Title: PSI 17/18 - Relativity - Lecture 7

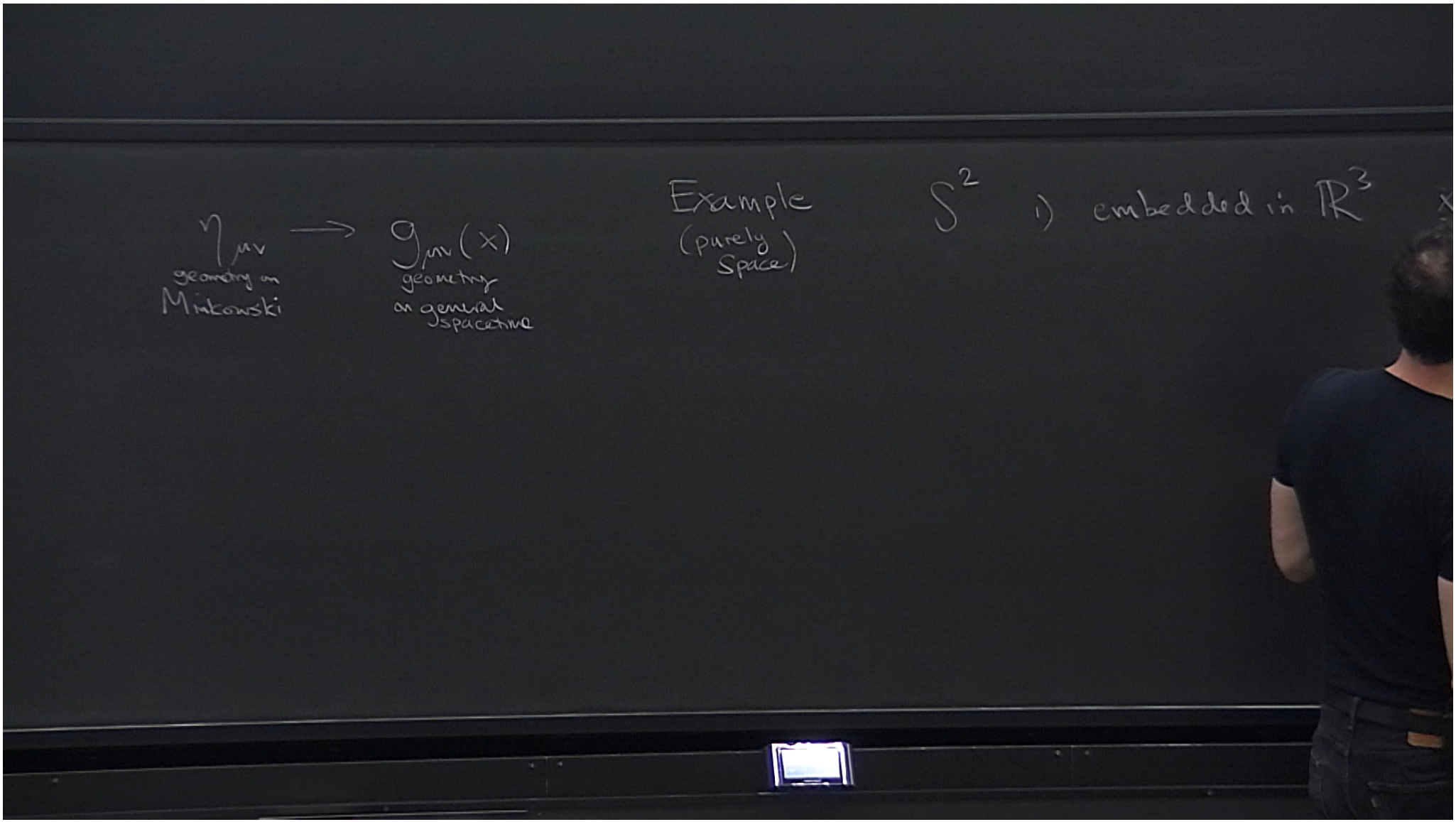
Date: Sep 13, 2017 09:00 AM

URL: http://pirsa.org/17090026

Abstract:

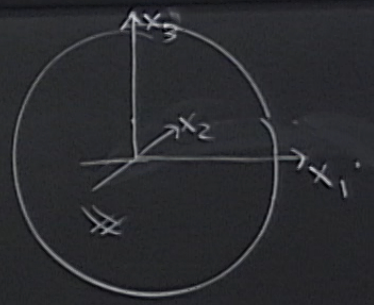
$\eta_{\mu\nu}$ geometry on Minkowski $\longrightarrow$ $g_{\mu\nu}(x)$ geometry on general spacetime

Example (purely Space) S^2 1) embedded in $\mathbb{R}^3$



S^2 1) embedded in $\mathbb{R}^3$ $\underline{x}^2 = 1$

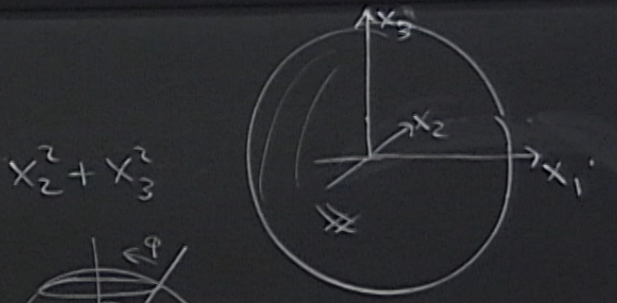
$= x_1^2 + x_2^2 + x_3^2$

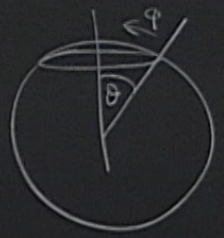


S^2 1) embedded in $\mathbb{R}^3$ $\underline{x}^2 = 1$

$= x_1^2 + x_2^2 + x_3^2$

2) intrinsically x_1





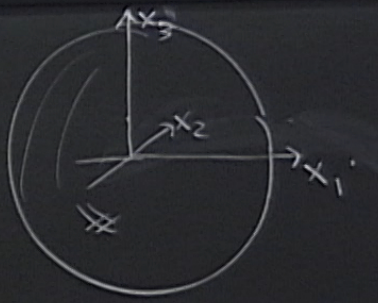
S^2 1) embedded in $\mathbb{R}^3$ $\quad \underline{x}^2 = 1$

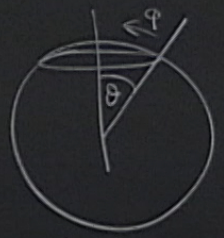
$= x_1^2 + x_2^2 + x_3^2$

2) intrinsically $\quad x_1 = \sin\theta \cos\varphi$

$x_2 = \sin\theta \sin\varphi$

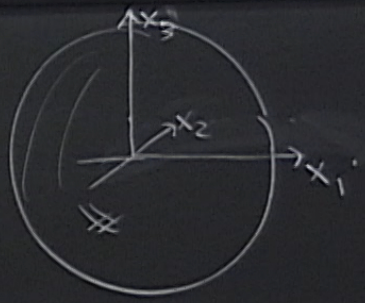
$x_3 = \cos\theta$





S^2 1) embedded in $\mathbb{R}^3$ $\quad \underline{x}^2 = 1$

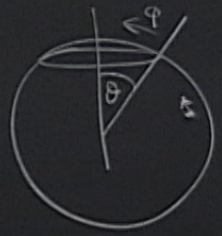
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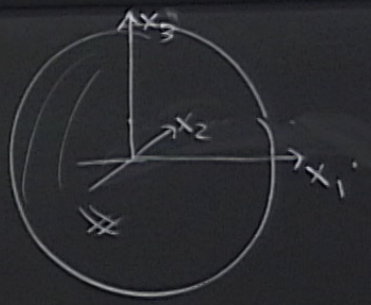
$x_2 = \sin\theta \sin\varphi$

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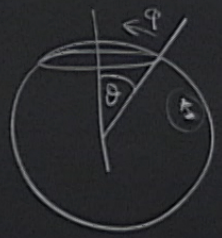


metric?

S^2 1) embedded in $\mathbb{R}^3$ $\underline{x}^2 = 1$
(Pythagoras) $= x_1^2 + x_2^2 + x_3^2$

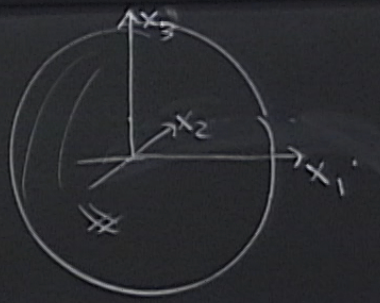


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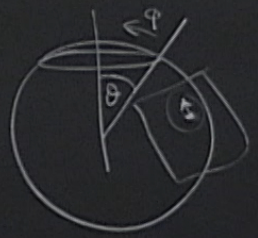


metric? $d\underline{x}^2$ in embedding space
(Pythagoras)

S^2 1) embedded in $\mathbb{R}^3$ $\quad \underline{x}^2 = 1$
(Pythagoras) $\quad = x_1^2 + x_2^2 + x_3^2$

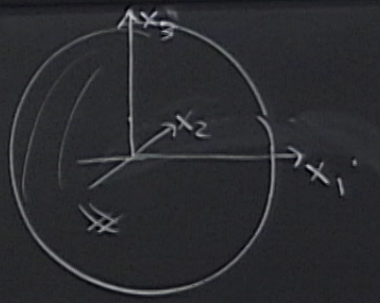


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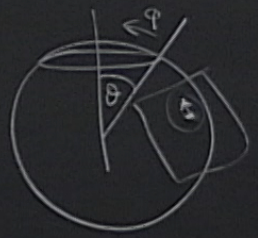


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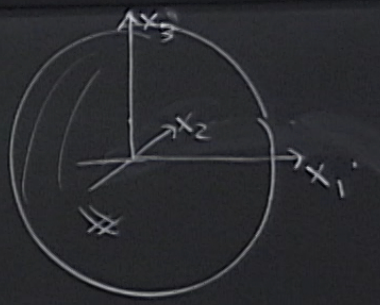


$dx_1 = \cos\theta\, d\theta\cos\varphi - \sin\theta\, d\varphi\sin\varphi$

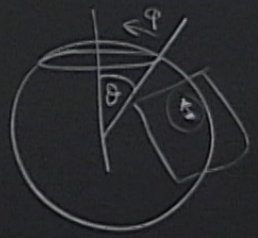
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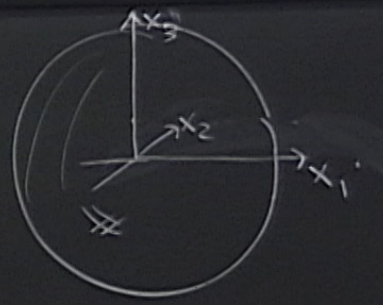


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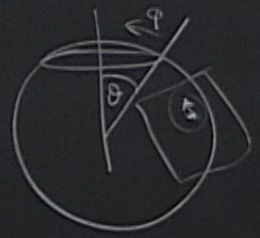
metric? $d\underline{x}^2$ in embedding space
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$dx_1^2 + dx_2^2 + dx_3^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ (Ex)

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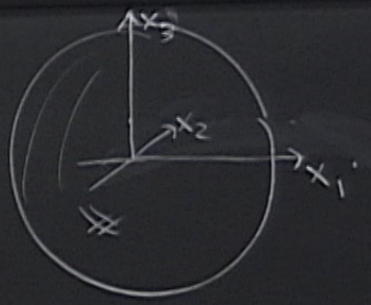
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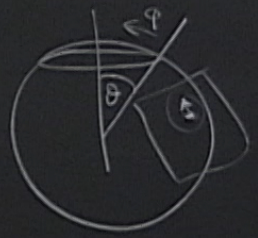
$dx_1^2 + dx_2^2 + dx_3^2 = d\theta^2 + \sin^2\theta d\varphi^2$ (Ex)

$= g_{ij} dx^i dx^j$ where $x^i = (\theta, \varphi)$ $i = 1, 2$

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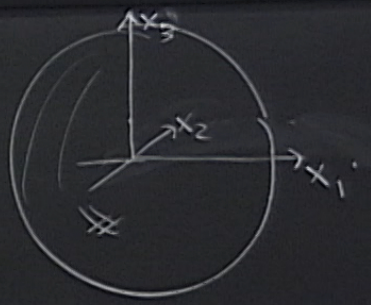
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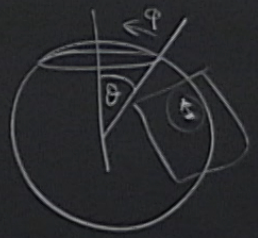
$dx_1^2 + dx_2^2 + dx_3^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ (Ex)

$= g_{ij}\, dy^i\, dy^j$ where $y^i = (\theta, \varphi)$ $i = 1, 2$

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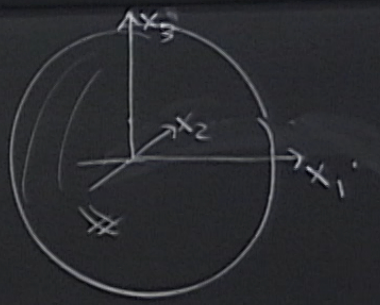
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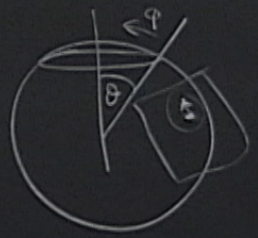
$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

S^2 1) embedded in $\mathbb{R}^3$ $\quad \underline{x}^2 = 1$
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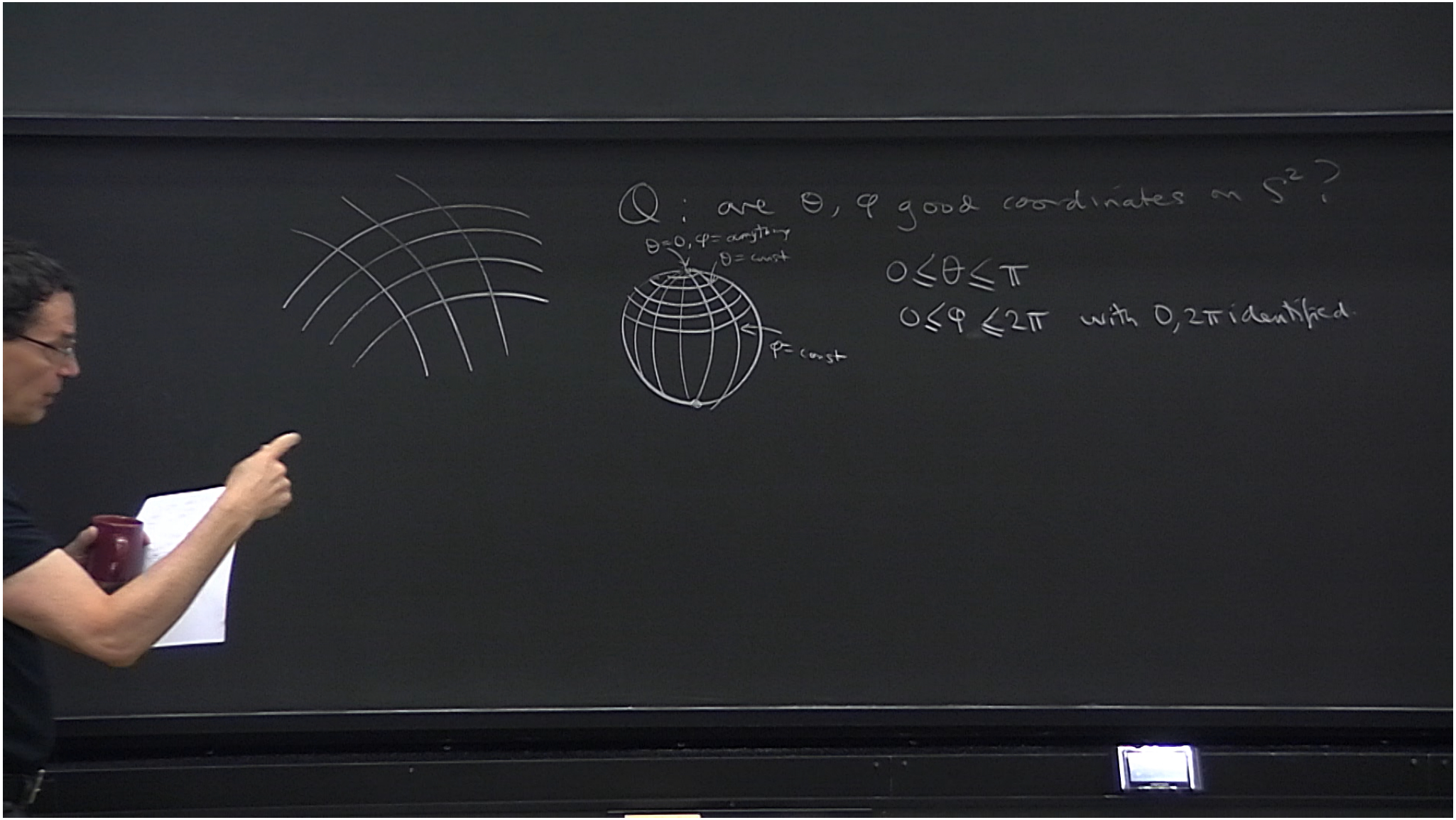


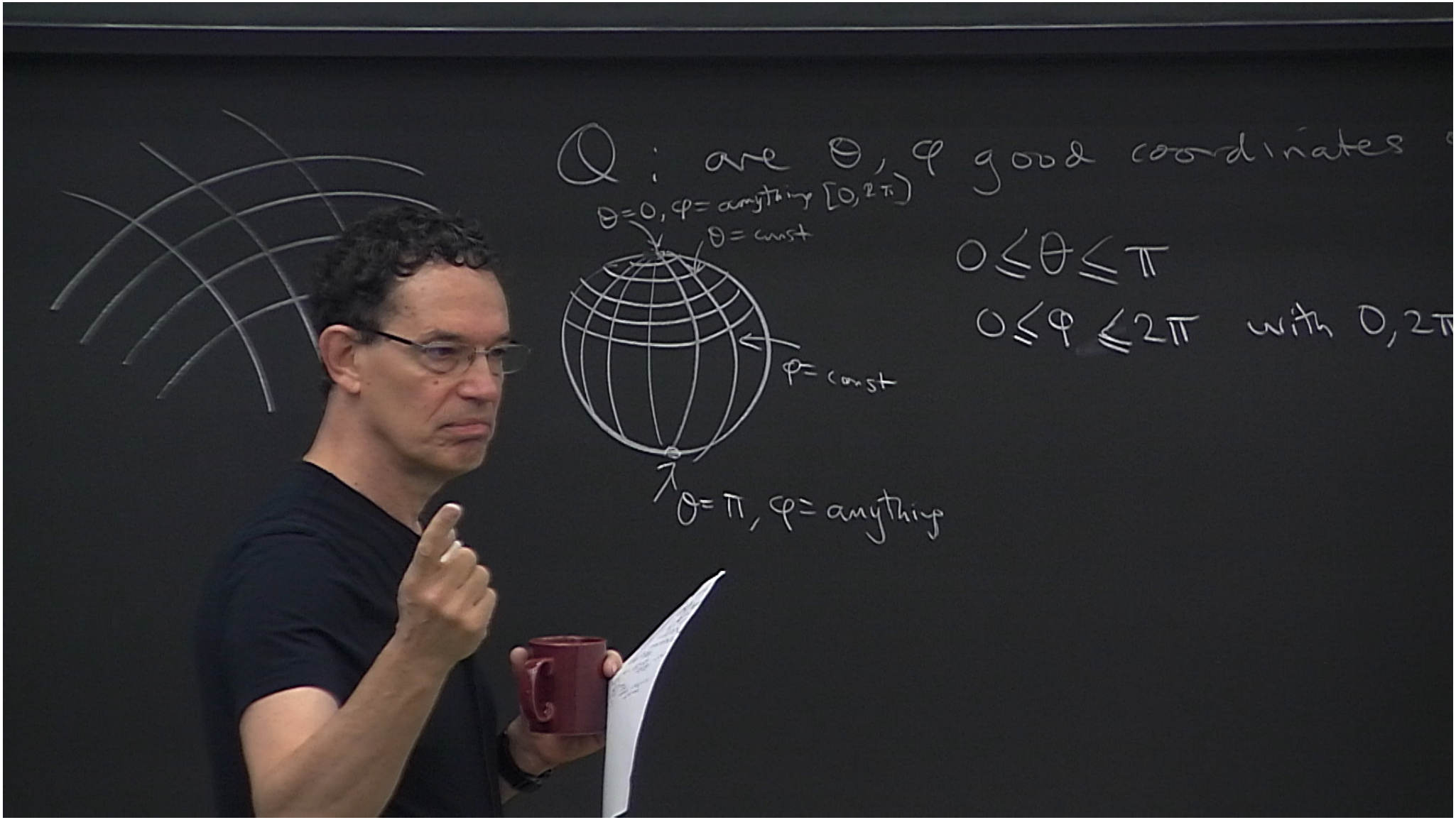
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$dx_1^2 + dx_2^2 + dx_3^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ (Ex)

$= g_{ij}\, dy^i\, dy^j$ where $y^i = (\theta, \varphi)$ $\quad i = 1, 2$

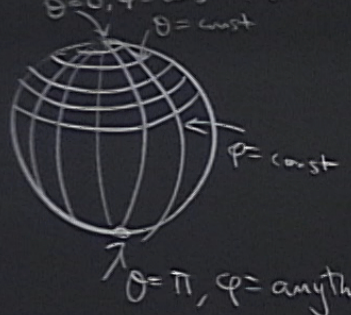
$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$ (metric induced from embedding metric)


Q: are θ, φ good coordinates on S^2?
θ=0, φ= anything
θ = const
φ= const
$0 \leq \theta \leq \pi$
$0 \leq \varphi \leq 2\pi$ with $0, 2\pi$ identified.


Q: are θ, φ good coordinates
θ=0, φ= anything [0, 2π]
θ= const
φ= const
θ=π, φ= anything
0≤θ≤π
0≤φ≤2π with 0, 2π

$(0 \ \sin^2\theta)$

Q: are θ, φ good coordinates on S^2?



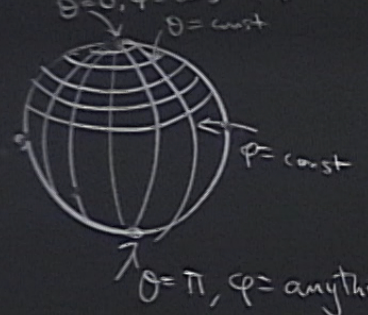
$0 \leq \theta \leq \pi$

$0 \leq \varphi < 2\pi$ with $0, 2\pi$ identified

NO: map $S^2 \to \{\theta, \varphi\}$ is 1-many at NP, S, P.

$(0, \sin^2\theta)$

Q: are θ, φ good coordinates on S^2?



$0 \le \theta \le \pi$

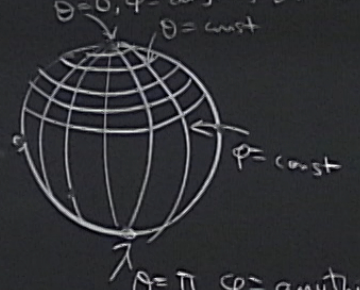
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Better system of coords: projective coords.



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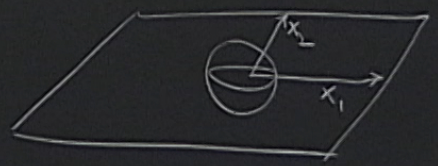


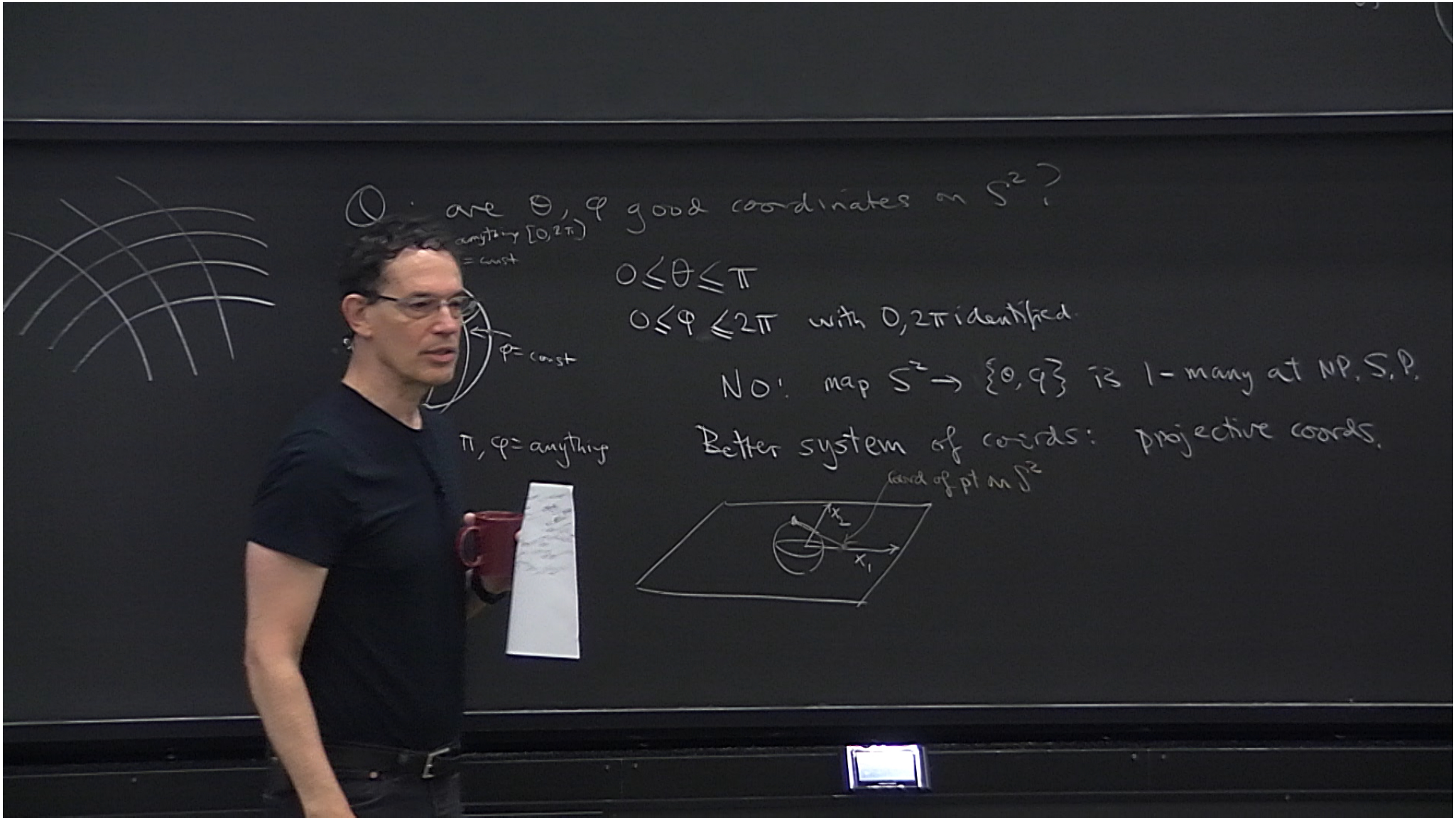
$0 \le \theta \le \pi$

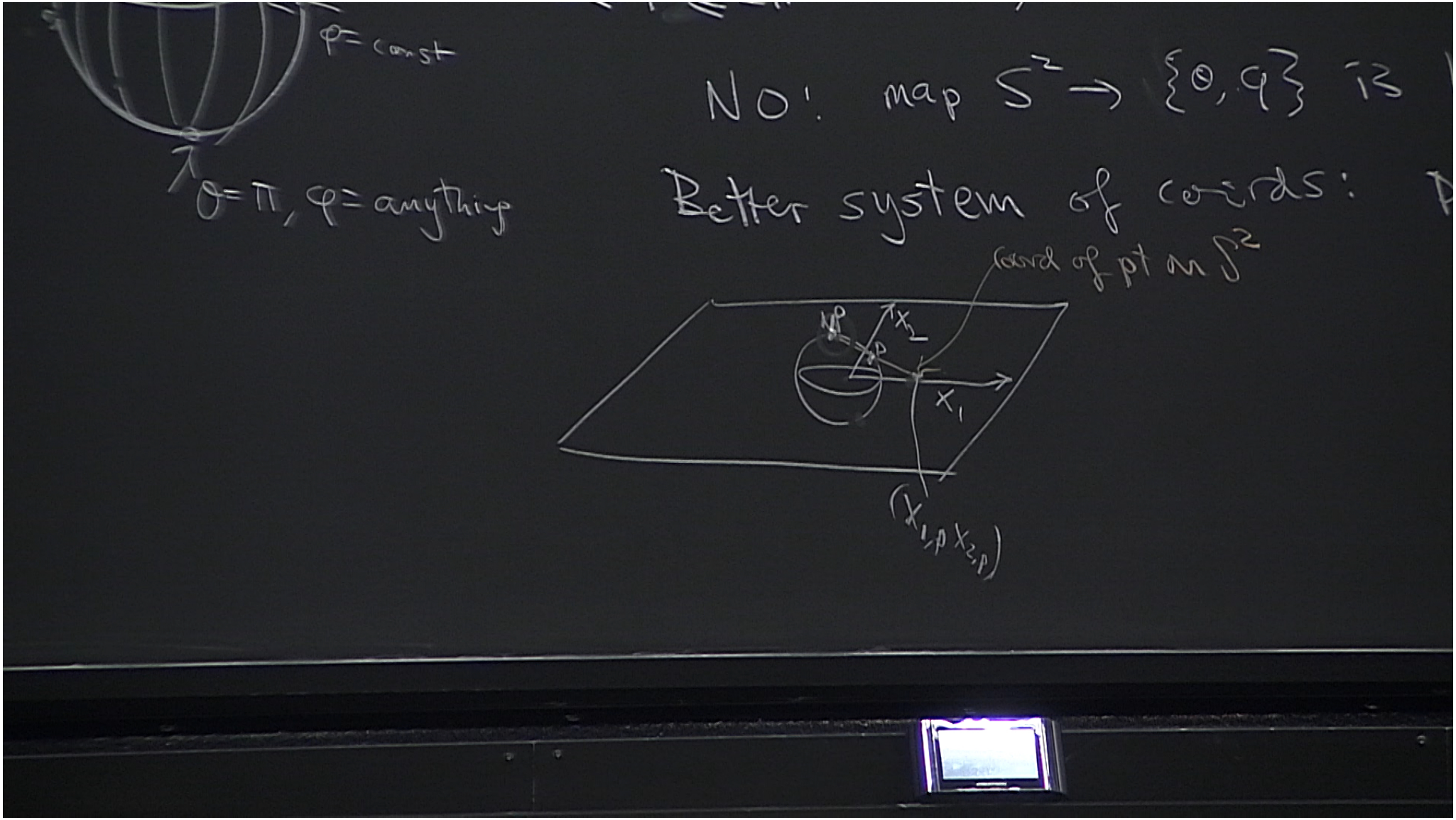
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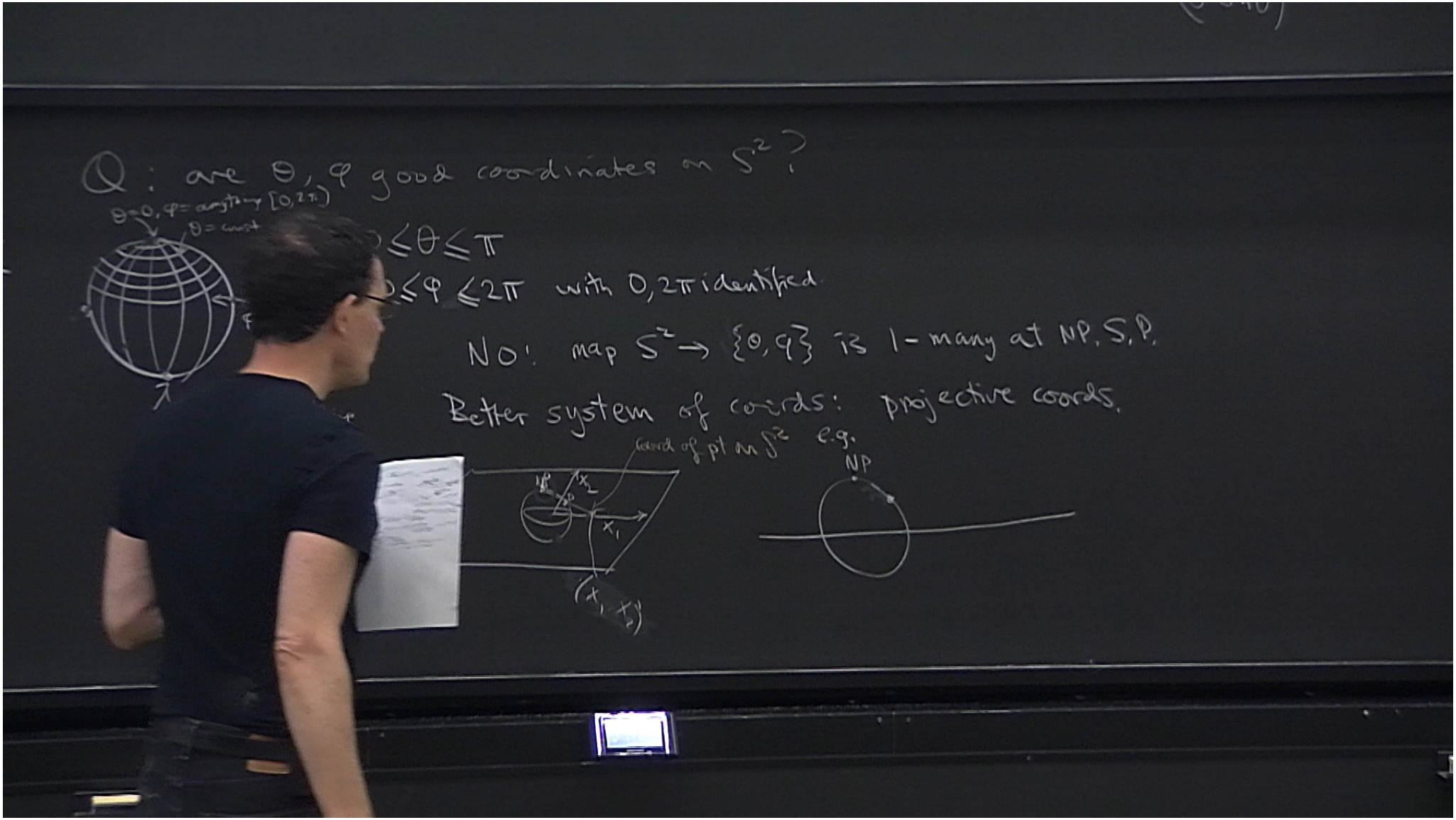
NO: map $S^2 \to \{\theta, \varphi\}$ is 1-many at NP, SP.

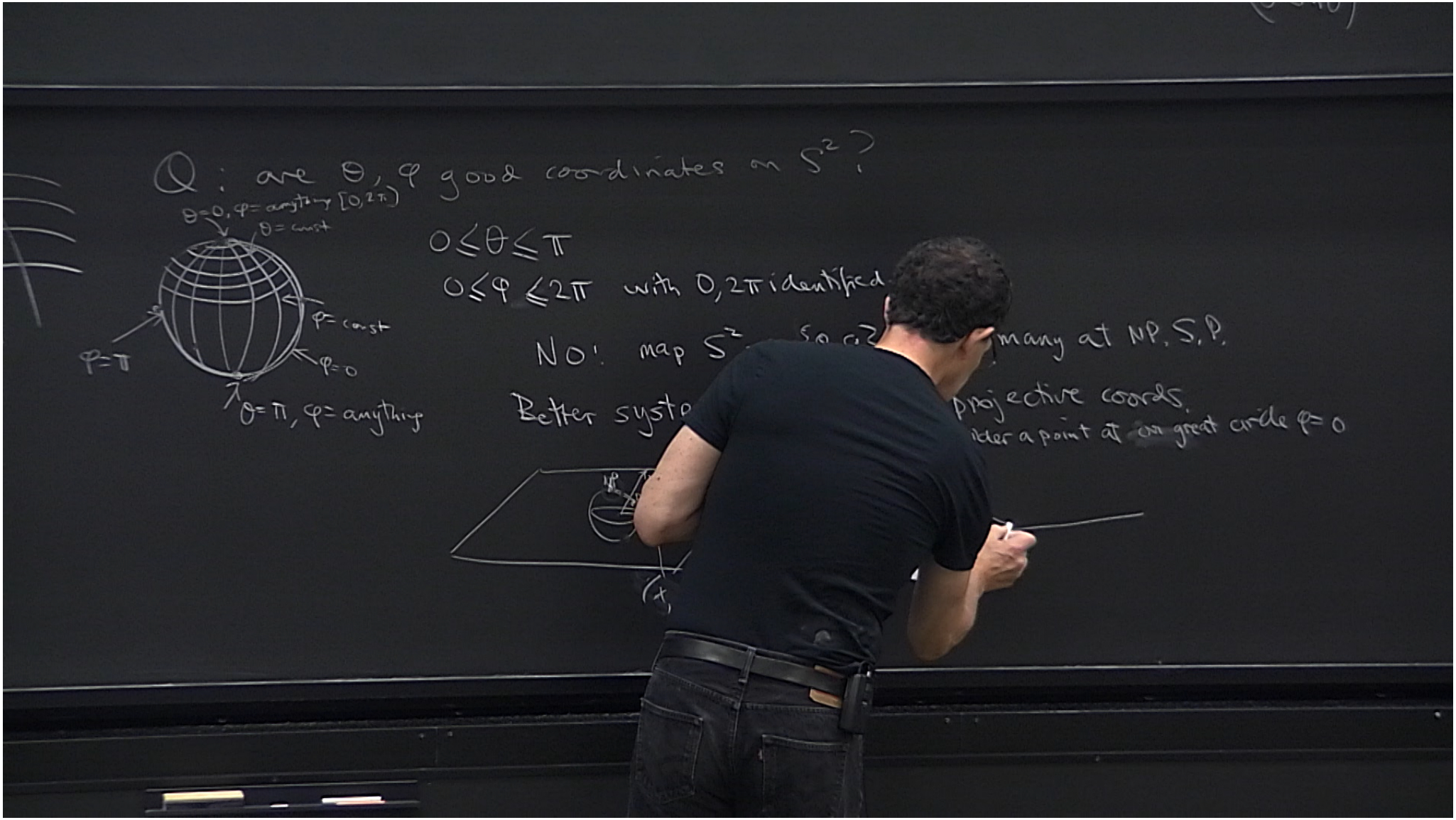
Better system of coords: projective coords.




Q: are θ, φ good coordinates on S^2?
anything $[0, 2\pi)$
= const
$0 \le \theta \le \pi$
$0 \le \varphi \le 2\pi$ with $0, 2\pi$ identified
φ= const
NO: map $S^2 \to \{\theta, \varphi\}$ is 1-many at NP, SP.
π, φ= anything
Better system of coords: projective coords.
coord of pt on S^2
x_2
x_1


φ = const
θ = π, φ = anything
NO: map $S^2 \to \{\theta, \varphi\}$ is
Better system of coords:
coord of pt on S^2
P
x_2
x_1
$(x_{1,P}, x_{2,P})$


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θ = const
$0 \leq \theta \leq \pi$
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Better system of coords: projective coords.
coord of pt on S^2
e.g.
NP


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φ= const
φ=π
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θ=π, φ= anything
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0≤φ<2π with 0,2π identified
NO!
many at NP, S.P.
projective coords.
Better syste
ider a point at
great circle φ=0

Q: are θ, φ good coordinates on S^2?

$\theta = 0, \varphi =$ anything $[0, 2\pi)$

$\theta =$ const

$\varphi = \pi$

$0 \leq \theta \leq \pi$

$0 \leq \varphi < 2\pi$ with $0, 2\pi$ identified

No: map $S^2 \to \{\theta, \varphi\}$ is 1-many at NP, SP.

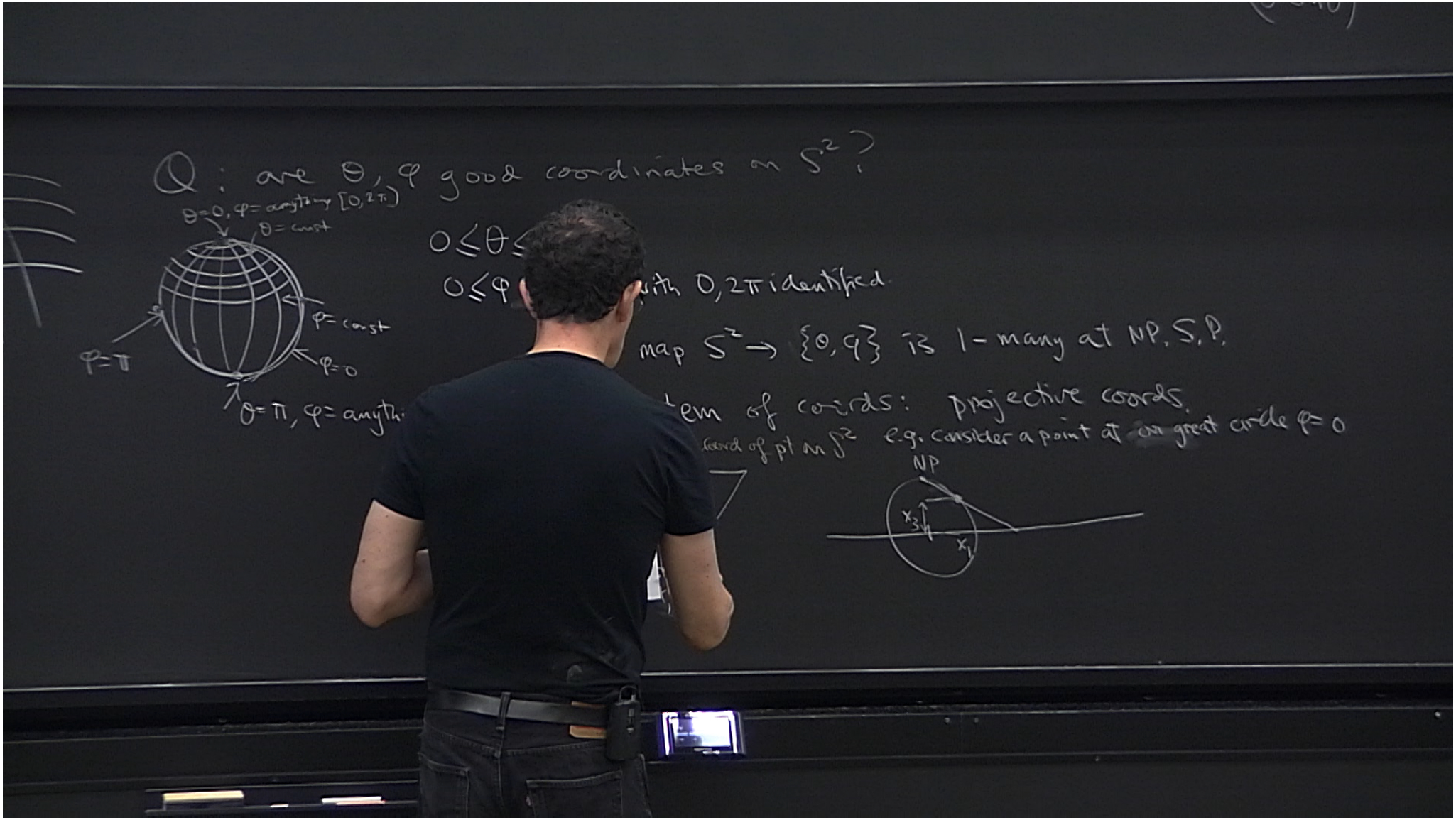
Better system of coords: projective coords.

e.g. consider a point at on great circle $\varphi = 0$

coord of pt on S^2

NP

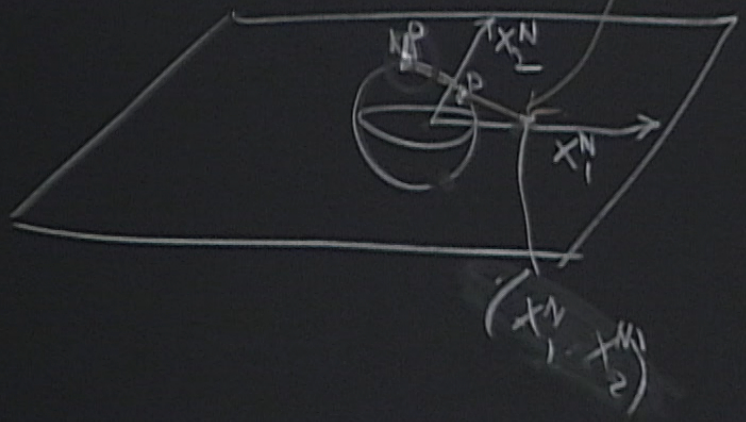
(x_1, x_2)

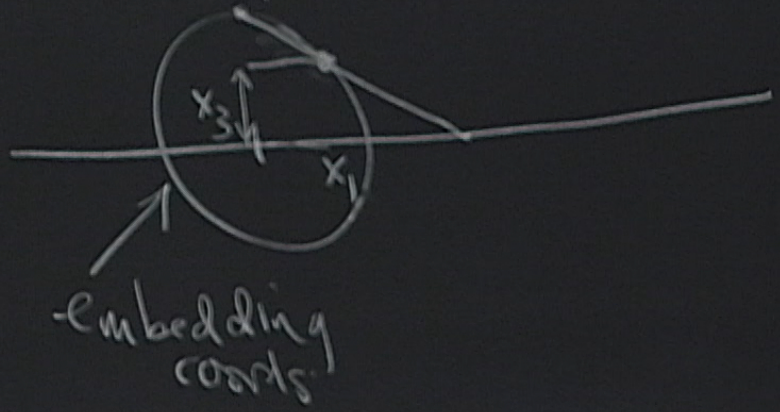

Q: are θ, φ good coordinates on S^2?
θ=0, φ= anything in [0, 2π)
θ= const
φ= const
φ=π
φ=0
θ=π, φ= anyth
0≤θ≤
0≤φ
with 0, 2π identified
map $S^2 \to \{0, q\}$ is 1-many at NP, S, P.
em of coords: projective coords.
coord of pt on S^2
e.g. consider a point at on great circle φ=0
NP
x_3
x_1

No: map $S^2 \to \{0,1\}$ is 1-many at NP, SP.

Better system of coords: projective coords.

coord of pt on S^2 e.g. consider a point at $x_2 = 0$ on great circ





- embedding coords.

Q: are θ, φ good coordinates on S^2?

$\theta = 0$, φ = anything $[0, 2\pi)$

θ = const

φ = const

$\varphi = 0$

$\theta = \pi$

$\leq \pi$

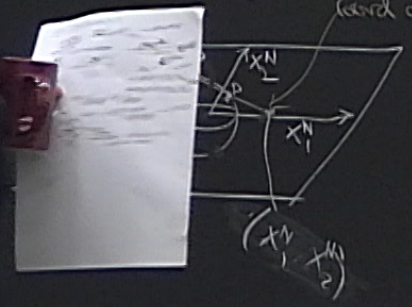
$\leq 2\pi$ with $0, 2\pi$ identified

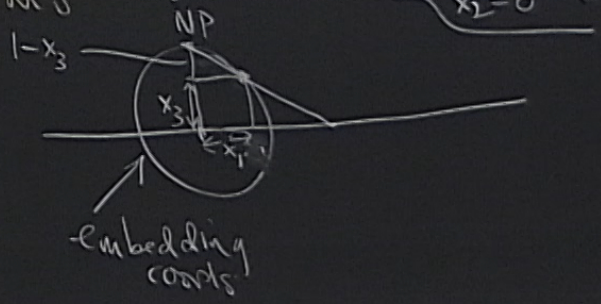
NO: map $S^2 \to \{\theta, \varphi\}$ is 1-many at NP, S, P.

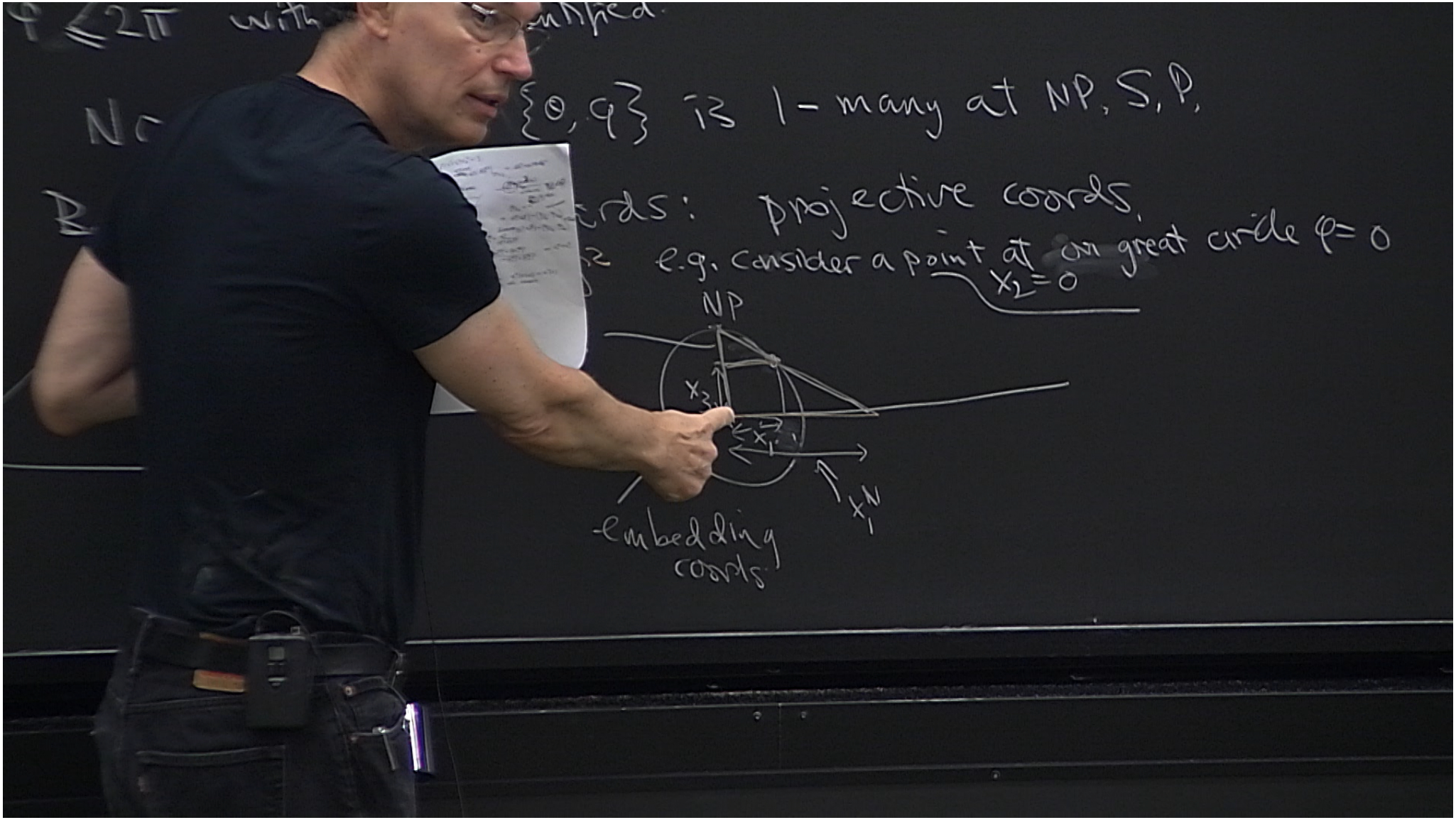
Better system of coords: projective coords.

e.g. consider a point at on great circle $\varphi = 0$ $x_2 = 0$

coord of pt on S^2





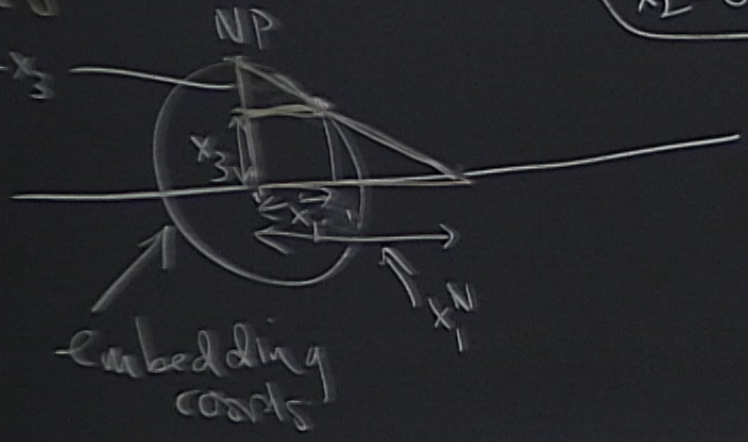

{0, q} is 1-many at NP, S, P.
projective coords.
e.g. consider a point at
great circle
NP
Embedding coords.

θ, φ is 1-many at NP, SP.

coords: projective coords.

e.g. consider a point at on great circle $\varphi = 0$

$x_2 = 0$



Similar Δs: $\frac{x_1^N}{1} = \frac{x_1}{1-x_3} \Rightarrow x_1^N = \frac{x_1}{1-x_3}$

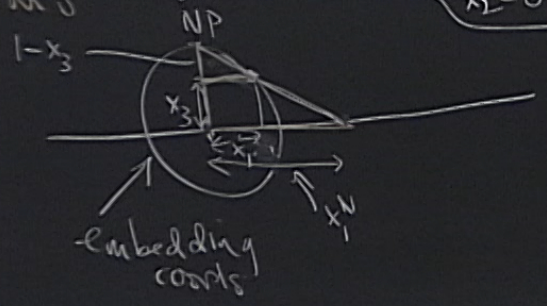
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map $S^2 \to \{0, q\}$ is 1-many at NP, S, P.

system of coords: projective coords.

e.g. consider a point at coord of pt on S^2 on great circle $\varphi = 0$, $x_2 = 0$



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generalise to arb! $x_1, x_2 \Rightarrow x_1^N = \frac{x_1}{1-x_3}$, $x_2^N = \frac{x_2}{1-x_3}$

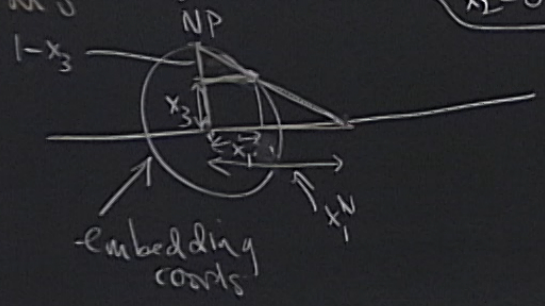
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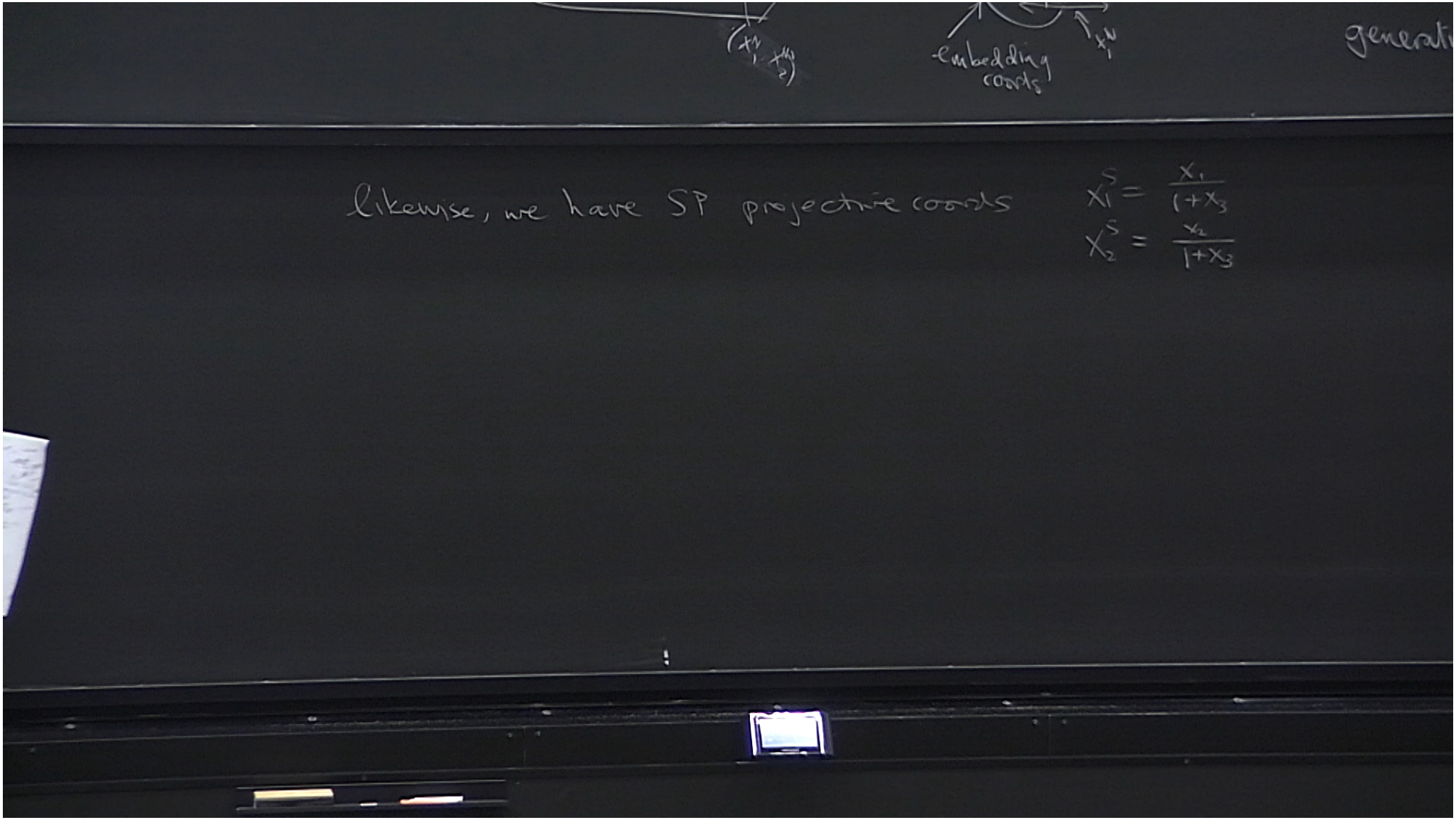
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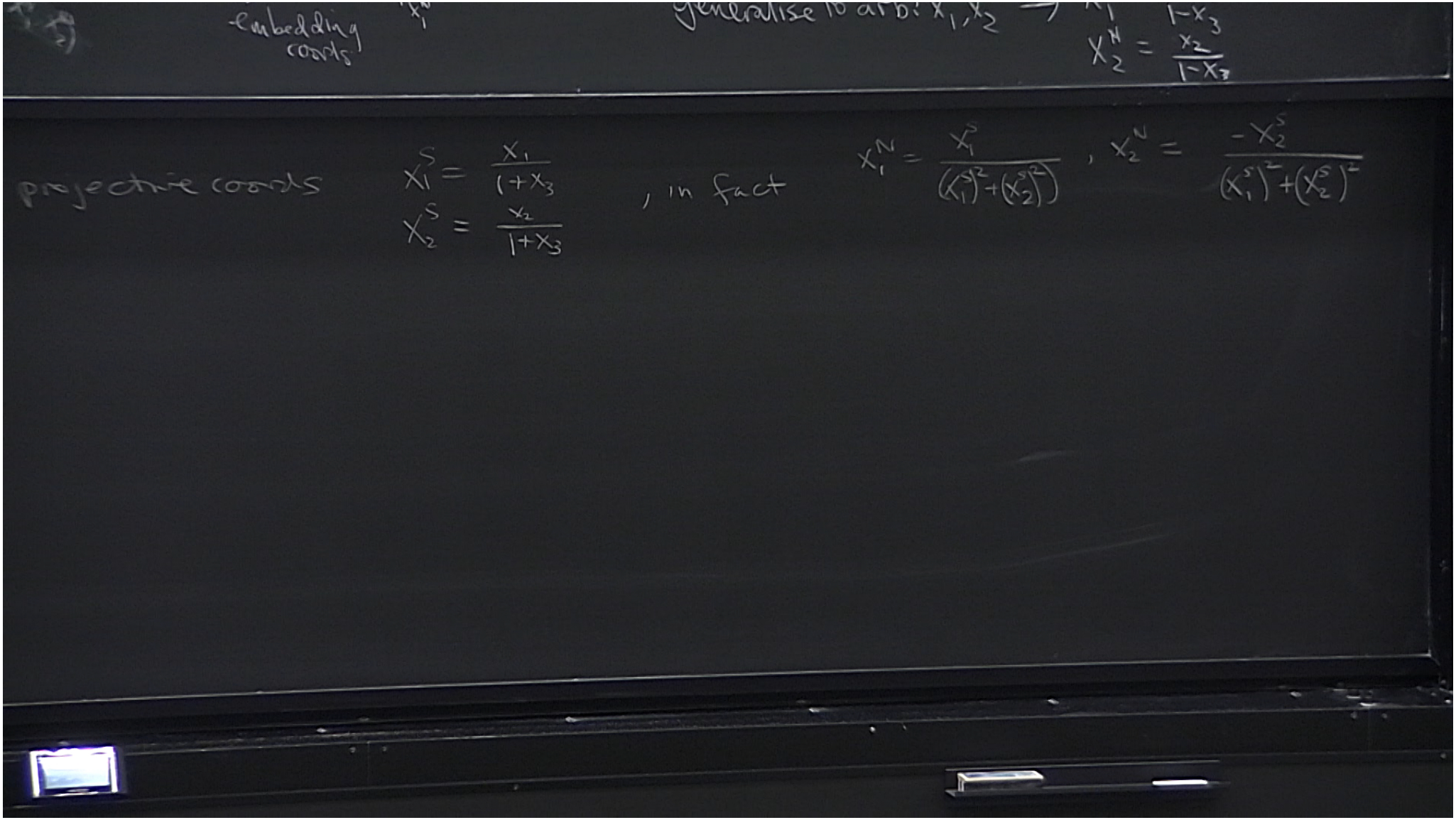


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$x_2^N = \frac{x_2}{1-x_3}$


embedding coords
likewise, we have SP projective coords
$X_1^S = \frac{X_1}{1+X_3}$
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embedding coords
generalise to arb. x_1, x_2
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projective coords
$X_1^S = \frac{x_1}{1+x_3}$
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, in fact
$X_1^N = \frac{X_1^S}{(X_1^S)^2+(X_2^S)^2}$, $X_2^N = \frac{-X_2^S}{(X_1^S)^2+(X_2^S)^2}$

embedding coords

generalise to arb. x_1, x_2 → $X_1^N = \frac{x_1}{1-x_3}$, $X_2^N = \frac{x_2}{1-x_3}$

projective coords $X_1^S = \frac{x_1}{1+x_3}$, $X_2^S = \frac{x_2}{1+x_3}$, in fact $X_1^N = \frac{X_1^S}{(X_1^S)^2+(X_2^S)^2}$, $X_2^N = \frac{X_2^S}{(X_1^S)^2+(X_2^S)^2}$

metric $d\underline{x}^2 = \frac{4\left((dx_1^N)^2+(dx_2^N)^2\right)}{\left(1+(x_1^N)^2+(x_2^N)^2\right)^2}$

embedding coords

generalise to arb. x_1, x_2

$$X_1^N = \frac{X_1}{1-X_3}, \quad X_2^N = \frac{X_2}{1-X_3}$$

projective coords

$$X_1^S = \frac{X_1}{1+X_3}, \quad X_2^S = \frac{X_2}{1+X_3}$$

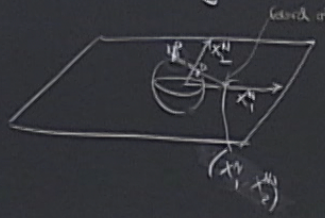
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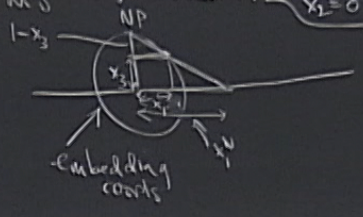
metric

$$d\underline{x}^2 = \frac{4\left((dx_1^N)^2+(dx_2^N)^2\right)}{\left(1+(X_1^N)^2+(X_2^N)^2\right)^2} = \frac{4\left((dx_1^S)^2+(dx_2^S)^2\right)}{\left(1+(X_1^S)^2+(X_2^S)^2\right)^2}$$

$\theta = \pi$, φ = anything Better system of coords: projective coords.



e.g. Consider a point at $x_2 = 0$ on great circle $\varphi = 0$



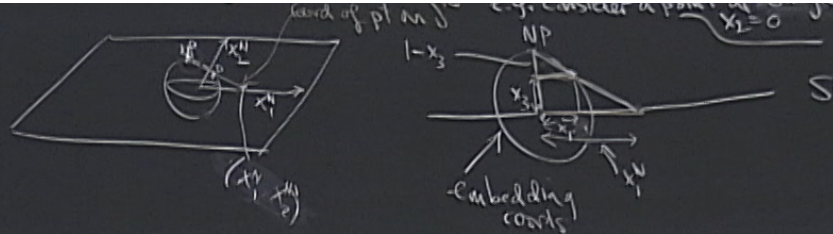
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Likewise, we have SP projective coords $X_1^S = \frac{X_1}{1+X_3}$, $X_2^S = \frac{X_2}{1+X_3}$, in fact $X_1^N = \frac{X_1^S}{(X_1^S)^2+(X_2^S)^2}$, $X_2^N = \frac{X_2^S}{(X_1^S)^2+(X_2^S)^2}$

$$\text{metric } d\underline{x}^2 = \frac{4\left((dx_1^N)^2+(dx_2^N)^2\right)}{\left(1+(X_1^N)^2+(X_2^N)^2\right)^2} = \frac{4\left((dx_1^S)^2+(dx_2^S)^2\right)}{\left(1+(X_1^S)^2+(X_2^S)^2\right)^2}$$

In general, we need a number of coordinate systems to fully describe a curved space, number depends on the topology.



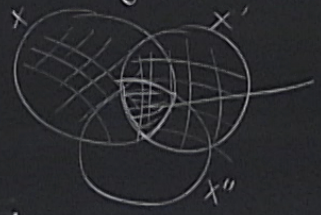
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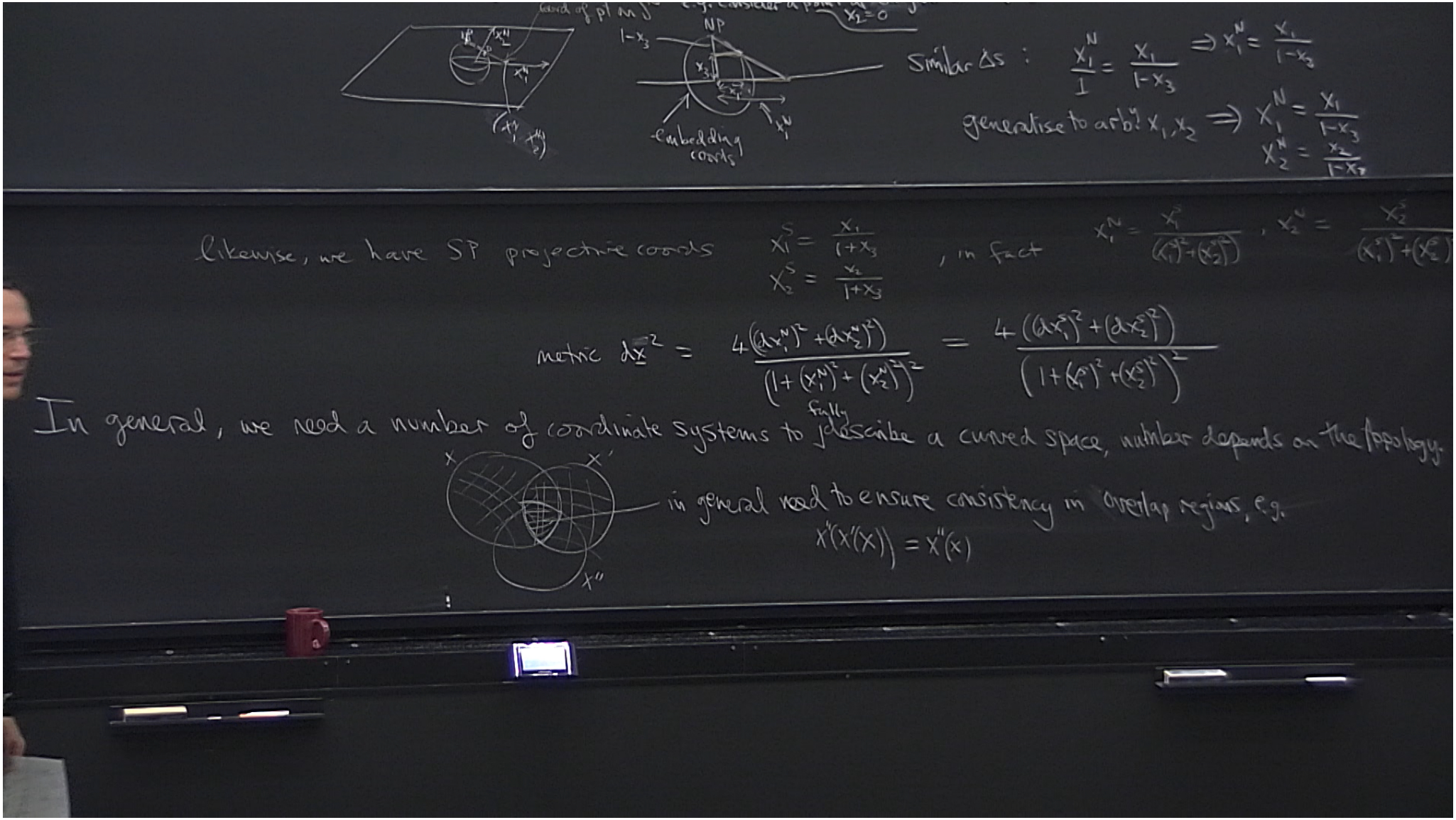
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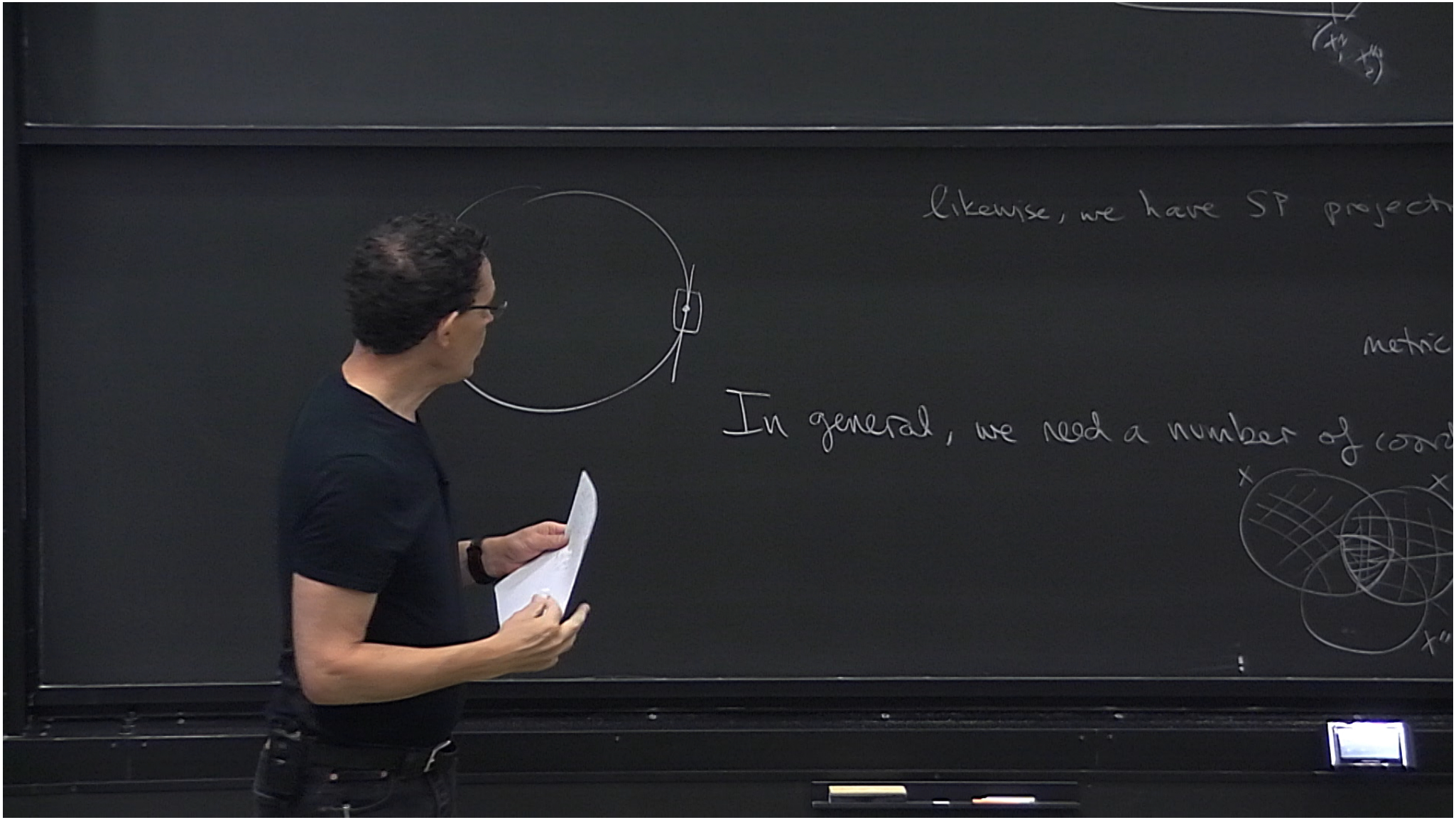
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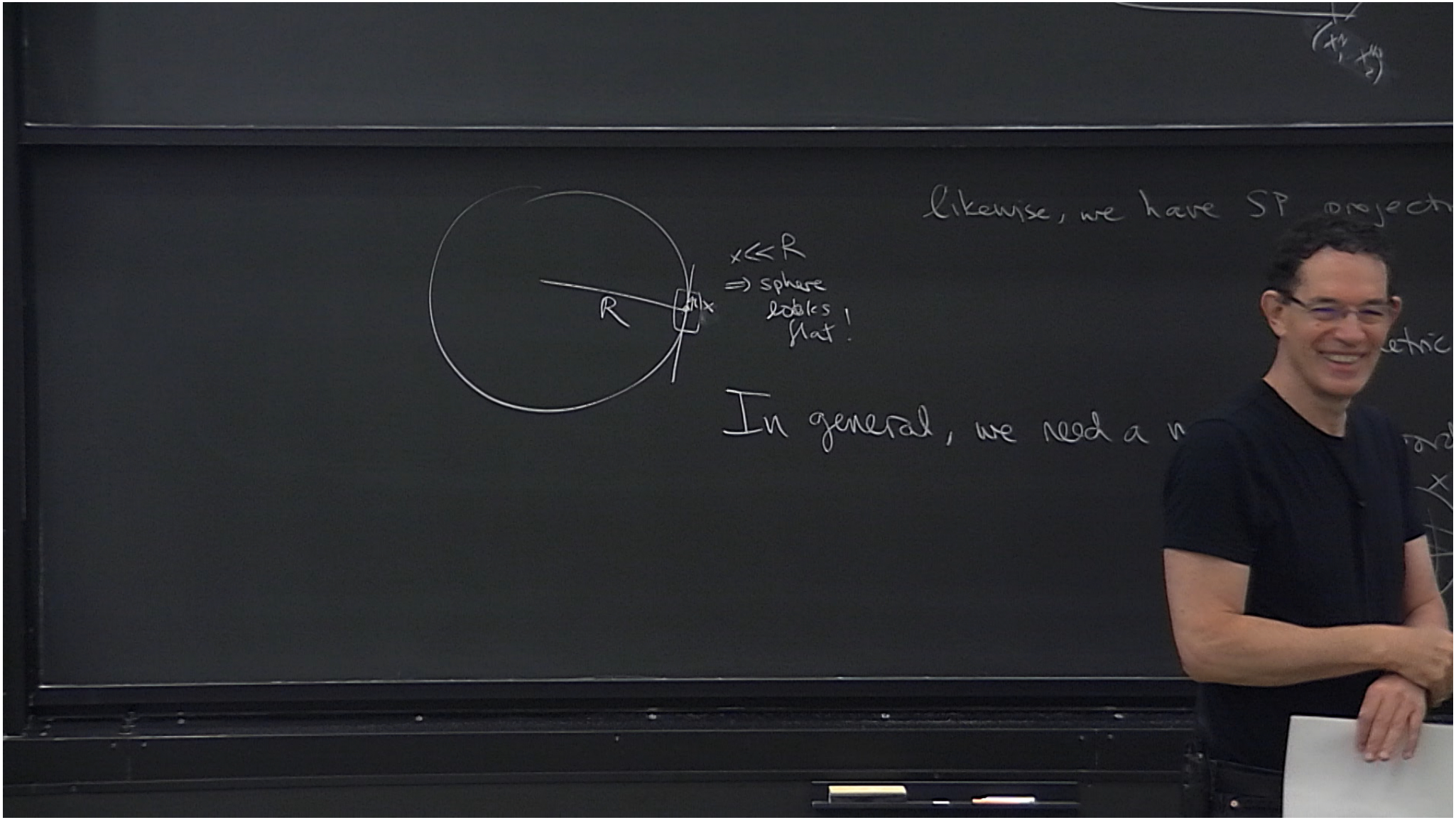
In general, we need a number of coordinate systems to (finally) describe a curved space, number depends on the topology.

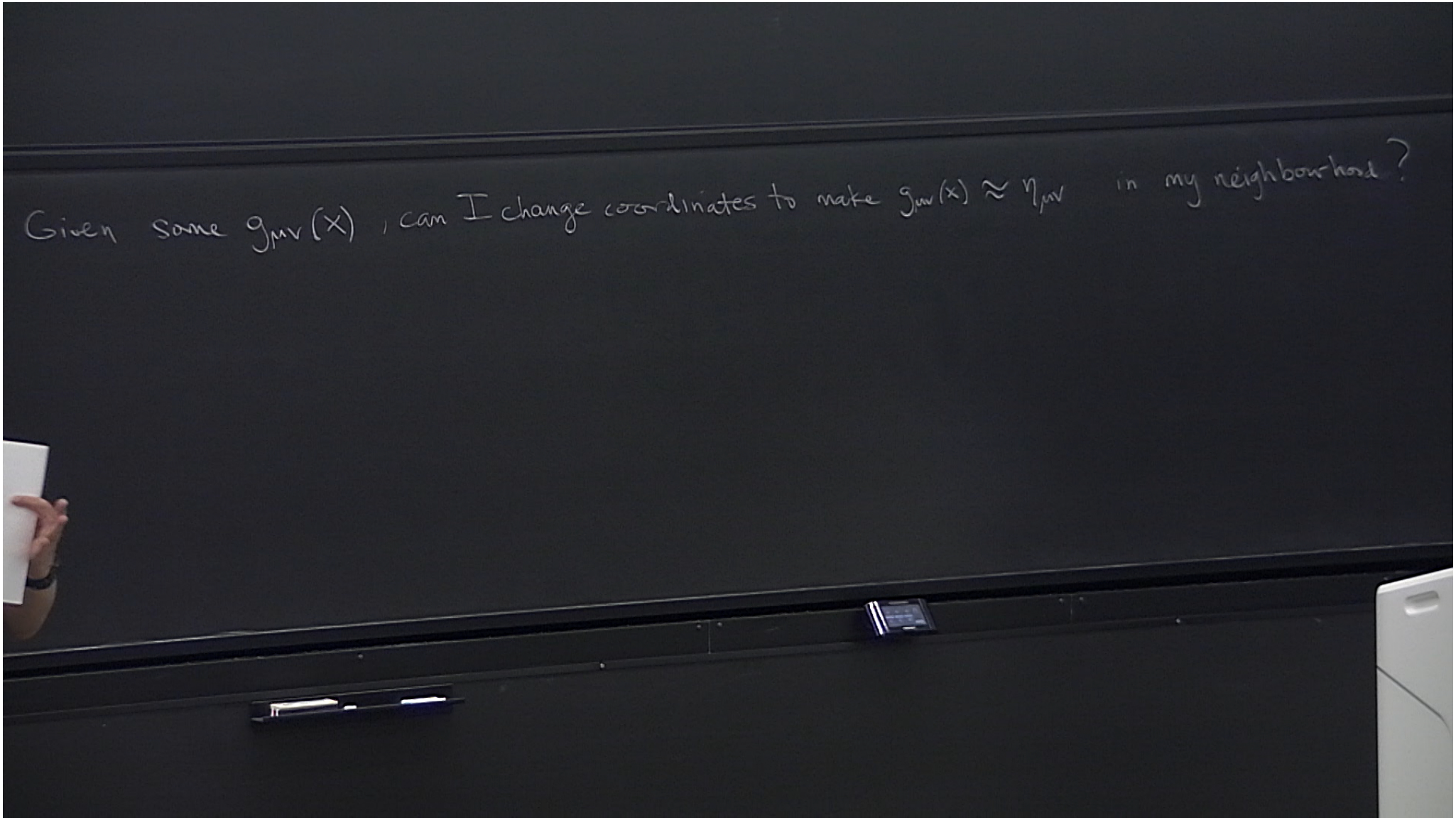


in general need to ensure consistency in overlap regions, e.g. $x''(x'(x))$


NP
$x_2=0$
$1-x_3$
x_3
x_1
x_1^N
embedding coords
Similar Δs : $\frac{x_1^N}{1} = \frac{x_1}{1-x_3} \Rightarrow x_1^N = \frac{x_1}{1-x_3}$
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In general, we need a number of coordinate systems to fully describe a curved space, number depends on the topology.
x
x'
x''
in general need to ensure consistency in overlap regions, e.g.
$x'(x''(x)) = x''(x)$


likewise, we have SP projecti
metric
In general, we need a number of coord


R
x
x ≪ R
⇒ sphere looks flat!
likewise, we have SP
In general, we need a


Given some $g_{\mu\nu}(x)$, can I change coordinates to make $g_{\mu\nu}(x) \approx \eta_{\mu\nu}$ in my neighbourhood?

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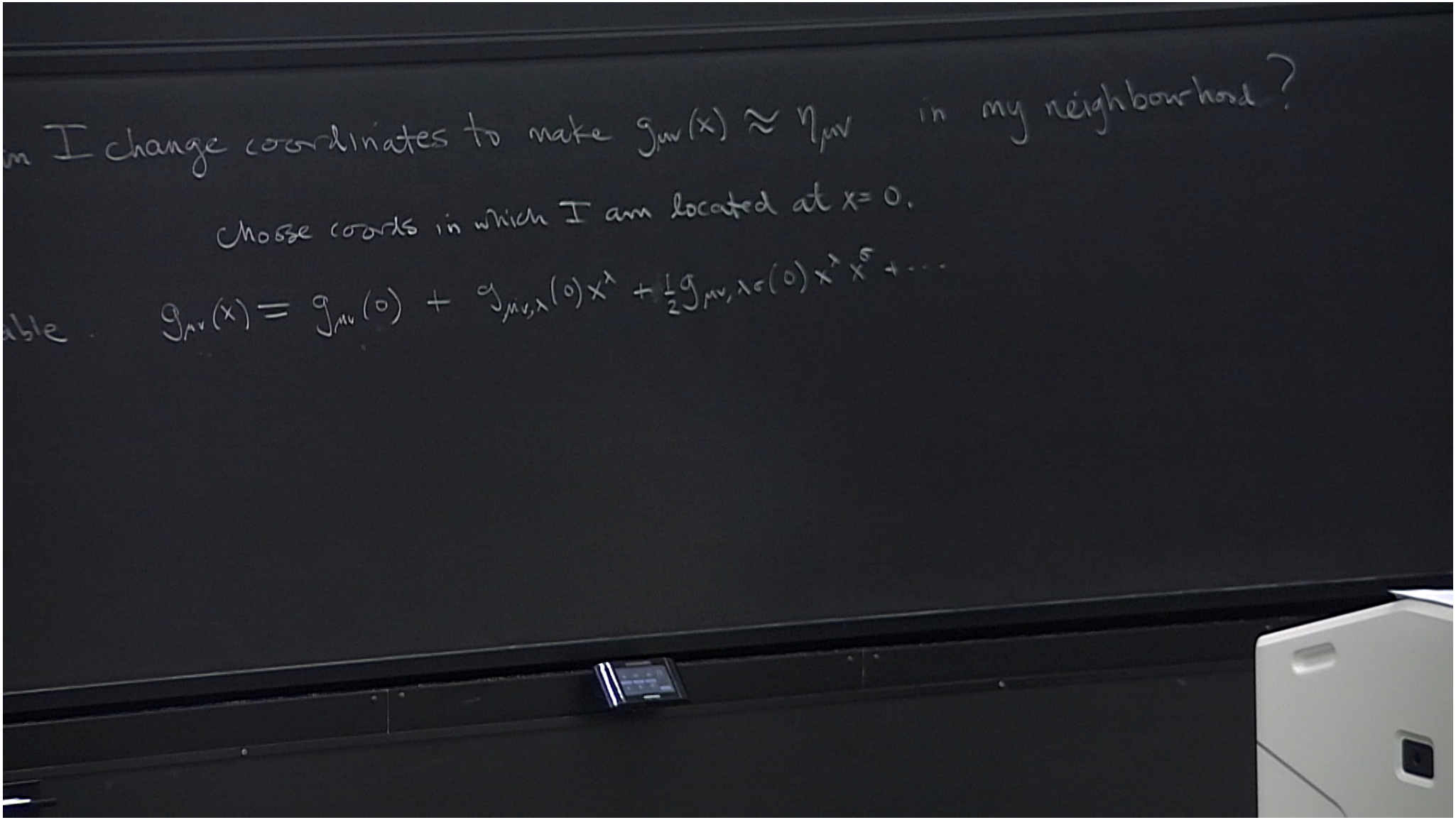
Yes (Riemann, 1830?)

Given some $g_{\mu\nu}(x)$, can I change coordinates to make $g_{\mu\nu}(x) \approx \eta_{\mu\nu}$

Yes (Riemann, 1830?)

Choose coords in which I am located at $x = 0$.

Assume $g_{\mu\nu}(x)$ is differentiable. $g_{\mu\nu}(x) = g_{\mu\nu}(0)$


n I change coordinates to make $g_{\mu\nu}(x) \approx \eta_{\mu\nu}$ in my neighbourhood?
Choose coords in which I am located at $x=0$.
able $g_{\mu\nu}(x) = g_{\mu\nu}(0) + g_{\mu\nu,\lambda}(0)x^\lambda + \frac{1}{2}g_{\mu\nu,\lambda\sigma}(0)x^\lambda x^\sigma + \dots$

change coordinates to make $g_{\mu\nu}(x) \approx \eta_{\mu\nu}$ in my neighbourhood?

Choose coords in which I am located at $x = 0$.

$x^\mu = (x^0, x^1 \ldots x^{D-1})$

Taylor expansion in D = # of indept coords.

$$g_{\mu\nu}(x) = g_{\mu\nu}(0) + g_{\mu\nu,\lambda}(0)x^\lambda + \frac{1}{2}g_{\mu\nu,\lambda\sigma}(0)x^\lambda x^\sigma + \ldots$$

Given some $g_{\mu\nu}(x)$, can I change coordinates to make $g_{\mu\nu}(x) \approx \eta_{\mu\nu}$

Yes (Riemann, 1830?)

Assume $g_{\mu\nu}(x)$ is differentiable.

Can I choose $x'^{\alpha}(x)$ s.t. $g_{\mu\nu}(0) = \eta_{\mu\nu}$

Choose coords in which I am located at $x=$

$$g_{\mu\nu}(x) = g_{\mu\nu}(0) + g_{\mu\nu,\lambda}(0)x^{\lambda} + \frac{1}{2}g_{\mu\nu,\lambda\sigma}(0)x^{\lambda}x^{\sigma}$$

Given some $g_{\mu\nu}(x)$, can I change coordinates to make $g_{\mu\nu}(x) \approx \eta_{\mu\nu}$

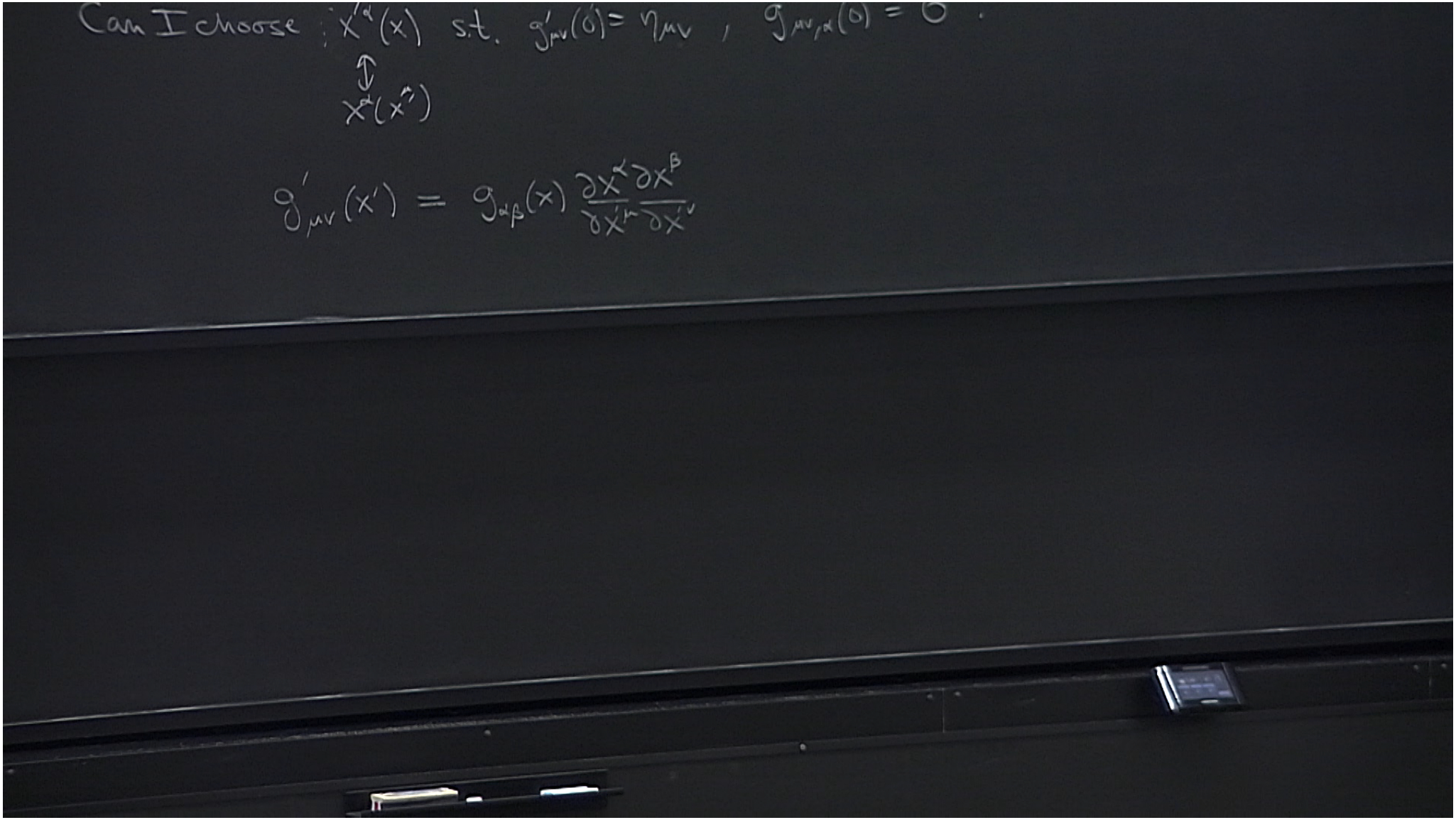
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Can I choose : $x'^{\alpha}(x)$ s.t. $g'_{\mu\nu}(0) = \eta_{\mu\nu}$, $g_{\mu\nu,\alpha}(0) = 0$.
$\Updownarrow$
$x^{\alpha}(x')$
$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}}$

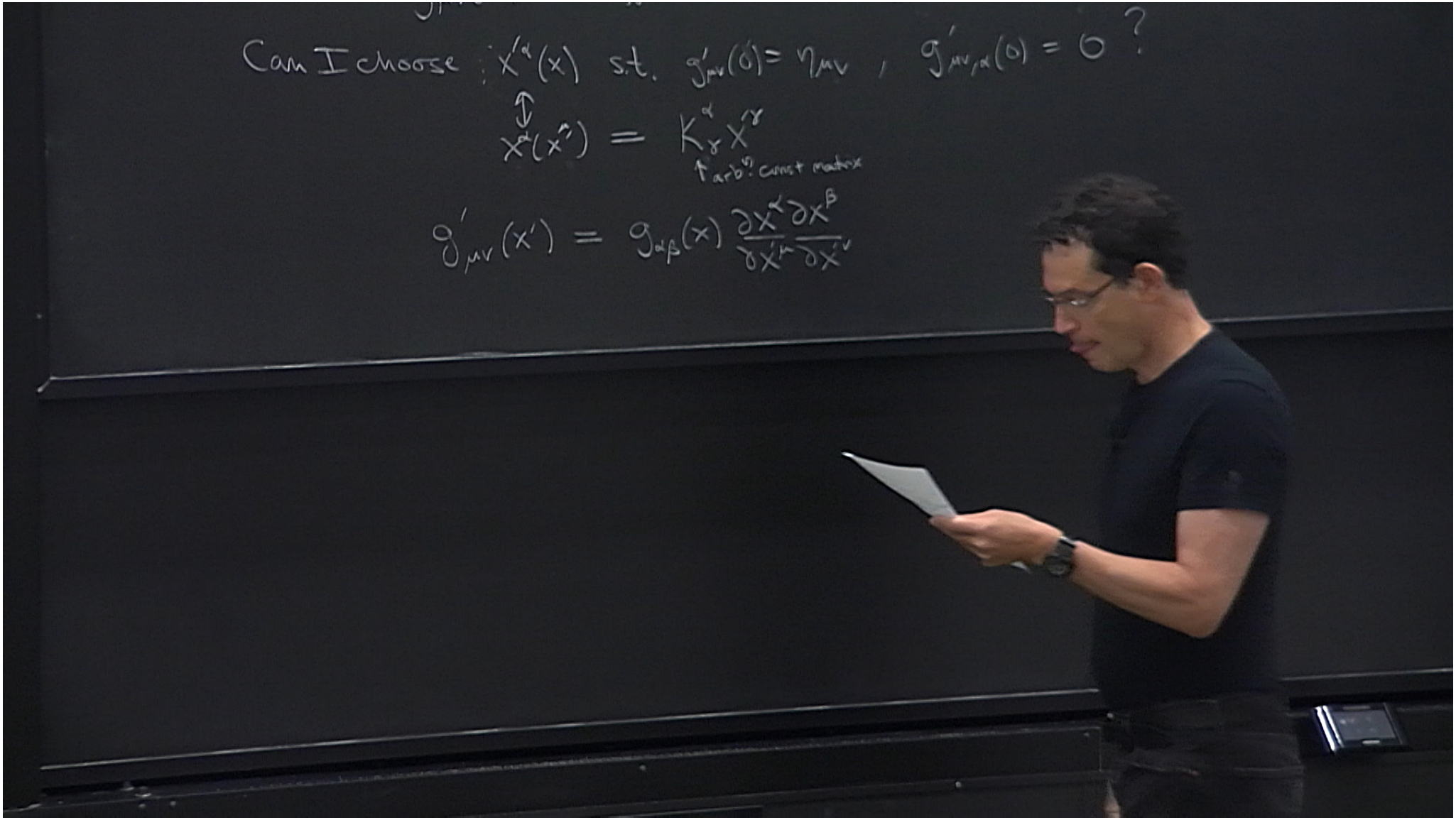
Assume $g_{\mu\nu}(x)$ is differentiable $\quad g_{\mu\nu}(x) = g_{\mu\nu}(0) + g_{\mu\nu,\lambda}(0)x^\lambda + \frac{1}{2}g_{\mu\nu,\lambda\sigma}(0)x^\lambda x^\sigma$

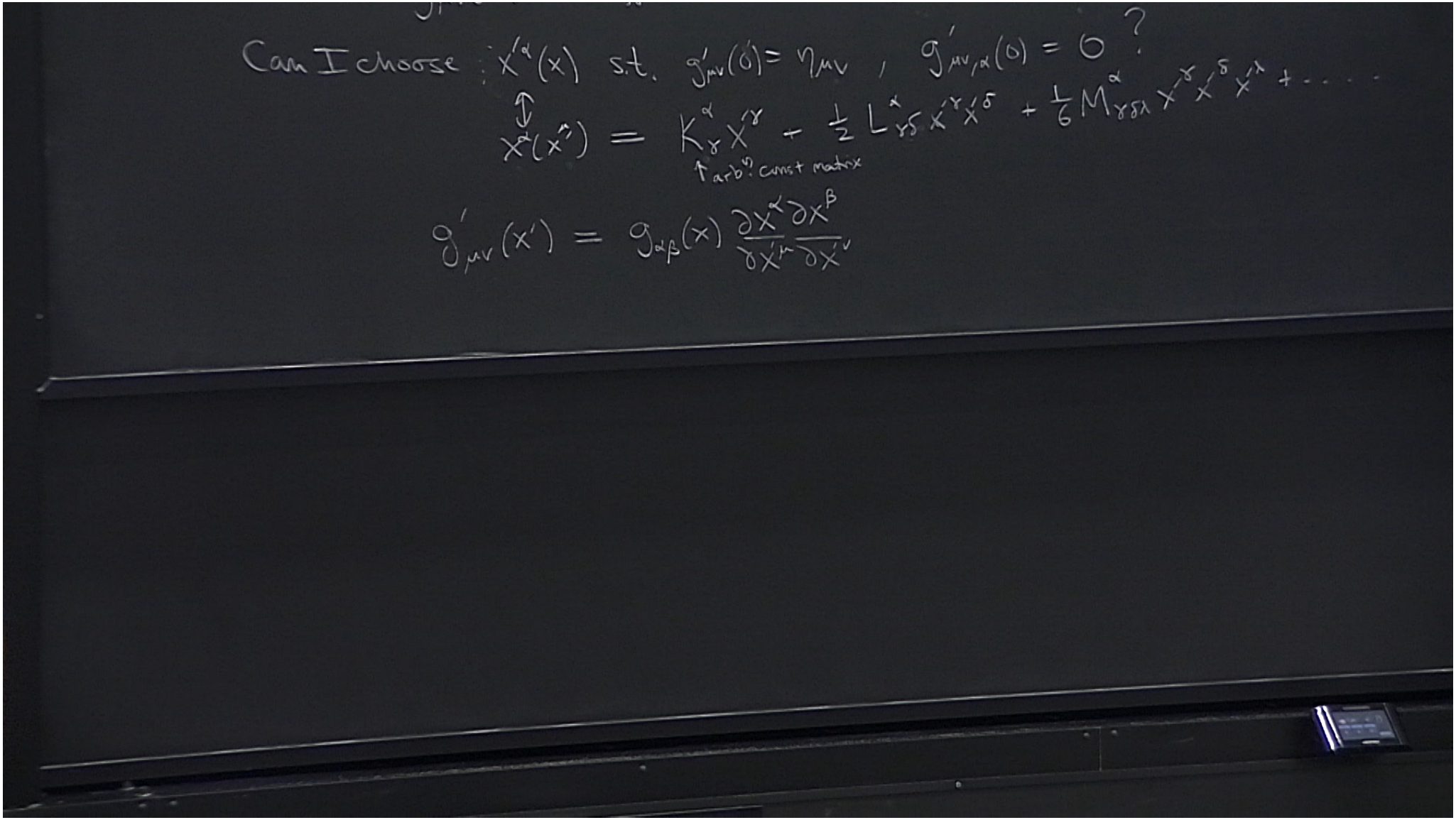
Can I choose $x'^\alpha(x)$ s.t. $g'_{\mu\nu}(0) = \eta_{\mu\nu}$, $g'_{\mu\nu,\alpha}(0) = 0$?

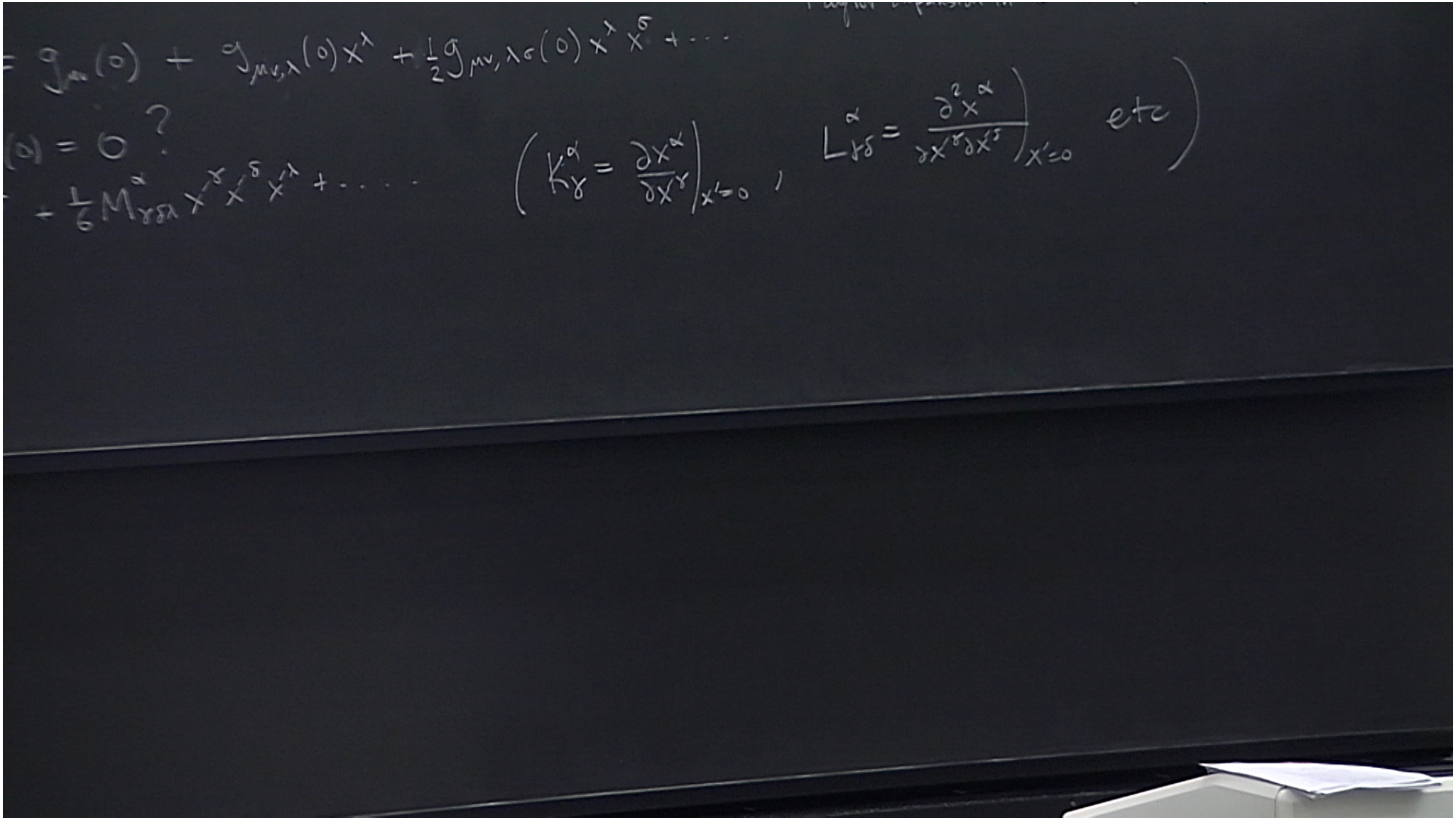
$\Updownarrow$

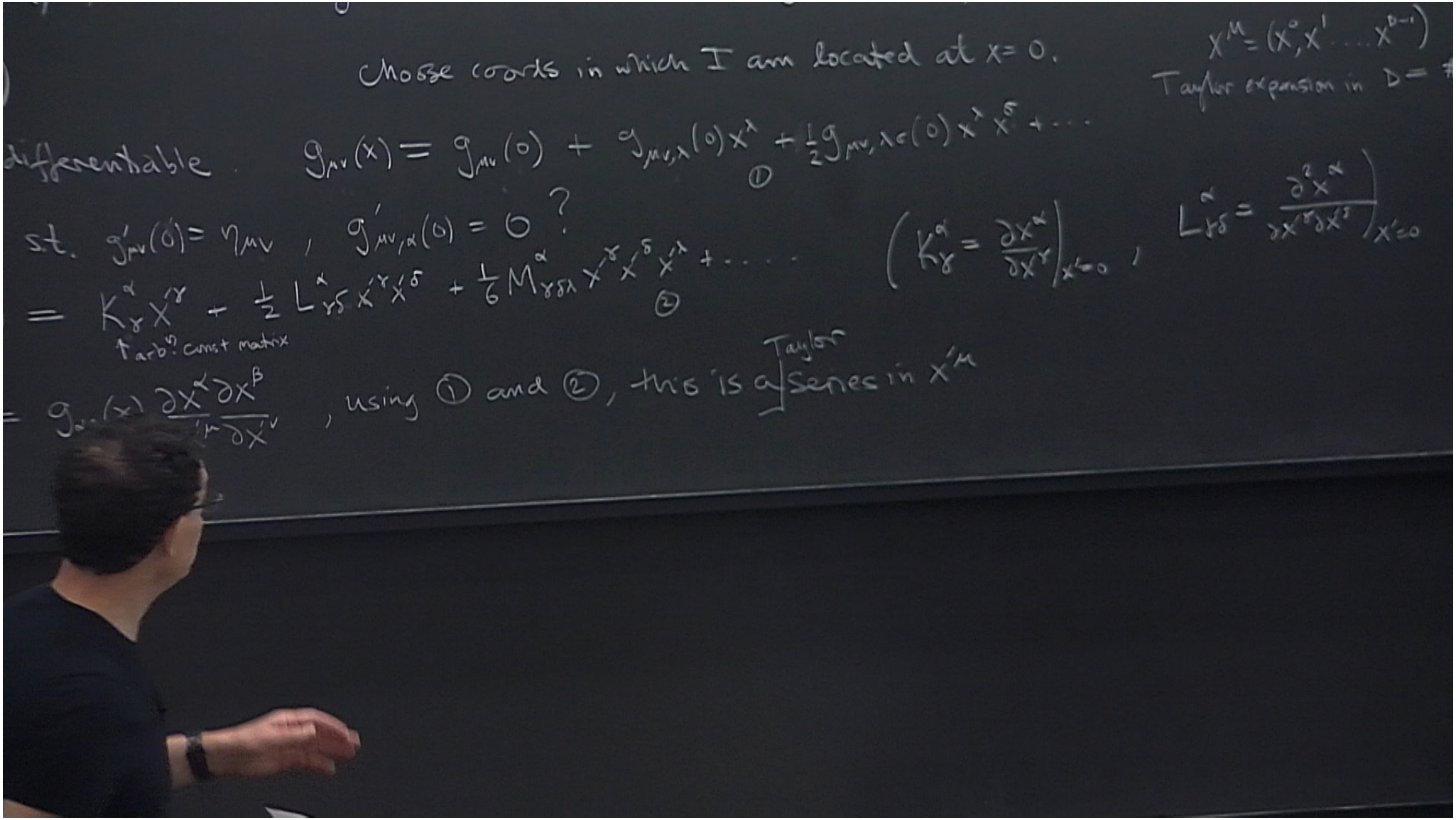
$x'^\alpha(x^\beta) =$

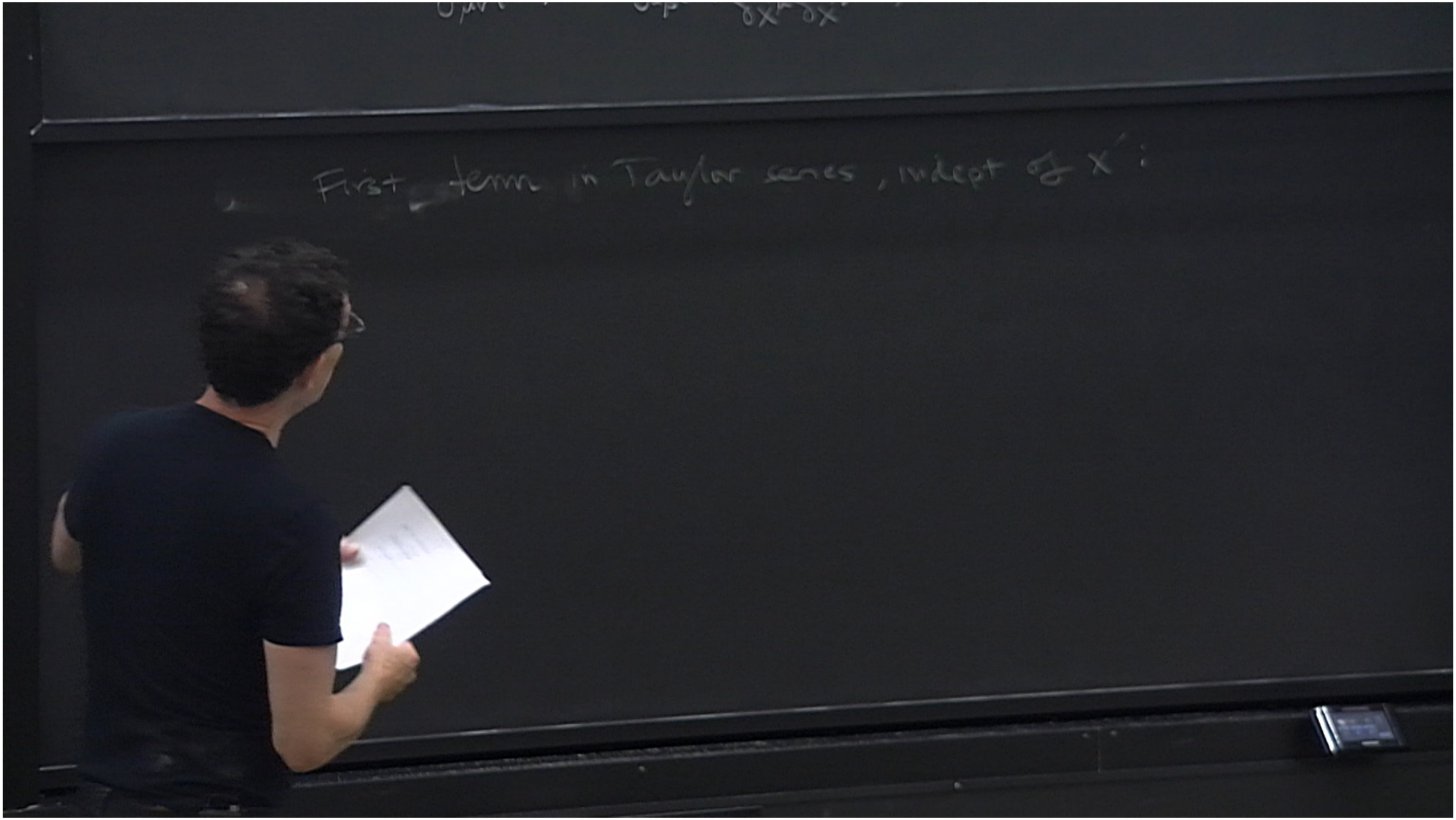
$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}$$


Can I choose : $x'^{\alpha}(x)$ s.t. $g'_{\mu\nu}(0) = \eta_{\mu\nu}$, $g'_{\mu\nu,\alpha}(0) = 0$?
$\Updownarrow$
$x'^{\alpha}(x^{\gamma}) = K^{\alpha}_{\gamma} x^{\gamma}$
↑ arb'y const matrix
$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}}$


Can I choose : $x'^{\alpha}(x)$ s.t. $g'_{\mu\nu}(0) = \eta_{\mu\nu}$, $g'_{\mu\nu,\alpha}(0) = 0$?
$x'^{\alpha}(x^{\gamma}) = K^{\alpha}_{\gamma} x^{\gamma} + \frac{1}{2} L^{\alpha}_{\gamma\delta} x^{\gamma} x^{\delta} + \frac{1}{6} M^{\alpha}_{\gamma\delta\lambda} x^{\gamma} x^{\delta} x^{\lambda} + \ldots$
arb'y const matrix
$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}}$


$= g_{\mu\nu}(0) + g_{\mu\nu,\lambda}(0)x^\lambda + \frac{1}{2}g_{\mu\nu,\lambda\sigma}(0)x^\lambda x^\sigma + \ldots$
$(0) = 0$?
$+ \frac{1}{6}M^\alpha_{\gamma\delta\lambda}x^\gamma x^\delta x^\lambda + \ldots$
$\left(K^\alpha_\gamma = \frac{\partial x^\alpha}{\partial x'^\gamma}\Big|_{x'=0}, \quad L^\alpha_{\gamma\delta} = \frac{\partial^2 x^\alpha}{\partial x'^\gamma \partial x'^\delta}\Big|_{x'=0}\right.$ etc $\left.\right)$


Choose coords in which I am located at x = 0.
$x^\mu = (x^0, x^1 \ldots x^{D-1})$
Taylor expansion in D = #
differentiable $g_{\mu\nu}(x) = g_{\mu\nu}(0) + g_{\mu\nu,\lambda}(0)x^\lambda + \frac{1}{2}g_{\mu\nu,\lambda\sigma}(0)x^\lambda x^\sigma + \ldots$
①
s.t. $g'_{\mu\nu}(0) = \eta_{\mu\nu}$, $g'_{\mu\nu,\alpha}(0) = 0$?
$= K^\alpha_\gamma x'^\gamma + \frac{1}{2}L^\alpha_{\gamma\delta}x'^\gamma x'^\delta + \frac{1}{6}M^\alpha_{\gamma\delta\lambda}x'^\gamma x'^\delta x'^\lambda + \ldots$
②
$\left(K^\alpha_\gamma = \frac{\partial x^\alpha}{\partial x'^\gamma}\right)_{x'=0}$, $L^\alpha_{\gamma\delta} = \left(\frac{\partial^2 x^\alpha}{\partial x'^\gamma \partial x'^\delta}\right)_{x'=0}$
↑ arb'y const matrix
$= g_{\cdot\cdot}(x)\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}$, using ① and ②, this is a Taylor series in x'^μ


First term in Taylor series, indept of x;

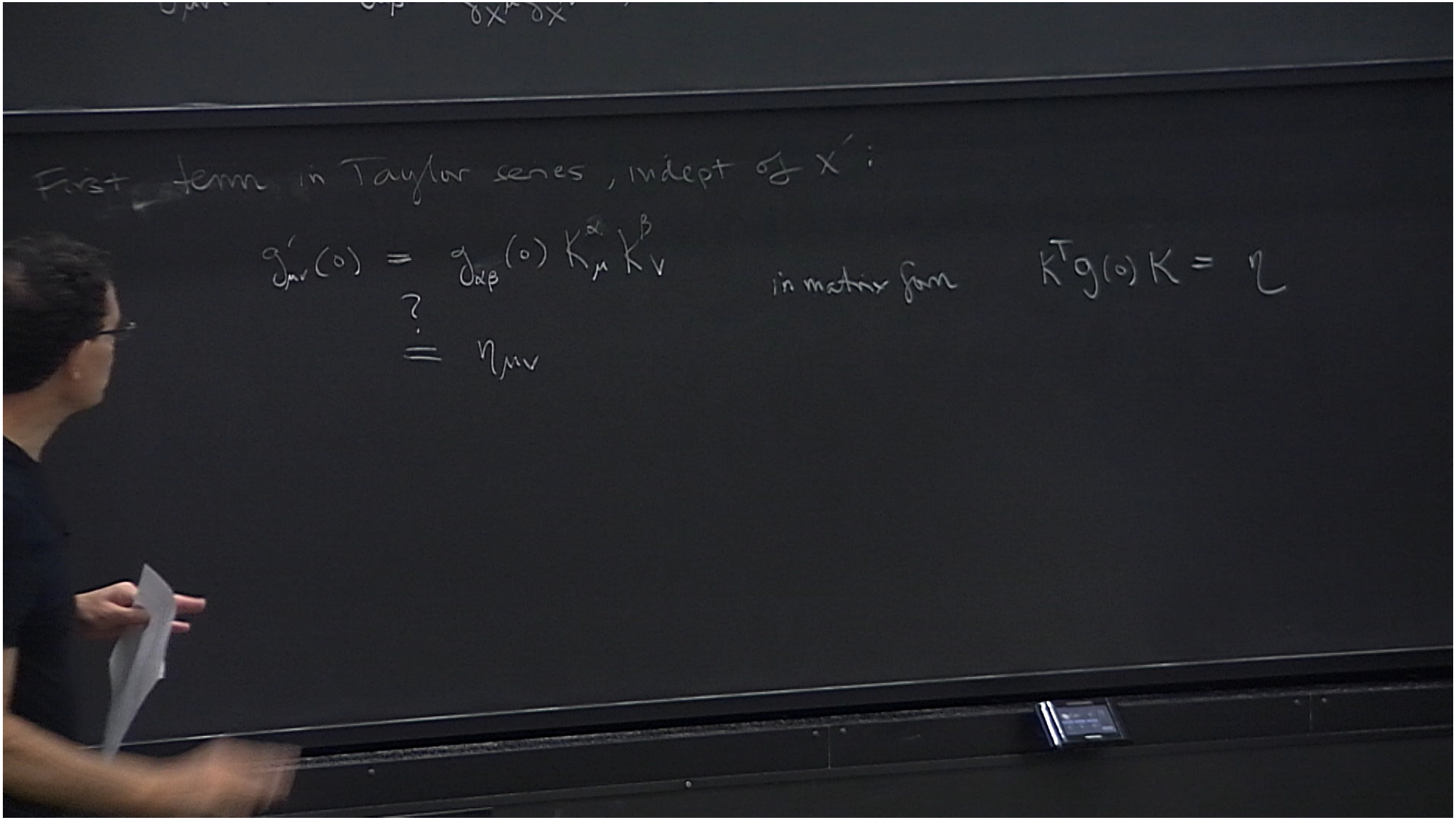
First term in Taylor series, indept of x':

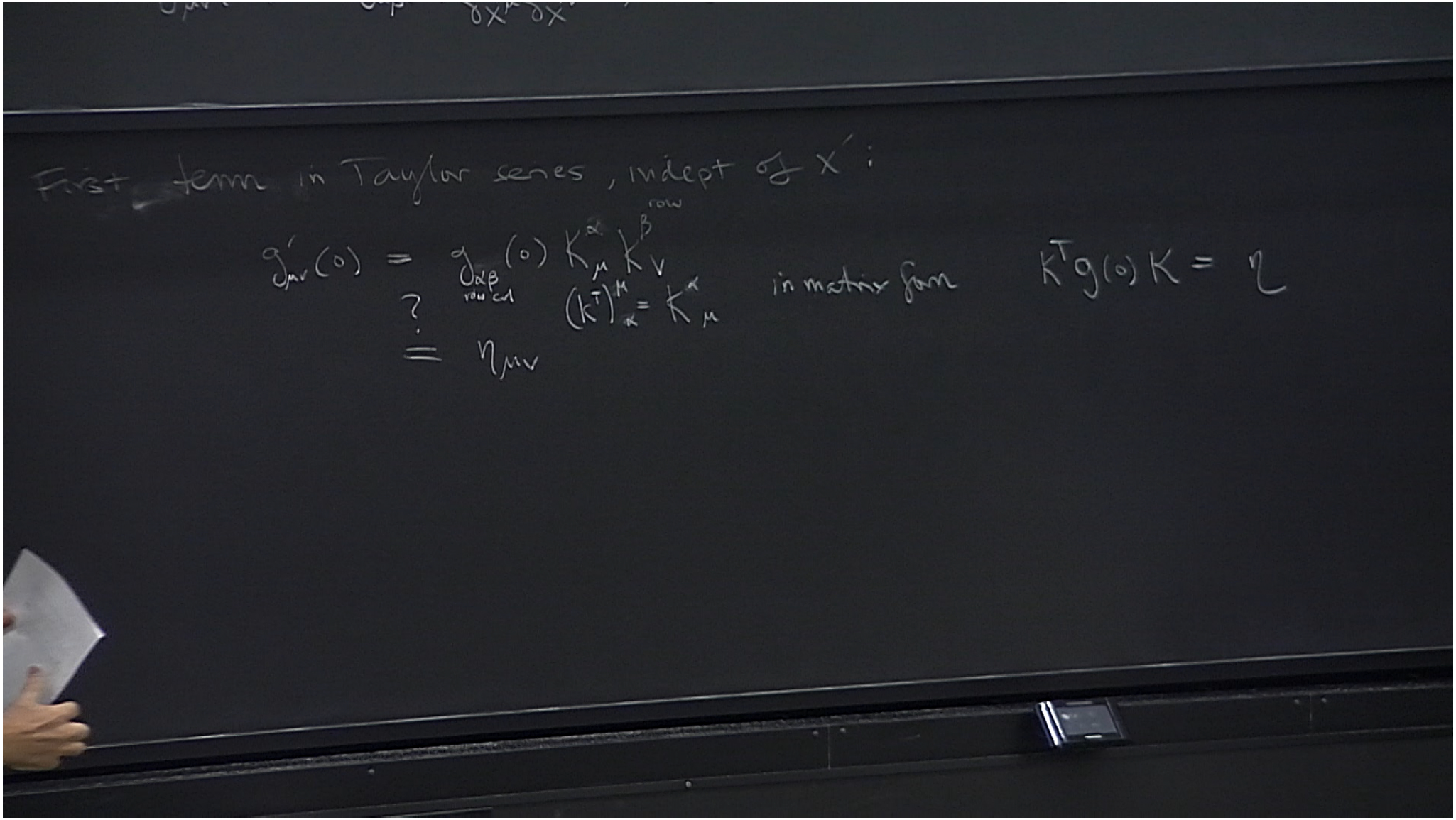
$$g'_{\mu\nu}(0) = g_{\alpha\beta}(0) K^{\alpha}_{\mu} K^{\beta}_{\nu}$$

First term in Taylor series, indept of x':

$$g'_{\mu\nu}(0) = g_{\alpha\beta}(0) K^{\alpha}_{\mu} K^{\beta}_{\nu}$$

$$\stackrel{?}{=} \eta_{\mu\nu}$$


First term in Taylor series, indept of x':
$g'_{\mu\nu}(0) = g_{\alpha\beta}(0) K^{\alpha}_{\mu} K^{\beta}_{\nu}$
in matrix form $K^T g(0) K = \eta$
$\stackrel{?}{=} \eta_{\mu\nu}$


First term in Taylor series, indept of x':
$g'_{\mu\nu}(0) = g_{\alpha\beta}(0) K^{\alpha}_{\mu} K^{\beta}_{\nu}$
row
$\overset{?}{=}$ $g_{\alpha\beta}$ row cd
$(K^T)^{\mu}_{\alpha} = K^{\alpha}_{\mu}$
$= \eta_{\mu\nu}$
in matrix form $K^T g(0) K = \eta$

term in Taylor series, indept of x':

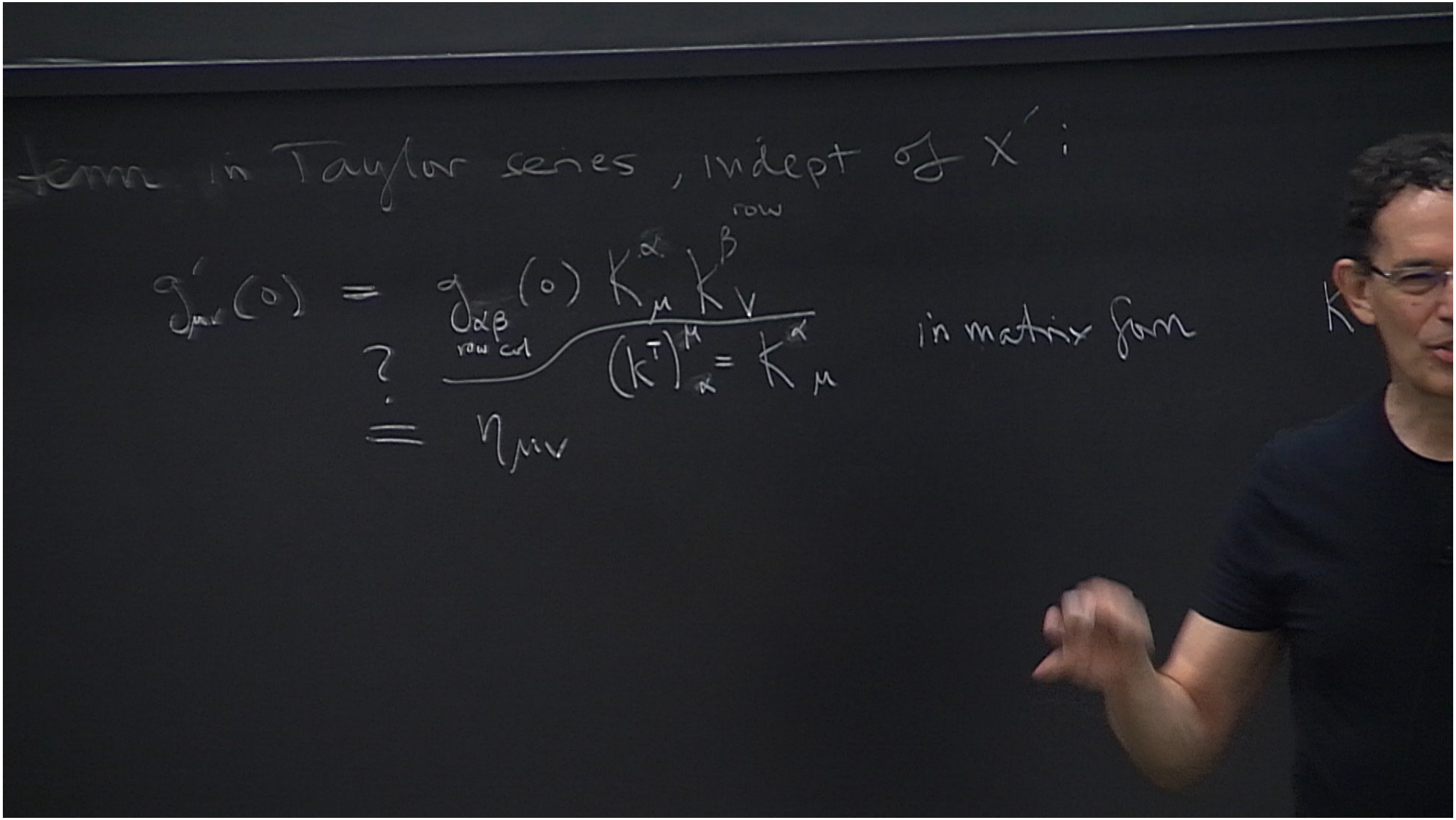
$$g'_{\mu\nu}(0) = g_{\alpha\beta}(0)\, K^{\alpha}_{\ \mu} K^{\beta}_{\ \nu}$$

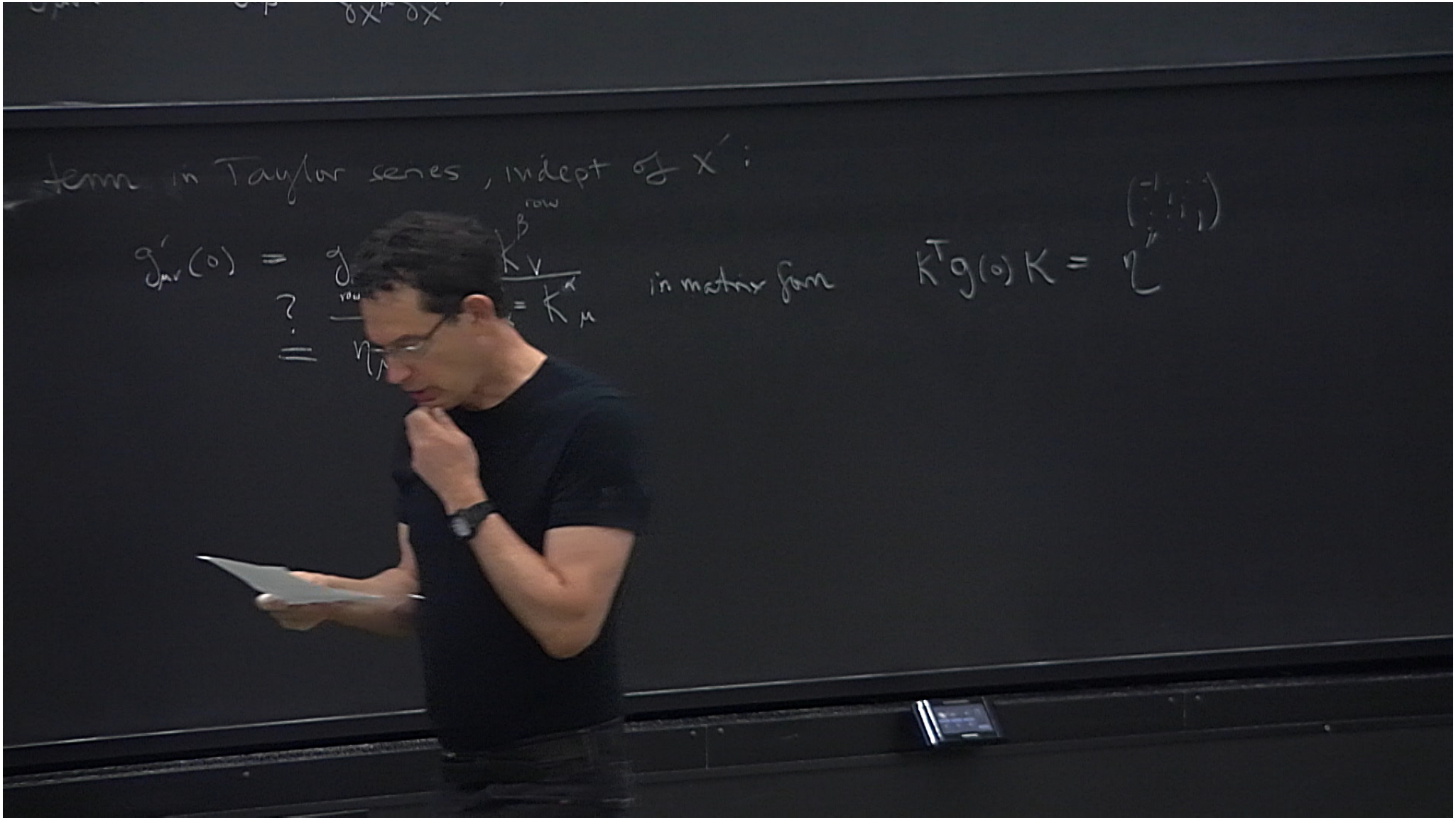
row col row

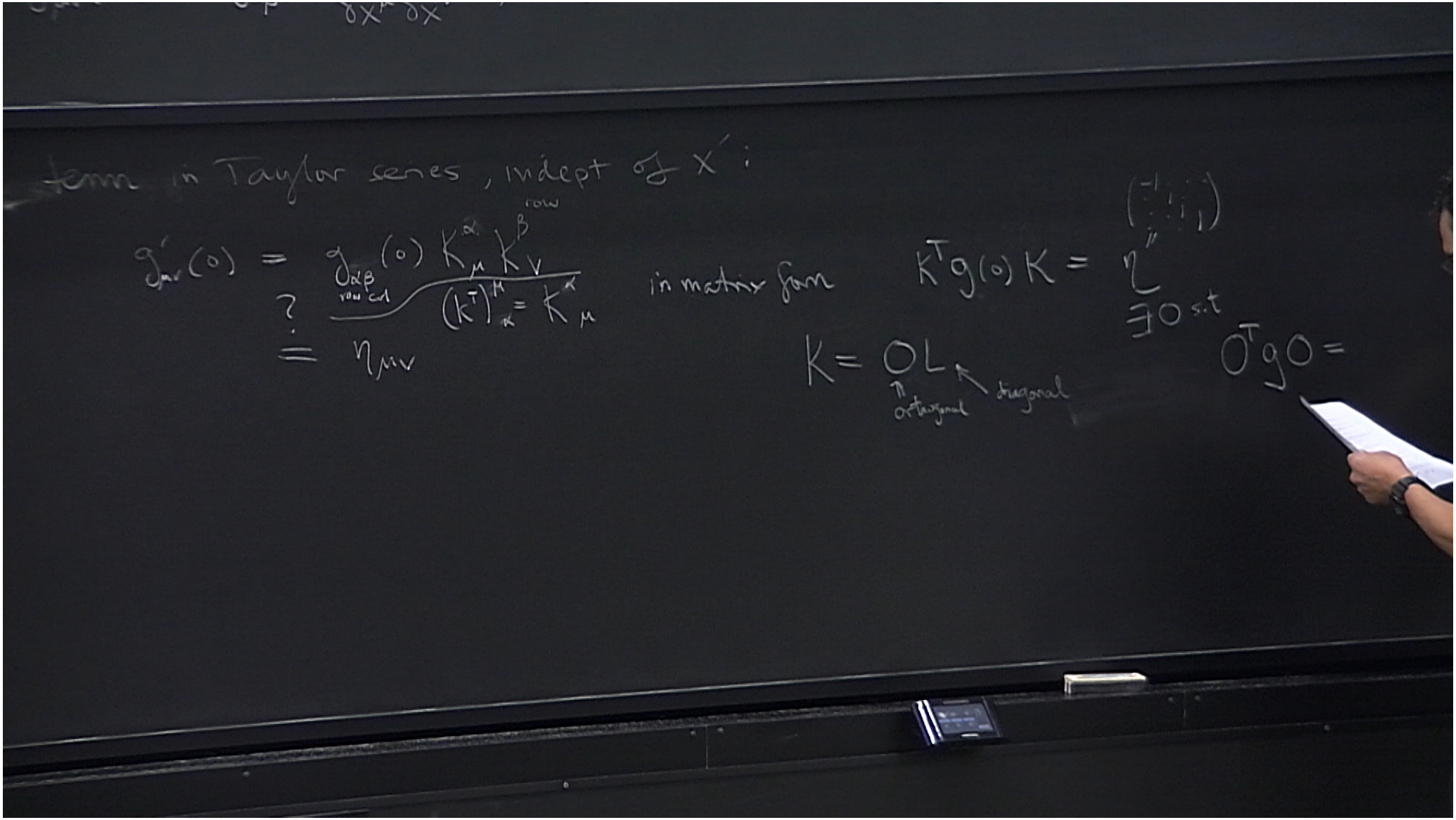
$$\stackrel{?}{=} \eta_{\mu\nu}$$

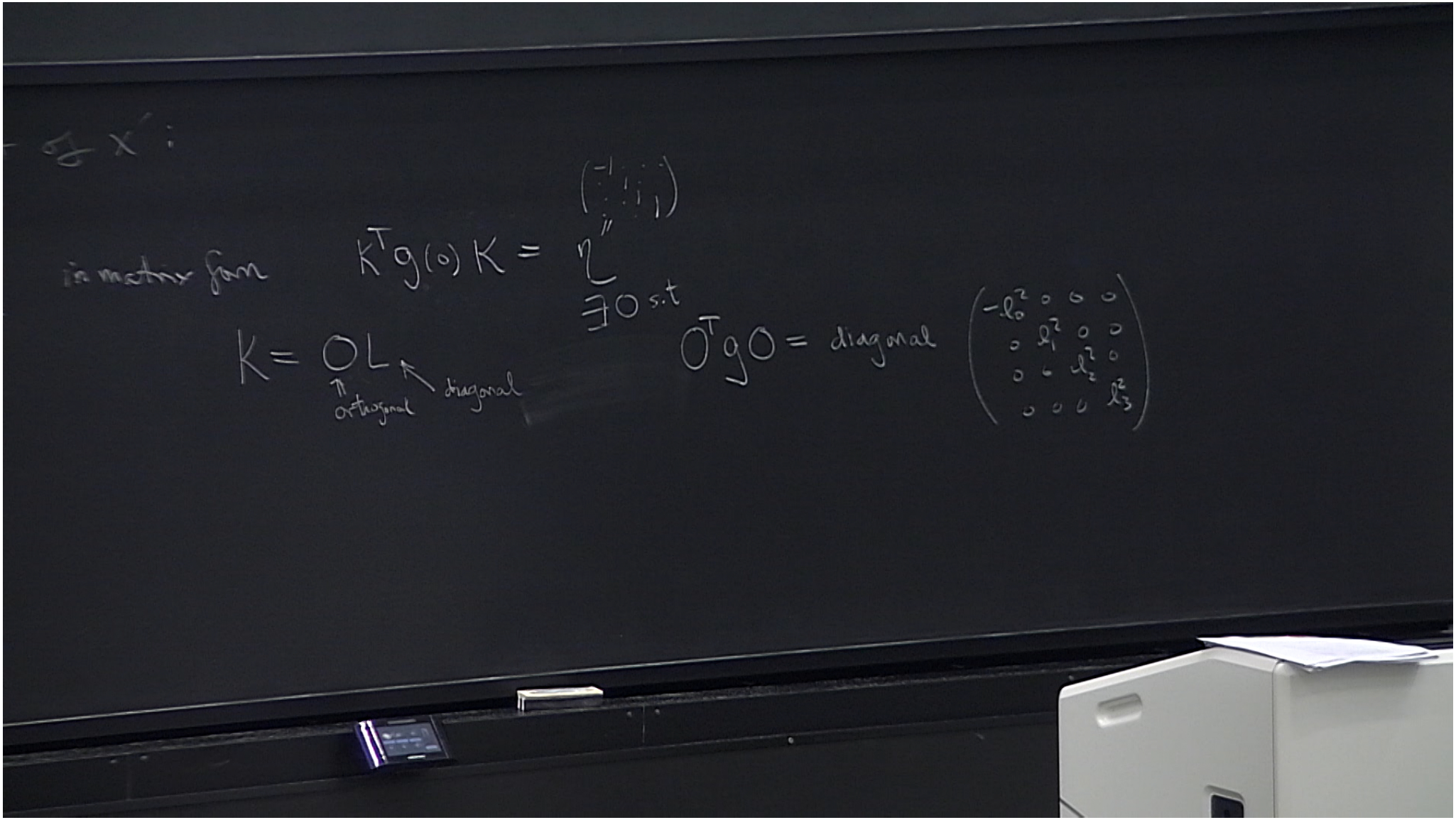
$$(K^T)^{\mu}_{\ \alpha} = K^{\alpha}_{\ \mu}$$

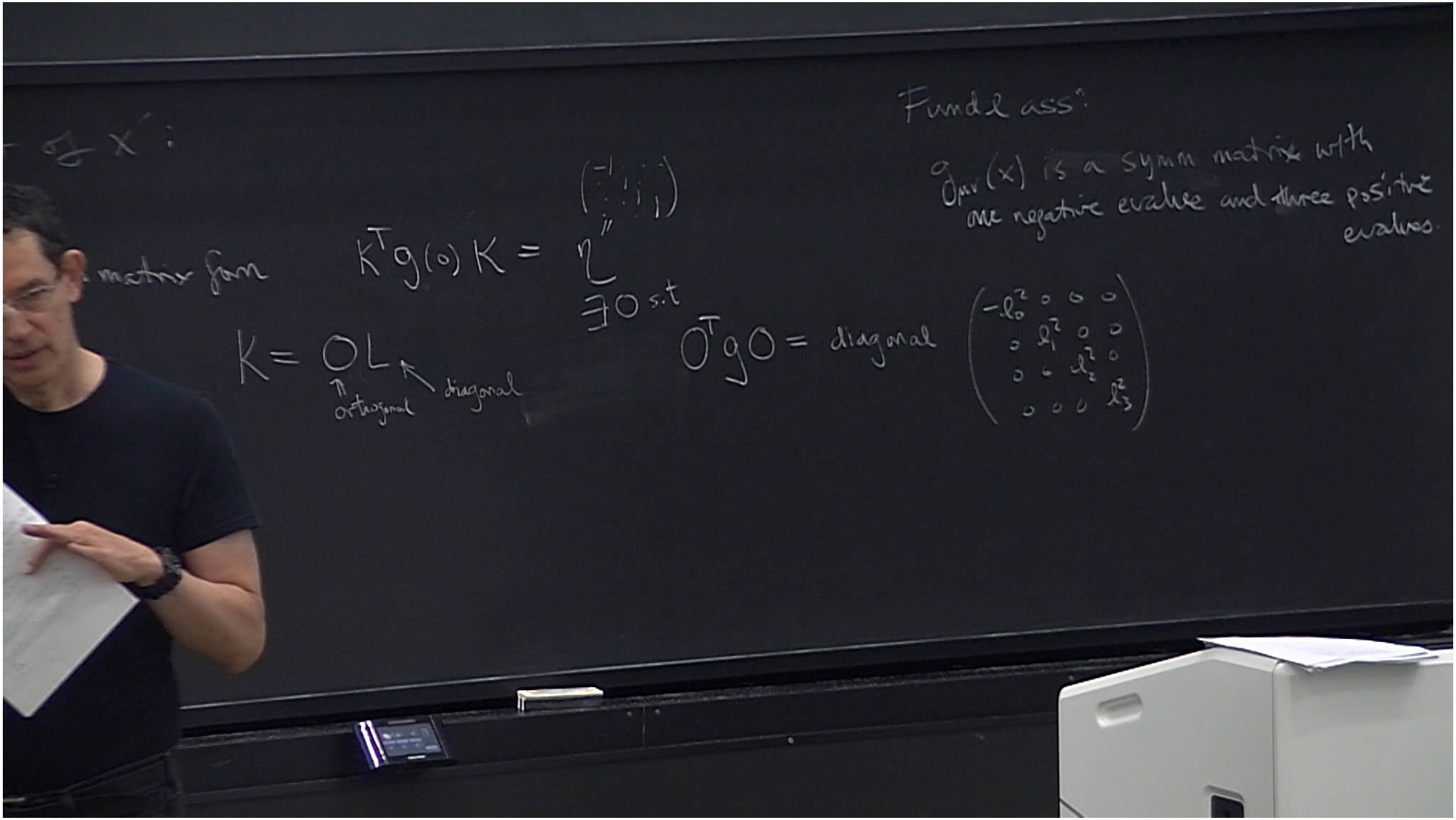
in matrix form $K^T g(0$

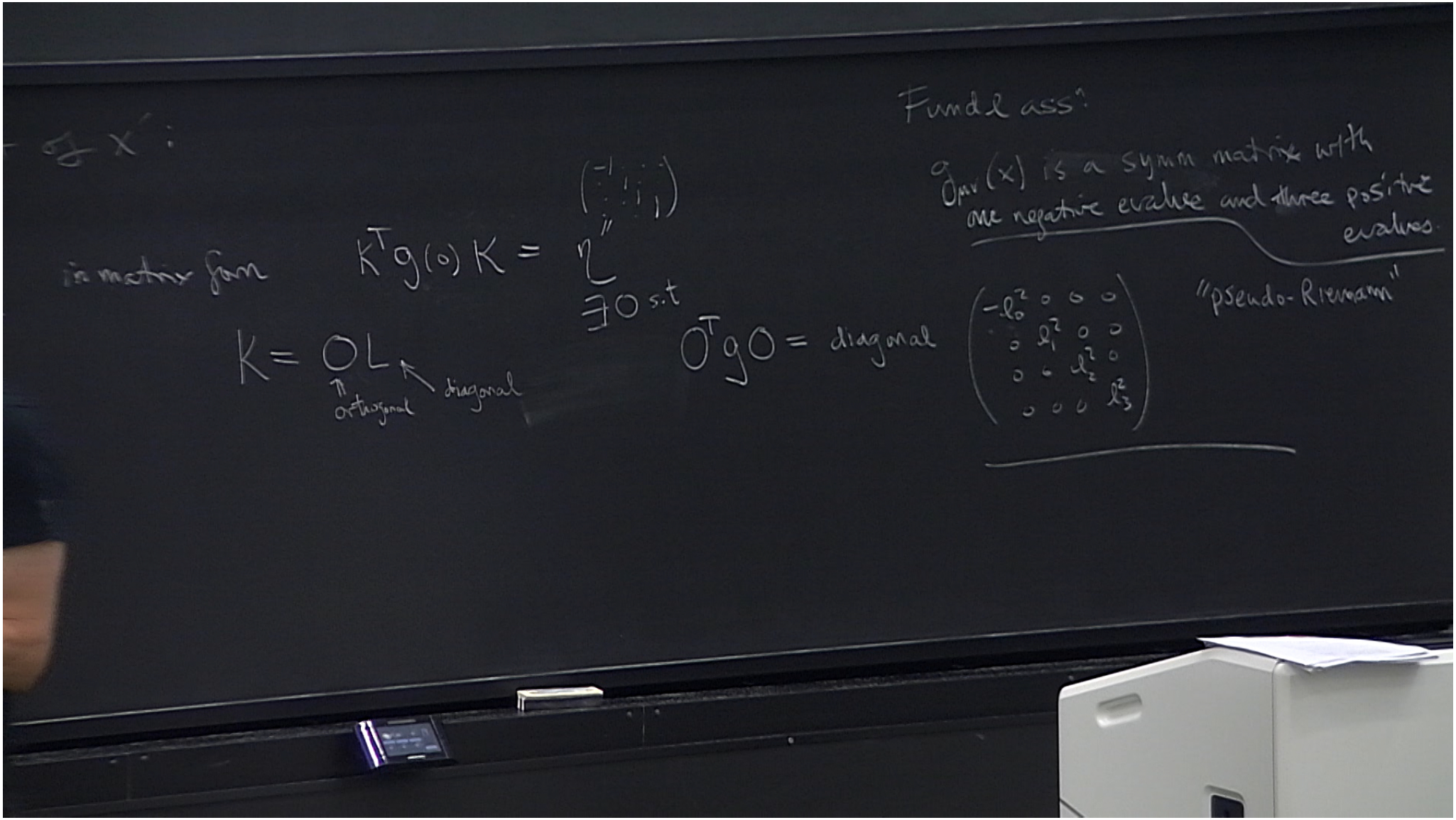

term in Taylor series, indept of x'
$g'_{\mu\nu}(0) = \frac{\partial_{\alpha\beta}}{\text{row col}}(0)\, K^{\alpha}_{\mu} K^{\beta}_{\nu}$
row
$(K^{-1})^{\mu}_{\alpha} = K^{\alpha}_{\mu}$
in matrix form
$\stackrel{?}{=} \eta_{\mu\nu}$

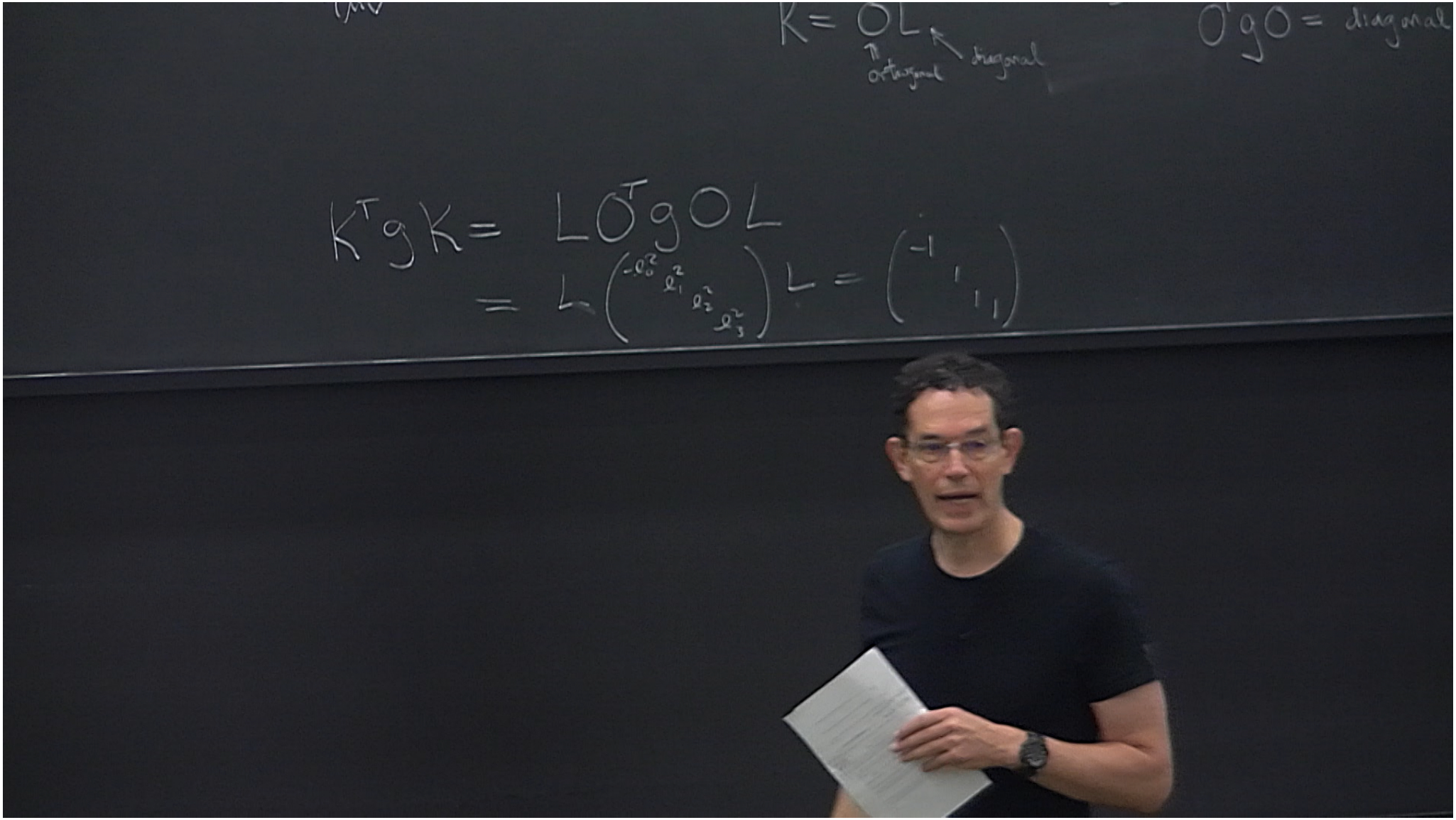

term in Taylor series, indept of x':
in matrix form
$K^T g(0) K =$

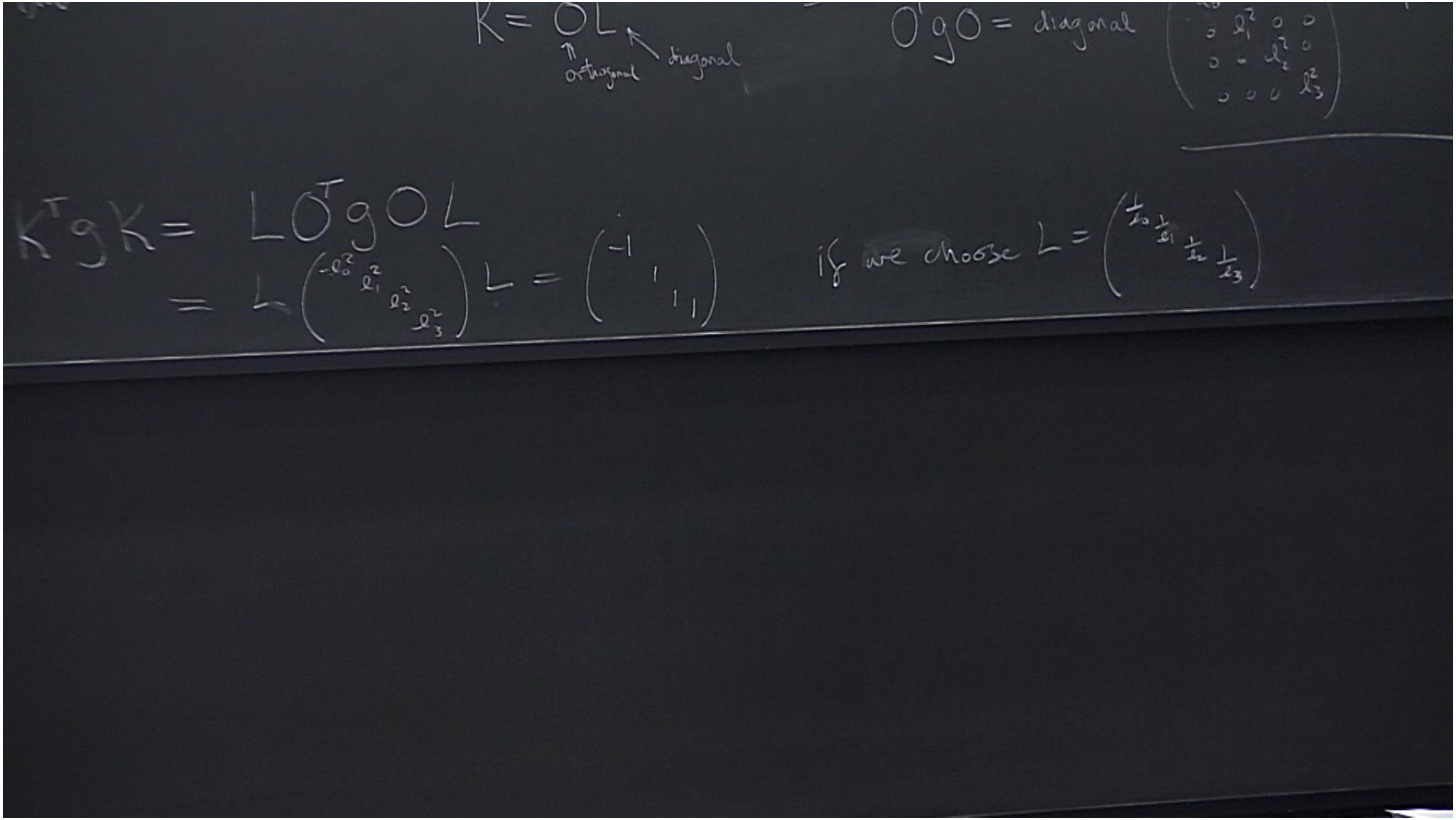

term in Taylor series, indept of x':
$g'_{\mu\nu}(0) = \frac{g_{\alpha\beta}(0)\, K^{\alpha}_{\mu} K^{\beta}_{\nu}}{(K^T)^{\mu}_{\alpha} = K^{\alpha}_{\mu}}$
row row col
? $= \eta_{\mu\nu}$
in matrix form $K^T g(0) K = \eta$
$\exists O$ s.t
$K = OL$
orthogonal
diagonal
$O^T g O =$

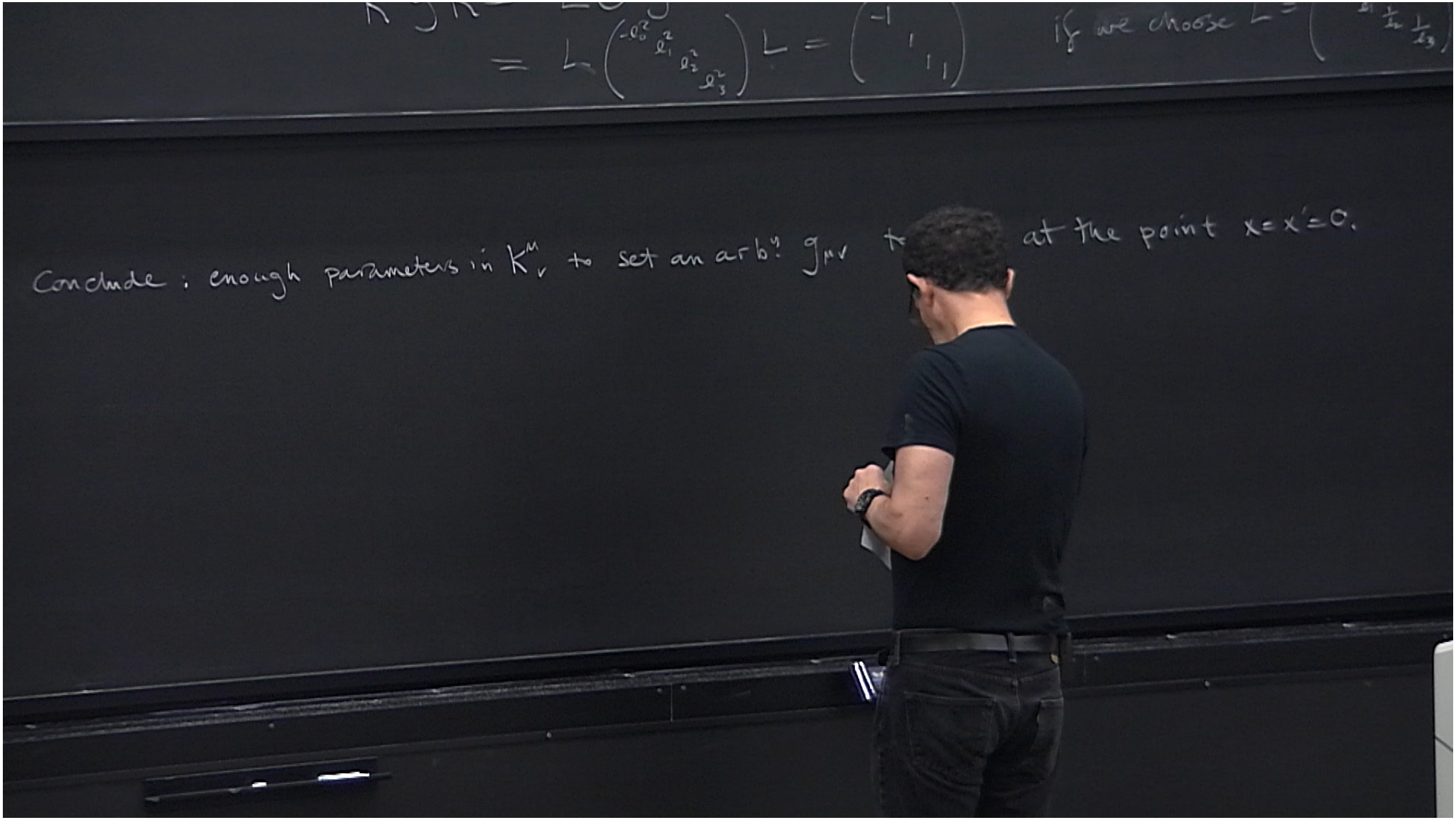

δx^i:
in matrix form $K^T g(0) K = \eta$
$\exists O$ s.t
$K = OL$
orthogonal
diagonal
$O^T g O$ = diagonal
$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$
$\begin{pmatrix} -l_0^2 & 0 & 0 & 0 \\ 0 & l_1^2 & 0 & 0 \\ 0 & 0 & l_2^2 & 0 \\ 0 & 0 & 0 & l_3^2 \end{pmatrix}$

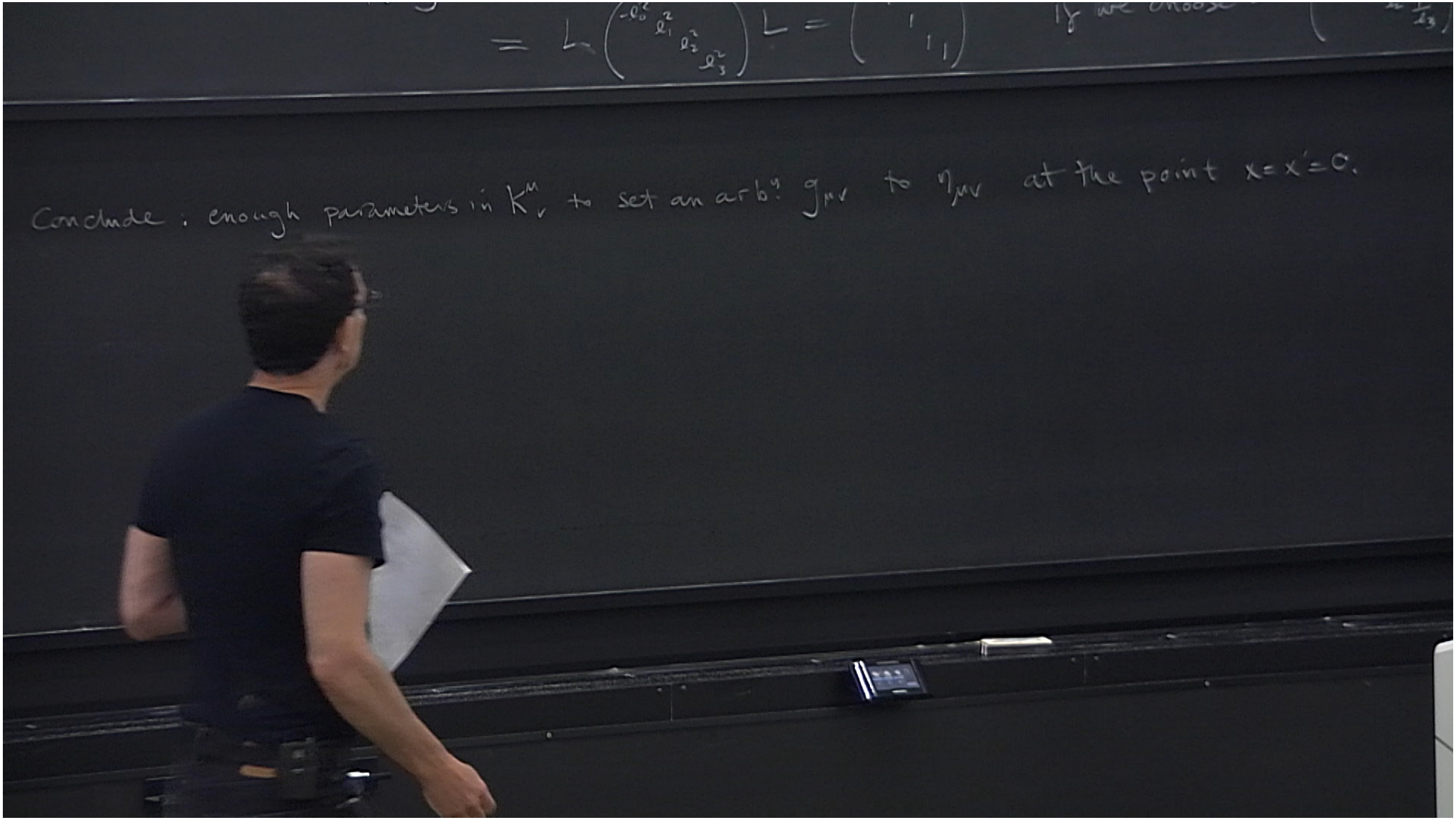

Fundl ass?
$g_{\mu\nu}(x)$ is a symm matrix with one negative evalue and three positive evalues.
$K^T g(0) K = \eta$
$\exists O$ s.t
$K = OL$
orthogonal
diagonal
$O^T g O =$ diagonal
$\begin{pmatrix} -l_0^2 & 0 & 0 & 0 \\ 0 & l_1^2 & 0 & 0 \\ 0 & 0 & l_2^2 & 0 \\ 0 & 0 & 0 & l_3^2 \end{pmatrix}$


Fundl ass?
$g_{\mu\nu}(x)$ is a symm matrix with one negative evalue and three positive evalues.
"pseudo-Riemann"
$K^T g(o) K = \eta$
$\exists O$ s.t
$K = OL$
orthogonal
diagonal
$O^T g O$ = diagonal
$\begin{pmatrix} -l_0^2 & 0 & 0 & 0 \\ 0 & l_1^2 & 0 & 0 \\ 0 & 0 & l_2^2 & 0 \\ 0 & 0 & 0 & l_3^2 \end{pmatrix}$


K = OL
orthogonal
diagonal
O^T g O = diagonal
K^T g K = L O^T g O L
= L (-Ω_0^2, Ω_1^2, Ω_2^2, Ω_3^2) L = (-1, 1, 1, 1)


K = OL
orthogonal
diagonal
O^T g O = diagonal
K^T g K = L O^T g O L
= L (-l_0^2, l_1^2, l_2^2, l_3^2) L = (-1, 1, 1, 1)
if we choose L = (1/l_0, 1/l_1, 1/l_2, 1/l_3)


Conclude : enough parameters in K^{μ}_{ν} to set an arb! $g_{\mu\nu}$ t at the point $x=x'=0$.


Conclude: enough parameters in K^{μ}_{ν} to set an arb. $g_{\mu\nu}$ to $\eta_{\mu\nu}$ at the point $x = x' = 0$.

$= L \begin{pmatrix} -\ell_0^2 & & & \\ & \ell_1^2 & & \\ & & \ell_2^2 & \\ & & & \ell_3^2 \end{pmatrix} L = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ If we choose

Conclude: enough parameters in $K^\mu{}_\nu$ ($\to 16$) to set an arb'y $g_{\mu\nu}$ ($\to 10$) to $\eta_{\mu\nu}$ at the point $x = x' = 0$.

6 parameters are not fixed $\to$ Lorentz transfn's (3 rotations + 3 boosts)

Conclude: enough parameters in K^μ_{ν} (16) to set an arb'y $g_{\mu\nu}$ (10) to

6 parameters are not fixed → Lorentz transfn's

Next question: can I set $g'_{\mu\nu,\lambda}(0) = 0$

$\frac{\partial}{\partial x'^\lambda} g'_{\mu\nu}(x')$

Conclude: enough parameters in K^μ_ν ($\downarrow 16$) to set an arb'y $g_{\mu\nu}$ ($\downarrow 10$) to

6 parameters are not fixed → Lorentz transfn's

Next question: can I set $g'_{\mu\nu,\lambda}(0) = 0$ using parameters in L^α_β

$\frac{\partial}{\partial x^\lambda} g'_{\mu\nu}(x')$

Yes (Riemann, 1830?)

Choose coords in which I am l

Assume $g_{\mu\nu}(x)$ is differentiable. $g_{\mu\nu}(x) = g_{\mu\nu}(0) + g_{\mu\nu,\lambda}(0)x^\lambda + \frac{1}{2}$ ①

Can I choose: $x'^\alpha(x)$ s.t. $g'_{\mu\nu}(0) = \eta_{\mu\nu}$, $g'_{\mu\nu,\alpha}(0) = 0$?

$$\updownarrow$$

$$x'^\alpha(x^\gamma) = K^\alpha_\gamma x^\gamma + \frac{1}{2} L^\alpha_{\gamma\delta} x^\gamma x^\delta + \frac{1}{6} M^\alpha_{\gamma\delta\lambda} x^\gamma x^\delta x^\lambda + \ldots$$

↑ arb^y const matrix ②

$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$$, using ① and ②, this is a Taylor ser

③

Conclude: enough parameters in [illegible] to set an arb^y $g_{\mu\nu}$ to

↓

10

6 parameters are [illegible] → Lorentz transf^n's

Yes (Riemann, 1830?)

Choose coords in which I am l

Assume $g_{\mu\nu}(x)$ is differentiable. $g_{\mu\nu}(x) = g_{\mu\nu}(0) + g_{\mu\nu,\lambda}(0)x^\lambda + \frac{1}{2}g$ ①

Can I choose: $x'^\alpha(x)$ s.t. $g'_{\mu\nu}(0) = \eta_{\mu\nu}$, $g'_{\mu\nu,\alpha}(0) = 0$?

$$\Updownarrow$$

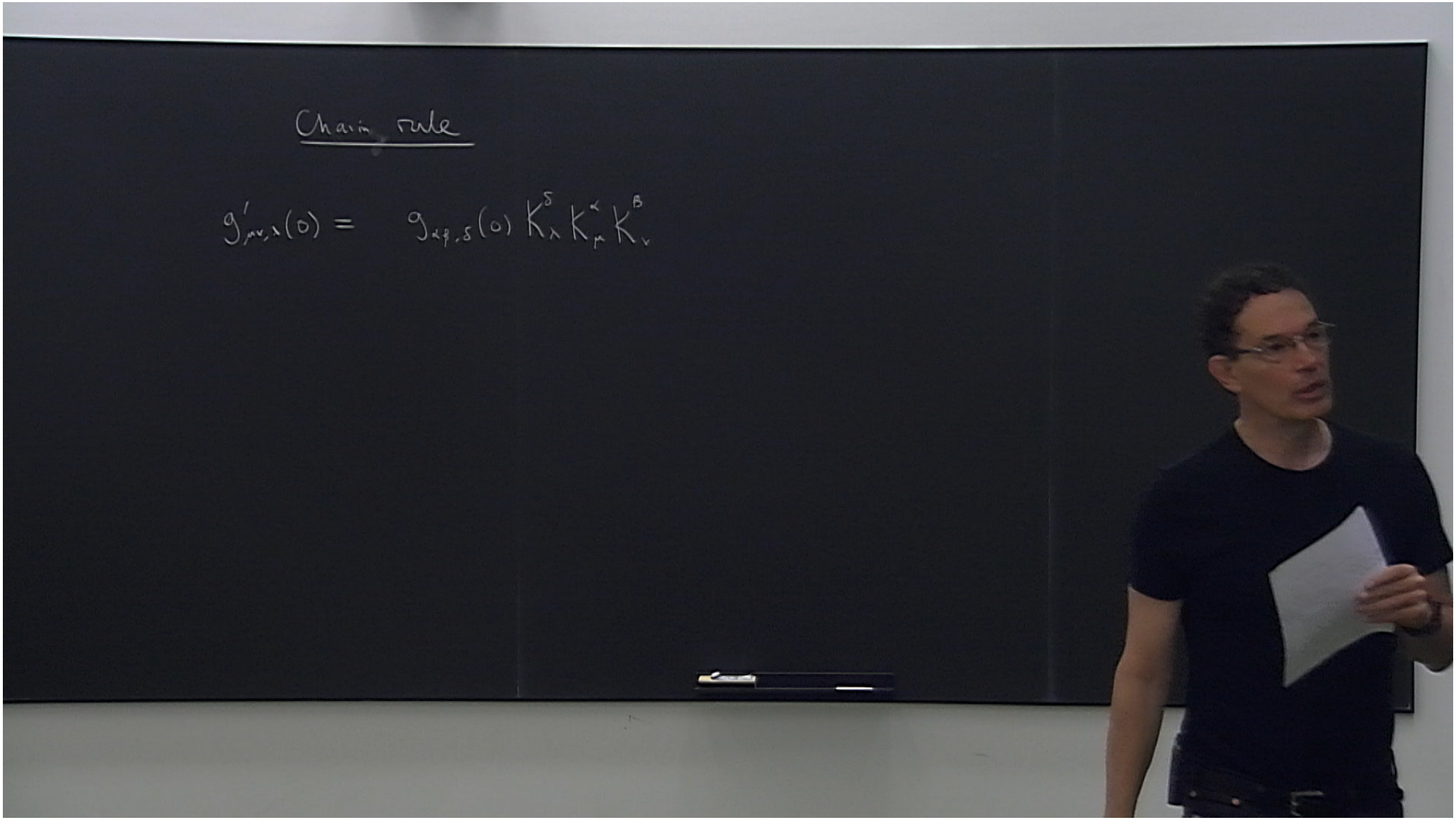
$$x'^\alpha(x^\gamma) = K^\alpha_\gamma x^\gamma - \frac{1}{2}L^\alpha_{\gamma\delta}x^\gamma x^\delta + \frac{1}{6}M^\alpha_{\gamma\delta\lambda}x^\gamma x^\delta x^\lambda + \dots$$

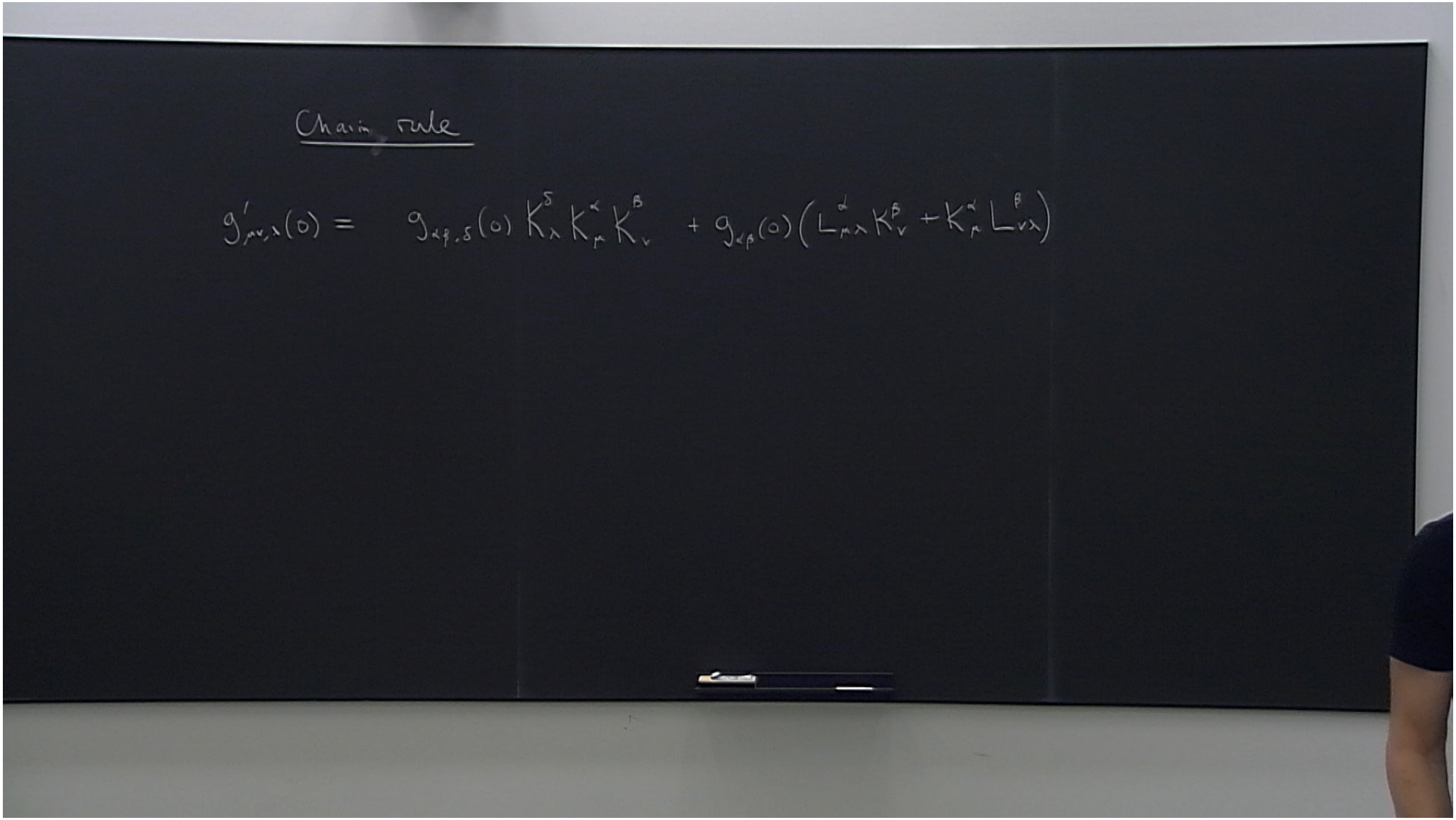
↑ arb'y const matrix ②

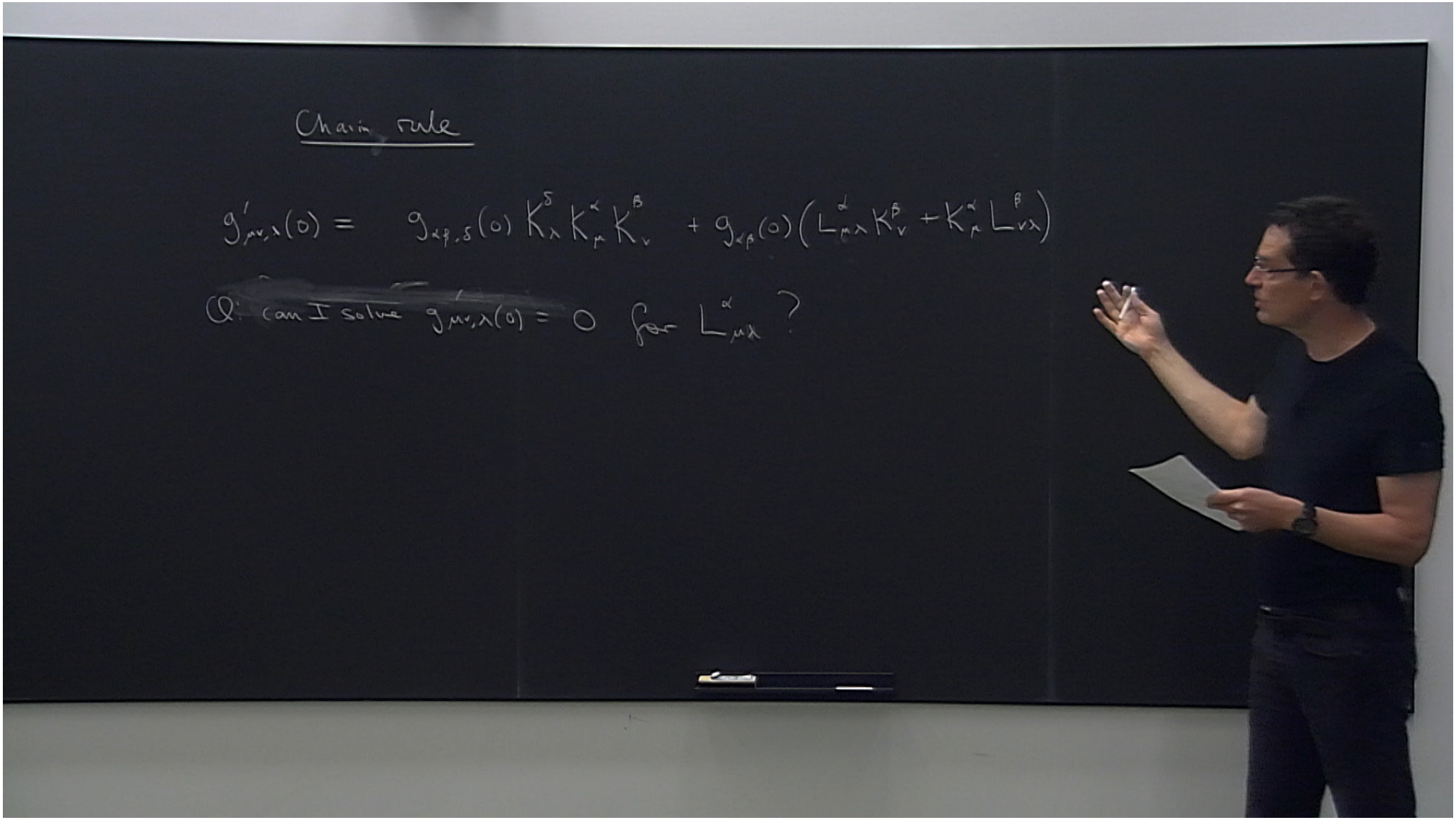
$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}$$ ③, using ① and ②, this is a Taylor ser

Conclude: enough parameters in K^μ_ν ($\downarrow$ 16) to set an arb'y $g_{\mu\nu}$ ($\downarrow$ 10) to

6 parameters are not fixed → Lorentz transfn's


Chain rule
$g'_{\mu\nu,\lambda}(0) = g_{\alpha\beta,\delta}(0) K^{\delta}_{\lambda} K^{\alpha}_{\mu} K^{\beta}_{\nu}$


Chain rule
$g'_{\mu\nu,\lambda}(0) = g_{\alpha\beta,\delta}(0) K^{\delta}_{\lambda} K^{\alpha}_{\mu} K^{\beta}_{\nu} + g_{\alpha\beta}(0)\left(L^{\alpha}_{\mu\lambda} K^{\beta}_{\nu} + K^{\alpha}_{\mu} L^{\beta}_{\nu\lambda}\right)$


Chain rule
$g'_{\mu\nu,\lambda}(0) = g_{\alpha\beta,\delta}(0) K^{\delta}_{\lambda} K^{\alpha}_{\mu} K^{\beta}_{\nu} + g_{\alpha\beta}(0)\left(L^{\alpha}_{\mu\lambda} K^{\beta}_{\nu} + K^{\alpha}_{\mu} L^{\beta}_{\nu\lambda}\right)$
Q: can I solve $g_{\mu\nu,\lambda}(0) = 0$ for $L^{\alpha}_{\mu\lambda}$?

$$K^{\delta}_{\lambda} K^{\alpha}_{\mu} K^{\beta}_{\nu} + g_{\alpha\beta}(0)\left(L^{\alpha}_{\mu\lambda} K^{\beta}_{\nu} + K^{\alpha}_{\mu} L^{\beta}_{\nu\lambda}\right)$$

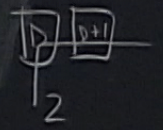
$= 0$ for $L^{\alpha}_{\mu\lambda}$?

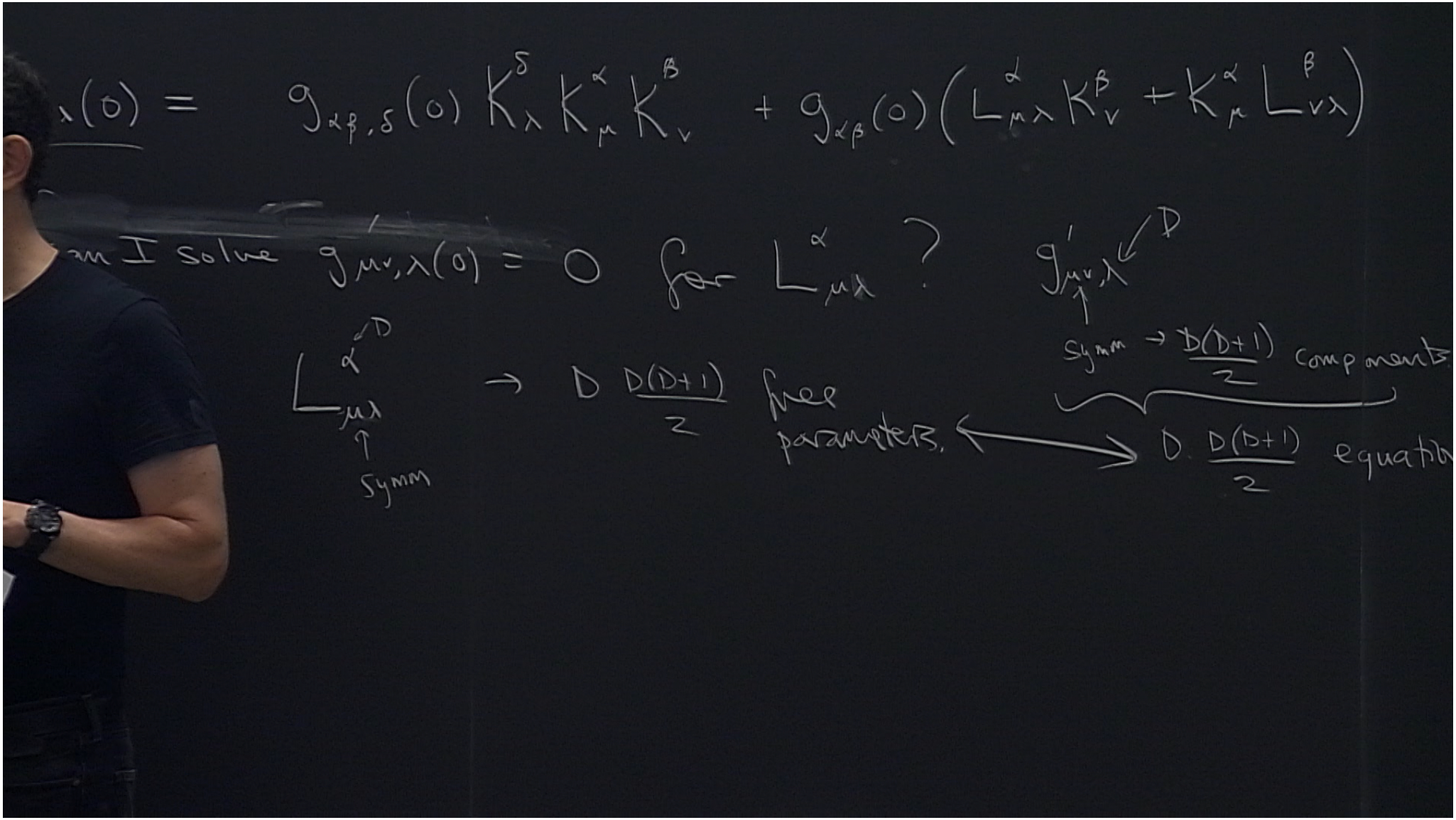
$g'_{\mu\nu,\lambda}$ ← D

Symm → $\frac{D(D+1)}{2}$ components

$D \cdot \frac{D(D+1)}{2}$ equations

$\frac{D(D+1)}{2}$




$\lambda(0) = g_{\alpha\beta,\delta}(0) K^{\delta}_{\lambda} K^{\alpha}_{\mu} K^{\beta}_{\nu} + g_{\alpha\beta}(0)\left(L^{\alpha}_{\mu\lambda} K^{\beta}_{\nu} + K^{\alpha}_{\mu} L^{\beta}_{\nu\lambda}\right)$
am I solve $g_{\mu\nu,\lambda}(0) = 0$ for $L^{\alpha}_{\mu\lambda}$?
$g'_{\mu\nu,\lambda}$
D
Symm → $\frac{D(D+1)}{2}$ components
$L^{\alpha}_{\mu\lambda}$
α D
Symm
→ $D \frac{D(D+1)}{2}$ free parameters.
$D \cdot \frac{D(D+1)}{2}$ equatio

Chain rule

$$g'_{\mu\nu,\lambda}(0) = g_{\alpha\beta,\delta}(0) K^{\delta}_{\lambda} K^{\alpha}_{\mu} K^{\beta}_{\nu} + g_{\alpha\beta}(0)\left(L^{\alpha}_{\mu\lambda} K^{\beta}_{\nu} + K^{\alpha}_{\mu} L^{\beta}_{\nu\lambda}\right)$$

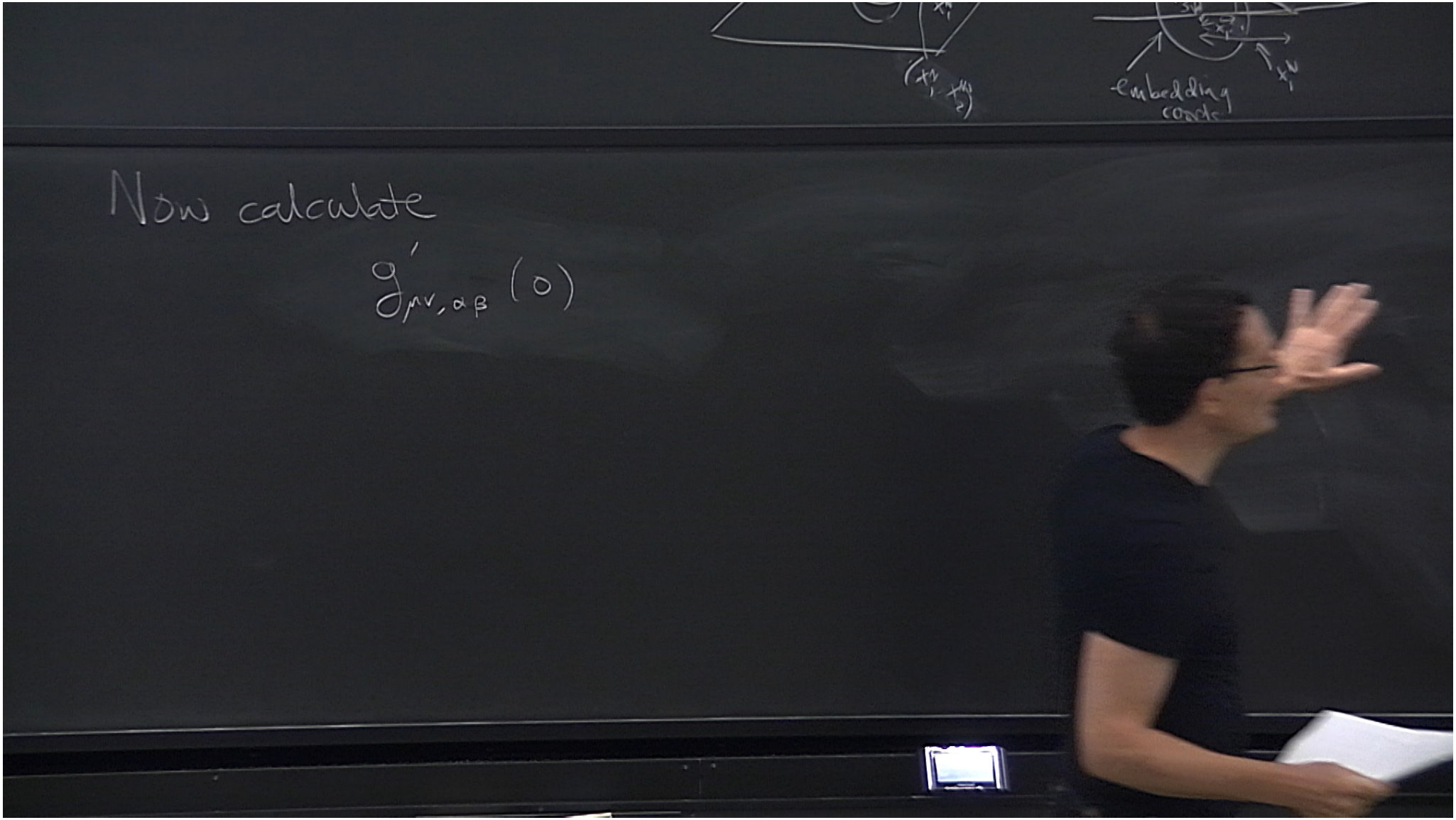
Q: can I solve $g'_{\mu\nu,\lambda}(0) = 0$ for $L^{\alpha}_{\mu\lambda}$?

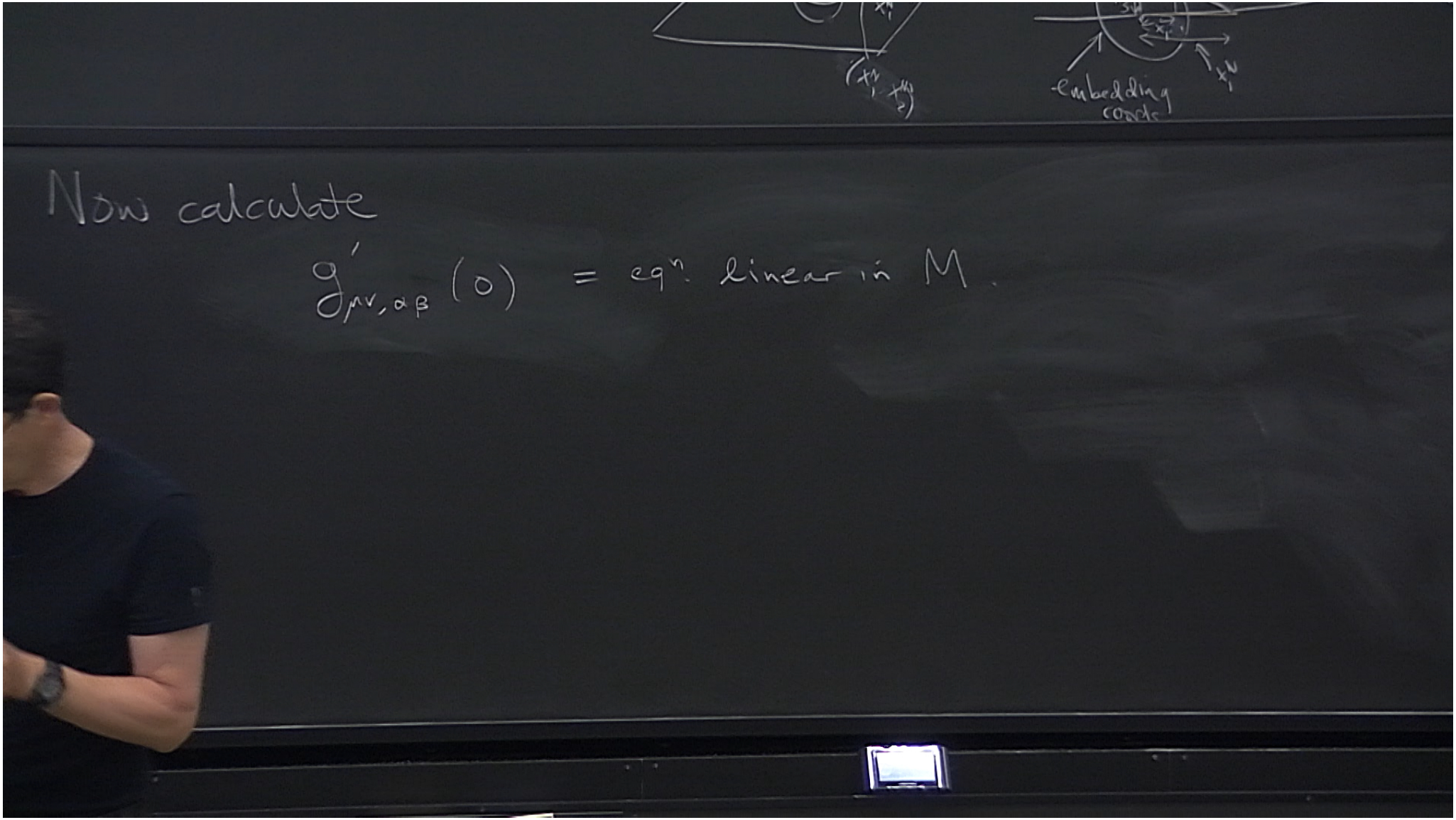
$L^{\alpha}_{\mu\lambda}$: α: D; $\mu\lambda$: symm → $D\,\frac{D(D+1)}{2}$ free parameters.

$g'_{\mu\nu,\lambda}$: $\mu\nu$: symm → $\frac{D(D+1)}{2}$ components; λ: D → $D\,\frac{D(D+1)}{2}$ equations

$\frac{D(D+1)}{2}$

(Riemann's trick from last time is exactly manipulation needed to solve explicitly for L)


Now calculate
$g'_{\mu\nu,\alpha\beta}(0)$
embedding coords

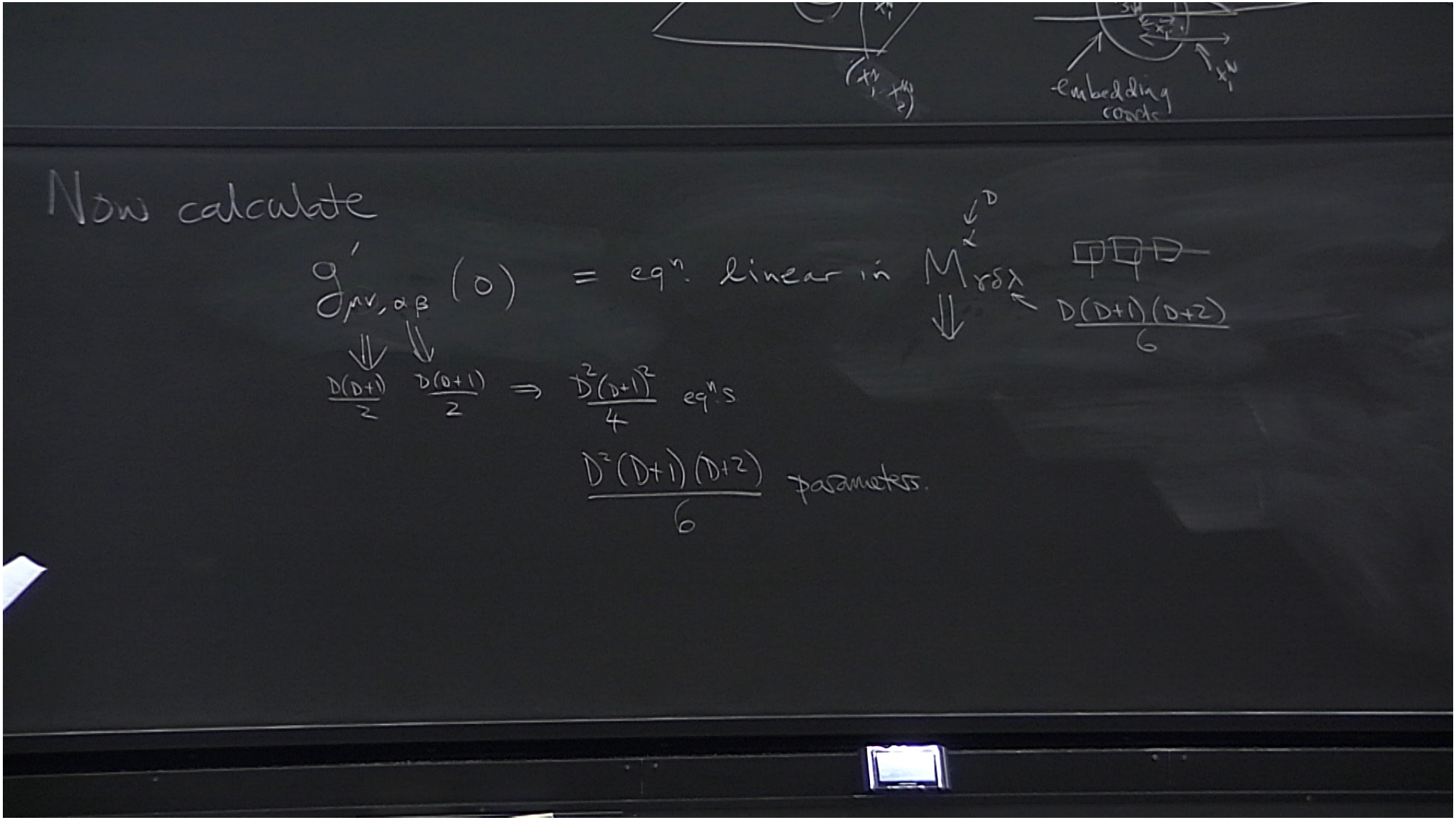

Now calculate
$g'_{\mu\nu,\alpha\beta}(0)$ = eqn. linear in M.
embedding coords

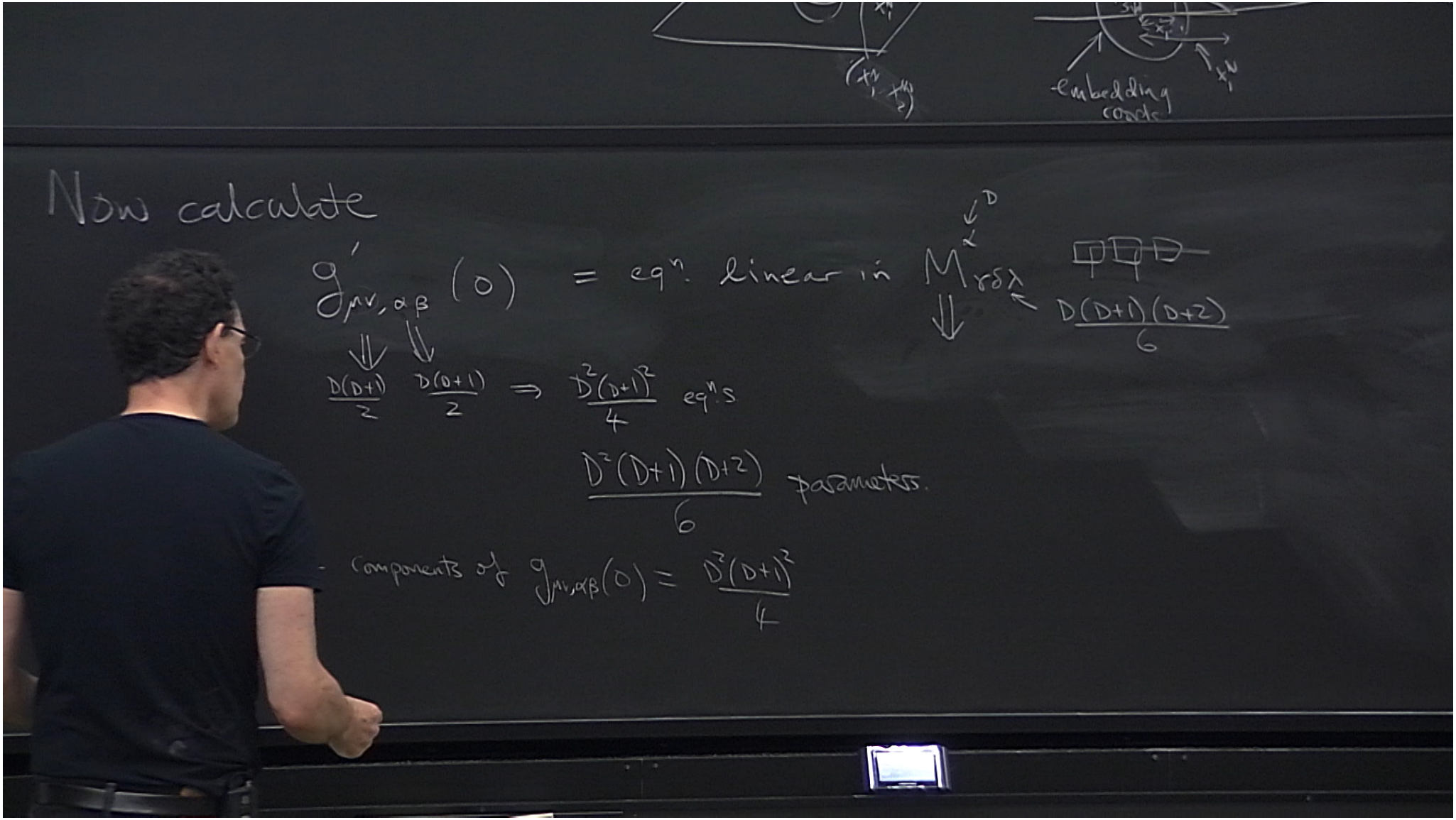


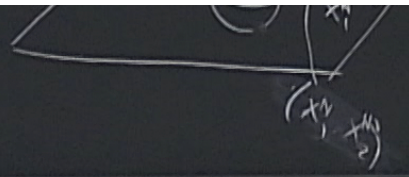
Now calculate

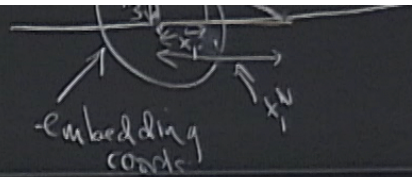
$$g'_{\mu\nu,\alpha\beta}(0) = \text{eqn. linear in } M.$$

$$\frac{D(D+1)}{2} \quad \frac{D(D+1)}{2} \Rightarrow \frac{D^2(D+1)^2}{4} \text{ eqn's}$$


Now calculate
$g'_{\mu\nu,\alpha\beta}(0)$ = eqn. linear in $M^{\lambda}_{\gamma\delta\lambda}$
D
$\frac{D(D+1)(D+2)}{6}$
$\frac{D(D+1)}{2}$ $\frac{D(D+1)}{2}$ ⇒ $\frac{D^2(D+1)^2}{4}$ eqn's
$\frac{D^2(D+1)(D+2)}{6}$ parameters.
embedding coords


Now calculate
$g'_{\mu\nu,\alpha\beta}(0)$ = eqn. linear in $M^{\lambda}_{\gamma\delta\lambda}$
D
$\frac{D(D+1)(D+2)}{6}$
$\frac{D(D+1)}{2}$ $\frac{D(D+1)}{2}$ $\Rightarrow$ $\frac{D^2(D+1)^2}{4}$ eqn's
$\frac{D^2(D+1)(D+2)}{6}$ parameters.
- Components of $g_{\mu\nu,\alpha\beta}(0) = \frac{D^2(D+1)^2}{4}$
embedding coords





1

generalise to

alculate

$g'_{\mu\nu,\alpha\beta}(0)$ = eqn. linear in $M^{\lambda}_{\gamma\delta}$ (λ: D; $\gamma\delta$: $\frac{D(D+1)}{2}$)

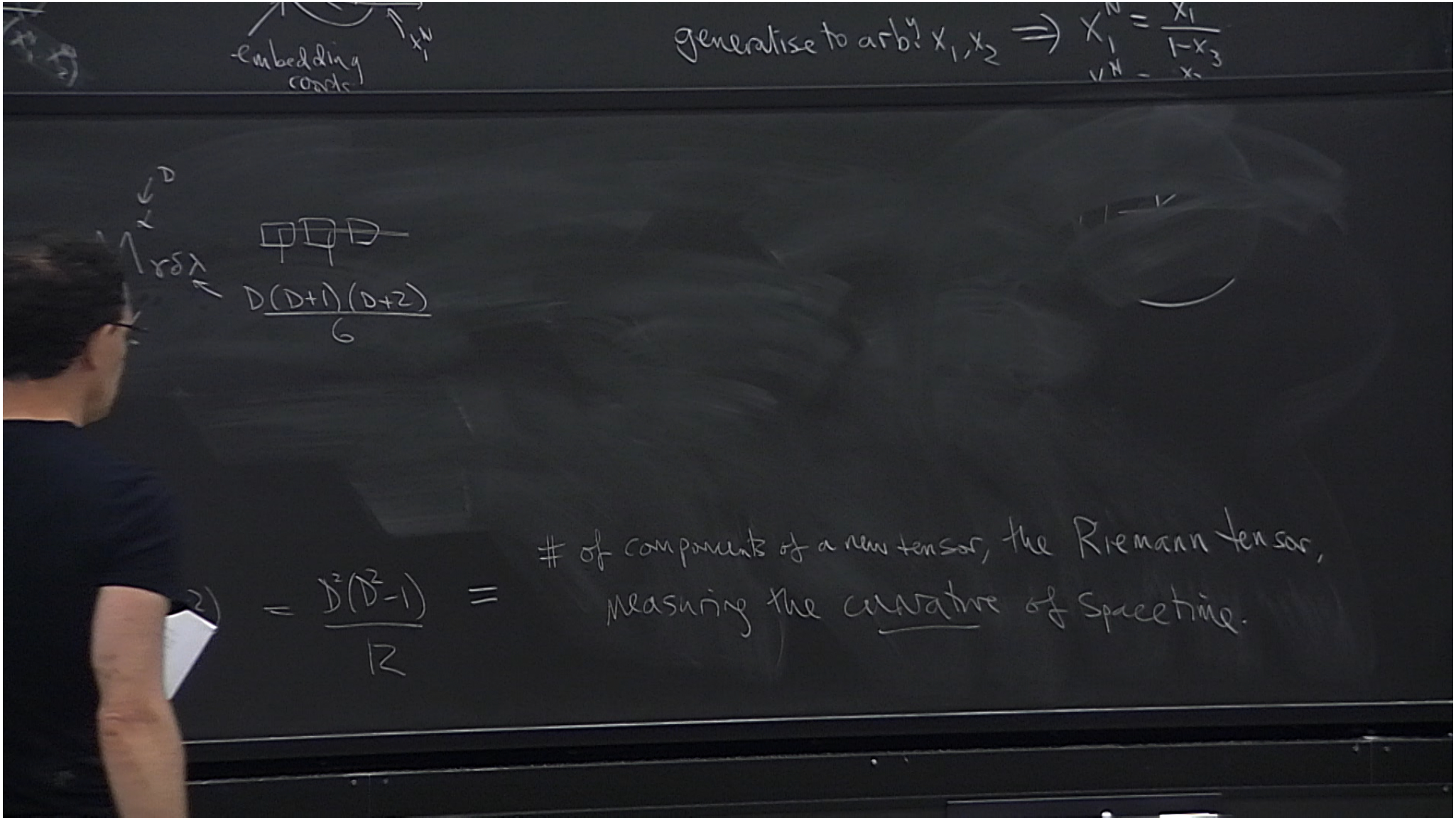
$D(D+2)$

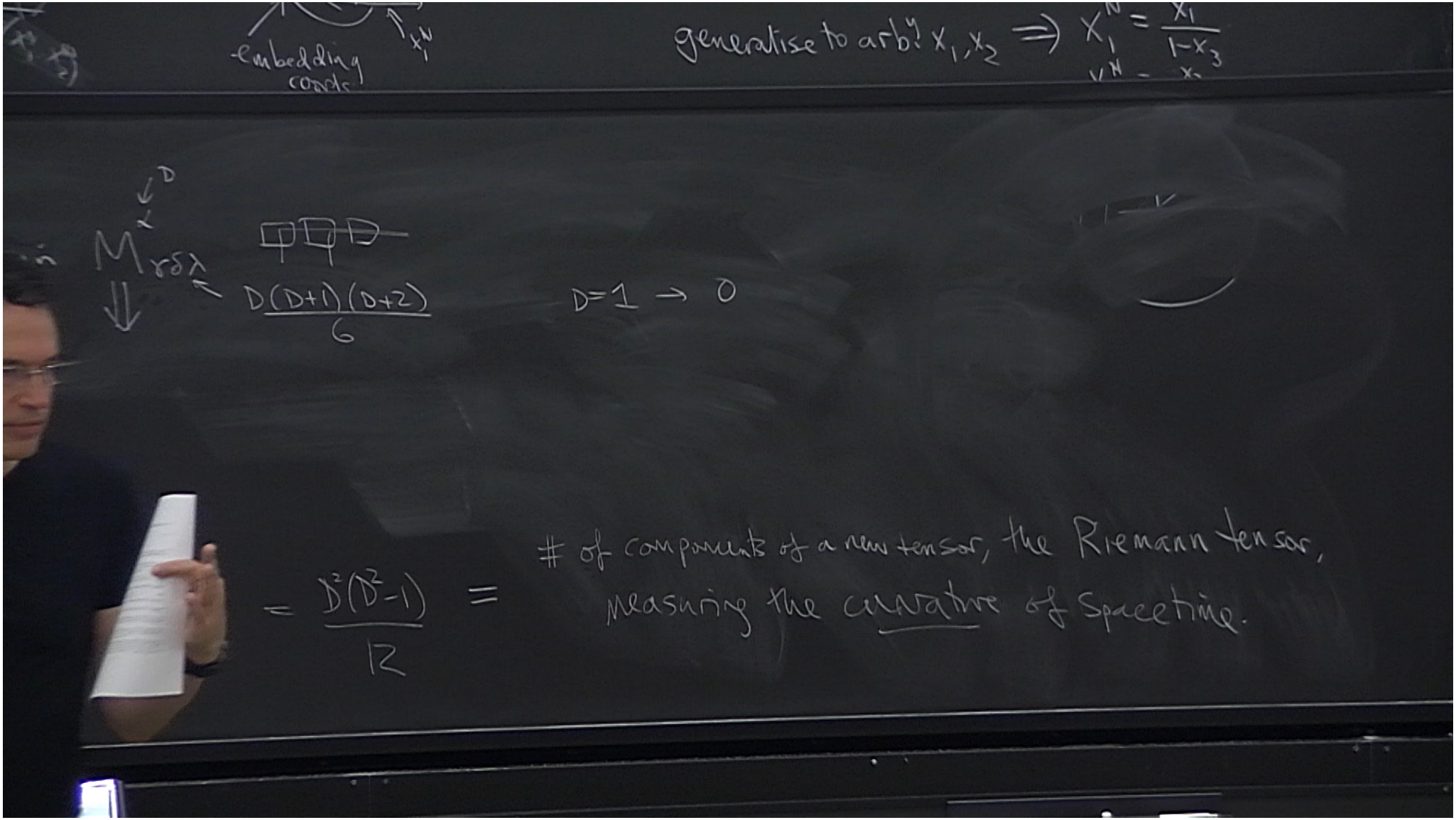
$\frac{D(D+1)}{6}$

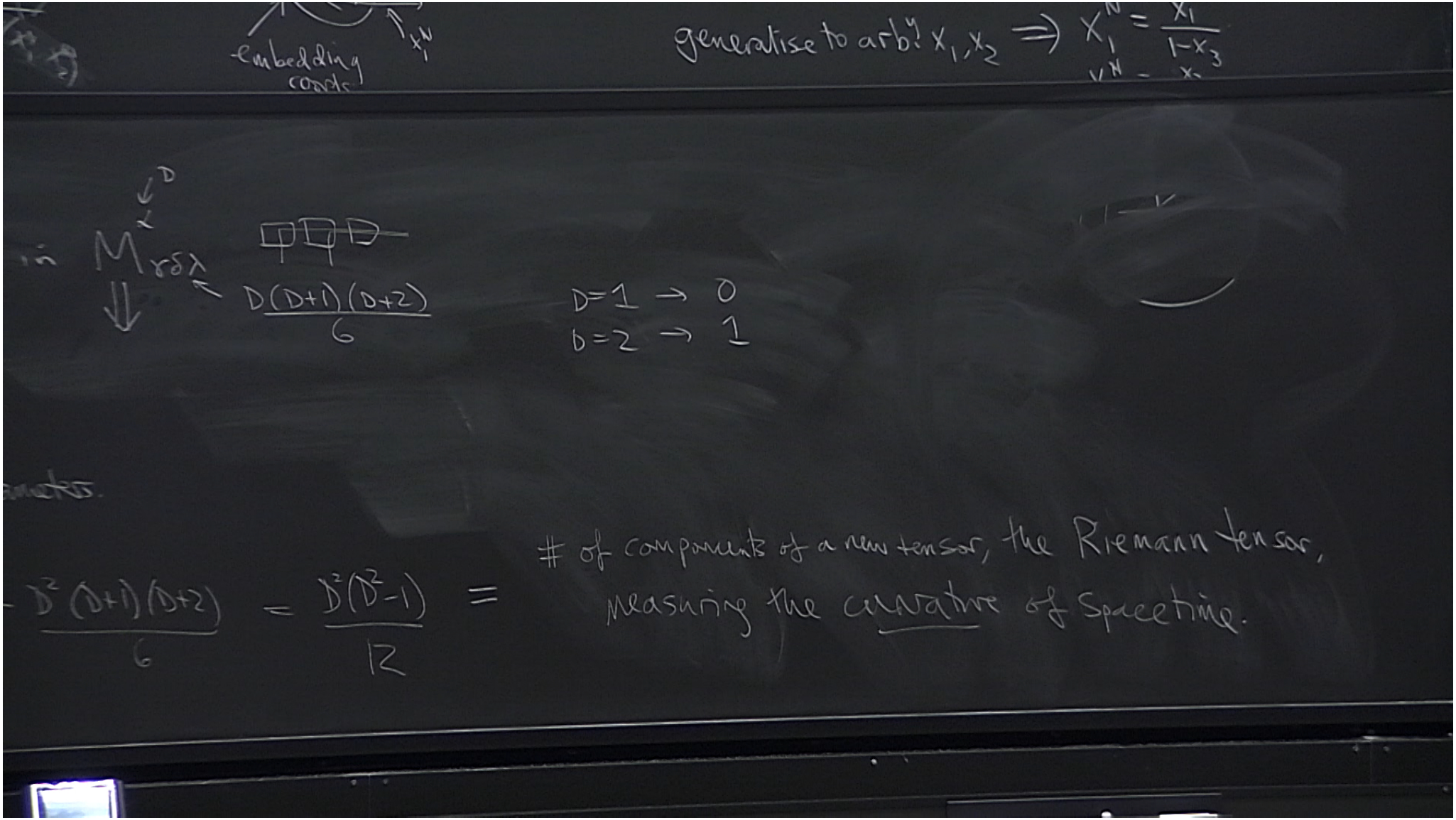
$\frac{D(D+1)}{2}$ $\frac{D(D+1)}{2}$ $\Rightarrow$ $\frac{D^2(D+1)^2}{4}$ eqns

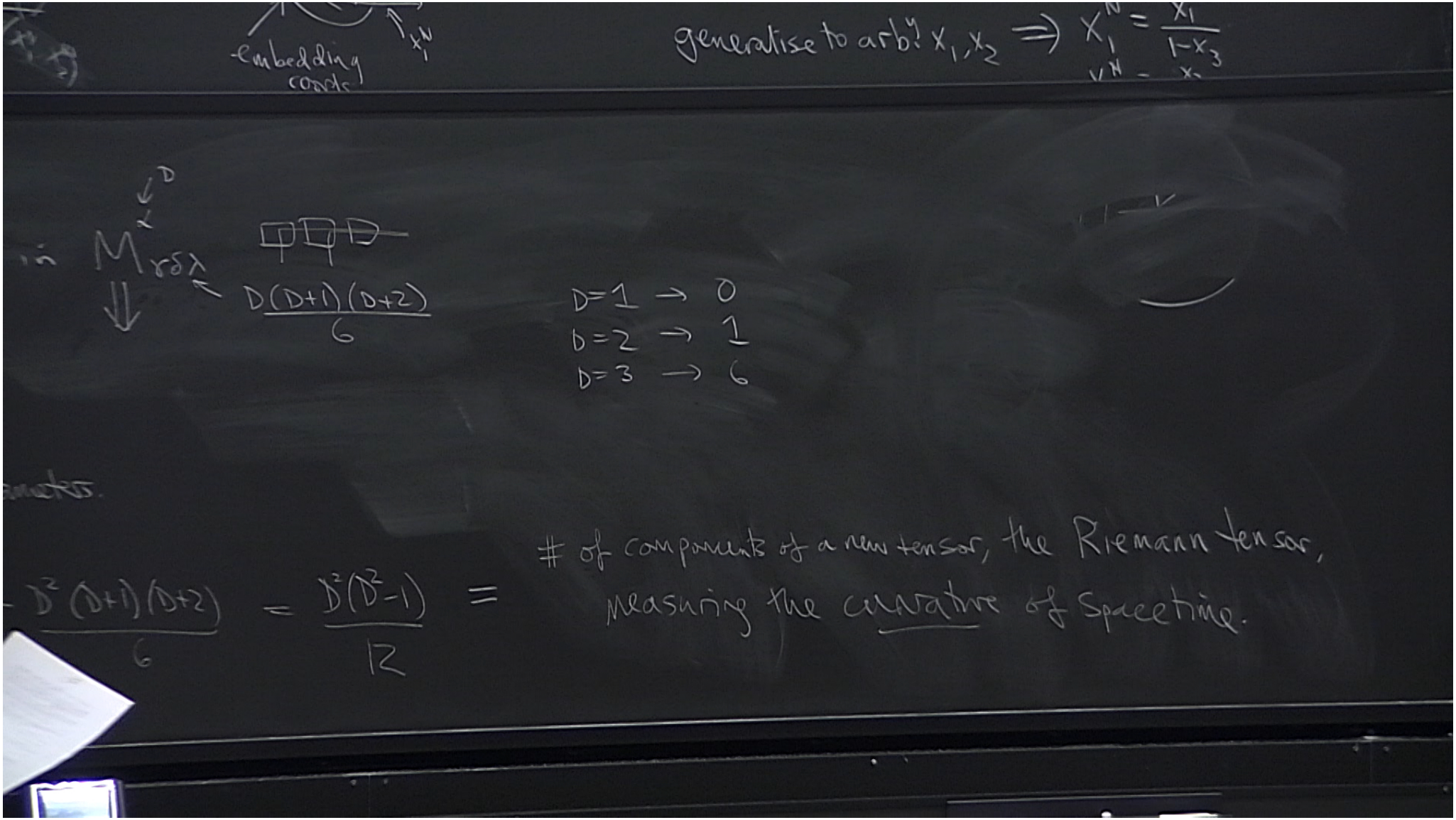
$\frac{D^2(D+1)(D+2)}{6}$ parameters.

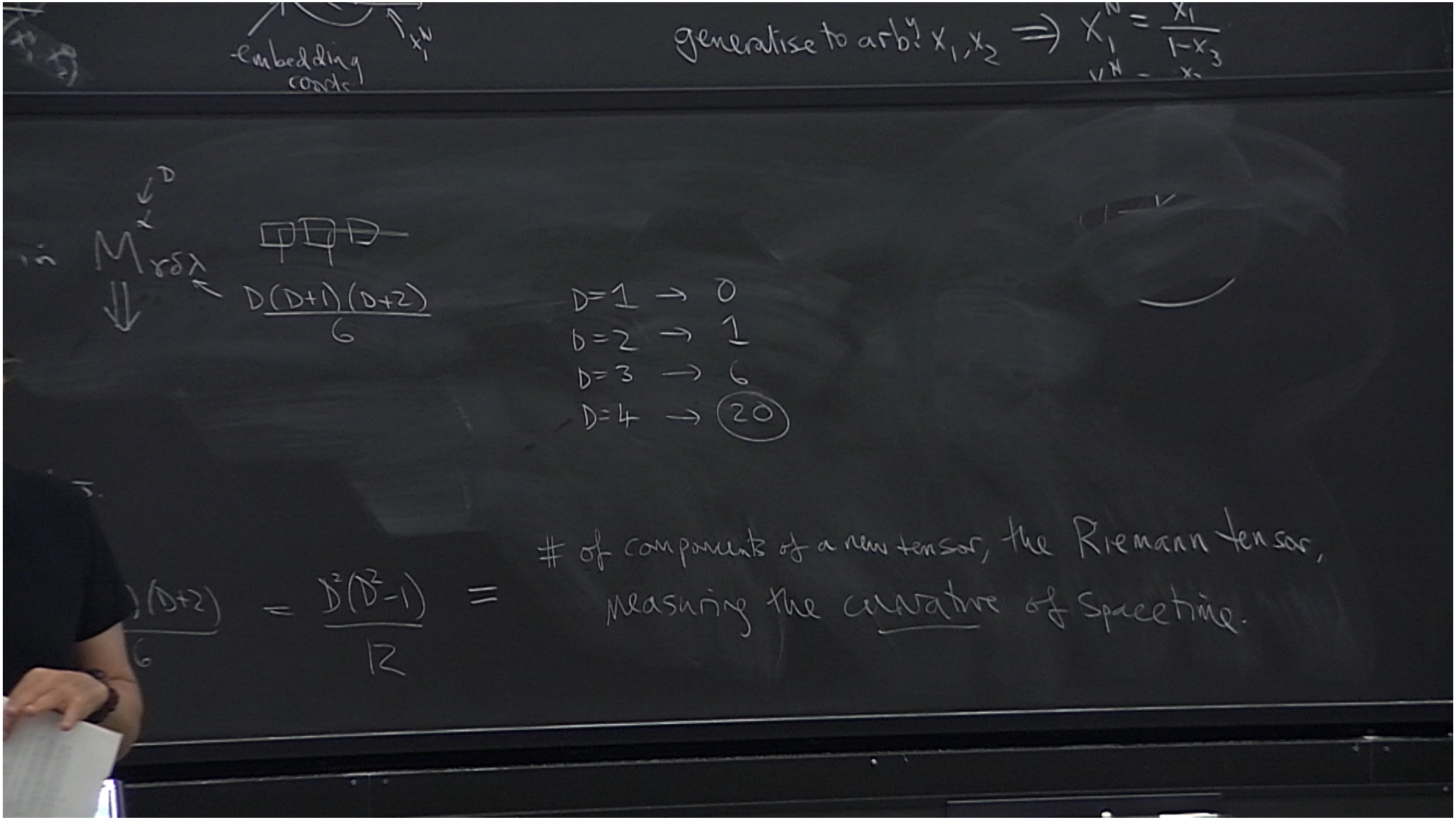
# of components of $g_{\mu\nu,\alpha\beta}(0)$ which cannot be cancelled $= \frac{D^2(D+1)^2}{4} - \frac{D^2(D+1)(D+2)}{6} = \frac{D^2(D^2-1)}{12}$ = # of components of a re

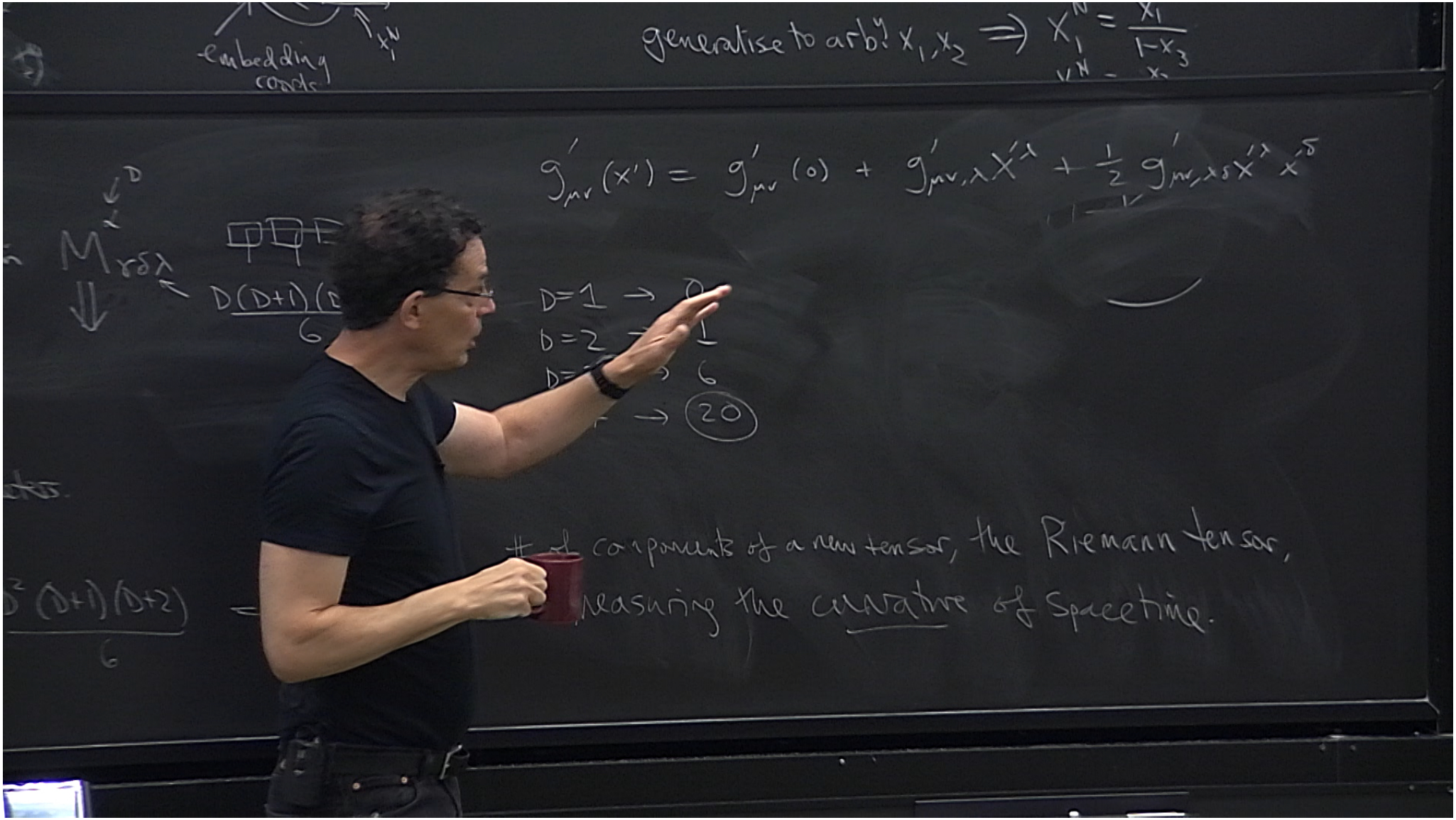

Embedding coords
generalise to arb? $x_1, x_2 \Rightarrow$
$D(D+1)(D+2)/6$
$= D^2(D^2-1)/12 \equiv$
# of components of a new tensor, the Riemann tensor,
measuring the curvature of Spacetime.

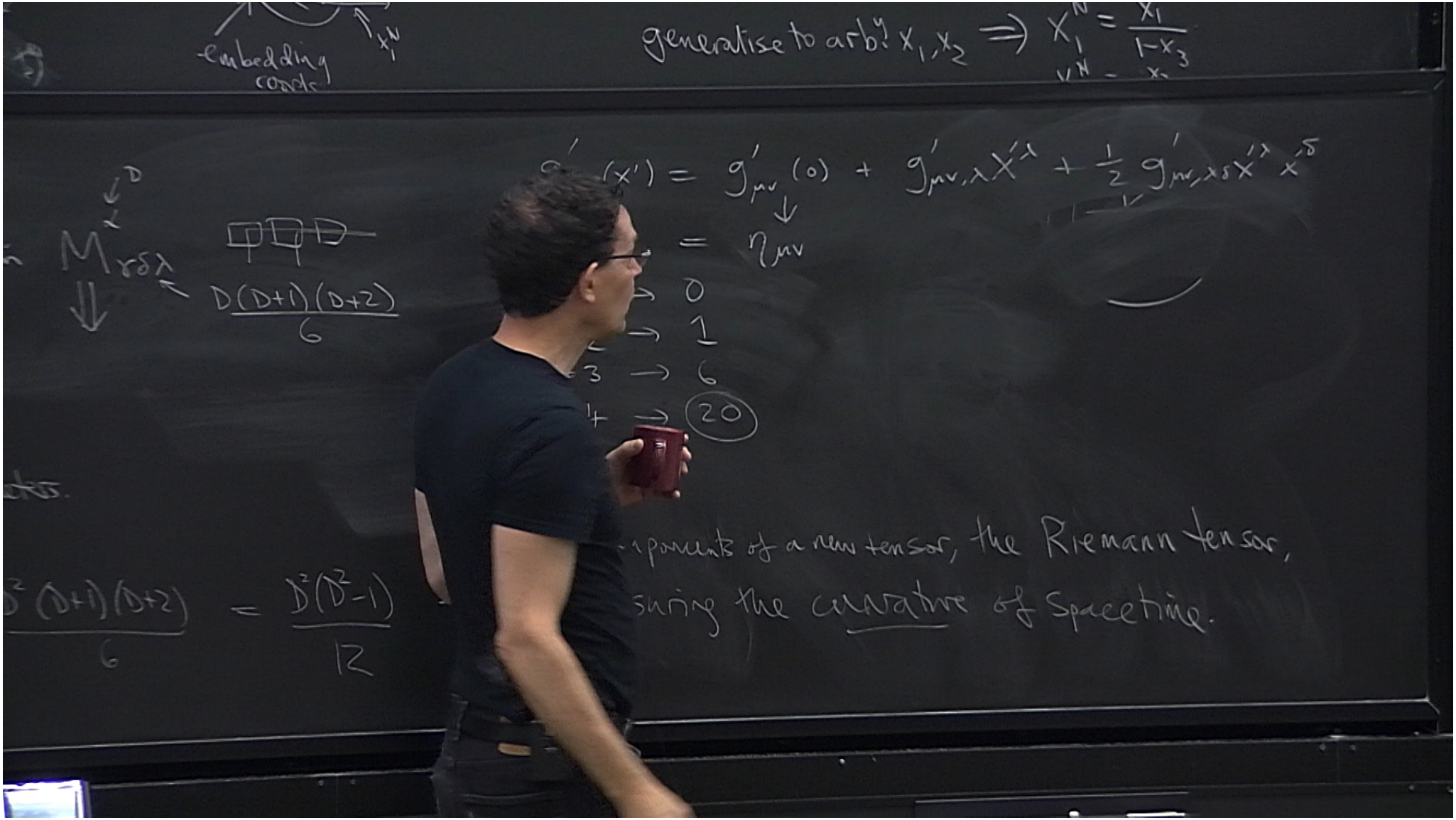

embedding coords
generalise to arb? $x_1, x_2 \Rightarrow$
D
$D(D+1)(D+2)/6$
$D=1 \to 0$
$= D^2(D^2-1)/12 =$
# of components of a new tensor, the Riemann tensor,
measuring the curvature of Spacetime.

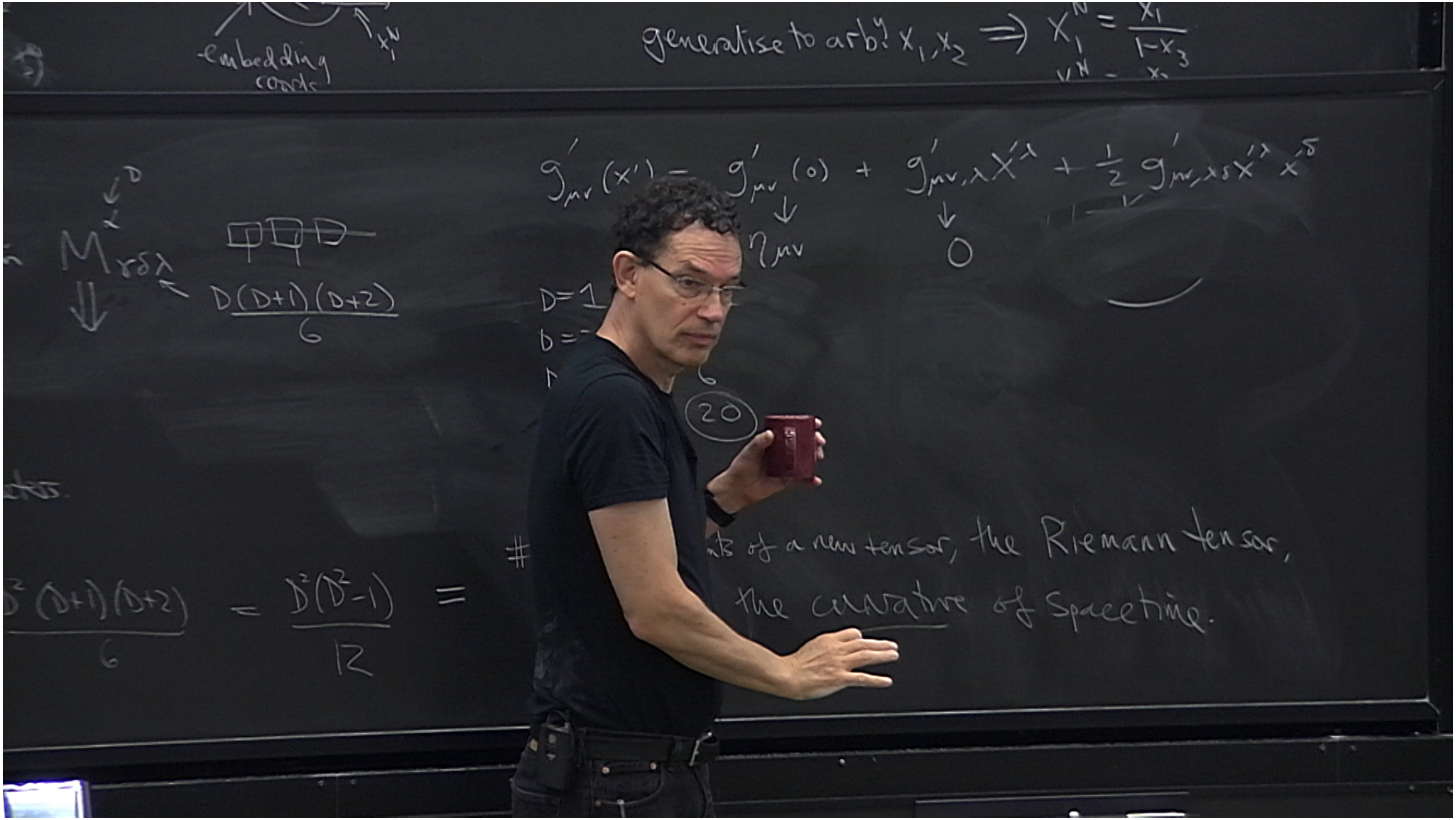

Embedding coords
generalise to arb? $x_1, x_2 \Rightarrow$
$D(D+1)(D+2)/6$
$D=1 \to 0$
$D=2 \to 1$
$D^2(D+1)(D+2)/6 = D^2(D^2-1)/12 \equiv$
# of components of a new tensor, the Riemann tensor,
measuring the curvature of Spacetime.

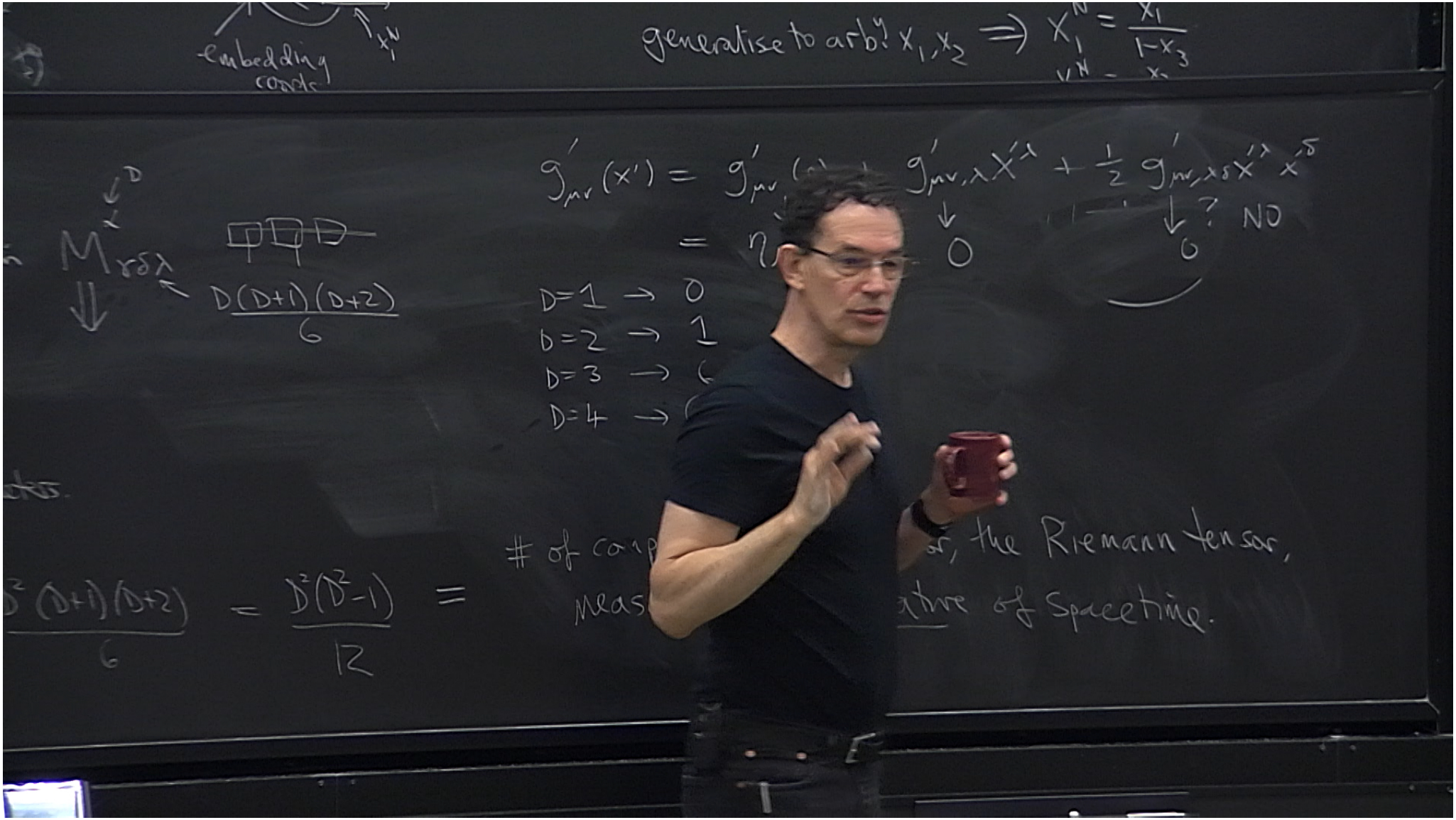

Embedding coords
generalise to arb? $x_1, x_2 \Rightarrow$
$D(D+1)(D+2)/6$
D=1 → 0
D=2 → 1
D=3 → 6
$D^2(D+1)(D+2)/6 = D^2(D^2-1)/12 =$
# of components of a new tensor, the Riemann tensor,
measuring the curvature of Spacetime.

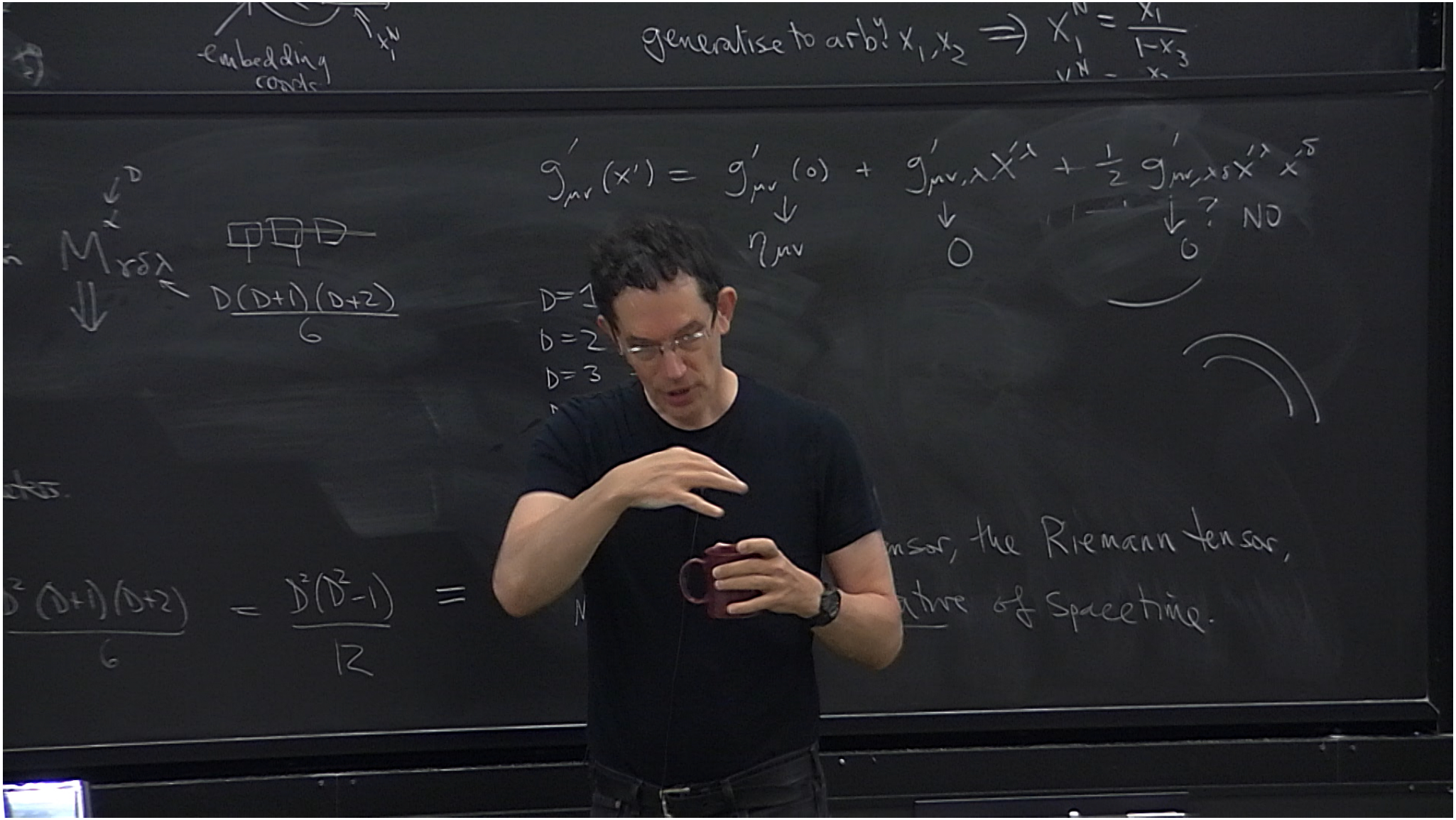

Embedding coords
generalise to arb? $x_1, x_2 \Rightarrow$
$D(D+1)(D+2)/6$
D=1 → 0
D=2 → 1
D=3 → 6
D=4 → 20
$= D^2(D^2-1)/12 \equiv$
# of components of a new tensor, the Riemann tensor,
measuring the curvature of Spacetime.


Embedding coords
generalise to arb? $x_1, x_2 \Rightarrow$
$g'_{\mu\nu}(x') = g'_{\mu\nu}(0) + g'_{\mu\nu,\lambda} x'^{\lambda} + \frac{1}{2} g'_{\mu\nu,\lambda\delta} x'^{\lambda} x'^{\delta}$
$M^{\mu\nu\lambda}$
$\frac{D(D+1)(D}{6}$
D=1 → 0
D=2 → 1
→ 6
→ 20
# of components of a new tensor, the Riemann tensor,
measuring the curvature of spacetime.


Embedding coords
generalise to arb? $x_1, x_2 \Rightarrow$
$= \frac{x_1}{1-x_3}$
$g'_{\mu\nu}(x') = g'_{\mu\nu}(0) + g'_{\mu\nu,\lambda} x'^{\lambda} + \frac{1}{2} g'_{\mu\nu,\lambda\delta} x'^{\lambda} x'^{\delta}$
$= \eta_{\mu\nu}$
$M_{\gamma\delta\lambda}$
$\frac{D(D+1)(D+2)}{6}$
$\rightarrow 0$
$\rightarrow 1$
$3 \rightarrow 6$
$\rightarrow 20$
components of a new tensor, the Riemann tensor,
$\frac{D^2(D+1)(D+2)}{6} = \frac{D^2(D^2-1)}{12}$
using the curvature of spacetime.


Embedding coords
generalise to arb? $x_1, x_2 \Rightarrow$
$g'_{\mu\nu}(x') = g'_{\mu\nu}(0) + g'_{\mu\nu,\lambda}x'^{\lambda} + \frac{1}{2}g'_{\mu\nu,\lambda\delta}x'^{\lambda}x'^{\delta}$
$\eta_{\mu\nu}$
0
$M^{\mu}_{\nu\delta\lambda}$
$\frac{D(D+1)(D+2)}{6}$
D=1
20
$\frac{D^2(D+1)(D+2)}{6} - \frac{D^2(D^2-1)}{12} =$
the curvature of spacetime.


Embedding coords
generalise to arb? $x_1, x_2 \Rightarrow$
$g'_{\mu\nu}(x') = g'_{\mu\nu}(a) + g'_{\mu\nu,\lambda} x'^{\lambda} + \frac{1}{2} g'_{\mu\nu,\lambda\delta} x'^{\lambda} x'^{\delta}$
$= \eta$
0
0
? NO
$M^{\lambda}_{\mu\delta}$
$\frac{D(D+1)(D+2)}{6}$
D=1 → 0
D=2 → 1
D=3 →
D=4 →
# of
the Riemann tensor,
Meas
ature of Spacetime.
$\frac{D^2(D+1)(D+2)}{6} - \frac{D^2(D^2-1)}{12} =$


embedding coords
generalise to arb? $x_1, x_2 \Rightarrow$
$g'_{\mu\nu}(x') = g'_{\mu\nu}(0) + g'_{\mu\nu,\lambda}x'^{\lambda} + \frac{1}{2} g'_{\mu\nu,\lambda\delta}x'^{\lambda}x'^{\delta}$
$\eta_{\mu\nu}$
0
0 ? NO
$M^{\mu\nu}_{\gamma\delta\lambda}$
$\frac{D(D+1)(D+2)}{6}$
D=1
D=2
D=3
$\frac{D^2(D+1)(D+2)}{6} - \frac{D^2(D^2-1)}{12} =$
nsor, the Riemann tensor,
ture of Spacetime.

Assume $g_{\mu\nu}(x)$ is differentiable. $g_{\mu\nu}(x) = g_{\mu\nu}(0) + g_{\mu\nu,\lambda}(0)x^\lambda + \frac{1}{2}$ ①

Can I choose: $x'^\alpha(x)$ s.t. $g'_{\mu\nu}(0) = \eta_{\mu\nu}$, $g'_{\mu\nu,\alpha}(0) = 0$?

$$\updownarrow$$

$$x'^\alpha(x^\gamma) = K^\alpha_\gamma x^\gamma + \frac{1}{2} L^\alpha_{\gamma\delta} x^\gamma x^\delta + \frac{1}{6} M^\alpha_{\gamma\delta\lambda} x^\gamma x^\delta x^\lambda + \ldots$$ ②

K^α_γ: ↑ arb? const matrix, 16

$L^\alpha_{\gamma\delta}$: $\frac{\delta(\delta+1)}{2}$

$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$$ ③, using ① and ②, this is a Taylor ser

Conclude: enough parameters in K^μ_ν to set an arb? $g_{\mu\nu}$ to

K^μ_ν: ↓ 16

$g_{\mu\nu}$: ↓ 10

6 paramet… fixed → Lorentz transfm's

Next question: can I set $g'_{\mu\nu}$, … parameters in $L^\alpha_{\gamma\delta}$