Title: PSI 17/18 - Relativity - Lecture 3

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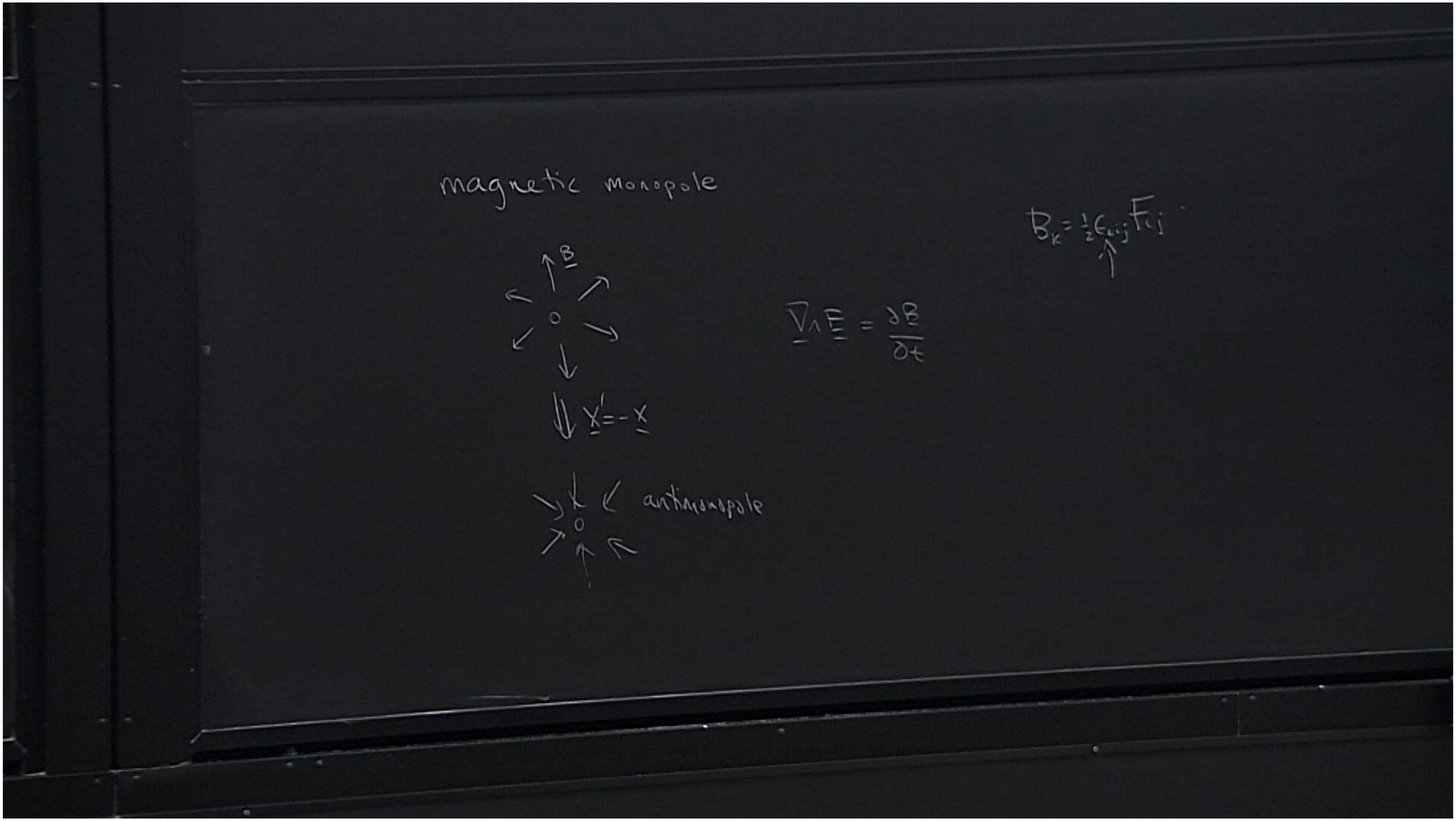
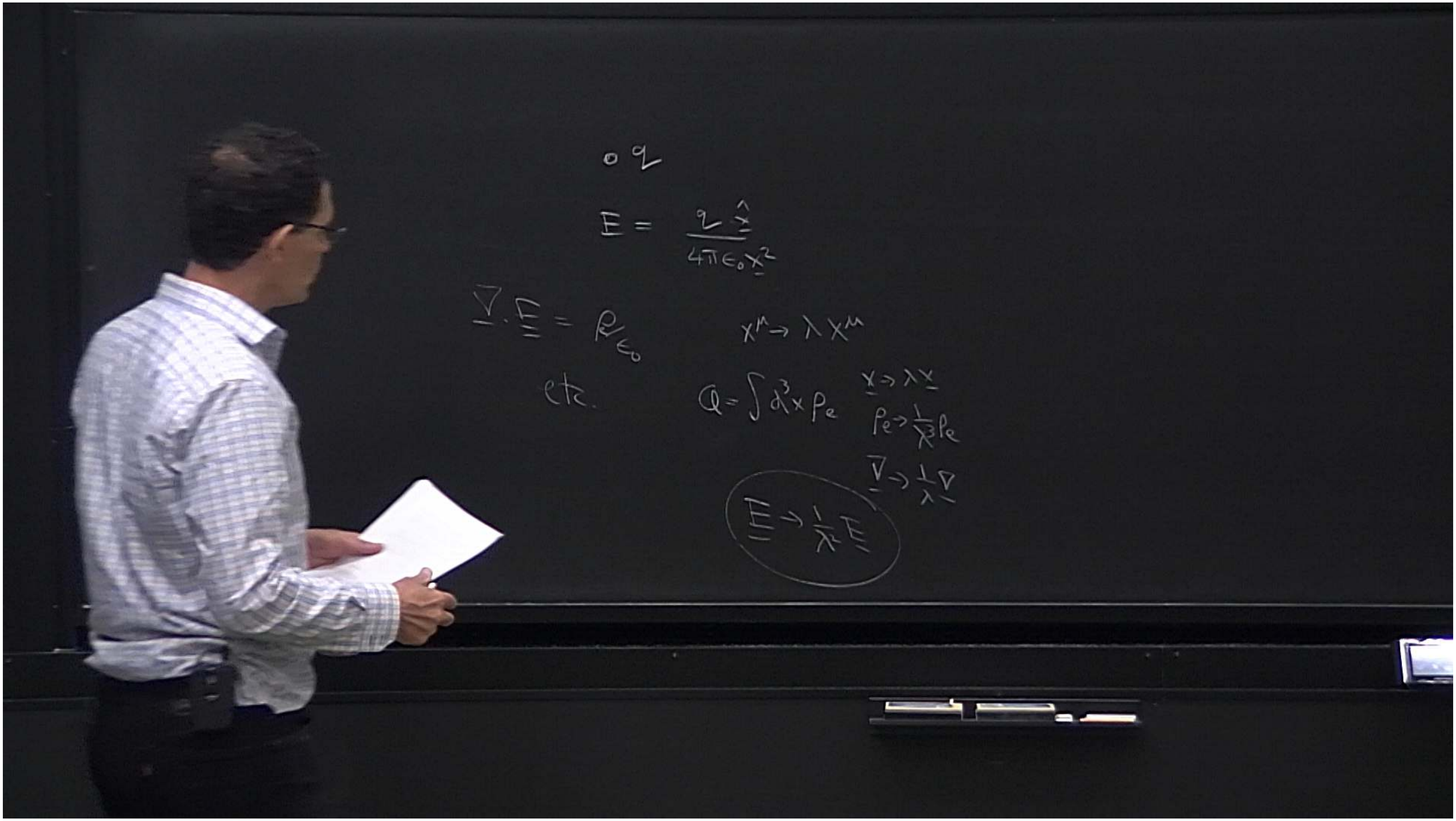
URL: http://pirsa.org/17090022

Abstract:


magnetic monopole
B
x' = -x
antimonopole
$\nabla \wedge E = \frac{\partial B}{\partial t}$
$B_k = \frac{1}{2}\epsilon_{kij}F_{ij}$


q
$\underline{E} = \dfrac{q\,\hat{\underline{x}}}{4\pi\epsilon_0 \underline{x}^2}$
$\underline{\nabla}\cdot\underline{E} = \rho/\epsilon_0$
etc.
$x^\mu \to \lambda x^\mu$
$Q = \int d^3x\, \rho_e$
$\underline{x} \to \lambda \underline{x}$
$\rho_e \to \frac{1}{\lambda^3}\rho_e$
$\underline{\nabla} \to \frac{1}{\lambda}\underline{\nabla}$
$\underline{E} \to \frac{1}{\lambda^2}\underline{E}$

$$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\eta^{\alpha\beta} = \eta^{\mu\nu}$$

$$x'^{\mu} = \overset{6}{\Lambda^{\mu}{}_{\nu}} x^{\nu} + \overset{4}{a^{\nu}}$$

$\circ\ q$

$$\underline{E} = \frac{q}{4\pi\epsilon_0 x^2}\hat{\underline{x}}$$

$$\underline{\nabla}\cdot\underline{E} = \rho/\epsilon_0$$

etc.

$$\overset{1}{x^{\mu} \to \lambda x^{\mu}}$$

$$Q = \int d^3x\, \rho_e \qquad \underline{x} \to \lambda \underline{x}, \quad \rho_e \to \frac{1}{\lambda^3}\rho_e, \quad \underline{\nabla} \to \frac{1}{\lambda}\underline{\nabla}$$

$$\underline{E} \to \frac{1}{\lambda^2}\underline{E}$$

$\circ\ q$

$$\underline{E} = \frac{q}{4\pi\epsilon_0 \underline{x}^2}\hat{\underline{x}}$$

$$\underline{\nabla}\cdot\underline{E} = \rho/\epsilon_0$$

etc.

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta} = \eta^{\mu\nu}$$

1

$$x^\mu \to \lambda x^\mu$$

$$Q = \int d^3x\, \rho_e$$

$$\underline{x} \to \lambda \underline{x}$$

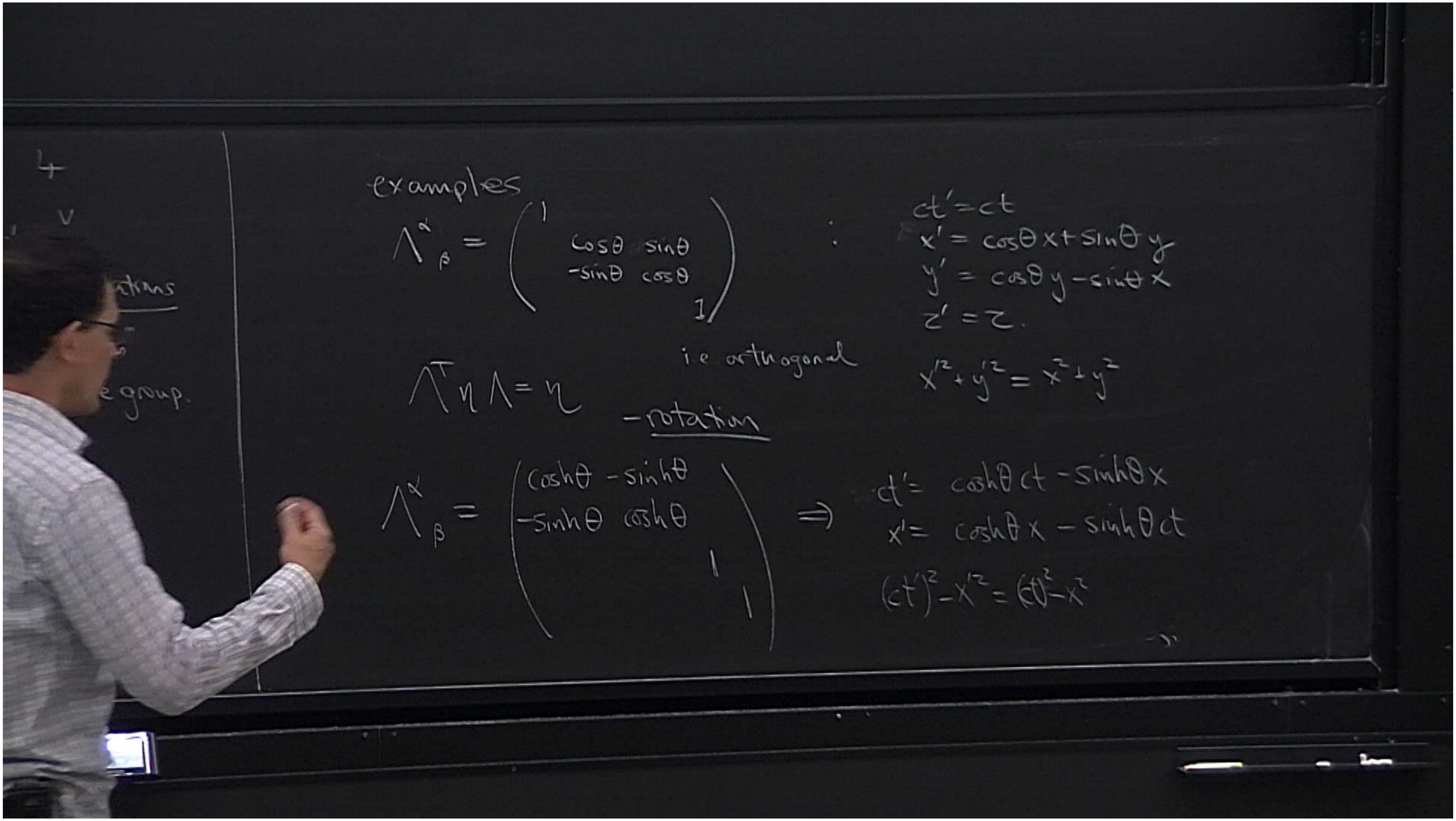
$$\rho_e \to \frac{1}{\lambda^3}\rho_e$$

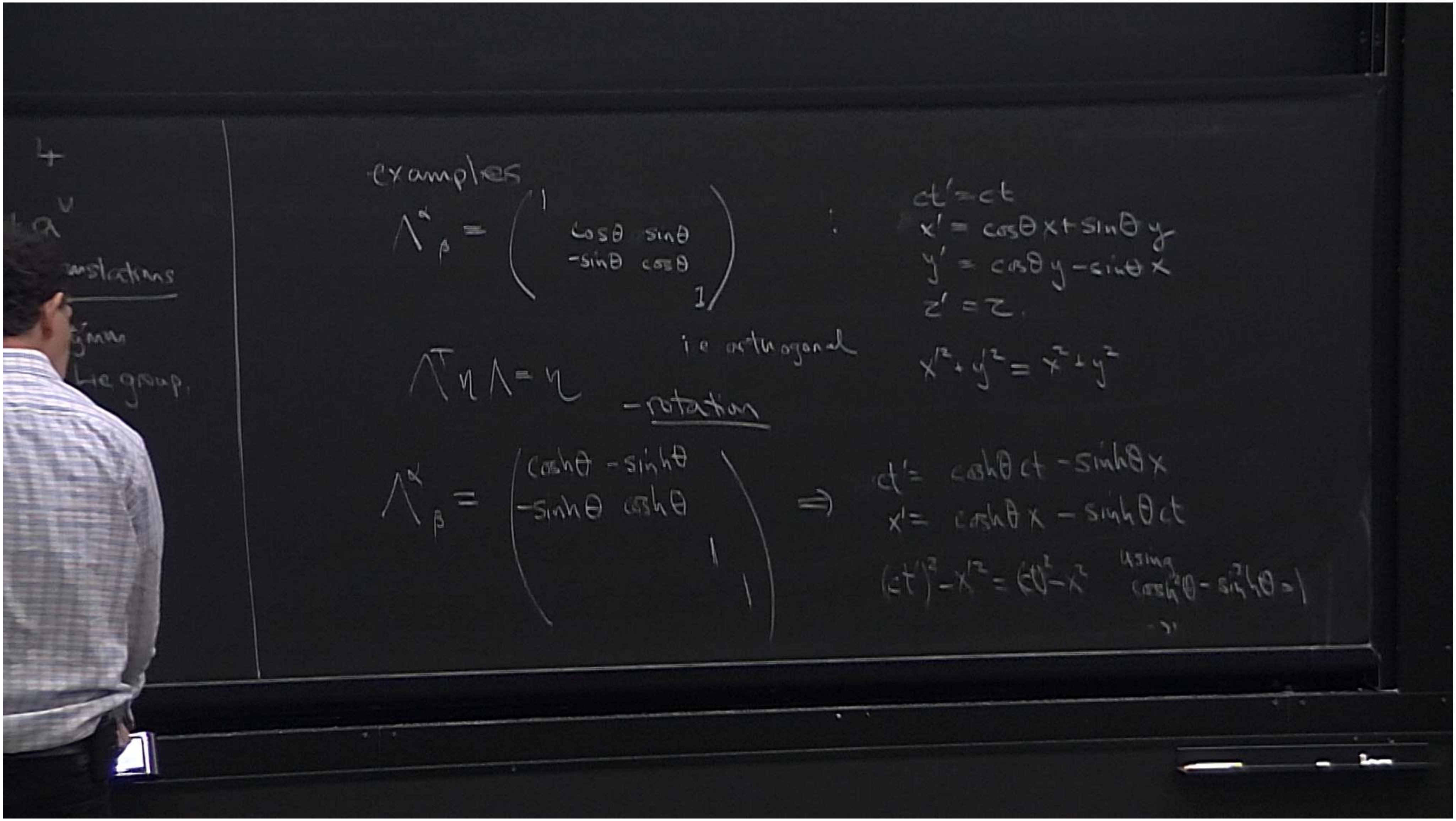
$$\underline{\nabla} \to \frac{1}{\lambda}\underline{\nabla}$$

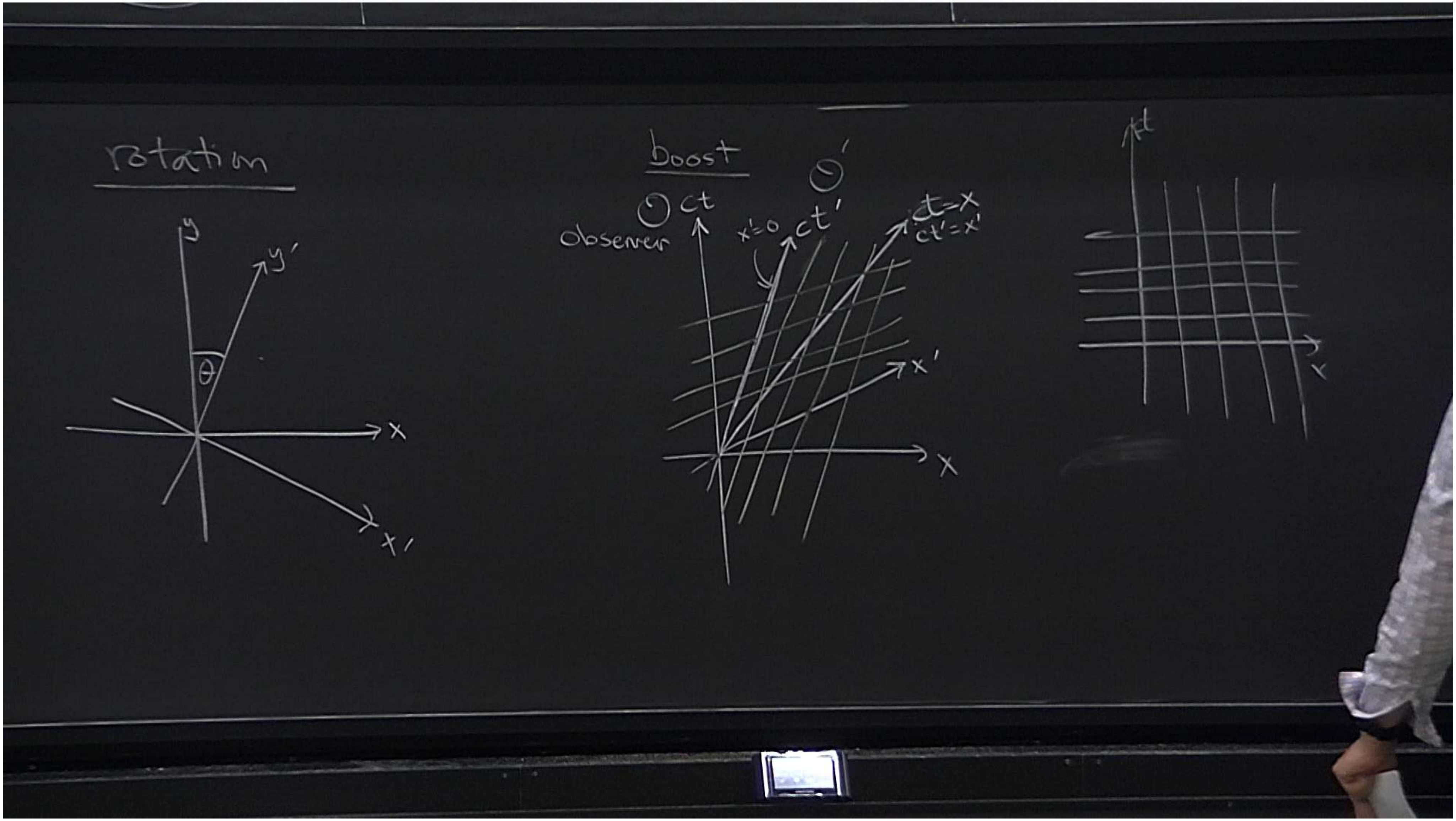
$$\underline{E} \to \frac{1}{\lambda^2}\underline{E}$$

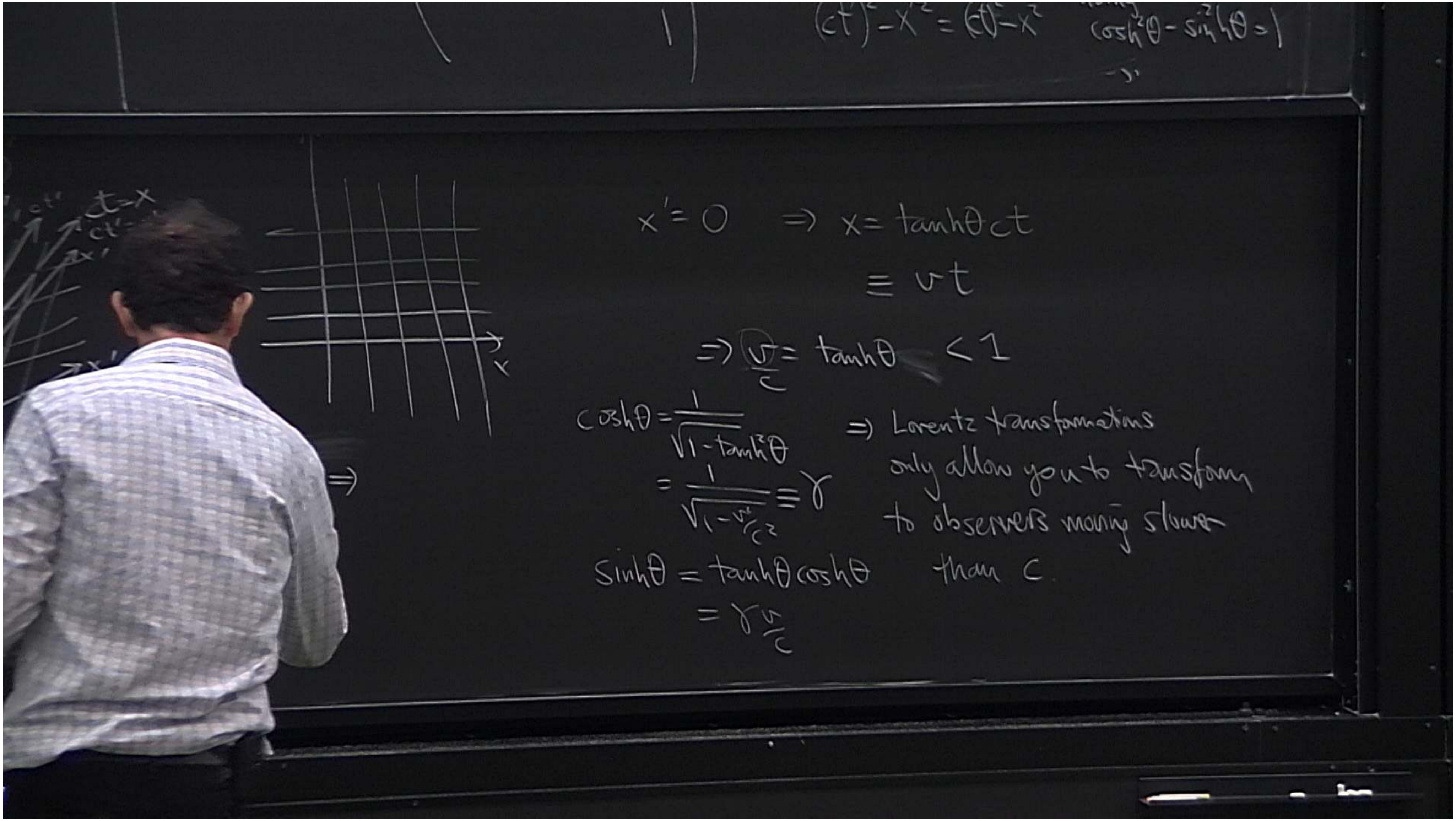
$$x'^\mu = \underbrace{\underbrace{\overset{6}{\Lambda^\mu{}_\nu x^\nu}}_{\text{Lorentz}} + \underbrace{\overset{4}{a^\nu}}_{\text{translations}}}_{\text{Poincaré symm}}$$

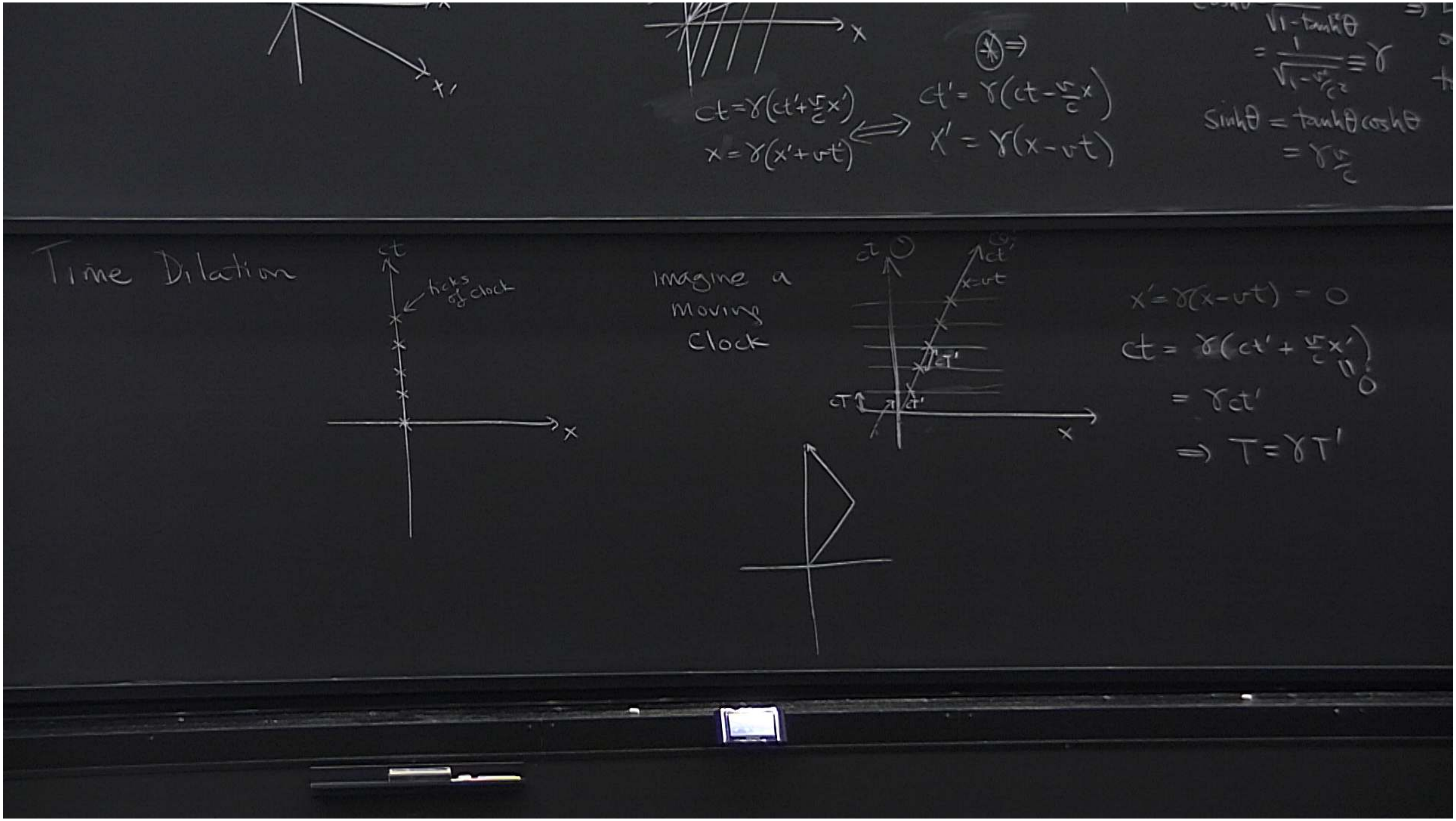
10-dim'l Lie group.

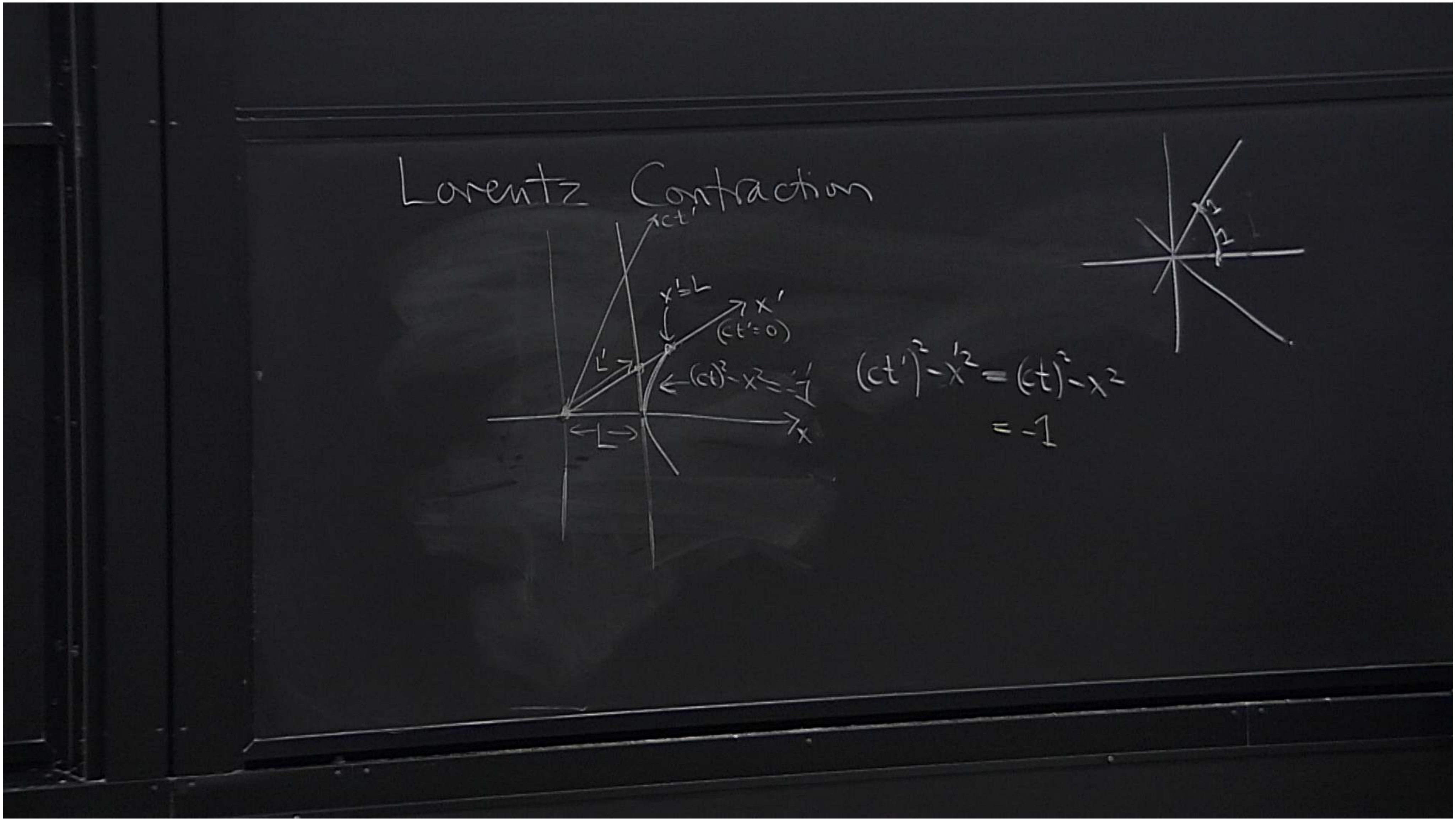

examples
Λ^α_β = (1, cosθ sinθ, −sinθ cosθ, 1)
ct′ = ct
x′ = cosθ x + sinθ y
y′ = cosθ y − sinθ x
z′ = z.
x′² + y′² = x² + y²
Λ^T η Λ = η
i.e. orthogonal
− rotation
Λ^α_β = (coshθ −sinhθ, −sinhθ coshθ, 1, 1)
⇒
ct′ = coshθ ct − sinhθ x
x′ = coshθ x − sinhθ ct
(ct′)² − x′² = (ct)² − x²
group.

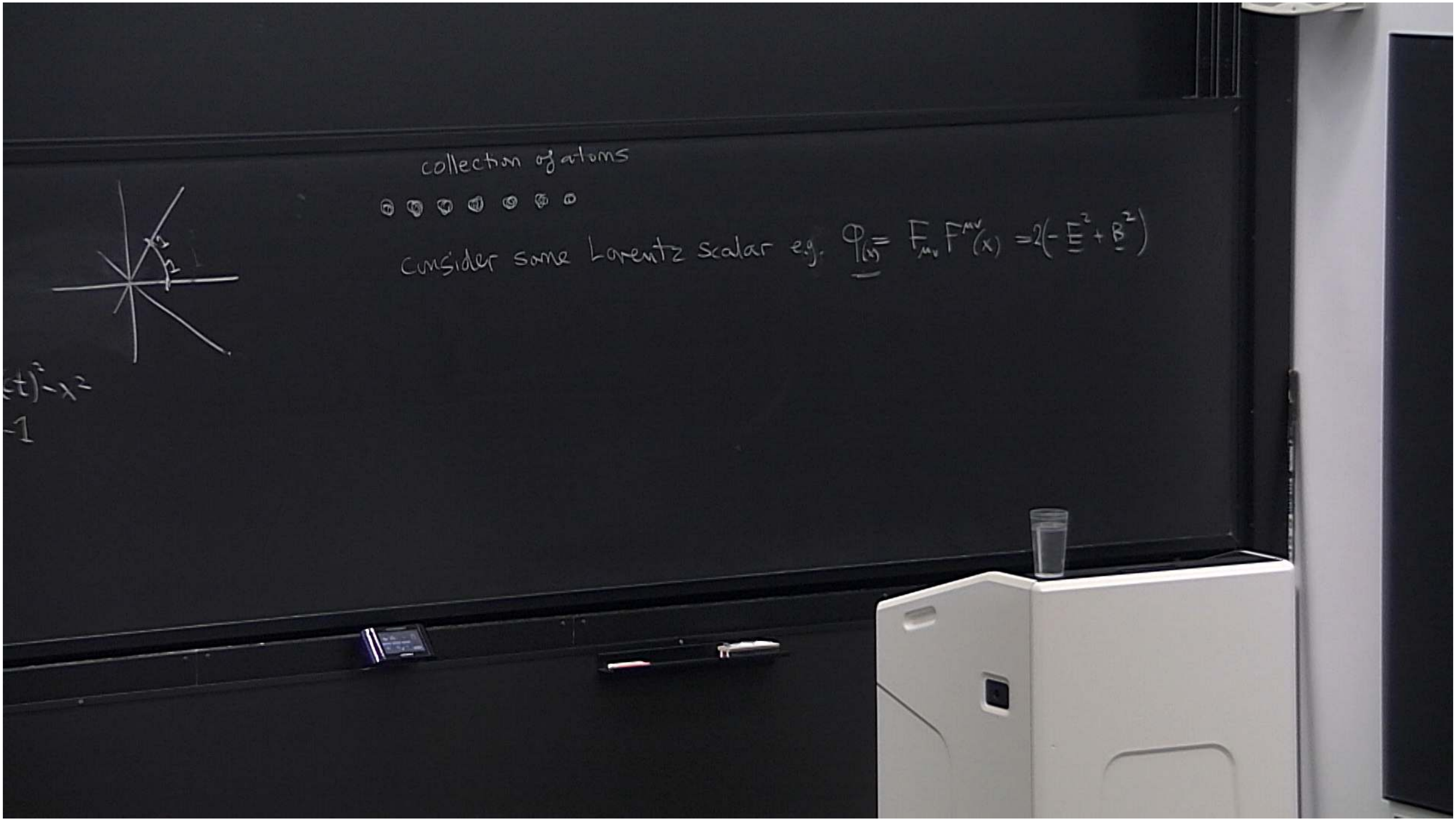

examples
$\Lambda^\alpha{}_\beta = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}$
$ct' = ct$
$x' = \cos\theta\, x + \sin\theta\, y$
$y' = \cos\theta\, y - \sin\theta\, x$
$z' = z$.
i.e. orthogonal
$x'^2 + y'^2 = x^2 + y^2$
$\Lambda^T \eta \Lambda = \eta$
– rotation
$\Lambda^\alpha{}_\beta = \begin{pmatrix} \cosh\theta & -\sinh\theta & & \\ -\sinh\theta & \cosh\theta & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Rightarrow$
$ct' = \cosh\theta\, ct - \sinh\theta\, x$
$x' = \cosh\theta\, x - \sinh\theta\, ct$
$(ct')^2 - x'^2 = (ct)^2 - x^2$
using $\cosh^2\theta - \sinh^2\theta = 1$
translations
Lie group.

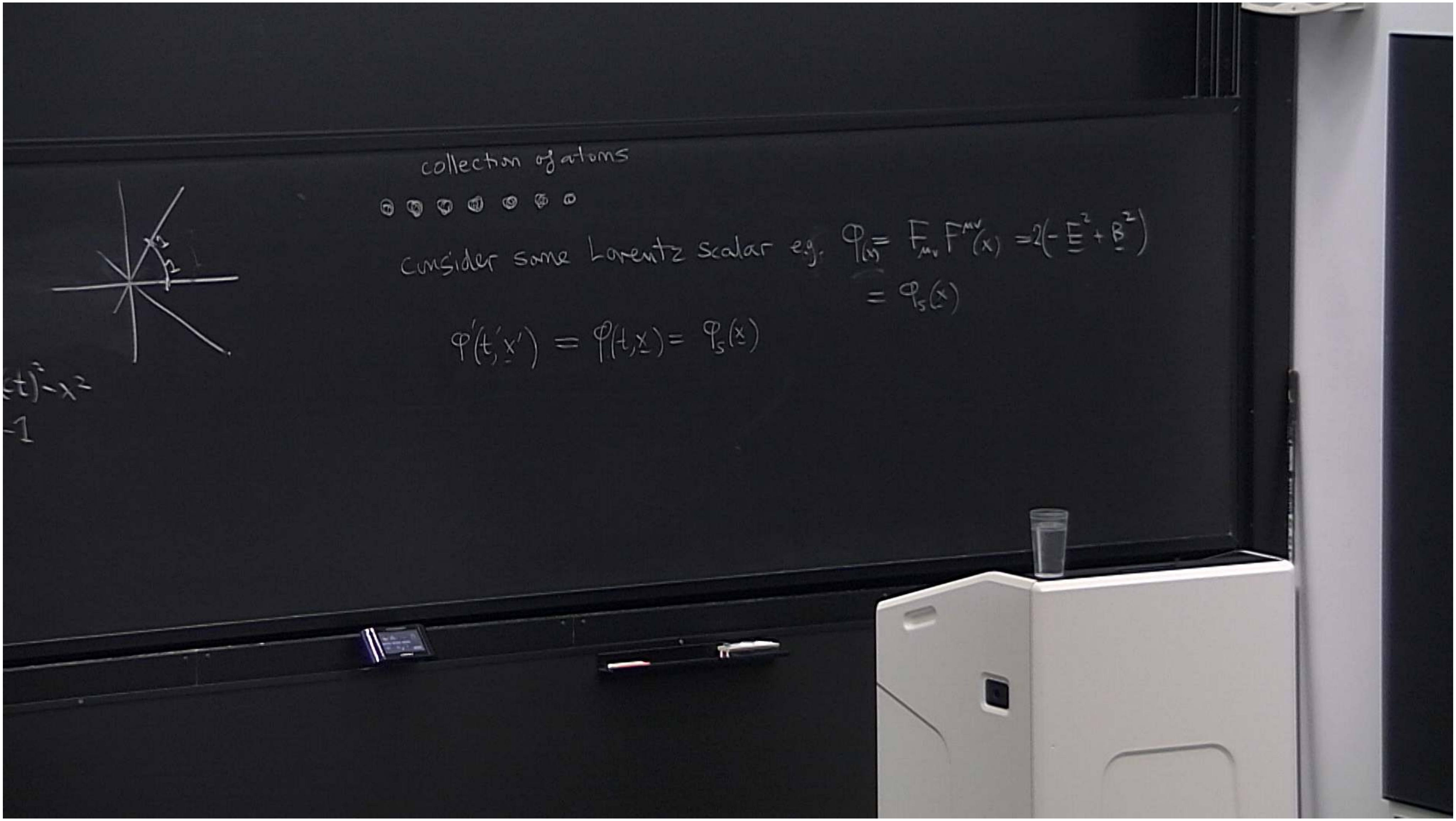

rotation
y
y'
θ
x
x'
boost
O'
O ct
observer
x'=0
ct'
ct=x
ct'=x'
x'
x
ct
x


$(ct')^2 - x'^2 = (ct)^2 - x^2$
$\cosh^2\theta - \sinh^2\theta = 1$
$x' = 0 \Rightarrow x = \tanh\theta \, ct$
$\equiv vt$
$\Rightarrow \frac{v}{c} = \tanh\theta < 1$
$\cosh\theta = \frac{1}{\sqrt{1-\tanh^2\theta}} = \frac{1}{\sqrt{1-v^2/c^2}} \equiv \gamma$
$\Rightarrow$ Lorentz transformations only allow you to transform to observers moving slower than c.
$\sinh\theta = \tanh\theta \cosh\theta = \gamma \frac{v}{c}$


ct=γ(ct'+v/c x')
x=γ(x'+vt')
ct'=γ(ct−v/c x)
x'=γ(x−vt)
=1/√(1−v²/c²)≡γ
sinhθ = tanhθ coshθ
=γ v/c
Time Dilation
ct
ticks of clock
x
Imagine a moving clock
ct
ct'
x=vt
cT'
cT
x'=γ(x−vt)=0
ct= γ(ct'+v/c x')
=γct'
⇒ T=γT'


Lorentz Contraction
ct'
x' = L
x'
(ct' = 0)
L'
(ct)² - x² = -1
L
x
(ct')² - x'² = (ct)² - x²
= -1


collection of atoms
consider some Lorentz scalar e.g. $\varphi_{(x)} = F_{\mu\nu} F^{\mu\nu}(x) = 2(-\underline{E}^2 + \underline{B}^2)$


collection of atoms
consider some Lorentz scalar e.g. φ(x) = F_μν F^μν(x) = 2(−E² + B²)
= φ_s(x)
φ′(t′, x′) = φ(t, x) = φ_s(x)
(ct)² − x²
−1

collection of atoms

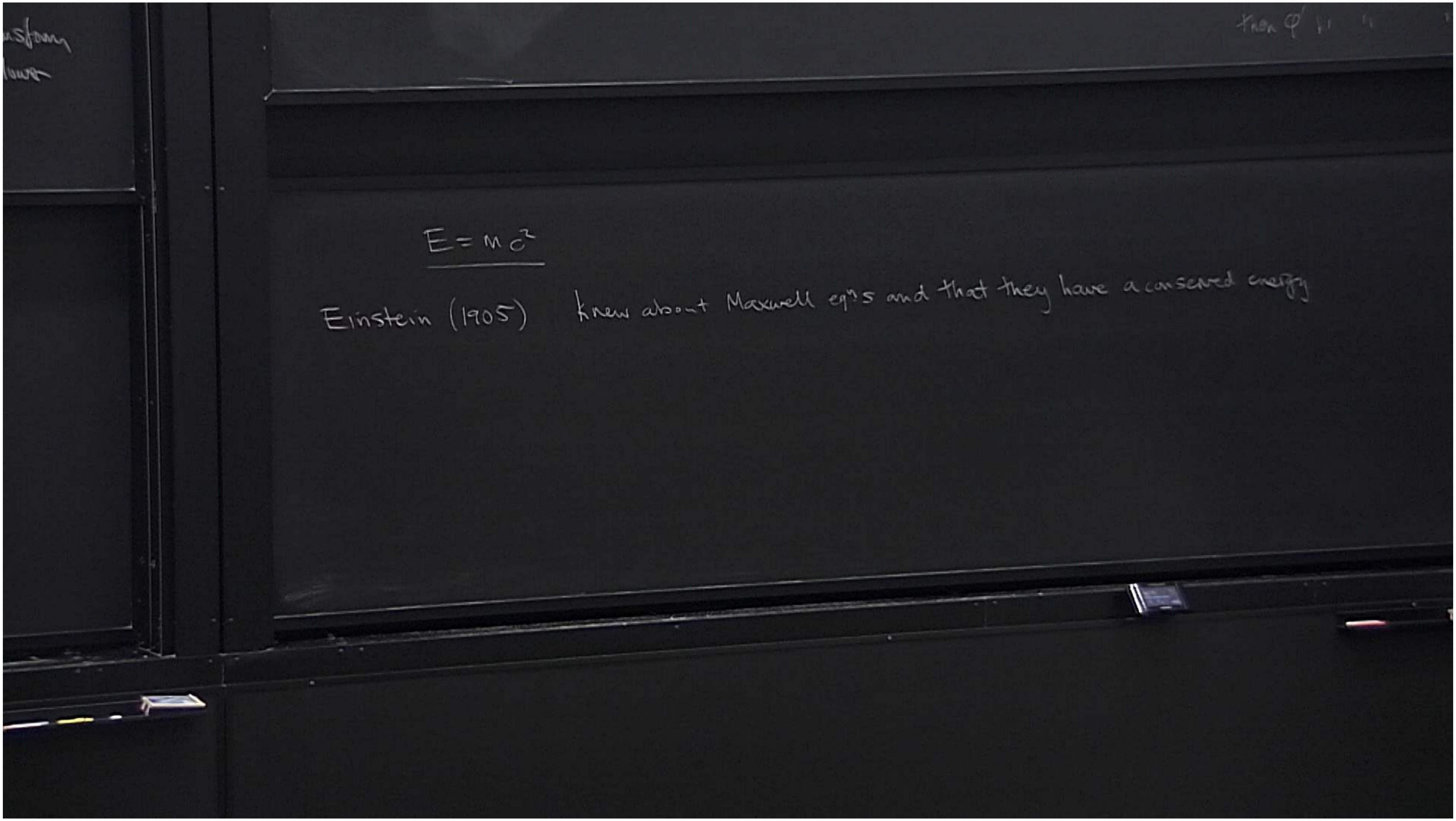
consider some Lorentz scalar e.g. $\varphi(x) = F_{\mu\nu}F^{\mu\nu}(x) = 2(-\underline{E}^2 + \underline{B}^2)$

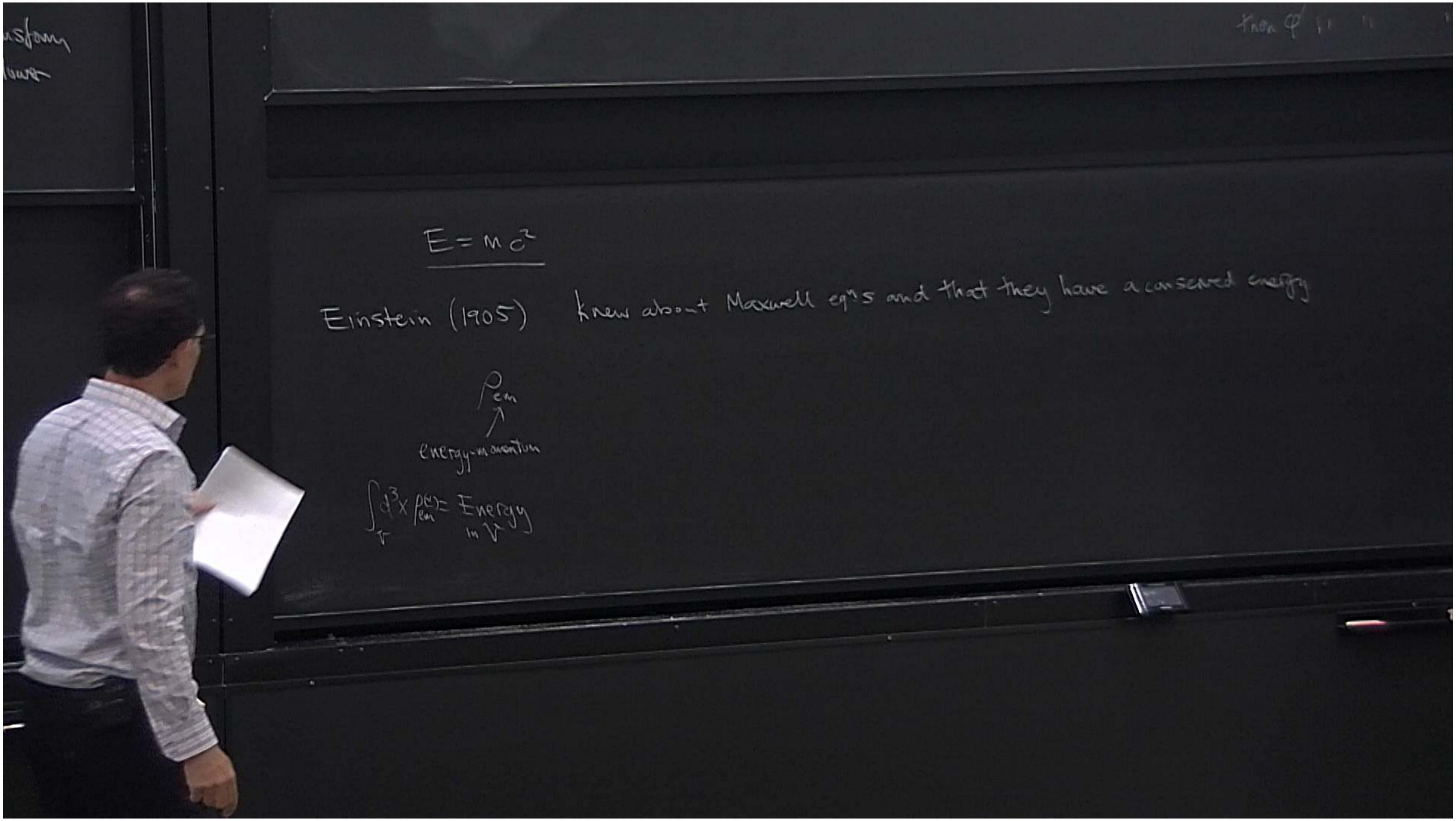
$$= \varphi_s(\underline{x})$$

$$\varphi'(t',\underline{x}') = \varphi(t,\underline{x}) = \varphi_s(\underline{x})$$
$$= \varphi_s(x, y, z)$$
$$= \varphi_s(\gamma(x'+vt'), y', z')$$

If φ_s has peaks separated by $\Delta x = L$

then φ' " " " " $\Delta x' = L/\gamma$


E = mc²
Einstein (1905) knew about Maxwell eqns and that they have a conserved energy


E = mc²
Einstein (1905) knew about Maxwell eqns and that they have a conserved energy
ρ_em
energy-momentum
∫_V d³x ρ_em = Energy in V

$E = mc^2$

Einstein (1905) knew about Maxwell eq^{n}s and that they have a conserved energy

$$\rho_{em} = \frac{1}{2}\epsilon_0 \underline{E}^2 + \frac{\underline{B}^2}{2\mu_0} \quad , \quad \text{flux } \underline{J}_{em} = \frac{1}{\mu_0}(\underline{E} \wedge \underline{B}) \quad \text{(Poynting)} \Rightarrow \frac{d\,\text{Energy}}{dt} = -\int_S \underline{J}_{em} \cdot d\underline{S}$$

energy-momentum

$$\int_V d^3x\, \rho_{em}^{(v)} = \underset{\text{in } V}{\text{Energy}}$$

$\Rightarrow$ 2-index symmetric tensor, energy-momentum tensor $T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\alpha\mu}F_\alpha{}^\nu - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)$:

...gy-momentum

$\Rightarrow$ 2-index symmetric tensor, energy-momentum tensor $T^{\mu\nu}_{em} = \frac{1}{\mu_0}\left(F^{\alpha\mu} F_{\alpha}{}^{\nu} \right.$

Energy in V

$\partial_\mu F^{\alpha\mu}$

If $J_c^\mu = 0$, then M.Es $\Rightarrow$ $\partial_\mu T^{\mu\nu}_{em} = 0$

(using ①)

$$\partial_\mu T^{\mu\nu}_{em} = \frac{1}{\mu_0}\left(F^{\alpha\mu} \partial_\mu F_\alpha{}^\nu - \frac{1}{2} F^{\alpha\beta} \partial^\nu F_{\alpha\beta} \right)$$

Energy-momentum

T^{00}_{em} = Energy in V

$\Rightarrow$ 2-index symmetric tensor, energy-momentum tensor $T^{\mu\nu}_{em} = \frac{1}{\mu_0}\left(F^{\alpha\mu}F_{\alpha}{}^{\nu} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)$: $T^{\mu\nu} = T^{\nu\mu} = J$

$\partial_\mu F^{\alpha\mu} = 0$

If $J_c^\mu = 0$, then M.Es $\Rightarrow$ $\boxed{\partial_\mu T^{\mu\nu}_{em} = 0}$

(using ①)

$$\partial_\mu T^{\mu\nu}_{em} = \frac{1}{\mu_0}\left(\underbrace{F^{\alpha\mu}\partial_\mu F_{\alpha}{}^{\nu}} - \frac{1}{2}F^{\alpha\beta}\partial^\nu F_{\alpha\beta}\right)$$

$$\frac{1}{2}F^{\alpha\mu}\left(\partial_\mu F_\alpha{}^\nu - \partial_\alpha F_\mu{}^\nu\right)$$

$$= \frac{1}{2\mu_0}F^{\alpha\mu}\underbrace{\left(\partial_\mu F_\alpha{}^\nu + \partial_\alpha F^\nu{}_\mu + \partial^\nu F_{\mu\alpha}\right)}_{②} = 0$$

-index symmetric tensor, energy-momentum tensor $T^{\mu\nu}_{em} = \frac{1}{\mu_0}\left(F^{\mu\alpha} F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}\right)$ ... J_{em}

$$\partial_\mu F^{\alpha\mu} = 0$$

0, then M.E.s $\Rightarrow$ $\boxed{\partial_\mu T^{\mu\nu}_{em} = 0}$ $\Rightarrow$ $\partial_0 T^{00}_{em} = -\partial_i T^{i0}_{em}$ $\Rightarrow$ if $\text{Energy} \equiv \int d^3x\, T^{00}_{em}(t,\underline{x})$

$\partial_0 = \frac{1}{c}\partial_t$

then $\frac{d}{dt}\text{Energy} = \int \partial_t T^{00}(t,\underline{x})$

$= -c\int \partial_i T^{0i} d^3x$

$\overset{\text{div.thm.}}{=} -c\oint_S J^i dS^i$

$J^i = \frac{T^{0i}}{c}$.

(using ①)

$$= \frac{1}{\mu_0}\left(\underbrace{F^{\alpha\mu}\partial_\mu F_\alpha{}^\nu} - \frac{1}{2}F^{\alpha\beta}\partial^\nu F_{\alpha\beta}\right)$$

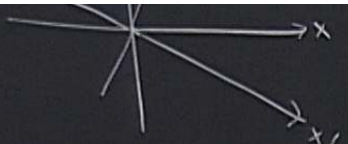
$$\frac{1}{2}F^{\alpha\mu}\left(\partial_\mu F_\alpha{}^\nu - \partial_\alpha F_\mu{}^\nu\right)$$

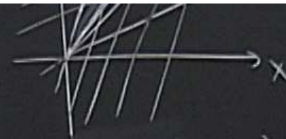
$$= \frac{1}{2\mu_0}F^{\alpha\mu}\left(\underbrace{\partial_\mu F_\alpha{}^\nu + \partial_\alpha F^\nu{}_\mu + \partial^\nu F_{\mu\alpha}}_{②}\right) = 0$$

and $\partial_0 T^{0i}_{em} = -\partial_j T^{ji}_{em}$

$p^i = \int d^3\underline{x}\, T^{0i}$

obeys a similar law.





$$ct = \gamma(ct' + \tfrac{v}{c}x') \qquad x = \gamma(x' + vt') \iff (*) \Rightarrow \quad ct' = \gamma(ct - \tfrac{v}{c}x) \qquad x' = \gamma(x - vt)$$

$$\cosh\theta = \frac{1}{\sqrt{1-\tanh^2\theta}} = \frac{1}{\sqrt{1-v^2/c^2}} \equiv \gamma$$

$$\sinh\theta = \tanh\theta\cosh\theta = \gamma\frac{v}{c}$$

$\Rightarrow$ Lorentz transformations only allow you to transform to observers moving slower than c.

These can be combined into P^μ (4-vector)

$$P^\mu = \frac{1}{6c}\int T^{\gamma\mu}\,\epsilon_{\mu\nu\alpha\beta}\,dx^\nu dx^\alpha dx^\beta$$

: Energy-Momentum 4-vector in EM field.

Consider plane wave $\underline{E} = (0, 0, f(x-ct))$, $\underline{B} = (0, -\frac{f(x-ct)}{c}, 0)$

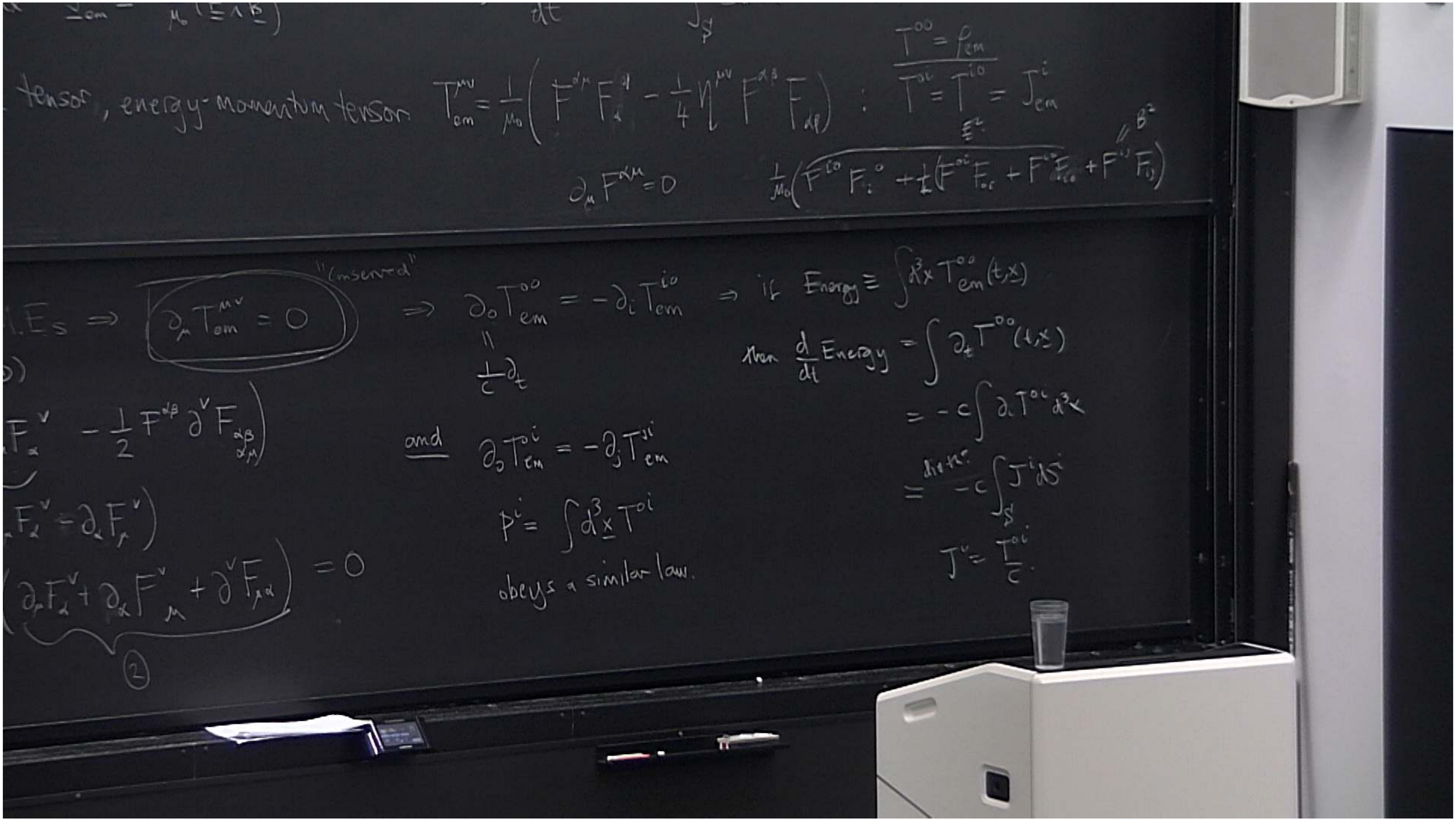
$$\Rightarrow T^{00} = \frac{f^2}{\mu_0 c^2}, \quad T^{0x} = \frac{f^2}{\mu_0 c^2}$$

$$\frac{1}{\mu_0 c^2} = \epsilon_0$$

$$\Rightarrow E = \text{Area}\int dx\, f^2 \epsilon_0, \quad P = \frac{\text{Area}}{c}\int dx\, f^2.$$

$\int dy\,dz$

$$\Rightarrow P^\gamma = (E/c, P, 0, 0) \quad \text{with } \underline{E = Pc}$$


tensor, energy-momentum tensor $T^{\mu\nu}_{em} = \frac{1}{\mu_0}\left(F^{\alpha\mu}F_\alpha{}^\nu - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)$
$T^{00} = \rho_{em}$
$T^{0i} = T^{i0} = J^i_{em}$
$\partial_\mu F^{\alpha\mu} = 0$
"conserved"
$\partial_\mu T^{\mu\nu}_{em} = 0$
$\Rightarrow \partial_0 T^{00}_{em} = -\partial_i T^{i0}_{em}$
$\frac{1}{c}\partial_t$
$\Rightarrow$ if Energy $\equiv \int d^3x\, T^{00}_{em}(t,x)$
then $\frac{d}{dt}$ Energy $= \int \partial_t T^{00}(t,x)$
$= -c\int \partial_i T^{0i} d^3x$
$= -c\int_S J^i dS^i$
$J^i = \frac{T^{0i}}{c}$
and $\partial_0 T^{0i}_{em} = -\partial_j T^{ji}_{em}$
$p^i = \int d^3x\, T^{0i}$
obeys a similar law.
$= 0$