

Title: PSI 17/18 - Relativity - Lecture 2

Date: Sep 06, 2017 09:00 AM

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Abstract:

Recall:

$$F_{\alpha\beta} = -F_{\beta\alpha} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & B_z & -B_y \\ (-) & & 0 & B_x \\ & & & 0 \end{pmatrix}$$

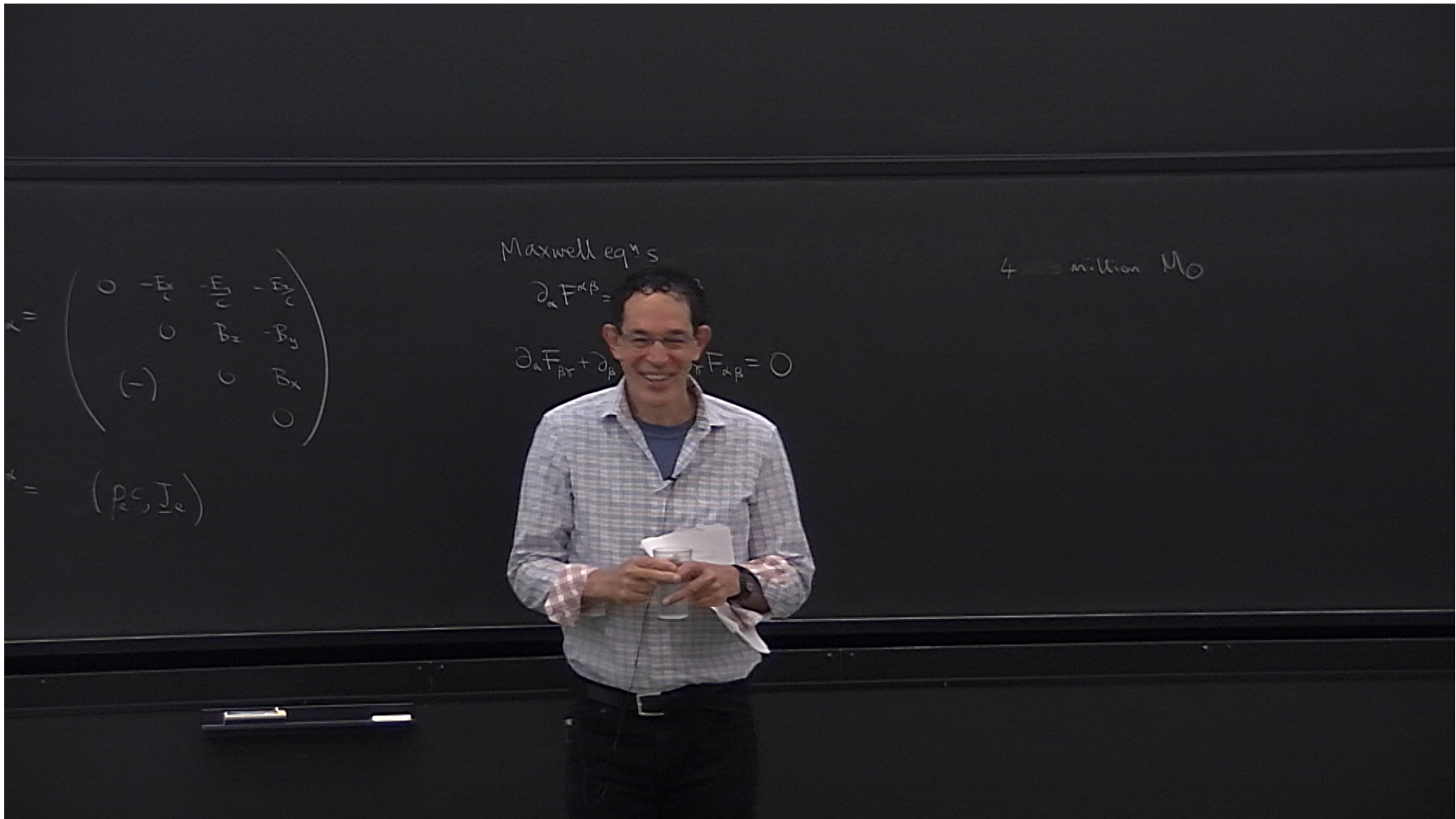
$$J^\alpha = (p_e^c, \underline{J}_e)$$

Maxwell eqⁿs

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$$

million



Maxwell eqⁿs

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$$

1) 4 million $M_\odot$

2) 200,000 $M_\odot$

$$\begin{pmatrix} -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ 0 & B_z & -B_y \\ (-) & 0 & B_x \\ & 0 & 0 \end{pmatrix}$$

Maxwell eqⁿs

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta \quad (1)$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \quad (2)$$

$$(2) \Rightarrow \partial_{[\alpha} F_{\beta\gamma]} = 0$$

"antisymmetrise"

$$A_{[2]} = \frac{1}{2!} (A_{2\beta} - A_{\beta 2})$$

Recall:

$$F_{\alpha\beta} = -F_{\beta\alpha} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ & 0 & B_z & -B_y \\ & (-) & 0 & B_x \\ & & & 0 \end{pmatrix}$$

"tensor"

$$J^\alpha = (p_e^c, \underline{J}_e)$$

4-current

Maxwell eq's

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta \quad (1)$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \quad (2)$$

$$(2) \Rightarrow \partial_{[\alpha} F_{\beta\gamma]} = 0$$

"antisymmetrise"

Maxwell eqⁿs

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta \quad (1)$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \quad (2)$$

$$(2) \Rightarrow \partial_{[\alpha} F_{\beta\gamma]} = 0$$

"antisymmetrise"

$$A_{[\alpha\beta]} = \frac{1}{2!} (A_{\alpha\beta} - A_{\beta\alpha})$$

$$A_{(\alpha\beta)} = \frac{1}{2!} (A_{\alpha\beta} + A_{\beta\alpha})$$

$$\left. \begin{array}{l} A_{[\alpha\beta]} \\ A_{(\alpha\beta)} \end{array} \right\} A_{\alpha\beta} = A_{[\alpha\beta]} + A_{(\alpha\beta)}$$

Maxwell eqⁿs

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta \quad (1)$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \quad (2)$$

$$\Rightarrow \partial_{[\alpha} F_{\beta\gamma]} = 0$$

"antisymmetrise"

$$\left. \begin{aligned} A_{[\alpha\beta]} &= \frac{1}{2!} (A_{\alpha\beta} - A_{\beta\alpha}) \\ A_{(\alpha\beta)} &= \frac{1}{2!} (A_{\alpha\beta} + A_{\beta\alpha}) \end{aligned} \right\} A_{\alpha\beta} = A_{[\alpha\beta]} + A_{(\alpha\beta)}$$

$$A_{[\alpha\beta\gamma]} = \frac{1}{3!} (A_{\alpha\beta\gamma} - A_{\alpha\gamma\beta} + A_{\beta\gamma\alpha} - A_{\beta\alpha\gamma} + A_{\gamma\alpha\beta} - A_{\gamma\beta\alpha})$$

$$A(\quad) = \quad + \quad + \quad + \quad + \quad +$$

$$\left. \begin{aligned} A_{[\alpha\beta]} &= \frac{1}{2!} (A_{\alpha\beta} - A_{\beta\alpha}) \\ A_{(\alpha\beta)} &= \frac{1}{2!} (A_{\alpha\beta} + A_{\beta\alpha}) \end{aligned} \right\} A_{\alpha\beta} = A_{[\alpha\beta]} + A_{(\alpha\beta)}$$

$$A_{[\alpha\beta\gamma]} = \frac{1}{3!} (A_{\alpha\beta\gamma} - A_{\alpha\gamma\beta} + A_{\beta\gamma\alpha} - A_{\beta\alpha\gamma} + A_{\gamma\alpha\beta} - A_{\gamma\beta\alpha})$$

$$A(\quad) = \quad + \quad + \quad + \quad + \quad +$$

$$\begin{pmatrix} -\frac{E_z}{c} \\ -B_x \\ B_y \\ 0 \end{pmatrix}$$

Maxwell eqⁿs

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta \quad (1)$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \quad (2)$$

$$(2) \Rightarrow \partial_{[\alpha} F_{\beta\gamma]} = 0$$

"antisymmetrise"

$$\begin{aligned} \partial_\alpha F_{\beta\gamma} &= \partial_\alpha F_{\gamma\beta} + \partial_\beta F_{\gamma\alpha} - \partial_\beta F_{\alpha\gamma} + \partial_\gamma F_{\alpha\beta} - \partial_\gamma F_{\beta\alpha} \\ &= 2(\partial_\alpha F_{\beta\gamma} + \text{cycles}) \end{aligned}$$

A_α
 A_α
 A_α
 A_α

4-current

$$\partial_\alpha F_{\beta\gamma} = \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} - \partial_\beta F_{\alpha\gamma} + \partial_\gamma F_{\alpha\beta} - \partial_\gamma F_{\beta\alpha}$$

$$= 2(\partial_\alpha F_{\beta\gamma} + \text{cycles})$$

$A(\quad) =$

then we can write (2) as $\frac{\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta}}{= 0} \quad (2')$

$$= \epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

$$a_\alpha b^\alpha = \eta_{\alpha\beta} a_\alpha b^\beta$$

$$= a^\alpha b_\alpha$$

$$\partial^\gamma (2) = \partial_\delta \partial_\alpha F_{\beta\gamma}$$

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha = \partial_\alpha A_\beta - \partial_\beta A_\alpha = \partial_\alpha A_\beta - \partial_\beta A_\alpha = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

$$= 2(\partial_\alpha A_\beta + \text{cycles})$$

then we can write (2) as $\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$ (2')

$$= \epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

$$a_\alpha b^\alpha = \eta^{\alpha\beta} a_\alpha b_\beta$$

$$= a^\beta b_\beta$$

$$\partial_\sigma F^{\alpha\sigma} = -\partial_\sigma F^{\sigma\alpha} = 0$$

$$\partial^\sigma (2) = \partial_\sigma \partial_\alpha F^{\sigma\alpha} + \underbrace{\partial_\beta \partial^\sigma F_{\sigma\alpha}}_{=0} + \underbrace{\partial^\sigma \partial_\alpha F_{\sigma\beta}}_{\text{wave operator}} = 0 \quad \text{if } T^\alpha = 0.$$

$$J^\mu = (p_e^c, \underline{J}_e)$$

4-current

$$\textcircled{2} \Rightarrow \partial_\alpha F_{\beta\gamma} = 0$$

"antisymmetric"

$$\begin{aligned} \partial_\alpha F_{\beta\gamma} &= \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} - \partial_\gamma F_{\alpha\beta} + \partial_\gamma F_{\alpha\beta} - \partial_\gamma F_{\alpha\beta} \\ &= 2(\partial_\alpha F_{\beta\gamma} + \text{cycles}) \end{aligned}$$

$$\begin{aligned} A_{\alpha\beta\gamma} &= \frac{1}{2!} (A_{\alpha\beta\gamma} + A_{\beta\alpha\gamma}) \\ A_{\alpha\beta\gamma} &= \frac{1}{3!} (A_{\alpha\beta\gamma} + A_{\beta\gamma\alpha} + A_{\gamma\alpha\beta} - \\ A_{\gamma\alpha\beta} &= \end{aligned}$$

then we can write $\textcircled{2}$ as $\boxed{\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0} \quad \textcircled{2'}$ i.e. 4-curl of $F_{\gamma\delta}$ is zero.

$$= \epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

$$\begin{aligned} a_\alpha b^\alpha &= \eta^{\alpha\beta} a_\alpha b_\beta \\ &= a^\beta b_\beta \end{aligned}$$

$$\partial_\gamma F^{\alpha\gamma} = -\partial_\gamma F^{\gamma\alpha} = 0$$

$$\partial^\gamma \textcircled{2} = \underbrace{\partial_\gamma \partial_\alpha F_\beta^\gamma}_{=0} + \underbrace{\partial_\beta \partial^\gamma F_{\gamma\alpha}}_{=0} + \underbrace{\partial^\gamma \partial_\gamma F_{\alpha\beta}}_{\text{wave operator}} = 0 \quad \text{if } J^\alpha = 0.$$



$$\begin{aligned} \nabla_\mu A^\mu(x) &= 0 \\ \phi &= \int_{x_0}^x A_\mu dx^\mu \end{aligned}$$

$$J^\mu = (p_e^c, \underline{J}_e)$$

4-current

$$\textcircled{2} \Rightarrow \partial_\mu F_{\mu\nu} = 0$$

"antisymmetric"

$$\begin{aligned} \partial_\mu F_{\mu\nu} &= \partial_\mu (F_{\mu\beta} + \frac{1}{2} F_{\mu\alpha} - \partial_\beta F_{\alpha\gamma} + \partial_\delta F_{\alpha\beta} - \partial_\delta F_{\alpha\gamma}) \\ &= 2(\partial_\mu F_{\mu\nu} + \text{cycles}) \end{aligned}$$

$$\begin{aligned} A_{(\mu\nu)} &= \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) \\ A_{[\mu\nu]} &= \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu} + A_{\nu\mu} - A_{\mu\nu}) \\ A_{(\quad)} &= \quad + \quad + \quad + \quad \end{aligned}$$

then we can write $\textcircled{2}$ as $\boxed{\epsilon^{\mu\nu\sigma\delta} \partial_\mu F_{\nu\sigma} = 0} \quad \textcircled{2'}$ ie 4-div of $F_{\nu\sigma}$ is zero.

$$= \epsilon^{\mu\nu\sigma\delta} \partial_\mu F_{\nu\sigma} = 0$$

$$\begin{aligned} a_\mu b^\mu &= \eta^{\mu\nu} a_\mu b_\nu \\ &= a^\mu b_\mu \end{aligned}$$

$$\partial_\nu F^{\mu\nu} = -\partial_\sigma F^{\sigma\mu} \stackrel{\textcircled{1}}{=} 0$$

$$\partial^\sigma \textcircled{2} = \partial_\nu \partial_\mu F_\nu^\sigma + \partial_\mu \partial_\nu F_\mu^\sigma + \underbrace{\partial_\sigma \partial_\nu F_{\mu\nu}}_{\text{same operator}} = 0 \quad \text{if } J^\mu = 0.$$

"0" "0" "0" if $J^\mu = 0$



$$\nabla_\mu A^\mu(x) = 0$$

$$\phi = \oint_C A_\mu dx^\mu$$

$$\oint_C A_\mu dx^\mu = 0$$

$$\nabla\phi = A$$

"antisymmetrise"

$$\partial_\alpha F_{\beta\gamma} = \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} - \partial_\beta F_{\alpha\gamma} + \partial_\gamma F_{\alpha\beta} - \partial_\gamma F_{\beta\alpha}$$

$$= 2(\partial_\alpha F_{\beta\gamma} + \text{cycles})$$

$$A_{(\alpha\beta\gamma)} = \frac{1}{3!}(A_{\alpha\beta\gamma} - A_{\alpha\gamma\beta} + A_{\beta\gamma\alpha} - A_{\beta\alpha\gamma} + A_{\gamma\alpha\beta} - A_{\gamma\beta\alpha})$$

$$A(\quad) = \quad + \quad + \quad + \quad + \quad +$$

we can write (2) as

$$\boxed{\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0} \quad (2')$$

$$= \epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

$$\partial_\gamma F^{\alpha\gamma} = -\partial_\gamma F^{\gamma\alpha} = 0 \quad (1)$$

$$\partial^\gamma (2) = \partial_\gamma \partial_\alpha F^{\alpha\gamma} + \underbrace{\partial_\beta \partial^\gamma F_{\gamma\alpha}}_{\text{wave operator}} + \underbrace{\partial^\gamma \partial_\gamma F_{\alpha\beta}}_{\text{if } J^\alpha = 0} = 0 \quad \text{if } J^\alpha = 0.$$

ie 4-curl of $F_{\gamma\delta}$ is zero.
 $\Rightarrow F$ is the gradient of something (Poincaré lemma)
 $F_{\gamma\delta} = \partial_\gamma A_\delta - \partial_\delta A_\gamma$

"antisymmetric"

$$\partial_\alpha F_{\beta\gamma} = \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} - \partial_\gamma F_{\alpha\beta} + \partial_\gamma F_{\alpha\beta} - \partial_\alpha F_{\beta\gamma} \\ = 2(\partial_\alpha F_{\beta\gamma} + \text{cycles})$$

$$A_{(\alpha\beta\gamma)} = \frac{1}{3!}(A_{\alpha\beta\gamma} - A_{\alpha\gamma\beta} + A_{\beta\gamma\alpha} - A_{\beta\alpha\gamma} + A_{\gamma\alpha\beta} - A_{\gamma\beta\alpha})$$

$$A(\) = \quad + \quad + \quad + \quad + \quad +$$

we can write (2) as

$$\boxed{\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0} \quad (2')$$

$$= \epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

$$\partial_\alpha F^{\alpha\gamma} = -\partial_\gamma$$

$$\partial^\alpha (2) = \partial_\alpha \partial_\alpha F^{\alpha\gamma} + \underbrace{\partial^\alpha \partial_\alpha F_{\alpha\beta}}_{\text{wave operator}} = 0 \quad \text{if } J^\alpha = 0.$$

i.e. 4-curl of $F_{\alpha\beta}$ is zero.
 $\Rightarrow F$ is the "gradient" of something (Poincaré lemma)

$$\underline{F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha}$$

$$A^\star = \left(\frac{\phi}{c}, \underline{A} \right)$$

$$B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} = \frac{1}{2} \epsilon_{ijk} (\partial_j A_k - \partial_k A_j) = \epsilon_{ijk} \partial_j A_k$$

$$\Rightarrow \underline{B} = \nabla \wedge \underline{A}$$

"antisymmetrise"

$$\partial_\alpha F_{\beta\gamma} = \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} - \partial_\beta F_{\alpha\gamma} + \partial_\gamma F_{\alpha\beta} - \partial_\gamma F_{\beta\alpha}$$

$$= 2(\partial_\alpha F_{\beta\gamma} + \text{cycles})$$

$$A_{(\alpha\beta\gamma)} = \frac{1}{3!}(A_{\alpha\beta\gamma} - A_{\alpha\gamma\beta} + A_{\beta\gamma\alpha} - A_{\beta\alpha\gamma} + A_{\gamma\alpha\beta} - A_{\gamma\beta\alpha})$$

$$A_{(\quad)} = \quad + \quad + \quad + \quad + \quad +$$

we can write (2) as $\boxed{\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0} \quad (2')$

$$\partial_\delta F^{\alpha\gamma} = -\partial_\gamma F^{\delta\alpha} \quad (1)$$

$$= \epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

$$\partial^\delta (2) = \partial_\delta \partial_\alpha F^{\alpha\gamma} + \partial_\gamma \partial_\alpha F^{\alpha\delta} = 0 \quad \text{if } J^\alpha = 0.$$

"wave operator"

ie 4-curl of $F_{\gamma\delta}$ is zero.
 $\Rightarrow F$ is the gradient of something (Poincaré lemma)

$$F_{\gamma\delta} = \partial_\gamma A_\delta - \partial_\delta A_\gamma$$

$$A^\star = \left(\frac{\phi}{c}, \underline{A} \right)$$

$$B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} = \frac{1}{2} \epsilon_{ijk} (\partial_j A_k - \partial_k A_j) = \epsilon_{ijk} \partial_j A_k$$

$$\underline{E} = F^{0i} \Rightarrow \boxed{\underline{B} = \nabla \wedge \underline{A}}$$

$$\underline{E} = -\nabla \phi - \frac{\partial \underline{A}}{\partial t}$$

$$\partial_\alpha F_{\beta\gamma} = \partial_\alpha F_{\gamma\beta} + \partial_\beta F_{\gamma\alpha} - \partial_\beta F_{\alpha\gamma} + \partial_\gamma F_{\alpha\beta} - \partial_\gamma F_{\beta\alpha}$$

$$= 2(\partial_\alpha F_{\beta\gamma} + \text{cycles})$$

$$= \epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

$$\partial_\delta F^{\beta\gamma} = -\partial_\delta F^{\gamma\beta} \stackrel{①}{=} 0$$

$$\partial^\sigma \textcircled{2} = \underbrace{\partial_\delta \partial_\alpha F_{\beta\gamma}^\sigma}_{\text{"0" if } T^\alpha=0.} + \underbrace{\partial_\beta \partial^\sigma F_{\gamma\alpha}}_{\text{"0" if } T^\alpha=0.} + \underbrace{\partial^\sigma \partial_\gamma F_{\alpha\beta}}_{\text{wave operator.}} = 0 \quad \text{if } T^\alpha = 0.$$

$$\oint \underline{A} \cdot d\underline{x}$$

$$\nabla \phi = \underline{A}$$

$$\frac{\underline{E}}{c} = \underline{F}^{0i} \Rightarrow \boxed{\begin{aligned} \underline{B} &= \nabla \wedge \underline{A} \\ \underline{E} &= -\nabla \phi - \frac{\partial \underline{A}}{\partial t} \end{aligned}}$$

note that $\underline{A}_\alpha$ is not unique $\cdot \quad \underline{A}_\mu \rightarrow \underline{A}'_\mu = \underline{A}_\mu + \partial_\mu f$

$$\Rightarrow F_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = F_{\mu\nu} + (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) f = F_{\mu\nu}$$

"gauge symmetry"

$$AB \text{ phase} = \oint \underline{A} \cdot d\underline{x} \quad \circlearrowleft$$

$$\oint \underline{A}' \cdot d\underline{x} = \oint (\underline{A} \cdot d\underline{x} + \overset{0}{\nabla \phi \cdot d\underline{x}})$$

$$\oint \underline{A} \cdot d\underline{x}$$

$$\nabla \phi = \underline{A}$$

$$\frac{\underline{E}'}{c} = \underline{F}^{0i} \Rightarrow \boxed{\begin{aligned} \underline{B} &= \nabla \wedge \underline{A} \\ \underline{E} &= -\nabla \phi - \frac{\partial \underline{A}}{\partial t} \end{aligned}}$$

note that $\underline{A}_\alpha$ is not unique . $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu f$

$$\Rightarrow F_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = F_{\mu\nu} + (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) f = F_{\mu\nu}$$

"gauge
symmetry"

$$AB \text{ phase} = \oint \underline{A} \cdot d\underline{x} \quad \circlearrowleft$$

$$\oint \underline{A}' \cdot d\underline{x} = \oint (\underline{A} \cdot d\underline{x} + \overset{0}{\nabla \phi \cdot d\underline{x}})$$

$$\frac{\underline{E}'}{c} = \underline{F}^{0i}$$

 $\Rightarrow$

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}$$

unique

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu f$$

$$\Rightarrow F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = F_{\mu\nu} + (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)f = F_{\mu\nu}$$

"gauge
symmetry"

$$AB \text{ phase} = \oint \underline{A} \cdot d\underline{x} \quad \oplus$$

$$\oint \underline{A}' \cdot d\underline{x} = \oint (\underline{A} \cdot d\underline{x} + \overset{0}{\underline{\nabla}}\phi \cdot d\underline{x})$$

- this will generalise, in
general relativity, to the
freedom to choose
coordinates on spacetime.

Lorentz transformations

In classical mechanics write e.g. $m\ddot{\underline{x}} = -\nabla V(\underline{x})$

Coordinates are invariant under $\underline{x} \rightarrow \underline{x}' = \underline{O}\underline{x}$, where $\underline{O}$ is a rotation matrix.

$\underline{O}$ satisfies

$$\underline{O}_{ik}^T \underline{O}_{kj} = \delta_{ij}$$

$$\underline{O}_{ki} \underline{O}_{lj} = \delta_{ij}$$

$\Rightarrow$ distances and angles are invariant

$$\underline{O}_{ij} x_j \underline{O}_{ik} x_k = \delta_{jk} x_j x_k = \underline{x}^2$$

Lorentz transformations

In classical mechanics, we write e.g. $m\ddot{\underline{x}} = -\nabla V(\underline{x})$

Coordinates are arbitrary, equation is invariant under $\underline{x} \rightarrow \underline{x}' = \underline{O}\underline{x}$, where $\underline{O}$ is a rotation matrix.

$\underline{O}$ satisfies $\underline{O}^T \underline{O} = \underline{1} \Rightarrow$ distances and angles are invariant

$$\underline{O}_{ik}^T \underline{O}_{lj} = \delta_{ij} \quad \underline{x}^2 = \underline{x}'^T \underline{x}' = \underline{O}_{ij}^T \underline{x}_j \underline{O}_{ik} \underline{x}_k = \delta_{jk} \underline{x}_j \underline{x}_k = \underline{x}^2$$

$$\underline{O}_{ki} \underline{O}_{lj} = \delta_{ij}$$

Lorentz transformations

In classical mechanics, we write e.g. $m\ddot{\underline{x}} = -\underline{\nabla} V(\underline{x})$

Coordinates are arbitrary, equation is invariant under $\underline{x} \rightarrow \underline{x}' = \underline{O}\underline{x}$, where $\underline{O}$ is a rotation matrix.

$\underline{O}$ satisfies $\underline{O}^T \underline{O} = \underline{1}$. $\Rightarrow$ distances and angles are invariant

$$\underline{O}_{ik}^T \underline{O}_{lj} = \delta_{il} \delta_{kj}, \quad \underline{x}^2 = \underline{x}'_i \underline{x}'_i = \underline{O}_{ij} \underline{x}_j \underline{O}_{ik} \underline{x}_k = \delta_{jk} \underline{x}_j \underline{x}_k = \underline{x}^2.$$

$$\underline{O}_{ki} \underline{O}_{lj} = \delta_{il} \delta_{kj}$$

$$m\ddot{\underline{x}}' + \underline{\nabla}' V(\underline{x}') = \underline{O} (m\ddot{\underline{x}} + \underline{\nabla} V) = 0$$

Coordinates are arbitrary. equation is invariant under $x \rightarrow x' = \underline{O}x$, where $\underline{O}$ is a rotation matrix

$\underline{O}$ satisfies $\underline{O}^T \underline{O} = \mathbb{1} \Rightarrow$ distances and angles are invariant

$$O_{ik} O_{jl} = \delta_{ij} \quad x'^2 = x'_i x'_i = \underbrace{O_{ij} x_j}_{x_j} O_{ik} x_k = \delta_{jk} x_j x_k = x^2.$$

$$\underbrace{O_{ji} O_{kj}}_{\delta_{ij}} = \delta_{ij}$$

$$SO(3) \rightarrow O(3) \quad m\ddot{x} + \nabla V(x) = 0 \quad \underline{O}(m\ddot{x} + \nabla V) = 0$$

$$\downarrow$$

$$x'_i = O_{ij} x_j$$

in simplifying Maxwell, we combined space with time

Lorentz transformations $x'^M = \Lambda^M_{\nu} x^{\nu}$

$x^{\mu} = (ct, \underline{x})$, so it is natural to look for the analog of rotations in 4d.

$$\partial_{\mu} = \left(\frac{1}{c} \frac{\partial}{\partial t}, \underline{\nabla} \right)$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \eta^{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Coordinates are arbitrary. equation is invariant under $x \rightarrow x' = \underline{O}x$, where $\underline{O}$ is a rotation matrix

$\underline{O}$ satisfies $\underline{O}^T \underline{O} = \underline{1} \Rightarrow$ distances and angles are invariant

$$O_{ik} O_{lj} = \delta_{ij} \quad x^2 = x_i x_i = \underbrace{O_{ij} x_j O_{ik} x_k}_{\delta_{jk} x_j x_k} = \delta_{ij} x_j x_i = x^2$$

$$O_{ji} O_{ij} = \delta_{ij}$$

$$(SO(3) \rightarrow O(3)) \quad m \ddot{x} + \nabla V(x) = 0 \quad \rightarrow \quad O(m \ddot{x} + \nabla V) = 0$$

$$\downarrow$$

$$x'_i = O_{ij} x_j$$

in simplifying Maxwell, we combined space with time

Lorentz transformations $x'^M = \Lambda^M_\nu x^\nu$

$$\Lambda^T \eta \Lambda = \eta$$

$x^\mu = (ct, \underline{x})$, so it is natural to look for the analog of rotations in 4d.

$$\partial_\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, \underline{\nabla} \right)$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \eta^{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Coordinates are arbitrary, equation is invariant under $x \rightarrow x' = \underline{O}x$, where $\underline{O}$ is a rotation matrix

$\underline{O}$ satisfies $\underline{O}^T \underline{O} = \underline{1} \Rightarrow$ distances and angles are invariant

$$O_{ik}^T O_{ij} = \delta_{ij}, \quad x^2 = x_i x_i = \underbrace{O_{ij} x_j O_{ik} x_k}_{\delta_{ij} x_j x_k} = x^2.$$

$$\boxed{O_{ik}^T \delta_{km} O_{mj} = \delta_{ij}} \quad O_{ki} O_{kj} = \delta_{ij}$$

$$(\underline{O}^T \underline{O} = \underline{1}) \rightarrow \underline{O} \in U(3)$$

$$\downarrow$$

$$x'_i = O_{ij} x_j$$

in simplifying Maxwell, we combined space with time

Lorentz transformations $x'^\mu = \Lambda^\mu_\nu x^\nu$

analog of $\underline{O}^T \underline{O} = \underline{1}$ is $\Lambda^T \eta \Lambda = \eta$

$$m \ddot{x}' + \nabla' V(x') = \underline{O} (m \ddot{x} + \nabla V) = 0$$

$x^\mu = (ct, \underline{x})$, so it is natural to look for the analog of rotations in

$$\partial_\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right)$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \eta^{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Lorentz transformations $(X'^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu})$

analog of $O^T O = \mathbb{1}$ is $\Lambda^T \eta \Lambda = \eta$

$$U_{\mu} = (c, \vec{v})$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \eta^{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$x'^{\mu}(x^{\mu}) \leftrightarrow x^{\mu}(x'^{\mu})$$

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \Rightarrow \frac{\partial}{\partial x^{\alpha}} = \frac{\partial x'^{\beta}}{\partial x^{\alpha}} \frac{\partial}{\partial x'^{\beta}} = \Lambda^{\beta}_{\alpha} \frac{\partial}{\partial x'^{\beta}}$$

$$\Rightarrow \frac{\partial}{\partial x'^{\alpha}} = \hat{\Lambda}^{\beta}_{\alpha} \frac{\partial}{\partial x^{\beta}}$$

Lorentz transformations $(X'^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu})$

analog of $O^T O = \mathbb{1}$ is $\Lambda^T \eta \Lambda = \eta$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \eta = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

$$X'^{\mu}(x^{\mu}) \leftrightarrow X^{\mu}(x'^{\mu})$$

$$X'^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu} \Rightarrow \frac{\partial}{\partial x^{\alpha}} = \frac{\partial x'^{\beta}}{\partial x^{\alpha}} \frac{\partial}{\partial x'^{\beta}} = \Lambda^{\beta}_{\alpha} \frac{\partial}{\partial x'^{\beta}}$$

$$\Rightarrow \frac{\partial}{\partial x'^{\alpha}} = \tilde{\Lambda}^{\beta}_{\alpha} \frac{\partial}{\partial x^{\beta}} \quad \text{where} \quad \tilde{\Lambda}^{\alpha}_{\beta} \Lambda^{\beta}_{\gamma} = \Lambda^{\alpha}_{\rho} \tilde{\Lambda}^{\rho}_{\gamma} = \delta^{\alpha}_{\gamma}$$

$$\text{likewise } J'^{\alpha} = \Lambda^{\alpha}_{\beta} J^{\beta}$$

$$\text{and } F'^{\alpha\beta} =$$

Lorentz transformations $(X'^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu})$
 analog of $O^T O = \mathbb{1}$ is $\Lambda^T \eta \Lambda = \eta$

$$\eta_{\mu\nu} = (\epsilon, \delta_{ij}) \quad \eta^{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \eta = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$X'^{\mu}(x') \leftrightarrow X^{\mu}(x')$$

$$X'^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu} \Rightarrow \frac{\partial}{\partial x'^{\alpha}} = \frac{\partial X^{\beta}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\beta}} = \Lambda^{\beta}_{\alpha} \frac{\partial}{\partial x^{\beta}}$$

$$\Rightarrow \frac{\partial}{\partial x'^{\alpha}} = \tilde{\Lambda}^{\beta}_{\alpha} \frac{\partial}{\partial x^{\beta}} \quad \text{where} \quad \tilde{\Lambda}^{\alpha}_{\beta} \Lambda^{\beta}_{\gamma} = \Lambda^{\alpha}_{\rho} \tilde{\Lambda}^{\rho}_{\gamma} = \delta^{\alpha}_{\gamma}$$

likewise $J'^{\alpha} = \Lambda^{\alpha}_{\beta} J^{\beta}$

and $F'^{\alpha\beta} = \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} F^{\mu\nu}$, $\epsilon'^{\mu\nu\rho\sigma} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} \Lambda^{\rho}_{\gamma} \Lambda^{\sigma}_{\delta} \epsilon^{\alpha\beta\gamma\delta} =$

Lorentz transformations $(X'^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu})$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

analog of $O^T O = \mathbb{1}$ is $\Lambda^T \eta \Lambda = \eta$

$$\det \Lambda = \pm 1$$

$$\Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} \eta^{\alpha\beta} = \eta^{\mu\nu} \quad \tilde{\Lambda}^{\mu}_{\alpha} \tilde{\Lambda}^{\nu}_{\beta} \eta_{\mu\nu} = \eta_{\alpha\beta}$$

$$X'^{\mu}(x^{\mu}) \leftrightarrow X^{\mu}(x'^{\mu})$$

$$X'^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu} \Rightarrow \frac{\partial}{\partial x^{\alpha}} = \frac{\partial X^{\beta}}{\partial x^{\alpha}} \frac{\partial}{\partial x^{\beta}} = \Lambda^{\beta}_{\alpha} \frac{\partial}{\partial x^{\beta}}$$

$$\Rightarrow \frac{\partial}{\partial x'^{\alpha}} = \tilde{\Lambda}^{\beta}_{\alpha} \frac{\partial}{\partial x^{\beta}} \quad \text{where} \quad \tilde{\Lambda}^{\alpha}_{\beta} \Lambda^{\beta}_{\gamma} = \Lambda^{\alpha}_{\rho} \tilde{\Lambda}^{\rho}_{\gamma} = \delta^{\alpha}_{\gamma}$$

$$\text{likewise} \quad J'^{\alpha} = \Lambda^{\alpha}_{\beta} J^{\beta}$$

$$\text{and} \quad F'^{\alpha\beta} = \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} F^{\mu\nu}, \quad \epsilon'^{\mu\nu\rho\sigma} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} \Lambda^{\rho}_{\gamma} \Lambda^{\sigma}_{\delta} \epsilon^{\alpha\beta\gamma\delta} = \epsilon^{\mu\nu\rho\sigma} \det \Lambda = \pm \epsilon^{\mu\nu\rho\sigma}$$

Maxwell eqⁿs

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta \quad (1)$$

$$\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0 \quad (2)$$

$$\phi = \oint_{\gamma} \underline{A} \cdot d\underline{x}$$

$$\underline{\nabla} \phi = \underline{A}$$

$$\frac{\underline{E}}{c} = \underline{F}^{0i} \Rightarrow \begin{cases} \underline{B} = \underline{\nabla} \wedge \underline{A} \\ \underline{E} = -\underline{\nabla} \phi - \frac{\partial \underline{A}}{\partial t} \end{cases}$$

Maxwell eq's

$$\partial_{\alpha} F^{\alpha\beta} = -\mu_0 J^{\beta} \quad (1)$$

$$\epsilon^{\alpha\beta\gamma\delta} \partial_{\beta} F_{\gamma\delta} = 0 \quad (2)$$

If we transform $x \rightarrow x'$, $J \rightarrow J'$, $F \rightarrow F'$ via Lorentz transformations, then (1) and (2) are unchanged.

$$U_{\alpha}^{\prime} U^{\prime\alpha} = \tilde{\Lambda}^{\beta}_{\alpha} U_{\beta} \Lambda^{\alpha}_{\gamma} U^{\gamma}$$

$$\tilde{\Lambda} \Lambda = \delta = U_{\gamma} U^{\gamma}$$

10. If $F^{\alpha\beta}(x)$ and $J^{\alpha}(x)$ are solutions of M.E's

then $F^{\prime\alpha\beta}(x') = \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} F^{\mu\nu}(x)$ and $J^{\prime\alpha}(x') = \Lambda^{\alpha}_{\beta} J^{\beta}(x)$ are

$\eta_{\alpha\beta}$

$\propto \frac{\partial}{\partial x^\beta}$

where $\tilde{\Lambda}^\alpha_\beta \Lambda^\beta_\gamma = \Lambda^\alpha_\beta \tilde{\Lambda}^\beta_\gamma = \delta^\alpha_\gamma$

$$\Lambda = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \\ & & & 1 \end{pmatrix}$$

$ct' = -ct$ time reversal
 $\underline{x}' = \underline{x}$

$$\Lambda \Lambda^T = \eta$$

$$\epsilon^{\alpha\beta\gamma\delta} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \Lambda^\gamma_\rho \Lambda^\delta_\sigma \epsilon^{\mu\nu\rho\sigma} = \epsilon^{\alpha\beta\gamma\delta} \det \Lambda = \pm \epsilon^{\alpha\beta\gamma\delta}$$