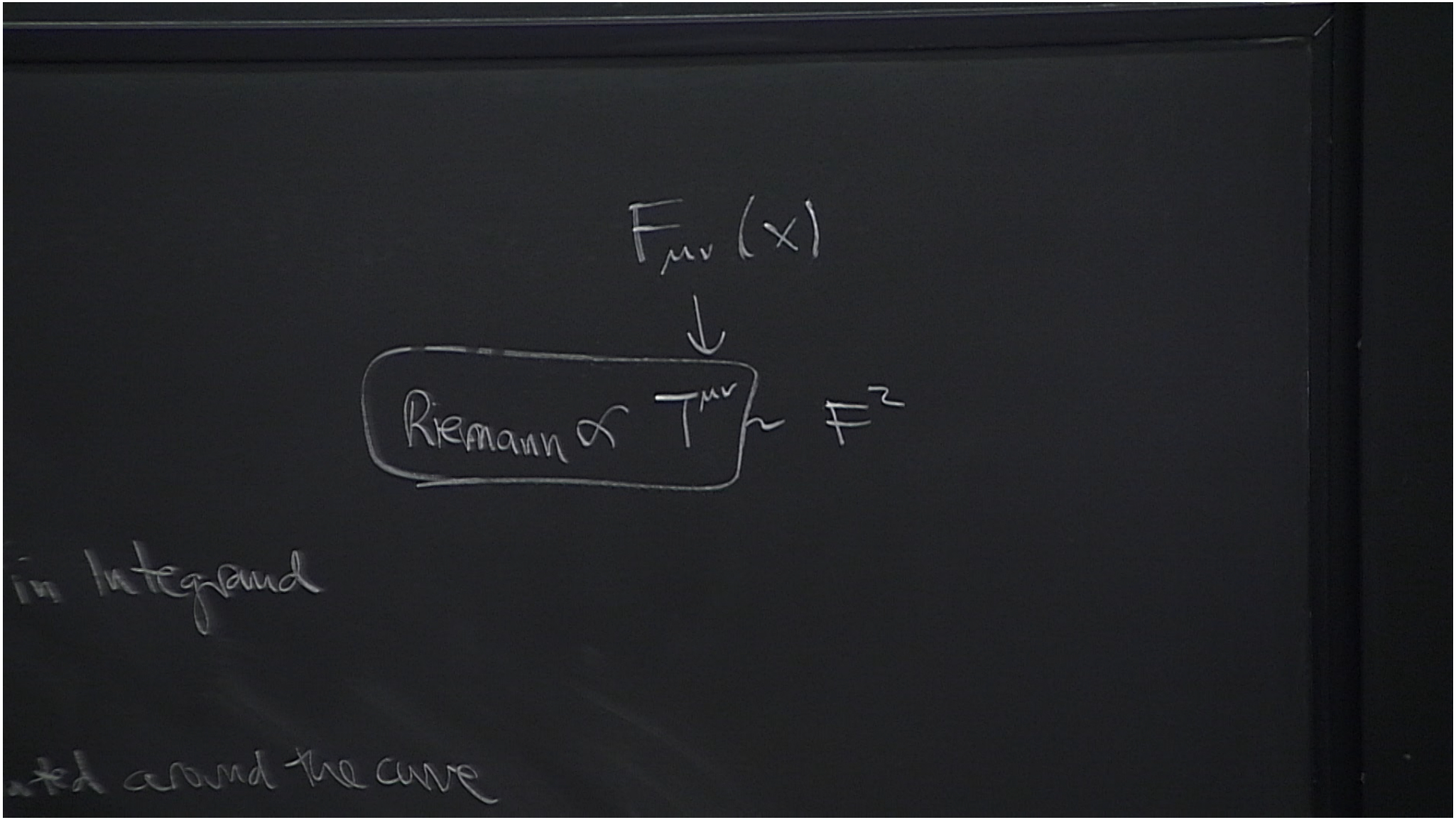
$$\Delta S_\mu = \oint \Gamma^\rho_{\mu\alpha} S_\rho \, dx^\alpha \approx \oint (\partial_\lambda \Gamma^\rho_{\mu\alpha}\Big|_0 S_\rho$$

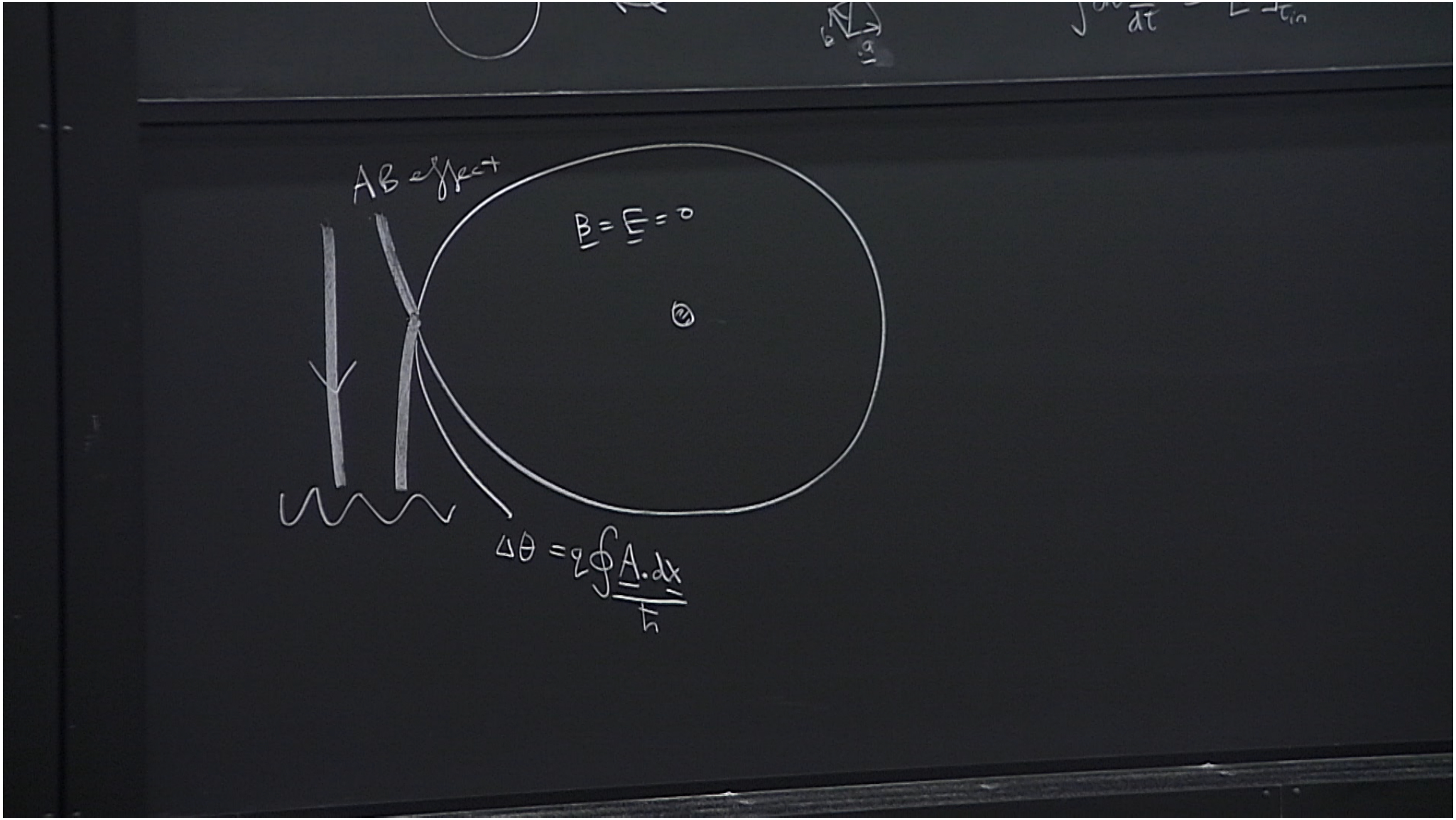


$$\cdots = \frac{1}{2} R^\rho_{\mu\lambda\alpha} S_\rho \alpha^{\lambda\alpha}$$

where $\boxed{R^\rho_{\mu\lambda\alpha} = \partial_\lambda \Gamma^\rho_{\mu\alpha} + \Gamma^\rho_{\mu}}$


$a^{\lambda\mu} = -a^{\mu\lambda}$
"area"
2nd order
(dropping $x^\lambda x^\nu$ in Integrand
$\Gamma^\alpha_{\rho\lambda} S_\alpha$ because S_ρ is parallel transported around the curve
is a tensor by construction, can check.


$F_{\mu\nu}(x)$
Riemann $\propto T^{\mu\nu} \sim F^2$
in Integrand
ated around the curve


AB effect
B = E = 0
Δθ = 2 ∮ A · dx / ħ