Title: PSI 17/18 - Relativity - Lecture 4

Date: Sep 07, 2017 09:00 AM

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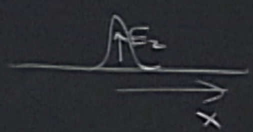
Abstract:

$$\left.\begin{array}{l}\partial_\mu F^{\mu\nu} \\ \epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta}\end{array}\right\} \Rightarrow \underline{\underline{\eta^{\mu\nu}}}\,\partial_\mu \partial_\nu F_{\alpha\beta} = 0$$

Recap: M.E.s $\rightarrow$ $F_{\mu\nu}$, J_ν $\eta^{\mu\nu}$ invariant

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$$\frac{\partial}{\partial x'^{\mu}} = \tilde{\Lambda}^{\alpha}{}_{\mu} \frac{\partial}{\partial x^{\alpha}}$$



$$p^{\mu} = \left(\frac{E}{c}, p^{x}, 0\ 0\right)$$

$p^{x} = E/c$ $\quad$ $|\underline{p}| = E/c$ in general

Einstein (1905, improved 1947)

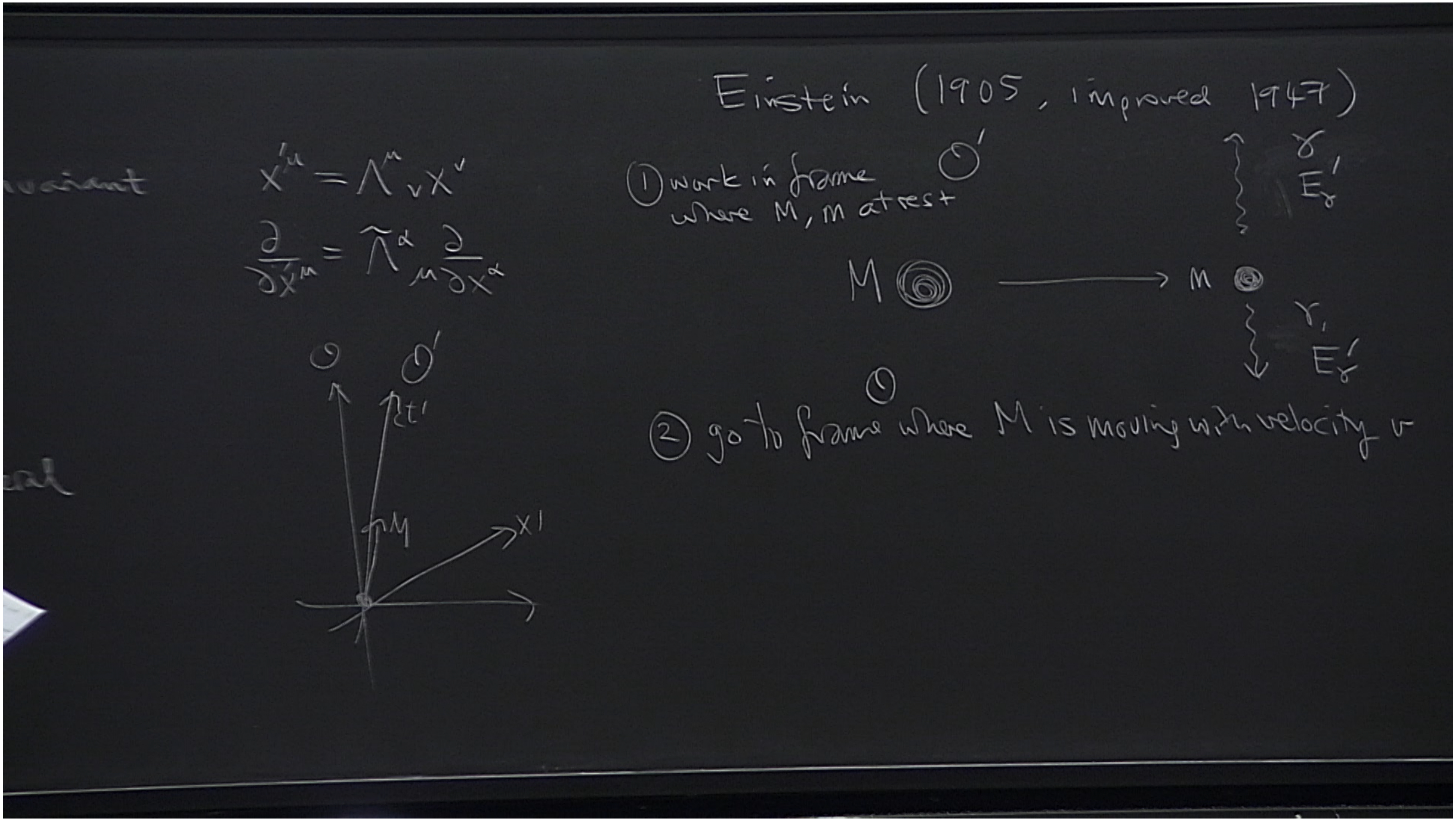
① work in frame O'
where M, m at rest

γ, E'_γ

$M \longrightarrow m$

γ, E'_γ

$$E'_M = E'_m + 2E'_\gamma$$


Einstein (1905, improved 1947)
$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$
$\frac{\partial}{\partial x'^{\mu}} = \tilde{\Lambda}^{\alpha}{}_{\mu} \frac{\partial}{\partial x^{\alpha}}$
① work in frame where M, m at rest
O'
γ, E'_γ
M
m
γ, E'_γ
O
② go to frame where M is moving with velocity v
O
O'
t'
M
x'

Einstein (1905, improved 1947)

① work in frame $\mathcal{O}'$ where M, m at rest

$M \longrightarrow m$

γ, E'_γ

γ, E'_γ

$$E'_M = E'_m + 2E'_\gamma$$

$$p'^{\mu} = \begin{array}{cccc} ct' & x' & y' & z' \\ \end{array}$$

$$p'^{\mu} = \left(\frac{E'_\gamma}{c}, 0, 0, \frac{E'_\gamma}{c}\right)$$

$$ct = \gamma\left(ct' + \frac{v x'}{c}\right)$$

$$p^{\mu} = \left(\gamma \frac{E'_\gamma}{c}, \dots\right) \Rightarrow E_\gamma = E'_\gamma \gamma$$

② go to frame $\mathcal{O}$ where M is moving with velocity v

Momentum cons^n in $\mathcal{O}$

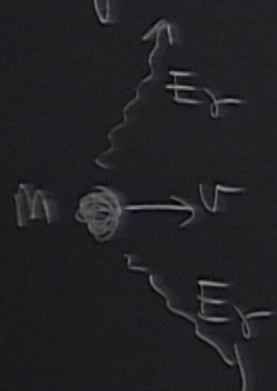
$M \rightarrow v$

$m \rightarrow v$, E_γ, E_γ

E_γ

$\gamma,\ E'_\gamma$

velocity v



$$E'_M = E'_m + 2E'_\gamma$$

$$\begin{array}{ccccc} & ct' & x' & y' & z' \\ p'^\mu = & \Big(\frac{E'_\gamma}{c}, & 0, & 0, & \frac{E'_\gamma}{c} \Big) \end{array}$$

$$ct = \gamma\left(ct' + \frac{v x'}{c}\right)$$

$$\Downarrow$$

$$p^\mu = \left(\gamma \frac{E'_\gamma}{c}, \ \ldots \right) \Rightarrow E_\gamma = E'_\gamma \cancel{\gamma}$$

$E_\gamma \approx E'_\gamma$ to order v^2.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \approx \cancel{1 + \frac{1}{2} v^2/c^2} + O\left((v/c)^2\right)$$

Einstein (1905, improved 1947)

① work in frame where M, m at rest $\mathcal{O}'$

$M \longrightarrow m$, γ, E'_γ ; γ, E'_γ

$\mathcal{O}$

② go to frame where M is moving with velocity v

$$E'_M = E'_m + 2E'_\gamma$$

$$\begin{array}{ccccc} & ct' & x' & y' & z' \\ p'^\mu = & \Big(\frac{E'_\gamma}{c}, & 0, & 0, & \frac{E'_\gamma}{c}\Big) \end{array}$$

$$ct = \gamma\left(ct' + \frac{vx'}{c}\right)$$

$$p^\mu = \left(\gamma \frac{E'_\gamma}{c}, \dots\right) \Rightarrow E_\gamma = E'_\gamma \gamma$$

$E_\gamma \approx E'_\gamma$ to order v.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}v^2/c^2 + O((v/c)^2)$$

$E_\gamma \leftrightarrow p_\gamma = \frac{E_\gamma}{c}$

θ, $m \to v$, E_γ

Momentum cons^n in $\mathcal{O}$

$M \to v$

(x-comp^{nt}) $$Mv = mv + 2\frac{E_\gamma}{c}\sin\theta$$

v

c $\sqrt{c^2+v^2}$

…stein (1905, improved 1947)

…frame O' M, m at rest

$\gamma,\ E'_\gamma$

M ⟶ m

$\gamma,\ E'_\gamma$

$$E'_M = E'_m + 2E'_\gamma$$

$$ct = \gamma\left(ct' + \frac{v x'}{c}\right)$$

$$\begin{array}{cccccc} & ct' & x' & y' & z' & \\ p'^{\mu} = & \Big(\frac{E'_\gamma}{c}, & 0, & 0, & \frac{E'_\gamma}{c} & \Big) \end{array}$$

O

…frame where M is moving with velocity v

$$p^\mu = \left(\gamma \frac{E'_\gamma}{c}, \ \ldots \right) \Rightarrow E_\gamma = E'_\gamma \gamma$$

$E_\gamma \longleftrightarrow p_\gamma = \frac{E_\gamma}{c}$

θ

$m \longrightarrow v$

E_γ

M

…momentum …ns… in O

v

$$E_\gamma \approx E'_\gamma \text{ to order } v^2 \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}v^2/c^2 + O((v/c)^2)$$

But $\sin\theta \approx \frac{v}{c}$ if $v \ll c$

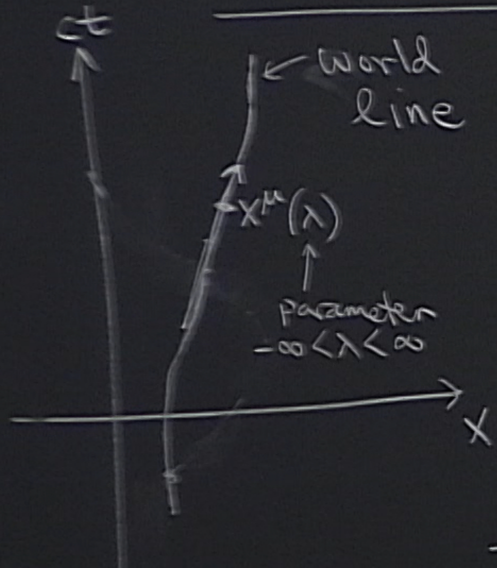
…-compt) $Mv = mv + 2\frac{E_\gamma}{c}\sin\theta$

$$\Rightarrow \quad Mv = mv + 2\frac{E_\gamma}{c}\cdot\frac{v}{c} \qquad \text{for } v \ll c$$

$$\div v \quad \Rightarrow \quad (M-m)c^2 = 2E_\gamma \quad \Rightarrow \quad \Delta M c^2 = \text{energy released in photons.}$$

in particular, if $m = 0$ (e.g. $\pi \to 2\gamma$) then E released $= Mc^2$.

Particles and their action



Lorentz invariant properties

$$\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \eta_{\mu\nu}\frac{dx'^\mu}{d\lambda}\frac{dx'^\nu}{d\lambda} \quad \text{under L.T.s.}$$

$< 0 \Rightarrow$ timelike

$-c^2dt^2 + d\underline{x}^2 \quad = 0 \Rightarrow$ null $\quad |d\underline{x}| = c\,dt$

$> 0 \Rightarrow$ spacelike. (tachyon $\not\exists$)

Massive particles follow timelike trajectories

can define "proper time" along world line $d\tau = \sqrt{d\cdots}$

– Lorentz invariant.

So can define $u^\mu = \frac{dx^\mu}{d\tau}$ – the 4-velocity.

$m_0 u^\mu = p^\mu$ the 4-momentum

m_0 ↑ rest mass (same in every frame)

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right)$$

$$\Rightarrow \eta_{\mu\nu} p^\mu p^\nu = -m_0^2 = -\left(\frac{E}{c}\right)^2 + |\underline{p}|^2$$

$$\Rightarrow E = p^0 c = \sqrt{\underline{p}^2 c^2 + m_0^2 c^4}$$

ld line $d\tau = \sqrt{dt^2 - \frac{dx^2}{c^2}} = dt\sqrt{1 - \left(\frac{dx}{dt}\right)^2 \frac{1}{c^2}} = dt\sqrt{1 - v^2/c^2}$

$$-\frac{d(t)^2}{d\tau^2} + \frac{\left(\frac{dx}{d\tau}\right)^2}{c^2} = -1$$

– the 4-velocity. $\eta_{\mu\nu}\frac{U^\mu U^\nu}{c^2} = -1$

the 4-momentum

$$\eta_{\mu\nu} p^\mu p^\nu = -m_0^2$$
$$= -\left(\frac{E}{c}\right)^2 + |\underline{p}|^2$$

$$= -\left(\frac{E}{c}\right)^2 + |\underline{p}|^2$$

If $\underline{p}^2 c^2 \ll m_0^2 c^4 \rightarrow E \approx \underbrace{m_0 c^2}_{\text{same old}} + \frac{1}{2}\,\underline{p}^2/m_0^2 + \dots$

$\frac{1}{2} m_0 \underline{v}^2$.

$\underline{p} = m_0$

$$\underline{p} = m_0 \underline{u} = m_0 \frac{d\underline{x}}{dt} \cdot \frac{dt}{d\tau} = \gamma m_0 \frac{d\underline{x}}{dt} = \gamma m_0 \underline{v}.$$

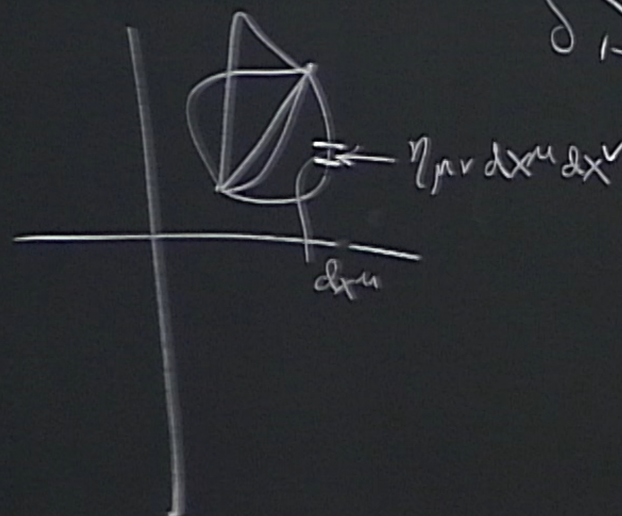
$$\underline{v} = \frac{d\underline{x}}{dt}$$

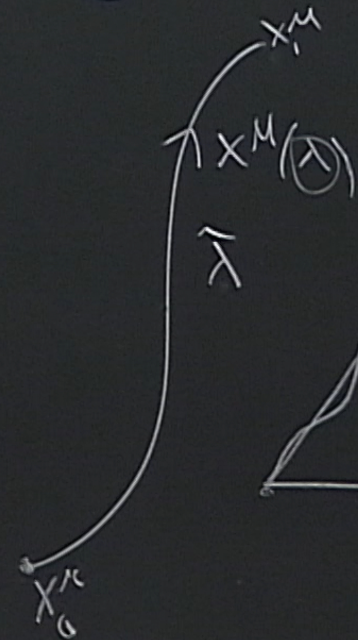
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2} v^2/c^2 + \cdots$$

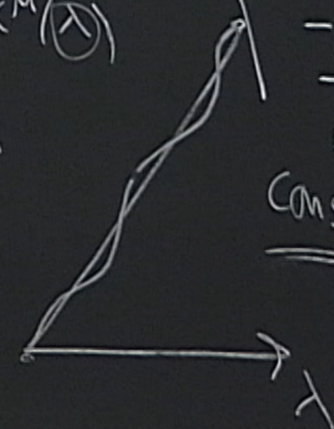
$$v \ll c \Rightarrow \underline{p} \approx m_0 \underline{v}$$

Action for a relativis

$$\delta S = 0 \implies \text{eq}^{\underline{n}}\text{s of motion} \quad , \quad \text{making}$$







$$S = -\underbrace{m_0 c}_{\text{const}} \int_{\lambda_0}^{\lambda_1} \underbrace{d\lambda \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}}_{d\tau}$$

$$S = \int d\tau$$

...vistic particle

...aking S invariant under some symmetry is the best way we knew to ensure

$+$

free
p'ticle

coupling to electromagnetic fields

$$F_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \qquad A_\mu \frac{dx^\mu}{d\lambda}$$

...on for a relativistic particle

... of motion , making S invariant under some symmetry is the ...

$$\ldots = -\underset{\text{const}}{m_0 c} \int_{\lambda_0}^{\lambda_1} \Big[\underbrace{d\lambda \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}}_{d\tau} + q A_\mu(x^\mu(\lambda)) \frac{dx^\mu}{d\lambda} \Big]$$

free ptcle

coupling to electromagnetic fields

$\to \lambda$

$$S = \int d\tau$$

$$F_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \qquad A_\mu \frac{dx^\mu}{d\lambda}$$

Lorentz force.

$$m\frac{d^2\underline{x}}{dt^2} = q\left(\underline{E} + \underline{v}\wedge\underline{B}\right)$$

er some symmetry is the best way we knew to ensure eq^{n}s of motion are invariant under that symm

$$\left.(\lambda)\right)\frac{dx^\mu}{d\lambda}\Bigg]$$

lectromagnetic fields

$$\mu\nu\,\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \qquad A_\mu\frac{dx^\mu}{d\lambda}$$

eq^{n}s of motion , making S invariant under some symmetry is

$$S = -m_0 c \int_{\lambda_0}^{\lambda_1} \underbrace{d\lambda \left[\sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \right.}_{d\tau} + \int d\lambda\, q A_\mu(x^\alpha(\lambda)) \frac{dx^\mu}{d\lambda} \Big]$$

const

$\Uparrow$ coupling to electromagnetic fields

$S = \int d\tau$ free ptcle

$$F_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \qquad A_\mu$$

λ

$$\sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = c\, d\tau$$

$d\lambda)$

$$\int d\tau \; m_0 \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau}(\delta x^\mu)$$

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\Big] + \int d\lambda\, q A_\mu(x^\mu(\lambda))\frac{dx^\mu(\lambda)}{d\lambda} \quad x^\mu + \delta x^\mu$$

free ptcle

coupling to electromagnetic fields

$$F_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \qquad A_\mu\frac{dx^\mu}{d\lambda}$$

$x \sim \lambda$

$F \sim \lambda^{-2}$

$A \sim \frac{1}{\lambda}$

$$\frac{\eta_{\mu\nu}\, dx^\mu\, d\delta x^\mu}{\sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu}}$$

$$\sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu} = c\, d\tau$$

$$m_0\, \eta_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{d}{d\tau}(\delta x^\mu) + q\, \overbrace{\partial_\nu A_\mu\, \delta x^\nu}^{\delta(A_\mu(x))}\frac{dx^\mu}{d\tau}\, d\tau + q A_\mu \frac{d}{d\tau}\delta x^\mu\, d\tau$$

$S = \int d\tau$ free ptcle

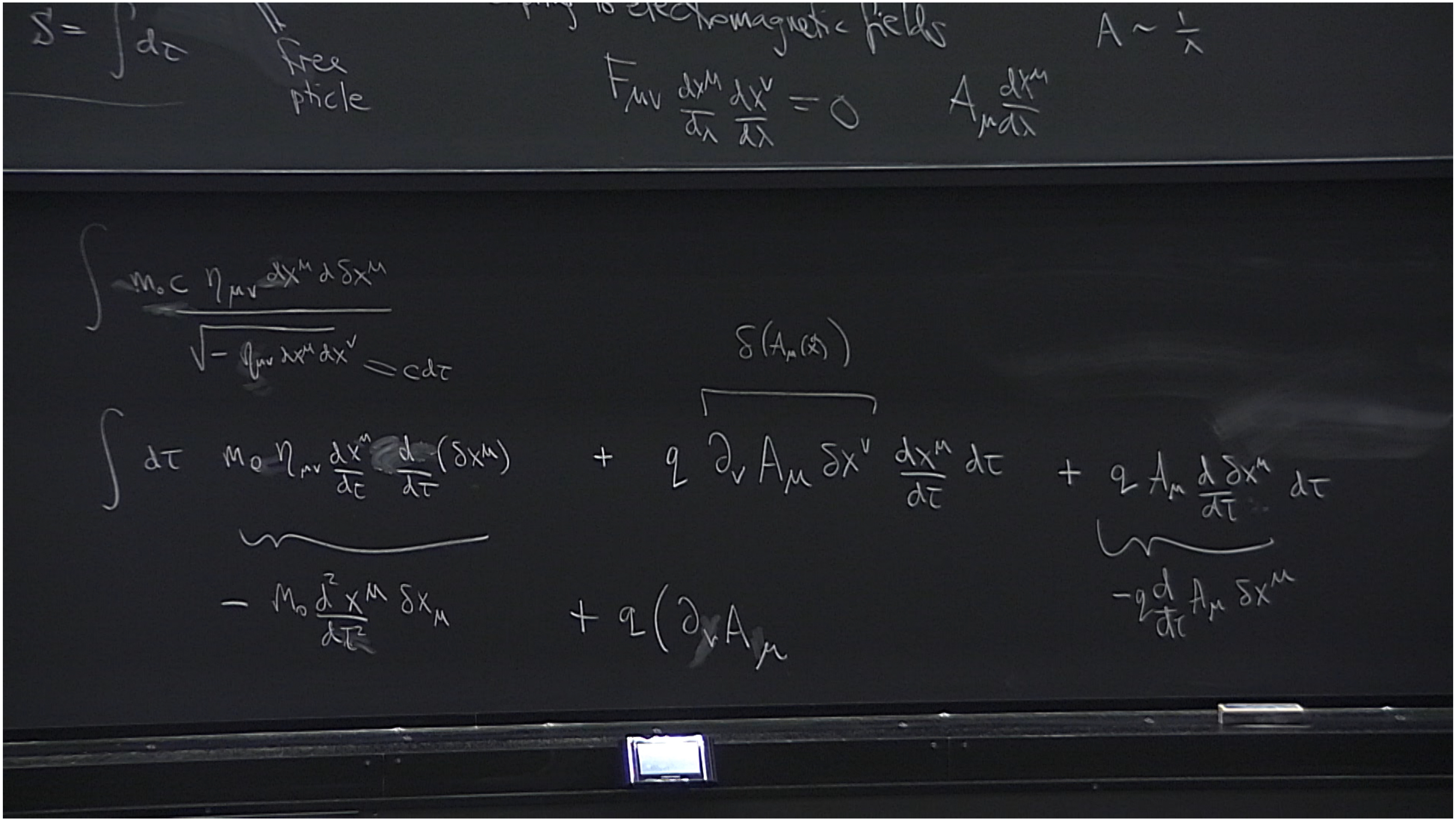
...ing to electromagnetic fields

$A \sim \frac{1}{\lambda}$

$F_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \qquad A_\mu \frac{dx^\mu}{d\lambda}$

$$\int \frac{m_0 c\, \eta_{\mu\nu}\, dx^\mu\, d\delta x^\mu}{\sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu} = c\, d\tau}$$

$$\int d\tau \quad \underbrace{m_0 \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau}(\delta x^\mu)}_{-\, m_0 \frac{d^2 x^\mu}{d\tau^2} \delta x_\mu} + q\, \overbrace{\partial_\nu A_\mu \delta x^\nu}^{\delta(A_\mu(x))} \frac{dx^\mu}{d\tau} d\tau + \underbrace{q A_\mu \frac{d\delta x^\mu}{d\tau}}\, d\tau$$


$S = \int d\tau$
free ptcle
fields
$A \sim \frac{1}{\lambda}$
$F_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$
$A_\mu \frac{dx^\mu}{d\lambda}$
$\int \frac{m_0 c\, \eta_{\mu\nu}\, dx^\mu\, d\delta x^\mu}{\sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}}$
$= c\,d\tau$
$\delta(A_\mu(x))$
$\int d\tau \quad m_0 \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau}(\delta x^\mu) + q\, \partial_\nu A_\mu \delta x^\nu \frac{dx^\mu}{d\tau} d\tau + q A_\mu \frac{d\delta x^\mu}{d\tau} d\tau$
$- m_0 \frac{d^2 x^\mu}{d\tau^2} \delta x_\mu$
$+ q(\partial_\nu A_\mu$
$-q \frac{d}{d\tau} A_\mu \delta x^\mu$

$$+ \; q \, \overbrace{\partial_\nu A_\mu \, \delta x^\nu}^{\delta(A_\mu(x))} \frac{dx^\mu}{d\tau} d\tau \; + \; \underbrace{q A_\mu \frac{d\,\delta x^\mu}{d\tau}}_{-q \frac{d}{d\tau} A_\mu \, \delta x^\mu} d\tau$$

$$-q \frac{d}{d\tau} A_\mu \, \delta x^\mu \longrightarrow \partial_\nu A_\mu \frac{dx^\nu}{d\tau} \delta x^\mu$$

$$+ \; q \left(\partial_\nu A_\mu \right.$$

$$\delta(A_\mu(x))$$

$$+ \; q\,\overbrace{\partial_\nu A_\mu\,\delta x^\nu}\,\frac{dx^\mu}{d\tau}\,d\tau \quad + \; \underbrace{q\,A_\mu\,\frac{d\,\delta x^\mu}{d\tau}}_{-q\frac{d}{d\tau}A_\mu\,\delta x^\mu}\,d\tau$$

$$-q\frac{d}{d\tau}A_\mu\,\delta x^\mu \longrightarrow \partial_\nu A_\mu \frac{dx^\nu}{d\tau}\,\delta x^\mu$$

$$+ \; q\left(\partial_\nu A_\mu - \partial_\mu A_\nu\right)\delta x^\nu \frac{dx^\mu}{d\tau}$$

$$= \int \frac{m_0 c\, \eta_{\mu\nu}\, dx^\mu\, d\delta x^\mu}{\sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu}} \qquad \sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu} = c\,d\tau$$

$$\int d\tau \;\; m_0\, \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau}(\delta x^\mu) \;\; + \;\; q\, \overbrace{\partial_\nu A_\mu\, \delta x^\nu}^{\delta(A_\mu(x))} \frac{dx^\mu}{d\tau}\, d\tau$$

$$- m_0 \frac{d^2 x_\nu}{d\tau^2}\, \delta x^\nu \qquad + q\left(\partial_\nu A_\mu - \partial_\mu A_\nu\right) \delta x^\nu \frac{d}{d}$$

$$c^4 \rightarrow E \approx \underbrace{m_0 c^2}_{\text{Same old}} + \frac{1}{2} \underline{p}^2 / m_0^2 + \cdots$$

$$\to \frac{1}{2} m_0 \underline{v}^2 .$$

$$\underline{p} = m_0 \underline{u} = m_0 \frac{dx}{dt} \cdot \frac{dt}{d\tau}$$

$\underline{v} << c$

$$-m_0 \frac{d^2 x_\nu}{d\tau^2} + q\left(\left(\partial_\nu A_\mu - \partial_\mu A_\nu \right) \frac{dx^\mu}{d\tau} \right) = 0 \quad \forall \delta x^\nu$$

$$\Rightarrow \boxed{m_0 \frac{d^2 x_\nu}{d\tau^2} = q F_{\nu\mu} \frac{dx^\mu}{d\tau}}$$

Lorentz Force Law