Title: General Relativity for Cosmology - Lecture 3

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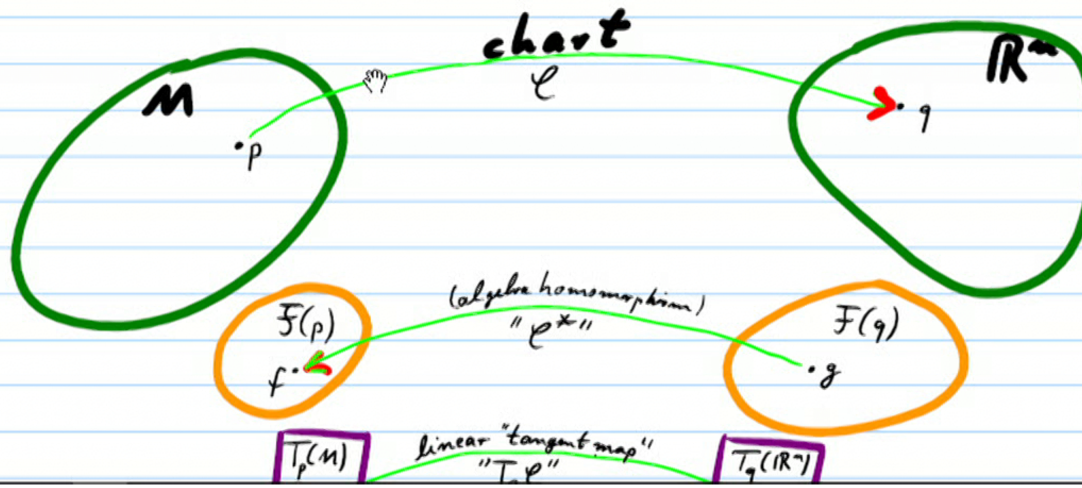
Abstract:

GR for Cosmology, Achim Kempf, Fall 17, Lecture 3

The "physicist's definition of $T_p(M)$"

Recall: We obtain concrete representations for $p \in M$ and $f \in \mathcal{F}(p)$ and $\xi \in T_p(M)$ using a chart $\varphi: M \to \mathbb{R}^n$:



Recall: Def's used

pre-composition:

$$\varphi^*[g] = g \circ \varphi$$

$$T_p\varphi[\xi] = \xi \circ \varphi^*$$

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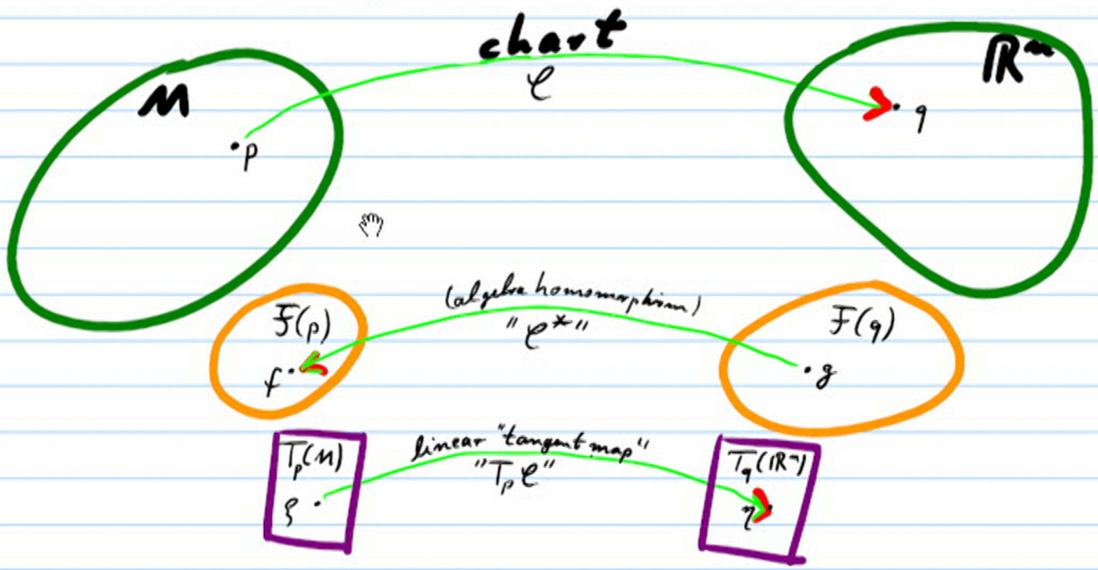

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Namely:

- Each $p \in M$ has now a concrete image $q \in \mathbb{R}^n$, i.e., it has 'coordinates'.
- Each $f \in \mathcal{F}(p)$ is the image of a concrete function germ $g \in \mathcal{F}(q)$.
- Each $\xi \in T_p(M)$ has now a concrete image

$$\eta \in T_q(\mathbb{R}^n)$$

which we know has the form:

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$$\eta \in T_q(\mathbb{R}^n)$$

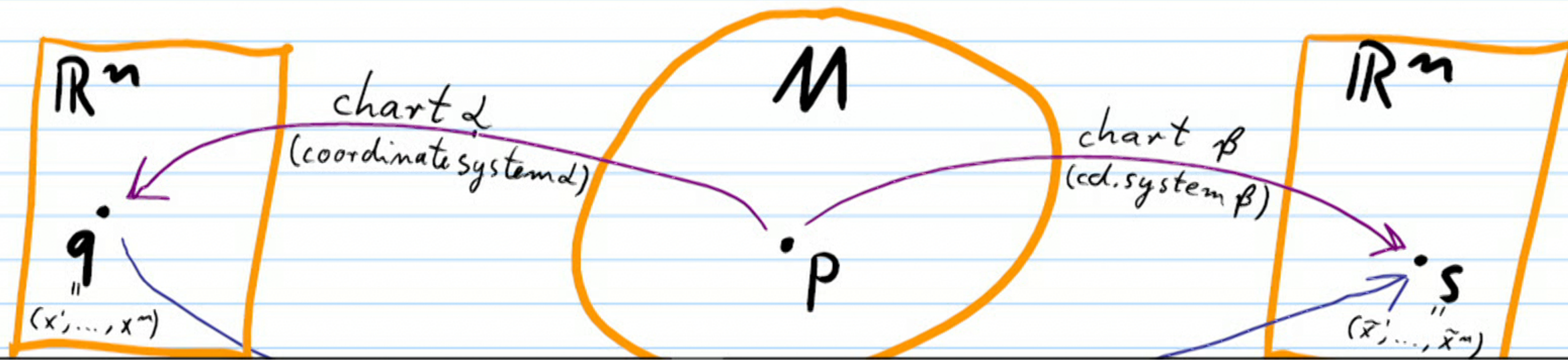
which we know has the form:

coefficients $\in \mathbb{R}$

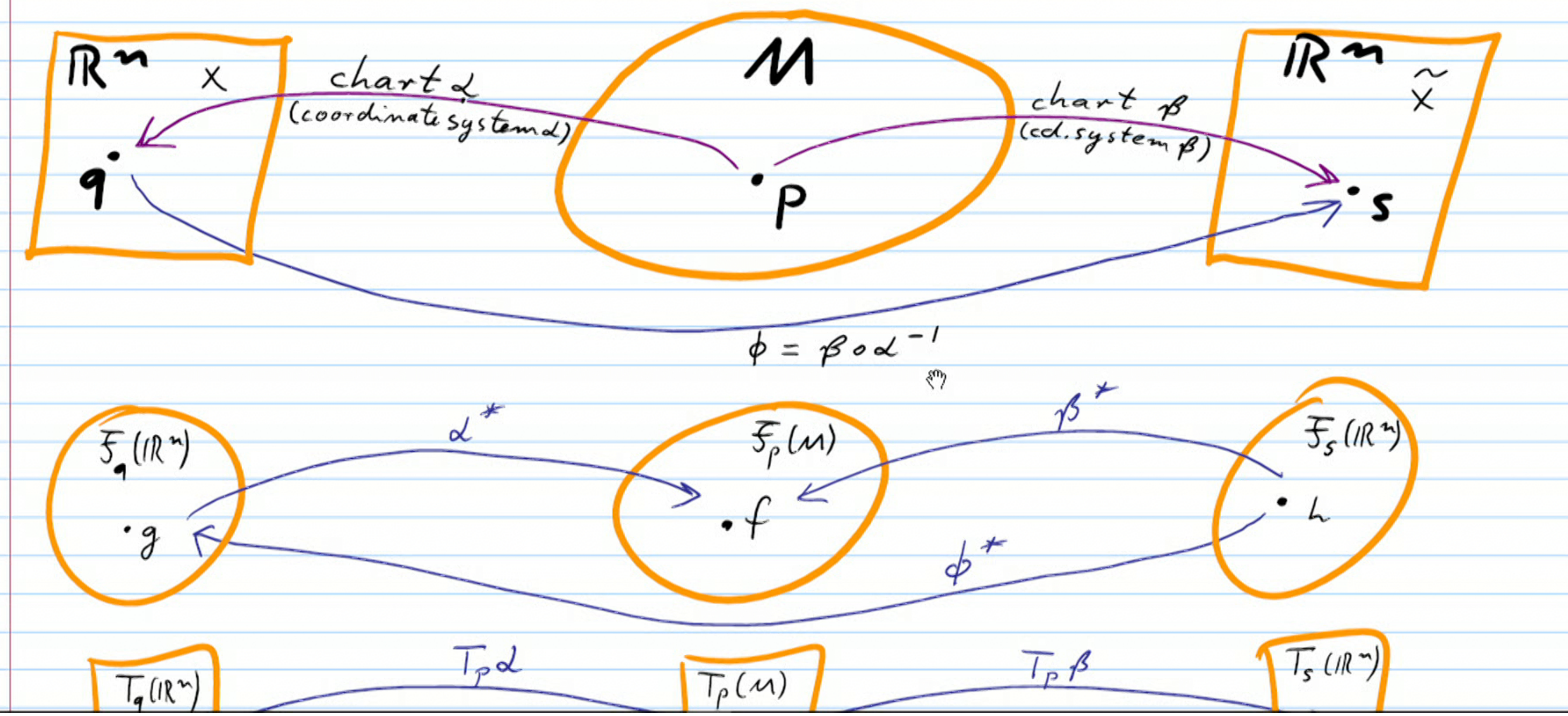
$$\eta = \sum_{i=1}^{n} \eta_i \frac{\partial}{\partial x^i}\bigg|_{x=q}$$

Question:

Given a $p \in M$ and a $\xi \in T_p(M)$, how do their coordinates and coefficients change under a change of charts?



⇒ When changing from chart α to chart β:



1. Every point $p \in \mathcal{M}$ now has 2 images,

$q = (x^1, \ldots, x^n)$ and $s = (\tilde{x}^1, \ldots, \tilde{x}^n)$

$$(\tilde{x}^1, \ldots, \tilde{x}^n) = \phi(x^1, \ldots, x^n)$$

Concretely: $\tilde{x}^i = \phi^i(x^1, \ldots, x^n)$.

2. Every function germ $f \in \mathcal{F}_p(\mathcal{M})$ has 2 pre-images,

$g \in \mathcal{F}_q(\mathbb{R}^n)$ and $h \in \mathcal{F}_s(\mathbb{R}^n)$, related by

$f(p) = g(q) = h(s) \quad (\in \mathbb{R})$ and by

$h(\tilde{x}^1, \ldots, \tilde{x}^n) = g(x^1, \ldots, x^n) \quad (*)$

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3. Every tangent vector $\xi \in T_p(M)$ now has 2 images, $\eta \in T_q(\mathbb{R}^n)$ and $\nu \in T_s(\mathbb{R}^n)$.

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By construction: (b/c of precomposition)

$$\eta(g) = \xi(f) = v(h) \qquad (\in \mathbb{R})$$

$\Rightarrow$ in particular:

$$\overbrace{\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} g(x^1, \ldots, x^n)\Big|_{x=q}}^{\eta(g)} = \overbrace{\sum_{j=1}^{n} v^j \frac{\partial}{\partial \tilde{x}^j} \underbrace{h(\tilde{x}^1, \ldots, \tilde{x}^n)}_{= g(x^1, \ldots, x^n) \text{ by } (*)}\Big|_{\tilde{x}=s}}^{v(h)}$$

$$\sum^{n} \; {}^{j} \, \partial x^k \big| \; \partial \; g(x^1, \ldots, x^n)\big|$$

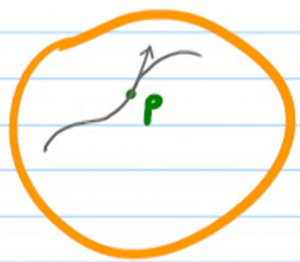
$\Rightarrow$ in particular:

$$\overbrace{\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} g(x^1,\dots,x^n)}^{\eta(g)}\Big|_{x=q} = \overbrace{\sum_{j=1}^{n} \nu^j \frac{\partial}{\partial \tilde{x}^j} \underbrace{h(\tilde{x}^1,\dots,\tilde{x}^n)}_{\overset{\text{by } (*)}{=}\; g(x^1,\dots,x^n)}}^{\nu(h)}\Big|_{\tilde{x}=s}$$

$$= \sum_{\substack{j=1\\k=1}}^{n} \nu^j \frac{\partial x^k}{\partial \tilde{x}^j}\Big|_{\tilde{x}=s} \frac{\partial}{\partial x^k} g(x^1,\dots,x^n)\Big|_{x=q}$$

The "geometric definition of $T_p(\mathcal{M})$":

Idea: Tangent vectors as tangents to paths.



Consider paths in $\mathcal{M}$ that pass through p:

$$\gamma : \mathbb{R} \to \mathcal{M}$$

$$\gamma(0) = p$$

Note: For any $f : \mathcal{M} \to \mathbb{R}$, we obtain:

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Define:

Two diffable paths, γ_a, γ_b are called equivalent,

if for all $f \in \mathcal{F}_p(\mathcal{M})$:

$$\frac{d}{dt}(f \circ \gamma_a)\Big|_{t=0} = \frac{d}{dt}(f \circ \gamma_b)\Big|_{t=0} \qquad (X)$$

Intuition: Two paths γ_a, γ_b are equivalent if they have the same 'velocity' at p:

Note: this includes speed and direction because (X) must hold for all $f \in \mathcal{F}_p(\mathcal{M})$.

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we'll usually mean $T_p^{(alg)}(M)$ when we write $T_p(M)$.

Are $T_p(M)^{(geom)}$ and $T_p(M)^{(alg)}$ equivalent?

Yes!

really: each equivalence class of diff'able paths through p

- Each path γ defines a linear map $\bar{\gamma}$:

$$\bar{\gamma} : \mathcal{F}(p) \longrightarrow \mathbb{R}$$

$$\bar{\gamma} : f \rightarrow \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}$$

- These $\bar{\gamma}$ obey the Leibniz rule:

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The "Cotangent Space" $T_p(M)^*$:

Recall:

Given an n-dimensional vector space V, the set of linear maps $\omega: V \to \mathbb{R}$ forms also an n-dim. vector space. It is called the "dual space", and denoted V^*.

Definition:

The dual vector space to $T_p(M)$

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We notice:

For every (germ of a) function at p,

$$f \in \mathcal{F}(p)$$

one naturally obtains an element

$$\text{"}df\text{"} \in T_p(M)^*$$

called the "<u>differential</u> of f."

Namely:

$df : T_p(M) \to \mathbb{R}$ is the linear map:

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Concretely: in a cds., i.e., in a chart,

the abstract $\xi \in T_p(M)$ and $f \in \mathcal{F}(p)$

correspond to some $\eta \in T_q(\mathbb{R}^n)$ and $g \in \mathcal{F}(q)$.

$T_p(M)^*$

Ψ

Then: $dg : T_q(\mathbb{R}^n) \to \mathbb{R}$

$$dg : \eta \to \eta(g) = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} g(x^1, \dots, x^n)\Big|_{x=q}$$

Recall: Since all $\eta \in T_q(\mathbb{R}^n)$ take the form $\eta = \sum \eta^i \frac{\partial}{\partial x^i}\Big|$

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Question: What is the dual basis in $T_q(\mathbb{R}^n)^*$?

- Consider the coordinate functions: $x^k: \mathbb{R}^n \to \mathbb{R}$.

- Their differentials $dx^k \in T_q(\mathbb{R}^n)^*$ obey:

$$dx^k: T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$dx^k: \left.\frac{\partial}{\partial x^i}\right|_{x=q} \longrightarrow \left.\frac{\partial}{\partial x^i} x^k\right|_{x=q} = \delta_i^k$$

$\Rightarrow$ The dual basis in $T_q(\mathbb{R}^n)^*$ is given by

Thus:

Every element $\omega \in T_q(\mathbb{R}^n)^*$ takes the form:

$$\omega = \sum_{i=1}^{n} \underbrace{\omega_i}_{\in \mathbb{R}} dx^i$$

and its action is:

$$\omega : T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$\omega : \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j} \to \sum_{i=1}^{n} \omega_i \, dx^i\left(\sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j}\right)$$

$$= \sum_{i=1}^{n} \omega_i \sum_{j=1}^{n} \eta^j \underbrace{\frac{\partial}{\partial x^j} x^i}_{= \delta^i_j}$$

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In particular: For arbitrary $g \in \mathcal{F}(q)$, its differential $dg \in T_q(\mathbb{R}^n)^*$ must be of the form:

$$dg = \sum_{k=1}^{n} \omega_k \, dx^k \quad \text{with suitable } \omega_k \in \mathbb{R}.$$

How to calculate them?

We know:

$$dg(\eta) = \eta(g) = \sum_{i=1}^{n} \eta^i \underbrace{\frac{\partial}{\partial x^i} g(x)}_{=\omega^i}\Big|_{x=q} \quad (\mathrm{II})$$

Compare I, II $\Rightarrow$ $\omega_i = \frac{\partial}{\partial x^i} g(x)\Big|$

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Compare I, II $\Rightarrow$ $\omega_i = \frac{\partial}{\partial x^i} g(x)\Big|_{x=q}$

$$\Rightarrow \boxed{dg = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x^i} g(x)\Big|_{x=q} \right) dx^i}$$

Exercise: (the "pull back" map)

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Assume that $\xi \in T_p(M)^*$, under two charts α, β, as above, corresponds to $\omega \in T_q(\mathbb{R}^n)^*$ and $\mu \in T_s(\mathbb{R}^n)^*$ with:

$$\omega = \sum_{i=1}^{n} \omega_i \, dx^i \quad \text{and} \quad \mu = \sum_{i=1}^{n} \mu_i \, d\tilde{x}^i$$

Show that

$$\mu_i = \sum_{j=1}^{n} \frac{\partial x^j}{\partial \tilde{x}^i}\bigg|_{\tilde{x}=s} \omega_j$$

Notice that this is the inverse of the Jacobian matrix of $\beta \circ \alpha^{-1}$ at q

Remark: The physicist's definition of $T_p(M)^*$ uses this.

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Some notation and terminology:

- Elements of $T_p(M)$ are called contravariant vectors
- Elements of $T_p(M)^*$ are called covariant vectors
- One often writes symbolically

$$\xi = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}\Big|_p \quad \text{for} \quad \xi \in T_p(M)$$

$$\omega = \sum_{i=1}^{n} \omega_i \, dx^i \quad \text{for} \quad \omega \in T_p(M)^*$$

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Def: A tensor, t, of rank (r,s) is an element of

$$T_p(M)^r_s := \underbrace{T_p(M)\otimes\ldots\otimes T_p(M)}_{r\text{ factors}}\otimes\underbrace{T_p(M)^*\otimes\ldots\otimes T_p(M)^*}_{s\text{ factors}}$$

In a chart:

$$t = \sum_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_s=1}}^{n} \underbrace{t^{i_1\ldots i_r}_{j_1\ldots j_s}}_{\in\,\mathbb{R}} \frac{\partial}{\partial x^{i_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_s}$$

Under chart change: (physicists, incl. Einstein, defined tensors this way)

$$\tilde{t}^{i_1\ldots i_r}_{j_1\ldots j_s} = \sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_s=1}}^{n} \frac{\partial\tilde{x}^{i_1}}{\partial x^{k_1}}\cdots\frac{\partial\tilde{x}^{i_r}}{\partial x^{k_r}}\frac{\partial x^{l_1}}{\partial\tilde{x}^{j_1}}\cdots\frac{\partial x^{l_s}}{\partial\tilde{x}^{j_s}}\, t^{k_1\ldots k_r}_{l_1\ldots l_s}$$

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$\mathbb{R}$

Under chart change: (physicists, incl. Einstein, defined tensors this way)

$$\tilde{t}^{i_1 \dots i_r}_{j_1 \dots j_s} = \sum_{\substack{i_1 \dots i_r \\ j_1 \dots j_s = 1}}^{m} \frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \tilde{x}^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{l_s}}{\partial \tilde{x}^{j_s}} \, t^{k_1 \dots k_r}_{l_1 \dots l_s}$$

Thus: $T_p(M) = T_p(M)'$ and $T_p(M)^* = T_p(M)$, i.e.:

- a tangent vector is a tensor of rank $(1,0)$
- a cotangent vector is a tensor of rank $(0,1)$

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Finally: From local to global!

a "base point"

Def: We call $T(M) := \bigcup_{p \in M} \left(p, T_p(M)\right)$,

a "fibre"

the **Tangent bundle**.

Note: $T(M)$ is itself a manifold. It is $2n$-dimensional.

Def: $T(M)$ is then also called the "Total Space".

Def: M is also called the "Base Space".

Recall that all $T_p(M)$ are n-dimensional

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Remark: One obtains other Fibre bundles by choosing other standard fibers.

E.g.:
- Co-tangent bundle $T^*(M)$
- (r,s)-tensor bundle $T^r_s(M)$
- Bundles for isospinors (vector bundles) and gauge groups (principle bundles)

Def: The map $\pi : T(M) \to M$

$$\pi : (p, T_p(M)) \to p \qquad \left(\text{i.e.: } \pi^{-1}(p) = T_p(M)\right)$$

is called the "Bundle Projection".

Def: A Section, σ, is a map, $\sigma : M \to T(M)$, which is a

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Notice: The graph of a "field" is a section of its fibre bundle.

Recall: The graph of a function $f: A \to B$ is:

$$\{(a, f(a))\}_{a \in A}$$

Def:

- A tangent vector field is a map $\xi: \underset{M}{p} \to \underset{T_p(M)}{\xi_p}$

  In a chart: $\xi = \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i}$

- A cotangent vector field is a map $\omega: \underset{M}{p} \to \omega_p \in T_p^*(M)$

  In a chart: $\omega = \sum_{i=1}^{n} \omega_i(x)\, dx^i$

$M \qquad T_p(M)_s^r$

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In a chart: $\omega = \sum_{i=1}^{n} \omega_i(x)\, dx^i$

Similarly, tensor fields: $t: \underset{\in M}{p} \to \underset{\in T_p(M)^r_s}{t_p}$

In a chart: $t = \sum t^{j_1 \ldots j_r}_{i_1 \ldots i_s}(x) \frac{\partial}{\partial x^{j_1}} \cdots \frac{\partial}{\partial x^{j_r}} dx^{i_1} \ldots dx^{i_s}$

So far, not a concern with GR, but it does come up with gauge theories.

Why then fibre bundles? To capture global nontriviality.

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Why then fibre bundles? To capture global nontriviality.

Fibre bundles are required to be locally trivial:

M can be covered with neighborhoods $\mathcal{U}_r$, so that

means: there exists a differentiable isomorphism

or other standard fibre for other fibre bundles.

$$\pi^{-1}(\mathcal{U}_r) \simeq \mathcal{U}_r \times \mathbb{R}^n$$

But fibre bundles are allowed to be globally nontrivial:

For a suitable vector bundle B, we can have

$$\pi^{-1}(\mathcal{U}_1) \simeq \mathcal{U}_1 \times \mathbb{R}^n$$

$$\pi^{-1}(\mathcal{U}_2) \simeq \mathcal{U}_2 \times \mathbb{R}^n$$

Definition: For the algebra of C^∞ functions $M \to \mathbb{R}$ we write $\mathcal{F}(M)$.

Note: One can show that contravariant vector fields are the derivations of the algebra $\mathcal{F}(M)$, i.e.:

If ξ is a contravariant vector field, then

$$\xi : \mathcal{F}(M) \to \mathcal{F}(M)$$

is linear and obeys the Leibniz rule:

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Next topic: Differential forms:

We already have covered some differential forms:

- The set $\Lambda_0 := \mathcal{F}(\mathcal{M})$ is called the set of 0-forms.

- The set of covariant vector fields is denoted Λ_1 and called the set of 1-forms.

- For $r = 2, 3, \ldots$ the set, Λ_r, of r-forms