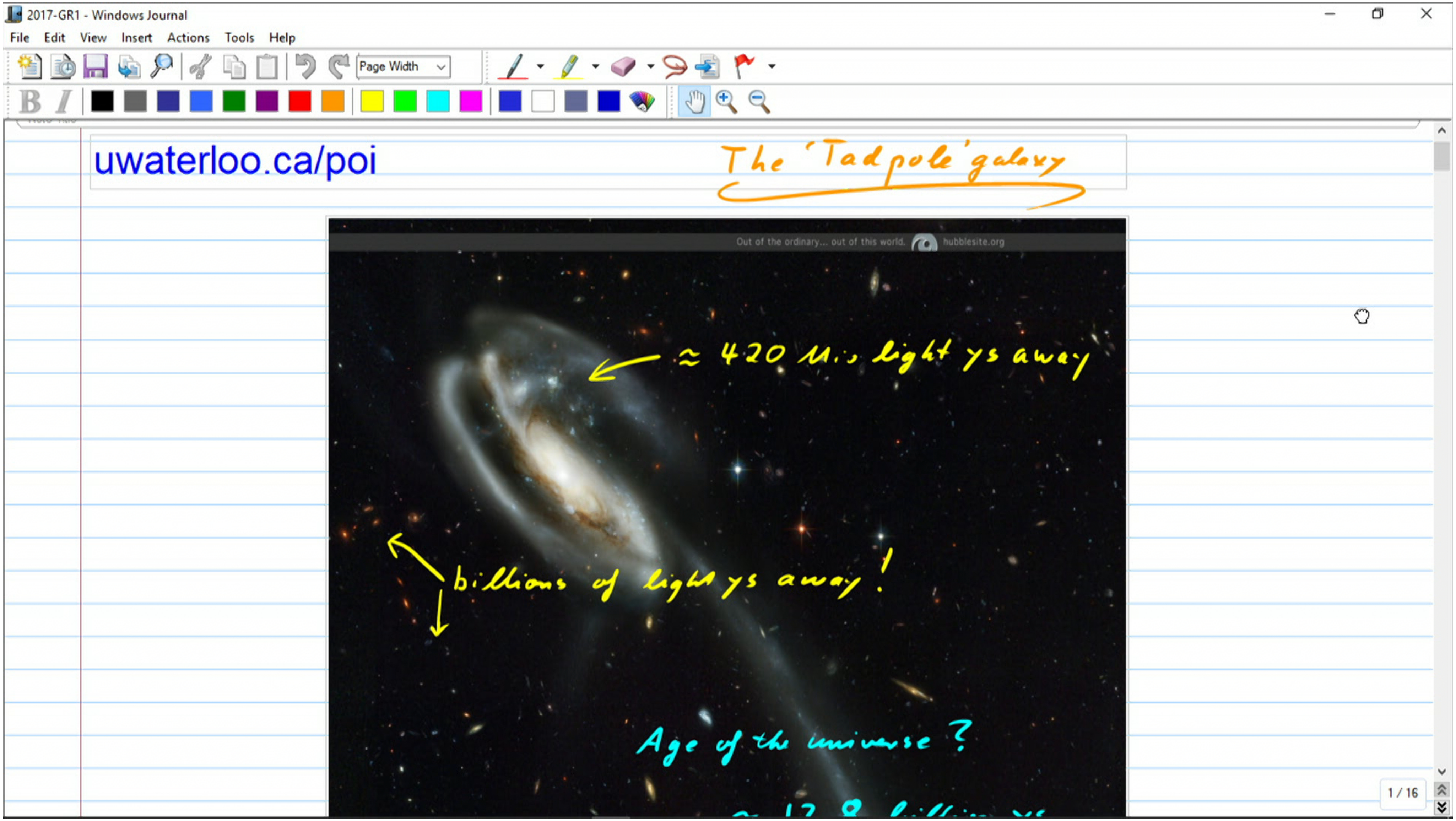
Title: General Relativity for Cosmology - Lecture 1

Date: Sep 08, 2017 04:00 PM

URL: http://pirsa.org/17090008

Abstract: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uwaterloo.ca/poi
The 'Tadpole' galaxy
Out of the ordinary... out of this world. hubblesite.org
≈ 420 Mio light ys away
billions of light ys away!
Age of the universe?
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HST: ABELL2218

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A. Math

Strategy: ☐ Start with a mere "set" of points (events), M

Then add structure:

☐ Define open neighborhoods (i.e., a "topology" on M)

☐ Define "separability" of points (i.e. Hausdorf condition)

☐ Define "continuity" (preimage of open sets is open)

☐ Define "differentiability" (via chart change diffability)

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- Define "differentiability" (via chart change diff'ability)

later:

- Define tangent & tensor spaces

⋮

$$\boxed{\text{Curvature} = \text{nontriviality of parallel transport}}$$

(Why consider others? May be useful

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$\boxed{\text{Curvature} = \text{nontriviality of parallel transport}}$

Other descriptions of curvature?

(Why consider others? May be useful for quantum gravity b/c what's on previous page is likely over idealized.)

☐ Curvature = sum of angles in triangle $\neq \pi$

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Other descriptions of curvature?

(for quantum gravity b/c what's on previous page is likely over idealized.)

- ☐ Curvature = sum of angles in triangle $\neq \pi$
- ☐ Curvature = nontriviality of Pythagoras law
- ☐ Curvature = tidal forces. Math of it: Sectional curvatures
- ☐ Curvature $\stackrel{?}{=}$ nontrivial sound of object when vibrating

This field is called Spectral Geometry.

Interesting b/c connects mathematical languages

of quantum theory (spectra etc) and general relativity.

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- Curvature = tidal forces. Math of it: Sectional curvatures

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Interesting b/c connects mathematical languages of quantum theory (spectra etc) and general relativity.

- Curvature $\overset{?}{=}$ nontrivial entanglement in vacuum fluctuations

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B) Structure and properties of General Relativity?

- Equations of motion

  for scalars, vectors, spinors and curvature

- Symmetries

  local and global conservation laws, if any!

- Tetrad formulation, GR as a gauge theory

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- Symmetries

  local and global conservation laws, if any!

- Tetrad formulation, GR as a gauge theory

- Singularities, and their unavoidability

C) Applications to Cosmology

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C) Applications to Cosmology

- Classification of exact solutions

- Models of cosmological matter

- FRW models, while using the tetrad formalism to exercise it. (e.g. for later use in quantum gravity)

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- FRW models, while using the tetrad formalism to exercise it. (e.g. for later use in quantum gravity)
- Cosmic inflation
- Black holes

A. Pseudo-Riemannian Differential Geometry

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A. Pseudo-Riemannian Differential Geometry

□ Differentiable Manifolds

(Riemann ≈ 1850s, Poincaré ≈ 1890s, Whitney ≈ 1930s...)

Def: An n-dimensional topological Manifold, M, is a Hausdorff

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Def: An n-dimensional topological Manifold, M, is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

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Here:

Def: A topological space, M, is a set, together with a specification of subsets U_i, which will be called "open subsets", which must obey $U_i \cap U_j$ is open, and $\bigcup_k U_k$ is open.

Def: A topological space M is called Hausdorff, if it is separable, i.e., if $x, y \in M$ and $x \neq y$ then x, y are elements

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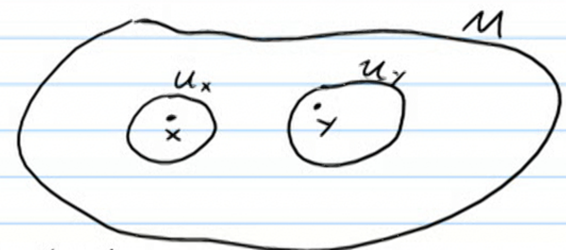
with a specification of subsets U_i, which will be called "open subsets",

which must obey $U_i \cap U_j$ is open, and $\bigcup_\lambda U_\lambda$ is open.

Def: A topological space M is called Hausdorff, if it is separable,

i.e., if $x, y \in M$ and $x \neq y$ then x, y are elements of some disjoint open sets.



$\forall x,y: x \neq y \; \exists U_x, U_y$ open : $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \{\}$

"for all" "there exist" empty set

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Notice: Now M has a topology consisting of open sets. And, of course, $\mathbb{R}^n$ also does.

Recall: If A, B are topol. spaces, then $f: A \to B$ is called continuous if $\forall V \subset A$, $U := f(V)$: (U open $\Rightarrow$ V open)

→ We can now express the idea that M is continuously parametrizable:

Def: M is called locally homeomorphic to $\mathbb{R}^n$, if each point, p, has a neighborhood

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continuous if $\forall V \subset \mathcal{A}$, $\mathcal{U} := f(V) : (\mathcal{U}$ open $\Rightarrow V$ open$)$

→ We can now express the idea that $\mathcal{M}$ is continuously parametrizable:

Def: $\mathcal{M}$ is called locally homeomorphic to $\mathbb{R}^n$, if each point, p, has a neighborhood $\mathcal{U}(p)$, and an invertible continuous map $h : \mathcal{U}(p) \to \mathbb{R}^n$.

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Def: A local homeomorphism,

$$h: \mathcal{U} \to \mathbb{R}^n \quad , \quad \mathcal{U} \subset M$$

called "domain"

is called a **chart** of M.

For any point $q \in \mathcal{U}$ its image

$$h(q) \in \mathbb{R}^n$$

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$$h: \mathcal{U} \to \mathbb{R}^n \, , \quad \mathcal{U} \subset \mathcal{M}$$

called "domain"

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For any point $q \in \mathcal{U}$ its image

$$h(q) \in \mathbb{R}^n$$

is a set of n numbers $(x_1, x_2, \dots, x_n)$

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For any point $q \in \mathcal{U}$ its image

$$h(q) \in \mathbb{R}^n$$

is a set of n numbers $(x_1, x_2, \ldots, x_n)$

called the coordinates of q.

Def: A chart, h, with domain $\mathcal{U}$,

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Local coordinate system for U.

Def: A collection of charts h_α with domains U_α is called an atlas if $\bigcup_\alpha U_\alpha = M$.

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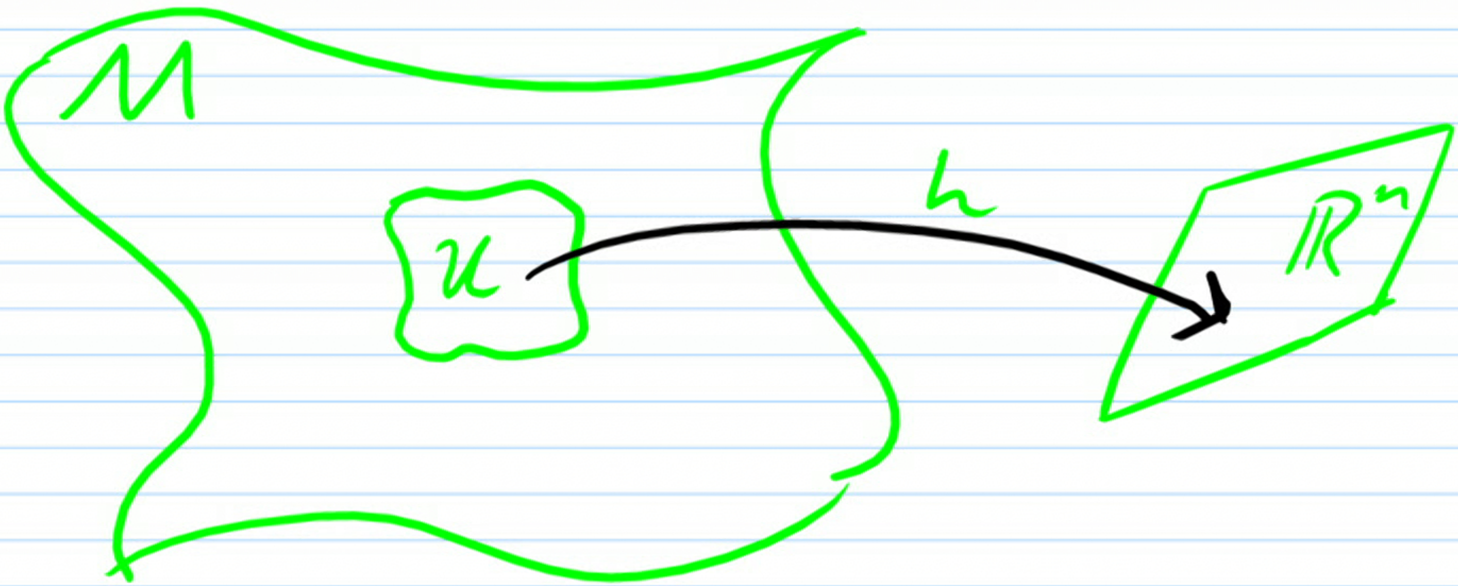

is also called a

local coordinate system for $\mathcal{U}$.

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Def: A collection of charts h_α with domains U_α is called an atlas if $\bigcup_\alpha U_\alpha = M$.

→ What, if we want to change coordinates, i.e. if we want to re-label the

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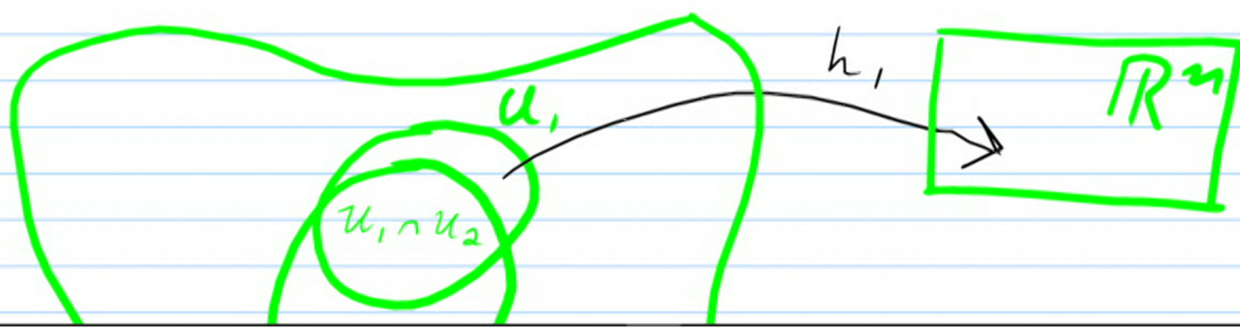
points of (e.g. a subset of) the manifold!

Consider 2 charts h_1, h_2, with intersecting domains $U_1 \cap U_2 \neq \emptyset$:



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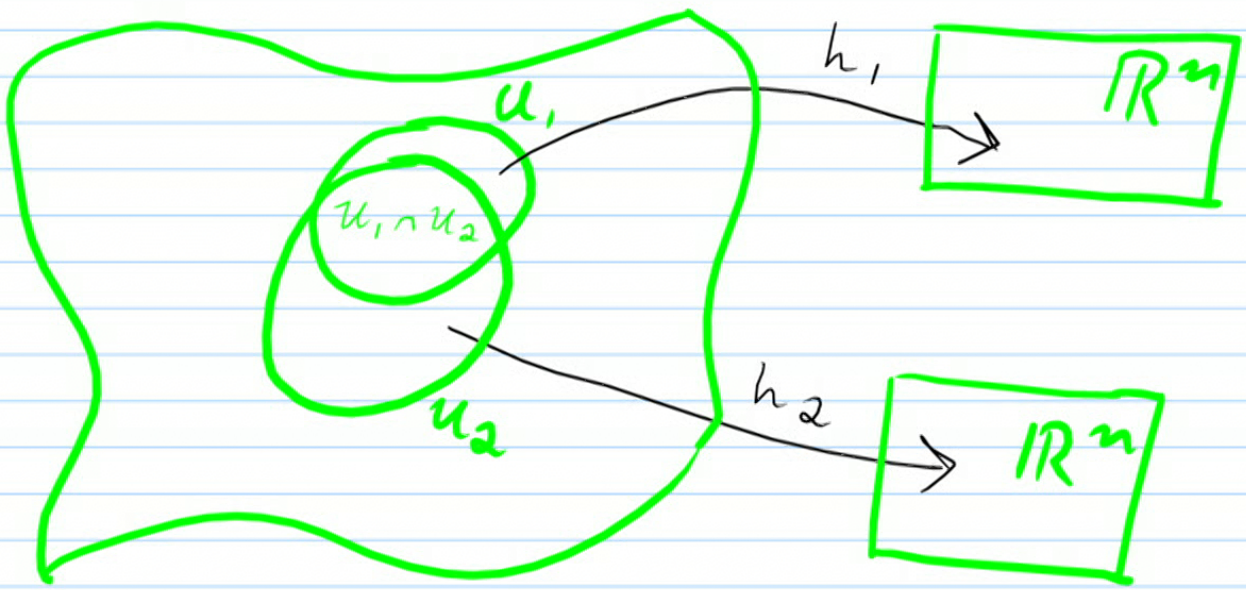
Consider 2 charts h_1, h_2, with intersecting domains $U_1 \cap U_2 \neq \phi$:



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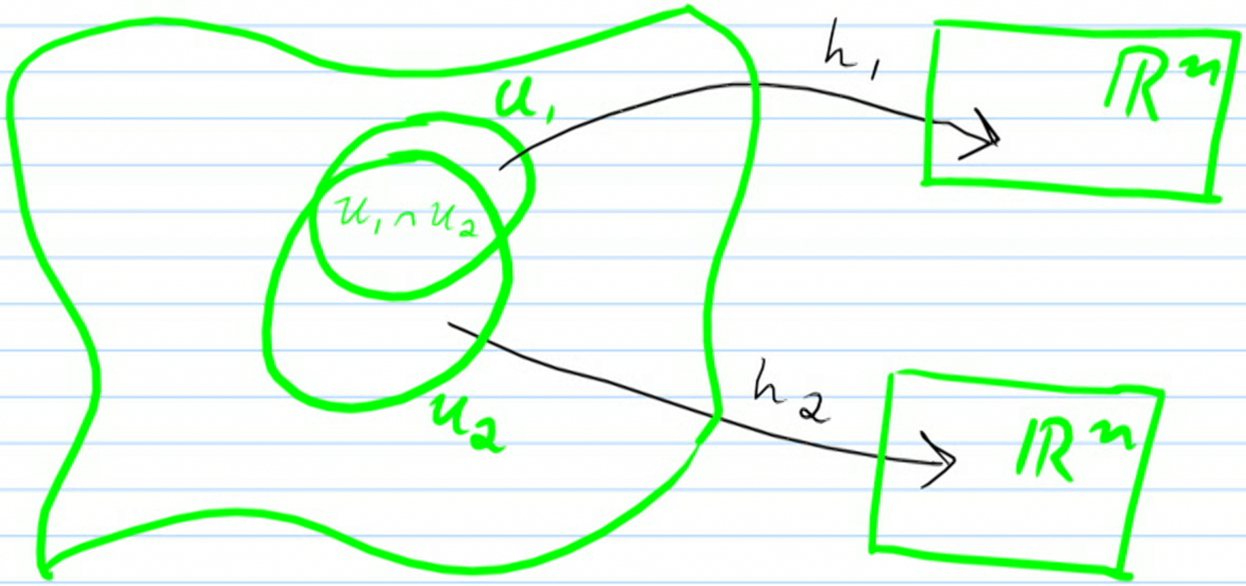

Then, $h_{12} = h_2 \circ h_1^{-1}$ is a continuous change of coordinates map $h_{12} : \mathbb{R}^n \to \mathbb{R}^n$

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Notice: For maps $\mathbb{R}^n \to \mathbb{R}^n$ we know what differentiability means!

Strategy: Let us define the diff'ability of an atlas through the diff'ability of its chart changes:

Def: An atlas is called C^r differentiable,

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Def: An atlas is called C^r differentiable, if all its coordinate changes, $h_{\alpha\beta}$, are C^r diffeomorphisms, i.e., r times continuous differentiable.

Strategy: Enlarge Atlas so every point of M is in multiple charts. Then, diffability of M is definable through atlas diffability

Def: Given a C^r differentiable atlas, A, we can generate a maximal

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Strategy: Enlarge atlas so every point of M is in multiple charts.
Then, diffability of M is definable through atlas diffability

Def: Given a C^r differentiable atlas, A, we can generate a maximal C^r differentiable atlas, $D(A)$, by adding all charts whose chart changes with charts in A are differentiable.

Def: $D(A)$ is also called a "Differentiable Structure" of class C^r for M.

Def: A differentiable manifold of class C^r

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Def: A differentiable manifold of class C^r is a topol. manifold with a maximal atlas of class C^r, i.e., with a differentiable structure of class C^r.

Theorem: (Whitney)

Every C^k structure with $k \geq 1$ is C^k equivalent to a C^∞ structure (i.e. there is always a suitable set of charts).

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Every C^k structure with $k \geq 1$ is C^k equivalent to a C^∞ structure (i.e. there is always a suitable set of charts).

I.e. any diff'able structure can be smoothened. Any lack of higher diff'ability is due to unlucky choice of chart.

Def: Since any C^1 manifold is also a C^∞ manifold, we also call diff'able manifolds simply smooth manifolds.

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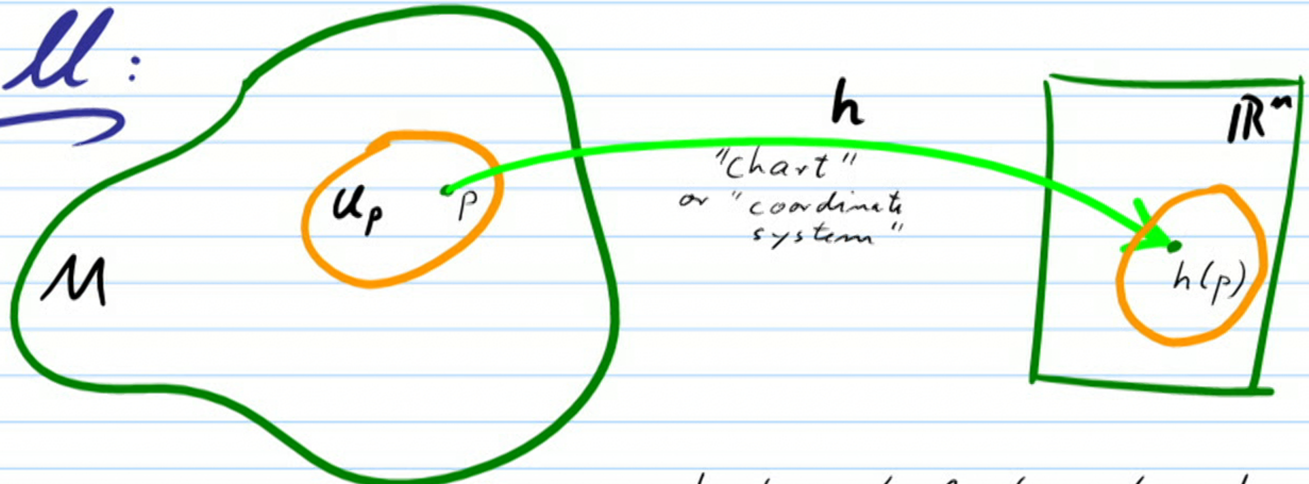
Note Title

GR for Cosmology, Fall 17, Achim Kempf, Lecture 2

Recall:



→ charts are tools to get a handle at the otherwise nameless abstract points of the manifold.

Problem:

How to define the abstract

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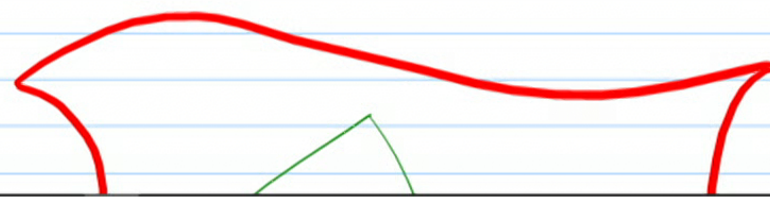
Problem:

How to define the abstract "Tangent space, $T_p(M)$," to a diffable mfld at a point p?

Intuition:



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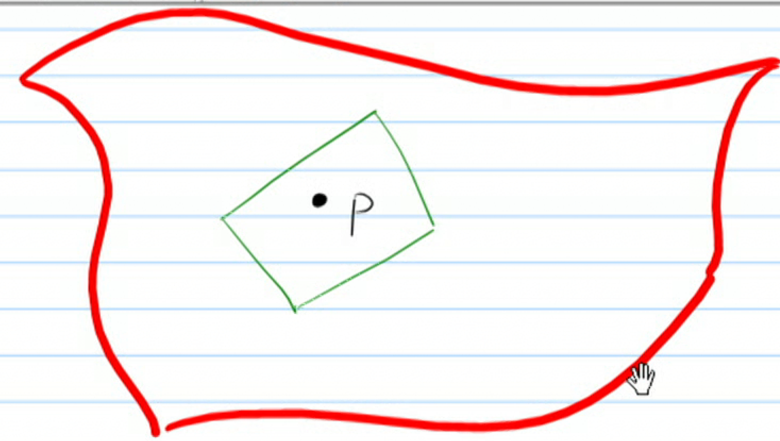

⇝ Proper definition should imply:

An n-dim mfld possesses for every point p an n-dim vector space of tangent vectors.

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3 equivalent definitions of $T_p(M)$:

1. "Algebraic" definition of $T_p(M)$:

Most powerful b/c no need for coordinates

Idea:
- A tangent vector = directional derivative,
- Derivatives definable through Leibniz rule:

$$(fg)' = f'g + fg'$$

2. "Physicist" definition of $T_p(M)$:

Idea: The elements of $T_p(M)$ are

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2. "Physicist" definition of $T_p(M)$:

Idea: The elements of $T_p(M)$ are to be vectors $\Rightarrow$ recognizable by how their components change with charts.

3. "Geometric" definition of $T_p(M)$:

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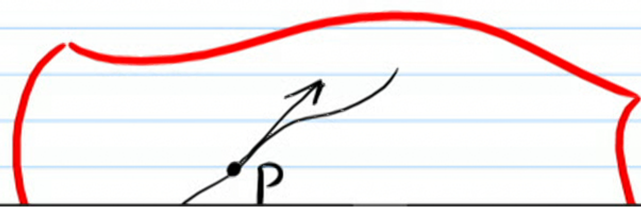
3. "Geometric" definition of $T_p(M)$:

Idea: The elements of $T_p(M)$ are to be actual tangent vectors of one-dim. paths in the manifold, that pass through p.



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→ we will do all 3:

1. Algebraic definition of $T_p(M)$

Idea: a) A tangent vector = directional derivative,

b) Derivatives definable through Leibniz rule:

$$(fg)' = f'g + fg'$$

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1. Algebraic definition of $T_p(M)$

Idea: a) A tangent vector = directional derivative,

b) Derivatives definable through Leibniz rule:

$$(fg)' = f'g + fg'$$

Key example: $M = \mathbb{R}^n$

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