Title: PSI 17/18 - Relativity - Lecture 8

Date: Sep 14, 2017 09:00 AM

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Abstract:

recall: last time, showed you can choose coords in which

$$g_{\mu\nu}(x) \approx \eta_{\mu\nu} + \frac{1}{2} g_{\mu\nu,\lambda\sigma}(0) x^\lambda x^\sigma + \cdots$$

"locally" flat

"curvature"

$g_{\mu\nu}(x) \approx \eta_{\mu\nu} +$

"locally" flat

Tensors under general coordinate transformations

e.g Scalar $\phi'(x') = \phi(x)$

$+ \frac{1}{2} g_{\mu\nu,\alpha}(0) x^\alpha x^\alpha + \cdots$

"curvature"

flat



$x'^\mu_P = x'^\mu(x^\alpha_P)$

$\phi'(x'_P) = \phi(x_P)$

recall: last time, showed you can cho...

$$g_{\mu\nu}(x) \approx \eta_{\mu\nu} + \frac{1}{2} g_{\mu\nu,\alpha\beta}(0) x^\alpha x^\beta + \dots$$

"locally" flat — "curvature"

Tensors under general coordinate transformations



e.g. Scalar $\phi'(x') = \phi(x)$

vectors $V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\alpha} V^\alpha$; $V'_\mu(x) = \frac{\partial x^\alpha}{\partial x'^\mu} V_\alpha(x)$ etc.

# of upstairs (contravariant) indices and # of downstairs

the type of tensor, e.g.

$$T^{\mu\nu}{}_{\lambda\rho} = \text{tensor of type } (2,3).$$

The metric $g_{\mu\nu}$ is a tensor of type (0,2).

Vectors $V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} V^{\alpha}$; $V'_{\mu}(x) = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} V_{\alpha}(x)$ etc.

# of upstairs (contravariant) indices and # of downstairs (

the type of tensor, e.g.

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The metric $g_{\mu\nu}$ is a tensor of type (0,2).

Tensors from Tensors : Add : $T_{\mu\nu} + S_{\mu\nu}$ is a tensor $M_{\mu\nu}$ of same type

Multiply: $T^{\mu\nu} S_{\alpha\beta}$ " $M^{\mu\nu}{}_{\alpha\beta}$ of type (2,2)

Contract $T^{\mu\nu} S_{\nu\beta}$ " $M^{\mu}{}_{\beta}$ " " (1,1)

Contract $T^{\mu\nu} S_{\nu\rho}$ " $M^{\mu}{}_{\rho}$ " "

Differentiate: $\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\alpha}}$

$$V_{\mu} = \frac{\partial \phi(x)}{\partial x^{\mu}}$$

$$V'_{\mu} = \frac{\partial \phi'(x')}{\partial x'^{\mu}} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\alpha}} \phi(x) = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} V_{\alpha} \quad \checkmark$$

Contract $T^{\dots}_{\dots\nu\rho}$ " M^{μ}_{ρ} " " $(1,1)$

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\alpha}}$$

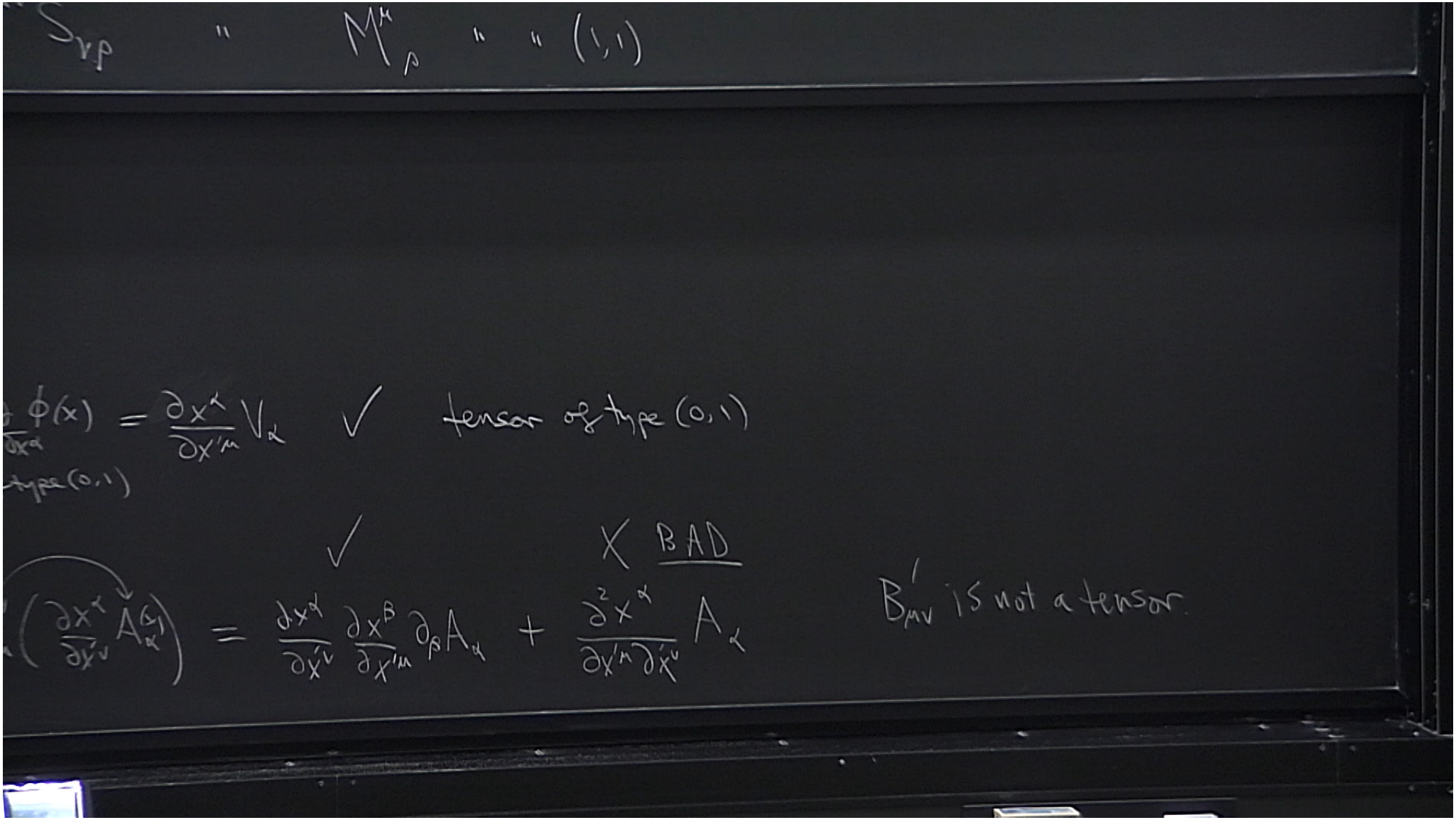
$$V_{\mu} = \frac{\partial \phi(x)}{\partial x^{\mu}}$$

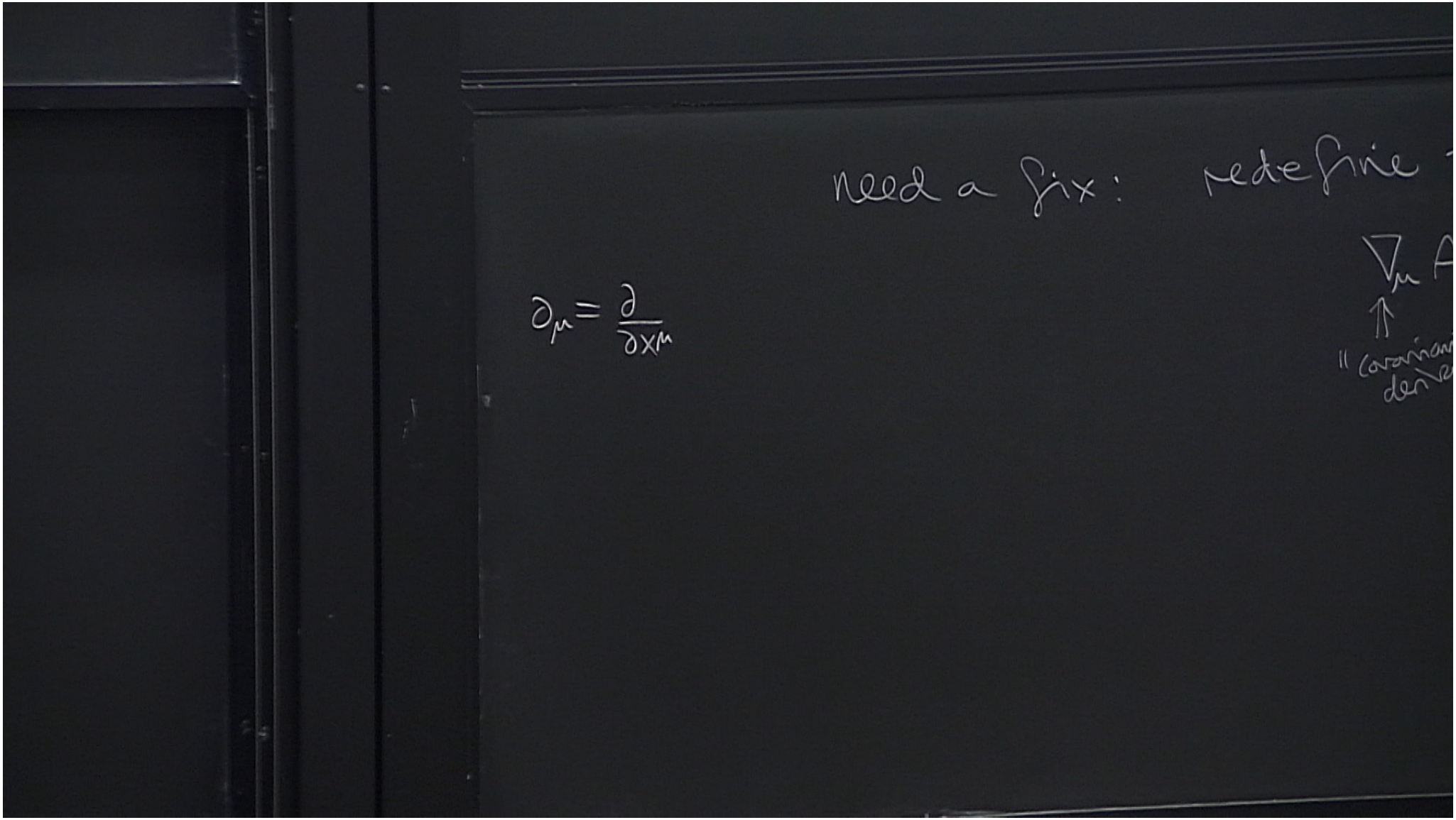
$$V'_{\mu} = \frac{\partial \phi'(x')}{\partial x'^{\mu}} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\alpha}} \phi(x) = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} V_{\alpha} \quad \checkmark \quad \text{tensor of type } (0,1)$$

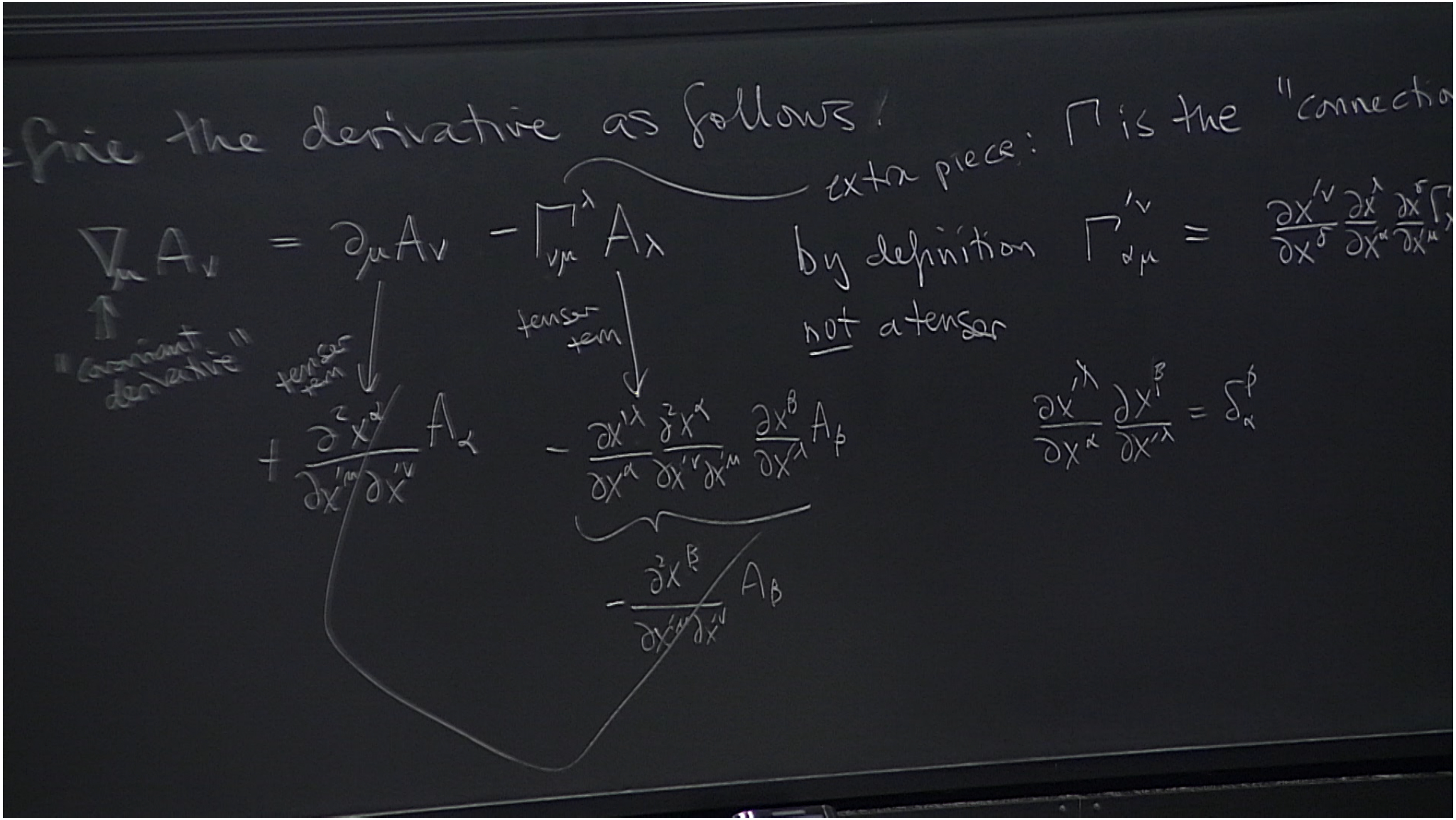
$$B_{\mu\nu} = \partial_{\mu} A_{\nu} \quad \leftarrow \text{tensor of type } (0,1)$$

$$B'_{\mu\nu} = \partial'_{\mu} A'_{\nu} = \partial'_{\mu}\left(\frac{\partial x^{\alpha}}{\partial x'^{\nu}} A_{\alpha}\right) = \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} \partial_{\beta} A_{\alpha} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} A_{\alpha}$$

$\checkmark$ $\quad$ $\times$ BAD


$S_{\nu\rho}$ " $M^{\mu}{}_{\rho}$ " " $(1,1)$
$\frac{\partial}{\partial x^\alpha}\phi(x) = \frac{\partial x'^\mu}{\partial x^\alpha} V_\mu$ ✓ tensor of type $(0,1)$
type $(0,1)$
✓ ✗ BAD
$\left(\frac{\partial x'^\mu}{\partial x^\nu} A^{(1)}_\mu\right) = \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\partial_\beta A_\alpha + \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} A_\alpha$
$B'_{\mu\nu}$ is not a tensor


need a fix: redefine
$\partial_\mu = \frac{\partial}{\partial x^\mu}$
∇_μ A
↑
" covariant
derivative


efine the derivative as follows!
$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\nu\mu} A_\lambda$
extra piece: Γ is the "connectio
by definition $\Gamma'^{\lambda}_{\alpha\mu} = \frac{\partial x'^\nu}{\partial x^\sigma}\frac{\partial x^\lambda}{\partial x'^\alpha}\frac{\partial x^\sigma}{\partial x'^\mu}\Gamma$
not a tensor
"covariant derivative"
tensor term
tensor term
$+\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} A_\alpha$
$-\frac{\partial x'^\lambda}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\nu \partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\lambda} A_\beta$
$\frac{\partial x'^\lambda}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\lambda} = \delta^\beta_\alpha$
$-\frac{\partial^2 x^\beta}{\partial x'^\mu \partial x'^\nu} A_\beta$

laws!

extra piece: Γ is the "connection"

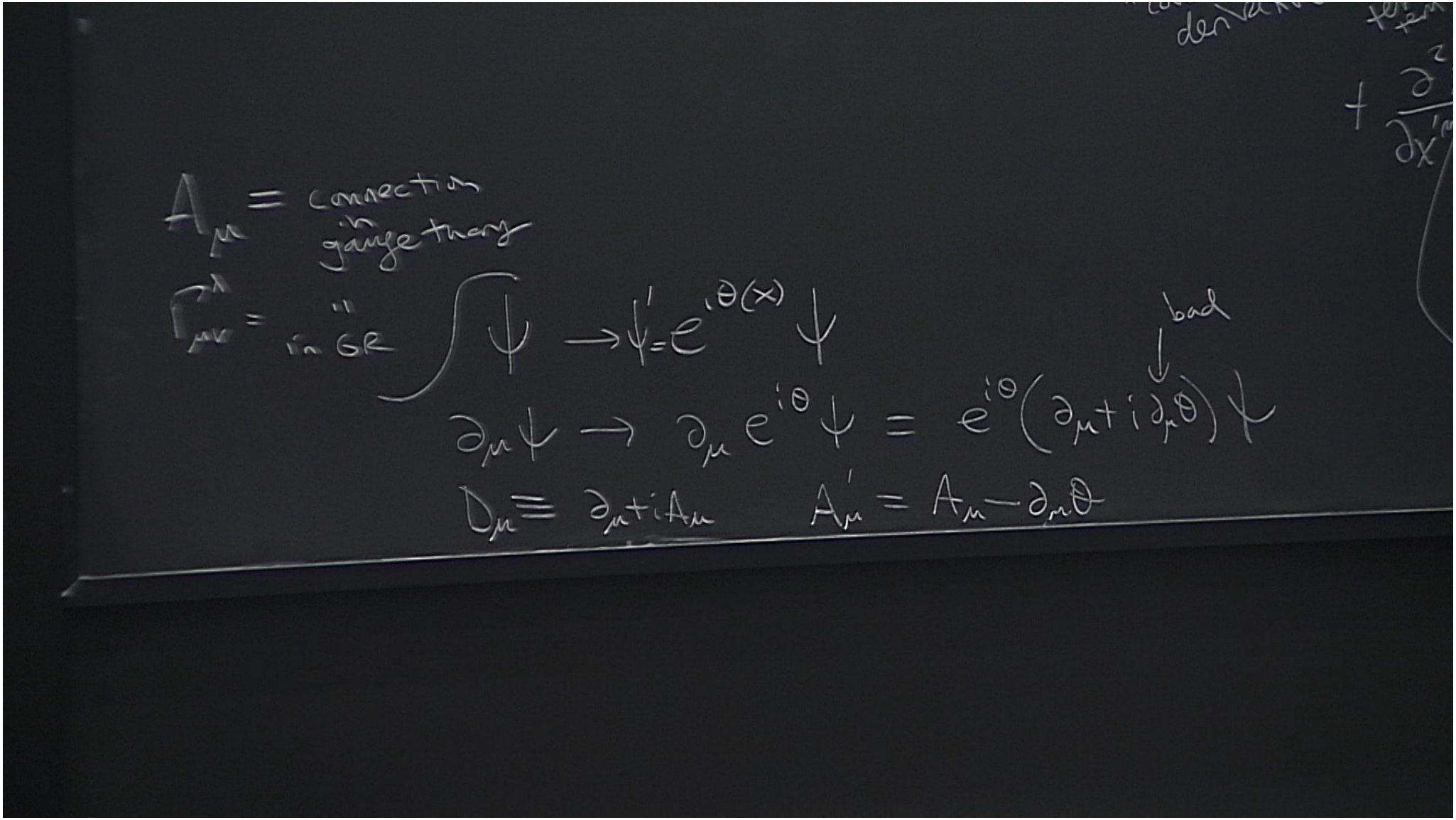
by definition $\Gamma'^{\nu}_{\alpha\mu} = \frac{\partial x'^{\nu}}{\partial x^{\delta}} \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \Gamma^{\delta}_{\lambda\sigma} + \frac{\partial x'^{\nu}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\alpha} \partial x'^{\mu}}$

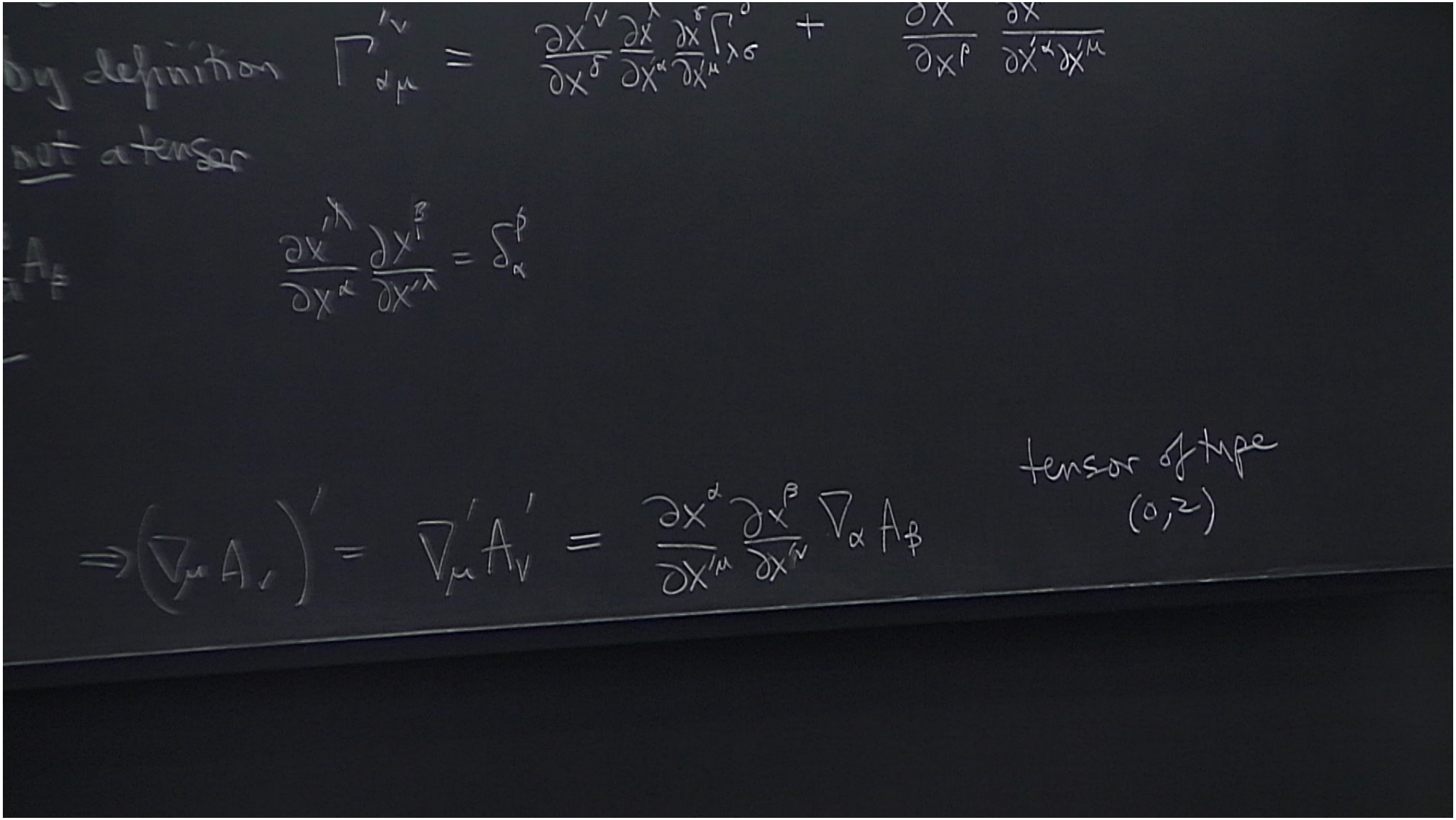
not a tensor

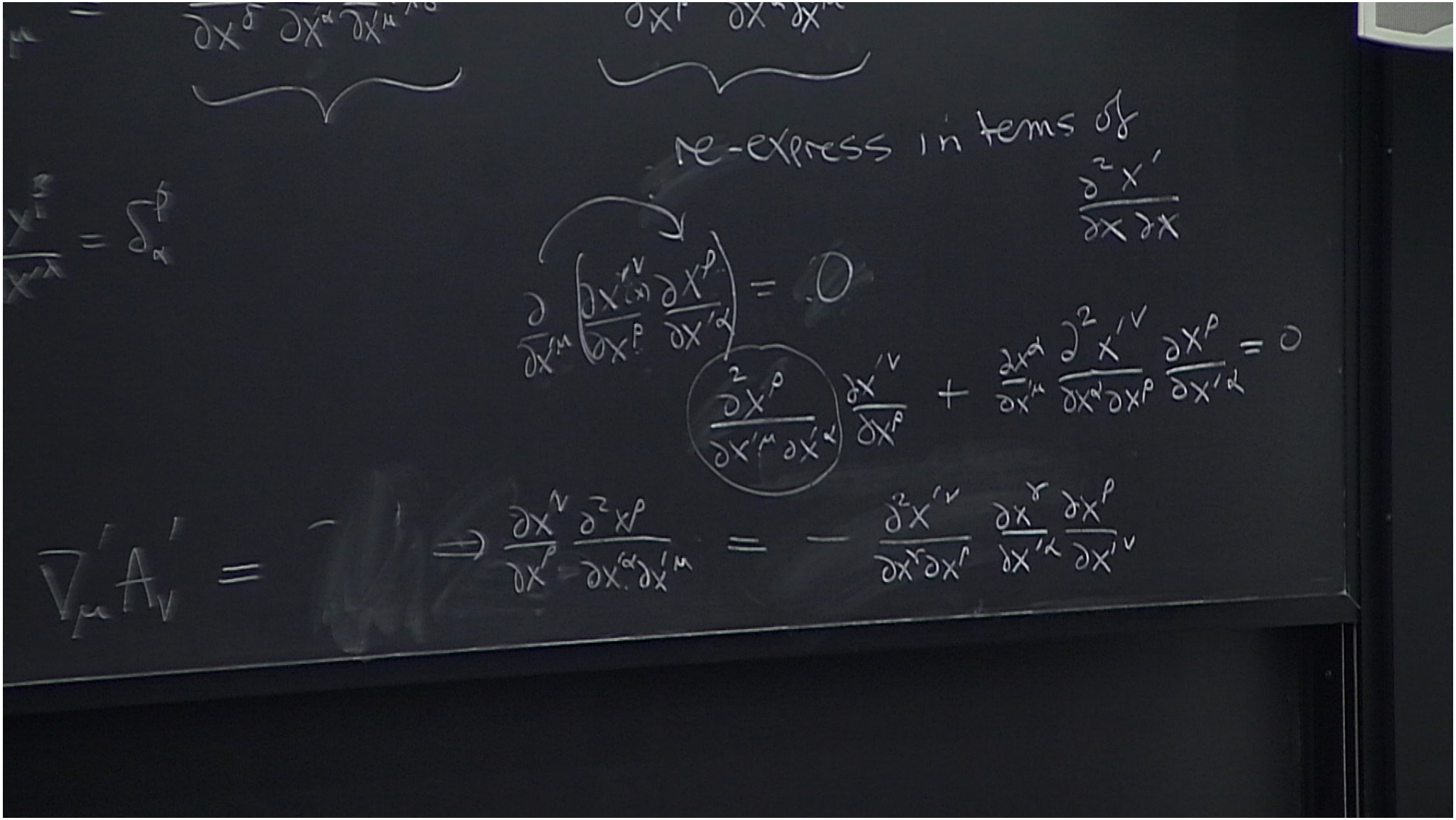
$\frac{\partial x^{\beta}}{\partial x'^{\lambda}} A_{\beta}$

$\frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} = \delta^{\beta}_{\alpha}$

A_{β}


A_μ = connection in gauge theory
$\Gamma^\lambda_{\mu\nu}$ = " in GR
$\psi \to \psi' = e^{i\theta(x)}\psi$
bad
$\partial_\mu \psi \to \partial_\mu e^{i\theta}\psi = e^{i\theta}(\partial_\mu + i\partial_\mu\theta)\psi$
$D_\mu \equiv \partial_\mu + iA_\mu$
$A'_\mu = A_\mu - \partial_\mu\theta$


by definition
not a tensor
tensor of type
(0,2)


Re-express in terms of
$\frac{\partial^2 x'}{\partial x \partial x}$
$\frac{\partial}{\partial x'^\mu}\left(\frac{\partial x'^\nu}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\alpha}\right) = 0$
$\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\alpha}\frac{\partial x'^\nu}{\partial x^\rho} + \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\alpha \partial x^\rho}\frac{\partial x^\rho}{\partial x'^\alpha} = 0$
$\nabla'_\mu A'_\nu = \cdots \Rightarrow \frac{\partial x^\nu}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\alpha \partial x'^\mu} = -\frac{\partial^2 x'^\nu}{\partial x^\gamma \partial x^\rho}\frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\rho}{\partial x'^\nu}$

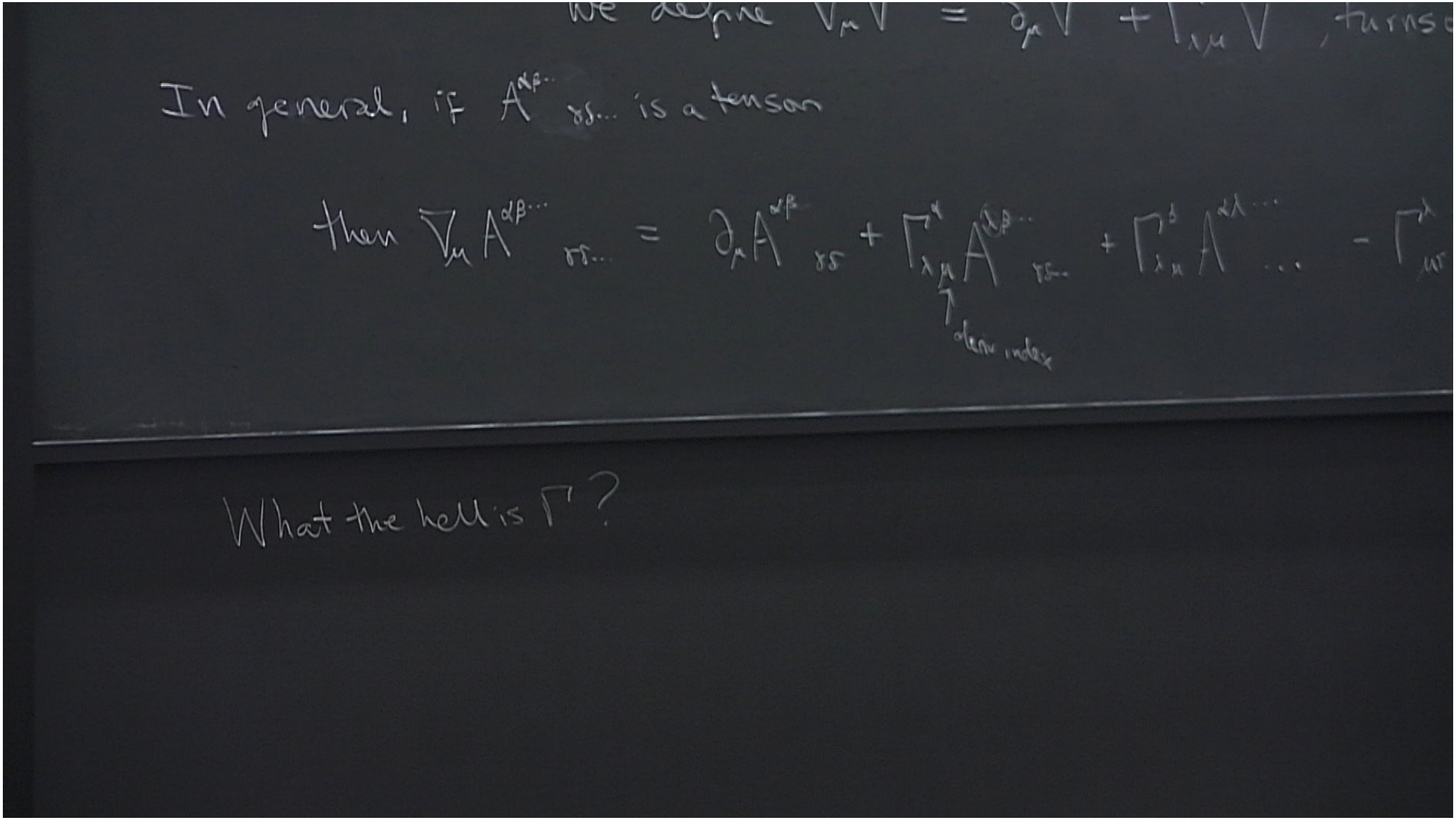
$\nabla_\mu V^\nu$. again $\partial_\mu V^\nu$ not a tensor, so introduce a

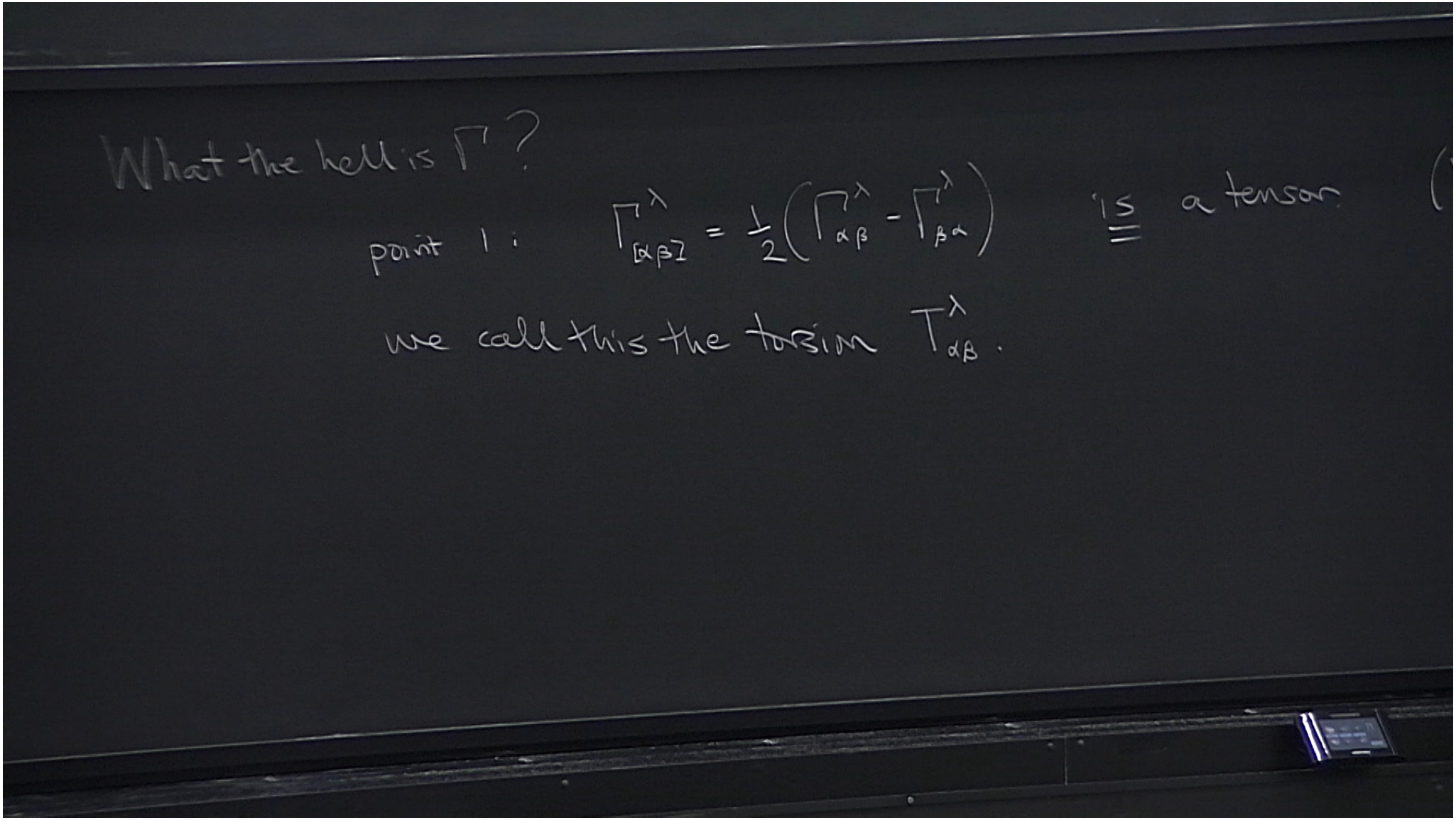
where $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\lambda\mu} V^\lambda$, turns out to be a tensor of t

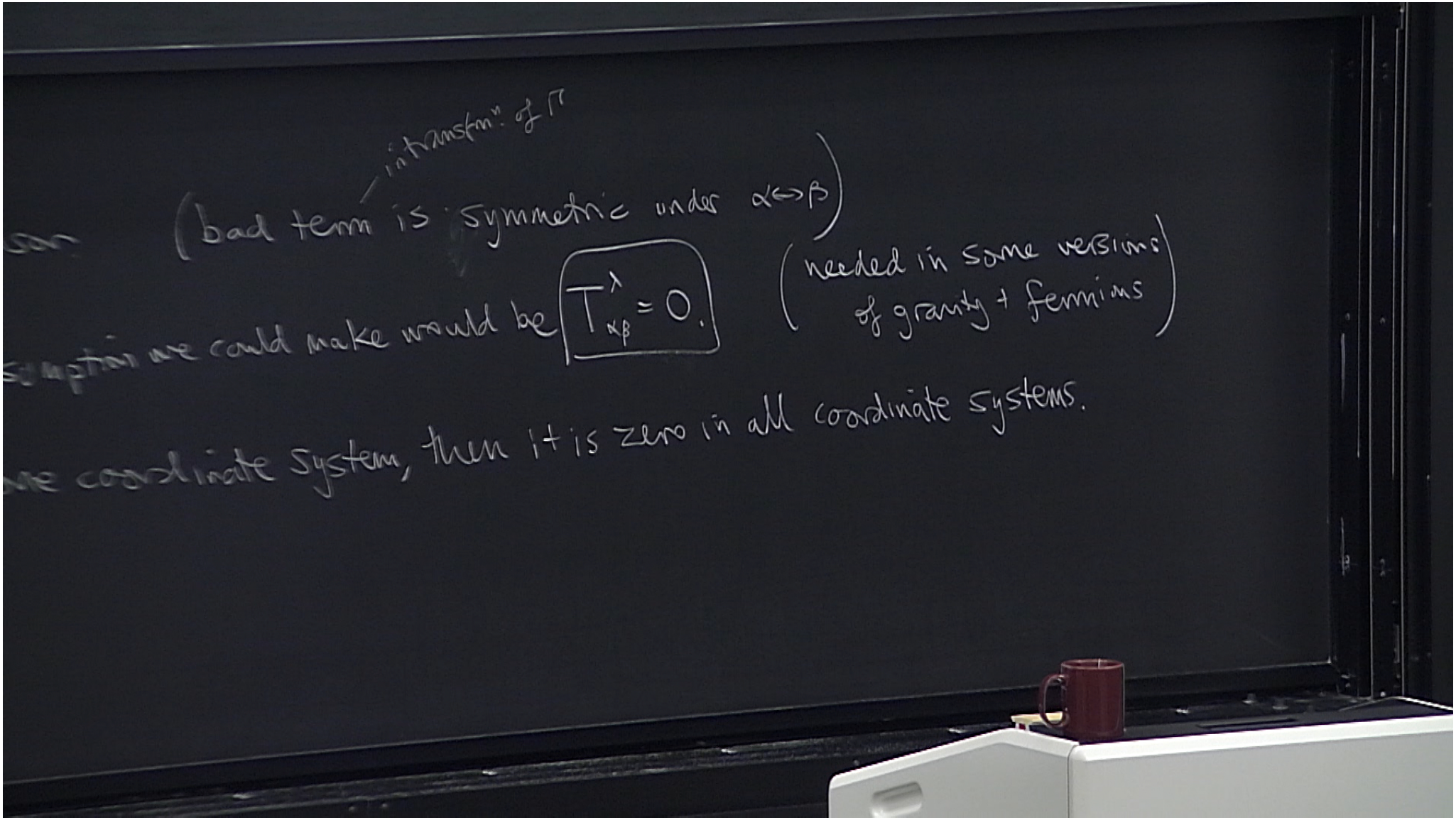
a tensor

$$\partial_\mu A^{\alpha\beta\ldots}_{\gamma\delta\ldots} + \Gamma^\alpha_{\lambda\mu} A^{\lambda\beta\ldots}_{\gamma\delta\ldots} + \Gamma^\beta_{\lambda\mu} A^{\alpha\lambda\ldots}_{\ldots} - \Gamma^\lambda_{\mu\gamma} A^{\alpha\beta\ldots}_{\lambda\delta\ldots} \quad \text{etc.}$$

↑ deriv. index


In general, if $A^{\alpha\beta\ldots}{}_{\gamma\delta\ldots}$ is a tensor
then $\nabla_\mu A^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} = \partial_\mu A^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} + \Gamma^{\alpha}_{\lambda\mu} A^{\lambda\beta\ldots}{}_{\gamma\delta\ldots} + \Gamma^{\beta}_{\lambda\mu} A^{\alpha\lambda\ldots}{}_{\gamma\delta\ldots} - \Gamma^{\lambda}_{\mu\gamma}$
dummy index
What the hell is Γ?


What the hell is Γ?
point 1: $\Gamma^{\lambda}_{[\alpha\beta]} = \frac{1}{2}\left(\Gamma^{\lambda}_{\alpha\beta} - \Gamma^{\lambda}_{\beta\alpha}\right)$ is a tensor
we call this the torsion $T^{\lambda}_{\alpha\beta}$.


(bad term is symmetric under α↔β)
(needed in some versions of gravity + fermions)
$T^{\lambda}_{\alpha\beta} = 0.$

If we assume $T^{\mu}_{\alpha\beta} = 0$, then $\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha}$, a

"METRICITY"

namely

$$g_{\alpha\beta} U^{\alpha} V^{\beta} = U.V$$

$$\nabla_{\mu} g_{\alpha\beta} = 0 \implies$$

Covariant derivative is distributive

if $\nabla_{\mu} U^{\alpha} = 0$ and $\nabla_{\mu} V^{\beta} = 0$

$L^{\lambda}_{\mu\nu}$

$$\partial_{\mu} g_{\alpha\beta} - \Gamma^{\lambda}_{\alpha\mu} g_{\lambda\beta} - \Gamma^{\lambda}_{\beta\mu} g_{\alpha\lambda} = 0$$

$$+ \partial g - \Gamma - \Gamma g$$

$$- \partial g + \Gamma + \Gamma g$$

$$\nabla_{\mu}(g_{\alpha\beta} U^{\alpha} V^{\beta}) = \partial_{\mu}(U.V)$$

$$= (\nabla_{\mu} g_{\alpha\beta}) U^{\alpha} V^{\beta} + g_{\alpha\beta} (\nabla_{\mu} U^{\alpha}) V^{\beta} + g_{\alpha\beta} U^{\alpha} \nabla_{\mu} V^{\beta}$$

$\alpha\beta = 0$, then $\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha}$, and Γ can be determined from a simple co

"METRICITY"

$$\nabla_\mu g_{\alpha\beta} = 0 \quad \Rightarrow \text{fixes } \Gamma$$

$$\partial_\mu g_{\alpha\beta} - \Gamma^{\lambda}_{\alpha\mu} g_{\lambda\beta} - \Gamma^{\lambda}_{\beta\mu} g_{\alpha\lambda} = 0$$

, need to solve for Γ.

$+\partial g \quad -\Gamma \quad -\Gamma g$

$-\partial g \quad +\Gamma \quad +\Gamma g$,

$$\Rightarrow \Gamma^{\lambda}_{\alpha\mu} = \frac{1}{2} g^{\lambda\beta}\left(g_{\alpha\beta,\mu} + g_{\mu\beta,\alpha} - g_{\alpha\mu,\beta}\right) = \left\{ {\lambda \atop \alpha\mu} \right\}$$

Chr

Su

or

can be determined from a simple condition

line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

$f^2(x)\, dx^2 = dx'^2$

$\rightarrow x' = \int^x f(x)\, dx$

$ds'^2 = g'_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} dx^\mu dx^\nu$

$= \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} dx^\rho dx^\sigma$

$+ \Gamma^\lambda_{\mu\nu}$

$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu} \right) = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$ Christoffel

termined from a simple condition,

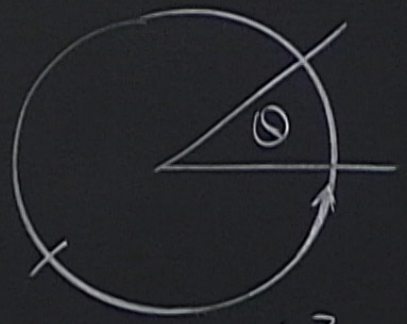
$dx^\mu dx^\nu$

$f^2(x)\,dx^2 = dx'^2$

$\rightarrow x' = \int^x f(x)\,dx$

can always choose coords

$2d \Rightarrow$



$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$

$\frac{\partial^2}{\partial x^\alpha \partial x^\beta} = g_{\mu\nu}\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}dx^\alpha dx^\beta$

$g_{\mu\nu} = \Omega^2(x)\,\eta_{\mu\nu}$

"conformally flat"

$+\Gamma^\lambda_{\alpha\beta}$

$+ g_{\mu\beta,\alpha} - g_{\alpha\mu,\beta}) = \left\{ {\lambda \atop \alpha\mu} \right\}$ Christoffel symbol.

If we assume $T^{\mu}_{\alpha\beta} = 0$

If I want
$\partial_\mu (U.V) = 0$
when $\nabla_\mu U^\alpha = \nabla_\mu V^\beta = 0$

namely

$g_{\alpha\beta} U^\alpha V^\beta = U.V$

∇_μ

$L^\lambda_{\mu\nu}$

Covariant derivative is distributive

if $\nabla_\mu U^\alpha = 0$ and $\nabla_\mu V^\beta = 0$

$$\nabla_\mu (g_{\alpha\beta} U^\alpha V^\beta) = \partial_\mu (U.V)$$

$$= (\nabla_\mu g_{\alpha\beta}) U^\alpha V^\beta + g_{\alpha\beta} (\nabla_\mu U^\alpha) V^\beta + g_{\alpha\beta} U^\alpha \nabla_\mu V^\beta$$

can be determined from a simple condition,

$$g_{11}(x) = f^2(x)$$

line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

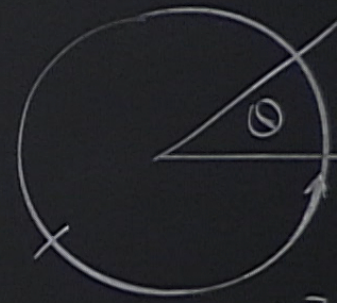
$$g'_{11}(x) = \frac{\partial x}{\partial x'} \frac{\partial x}{\partial x'} f^2(x) = dx'^2$$

$$\frac{\partial x}{\partial x'} = \frac{1}{f(x)} \rightarrow x' = \int^x f(x)\,dx$$

$$dx' = f(x)\,dx$$

can always choose coords

$2d \Rightarrow$



$$ds'^2 = g'_{\mu\nu} dx'^\mu dx'^\nu = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \frac{\partial x'^\mu}{\partial x^\delta} \frac{\partial x'^\nu}{\partial x^\delta} dx^\delta dx^\delta$$

$g_{\mu\nu} = \Omega^2(x)\,\eta_{\mu\nu}$

"conformally

$+ T^\lambda_{\mu\nu}$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} \left(g_{\beta\mu,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta} \right) = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$$ Christoffel Symbol