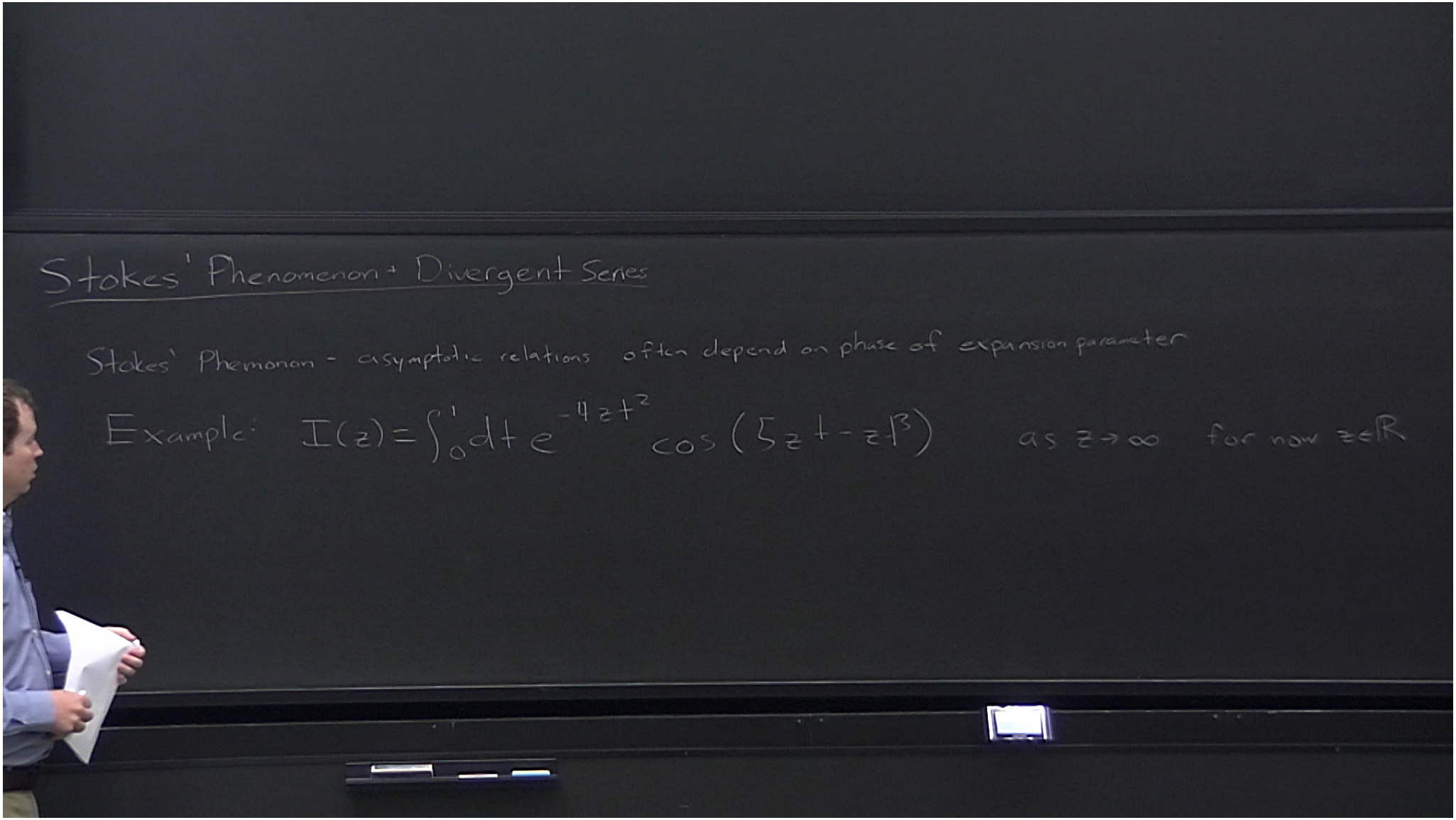
Title: PSI Front End - Functions, "Functions", etc.

Date: Aug 31, 2017 10:30 AM

URL: http://pirsa.org/17080070

Abstract:


Stokes' Phenomenon + Divergent Series
Stokes' Phenomenon - asymptotic relations often depend on phase of expansion parameter
Example: $I(z)=\int_0^1 dt\, e^{-4zt^2}\cos(5zt-z\beta)$ as $z\to\infty$ for now $z\in\mathbb{R}$

Stokes' Phenomenon + Divergent Series

Stokes' Phenomenon - asymptotic relations often depend on phase of expansion parameter

Example: $I(z) = \int_0^1 dt\, e^{-4zt^2} \cos(5zt - zt^3)$ as $z \to \infty$ for now $z \in \mathbb{R}$

$I(z) \overset{?}{\sim} \int_0^1 dt\, e^{-4zt^2} \sim \sqrt{\frac{\pi}{16z}}$ $z \to \infty$

when $t \sim \frac{1}{\sqrt{z}}$ arg of cos is large $\Rightarrow$ destructive interference

$I(z) \sim \int_0^1 dt\, e^{-4zt^2} \sim \sqrt{\frac{\pi}{16z}} \quad z \to +\infty$

when $t \sim \frac{1}{\sqrt{z}}$ arg of cos is large $\Rightarrow$ destructive interference

$$I(z) = \mathrm{Re} \int_0^1 dt\, e^{-4zt^2 + 5iz t - iz t^3}$$

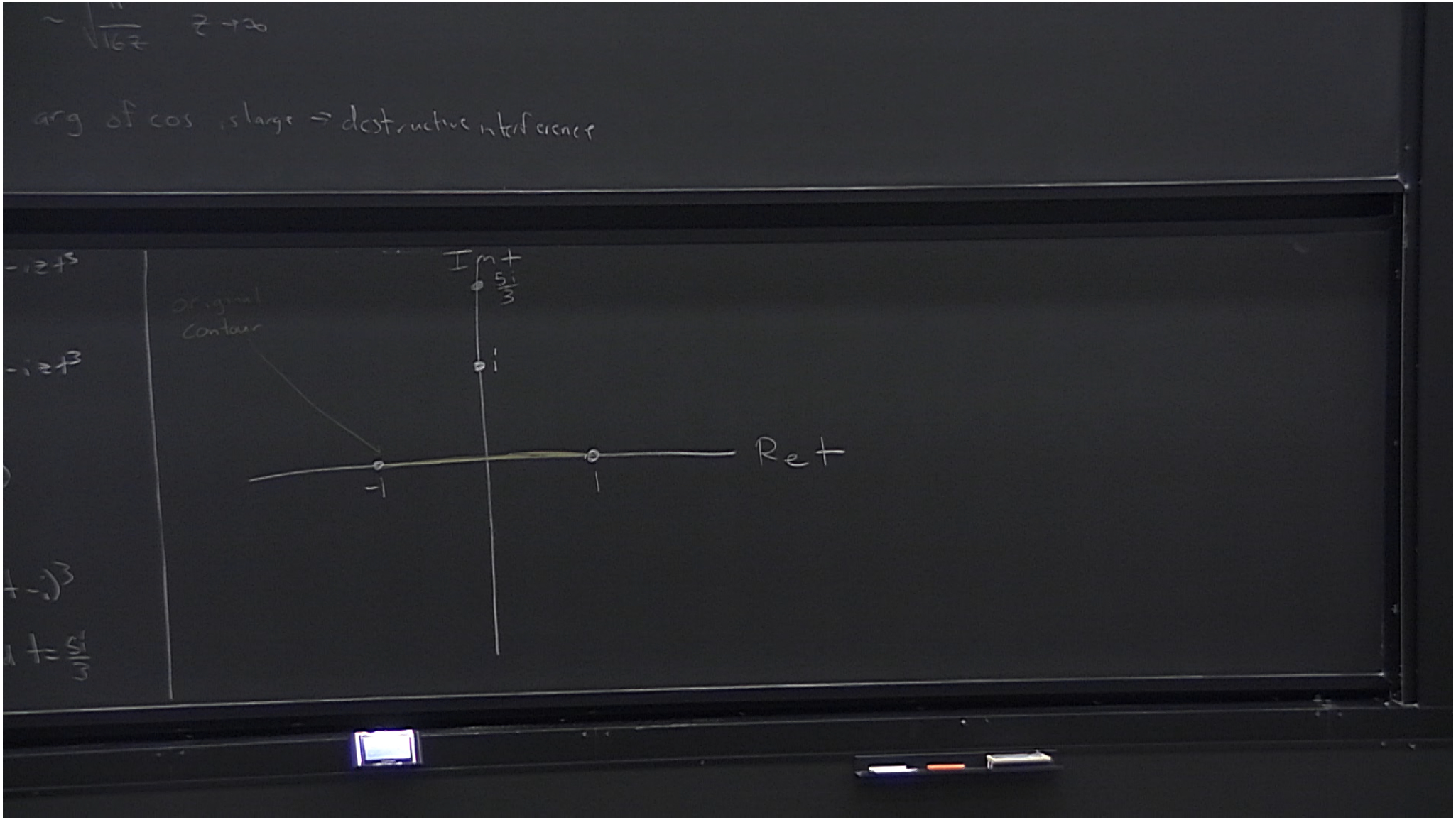
$$= \frac{1}{2} \int_{-1}^1 dt\, e^{-4zt^2 + 5izt - izt^3}$$

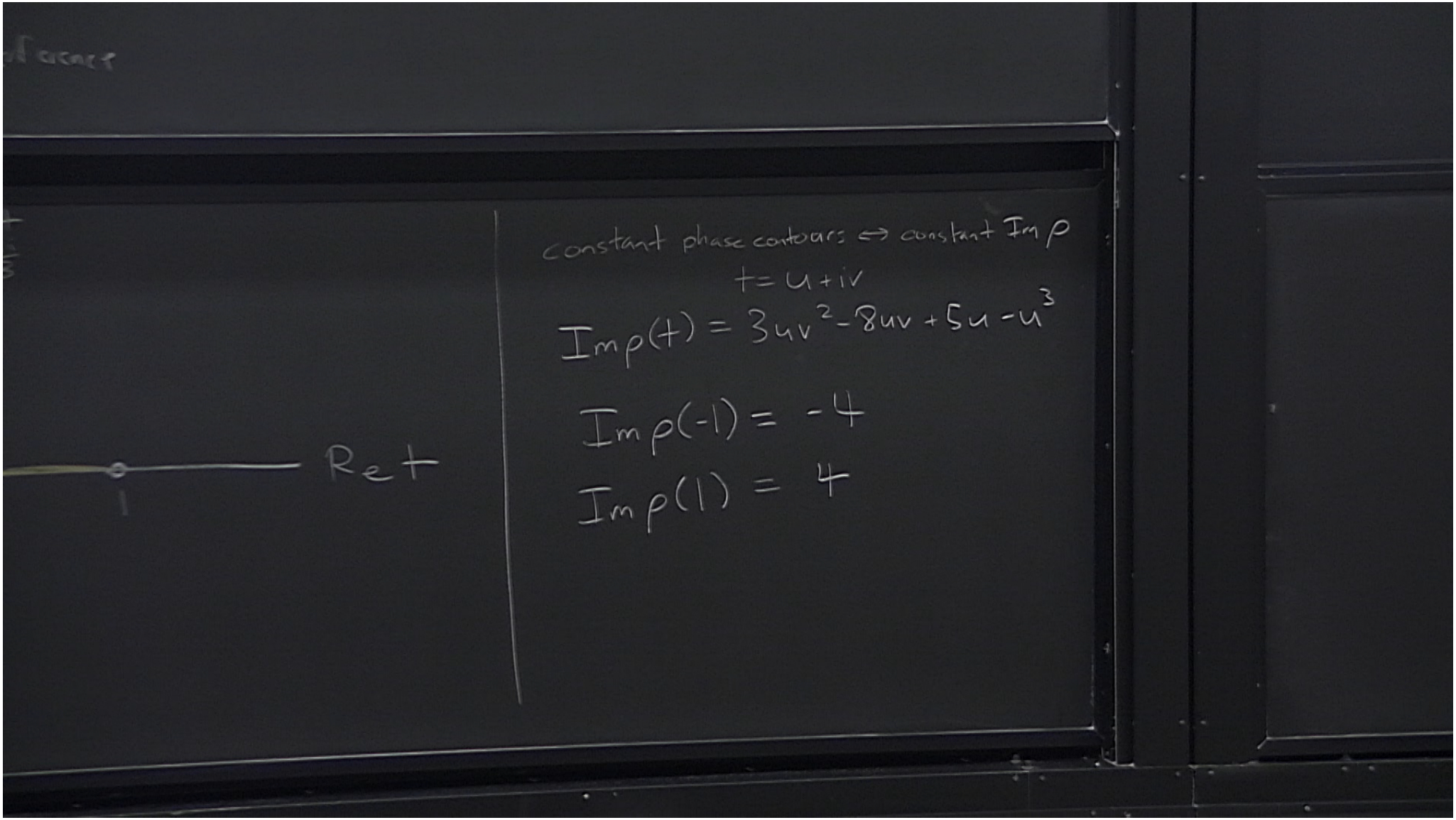
$$= \frac{1}{2} e^{-2z} \int_{-1}^1 dt\, e^{z\rho(t)}$$

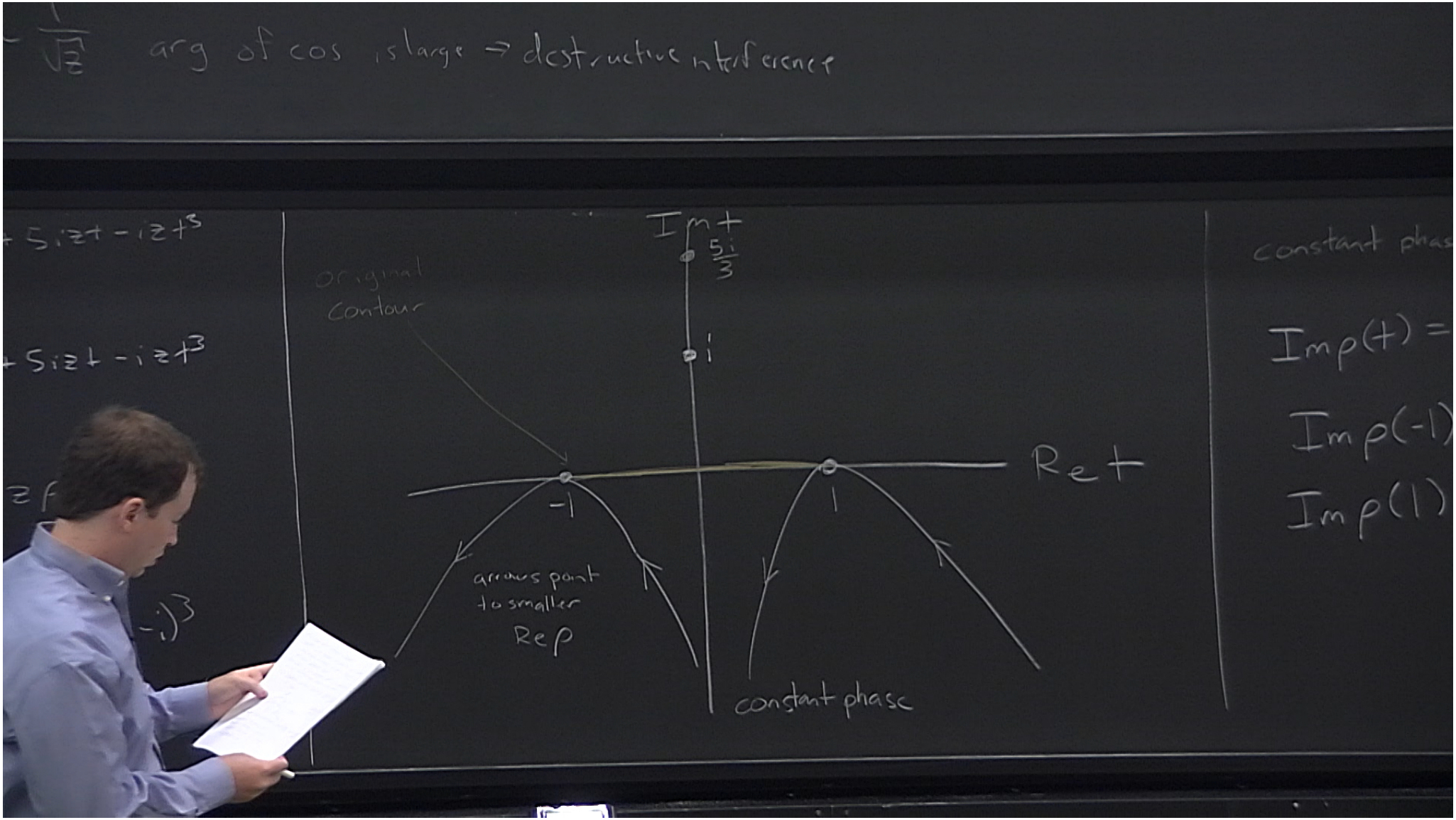
$$\rho(t) = -(t-i)^2 - i(t-i)^3$$

$\rho'(t) = 0$ at $t = i$ and $t = \frac{5i}{3}$




arg of cos is large ⇒ destructive interference
Im t
5i/3
i
original contour
-1
1
Re t


Re t
1
constant phase contours ⇔ constant Im ρ
t = u + iv
Im ρ(t) = 3uv² − 8uv + 5u − u³
Im ρ(−1) = −4
Im ρ(1) = 4


arg of cos is large ⇒ destructive interference
Im t
5i/3
i
Re t
-1
1
original contour
arrows point to smaller Re ρ
constant phase
constant phas
Im ρ(t) =
Im ρ(-1)
Im ρ(1)

$\sqrt{z}$ arg of cos is large ⇒ destructive interference

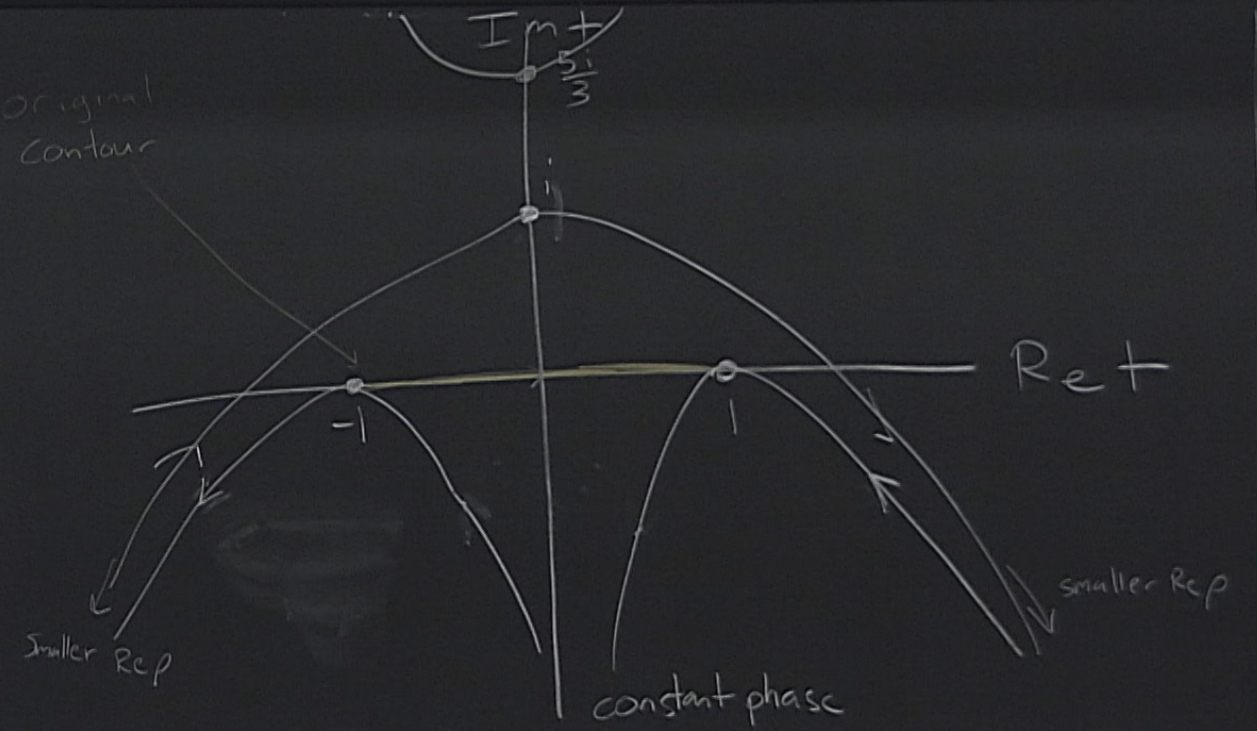
$+5iz+-iz+^3$

$+5iz+-iz+^3$

$z\rho(+)$

$^2 -i(+-i)^3$

$=i$ and $+=\frac{5i}{3}$

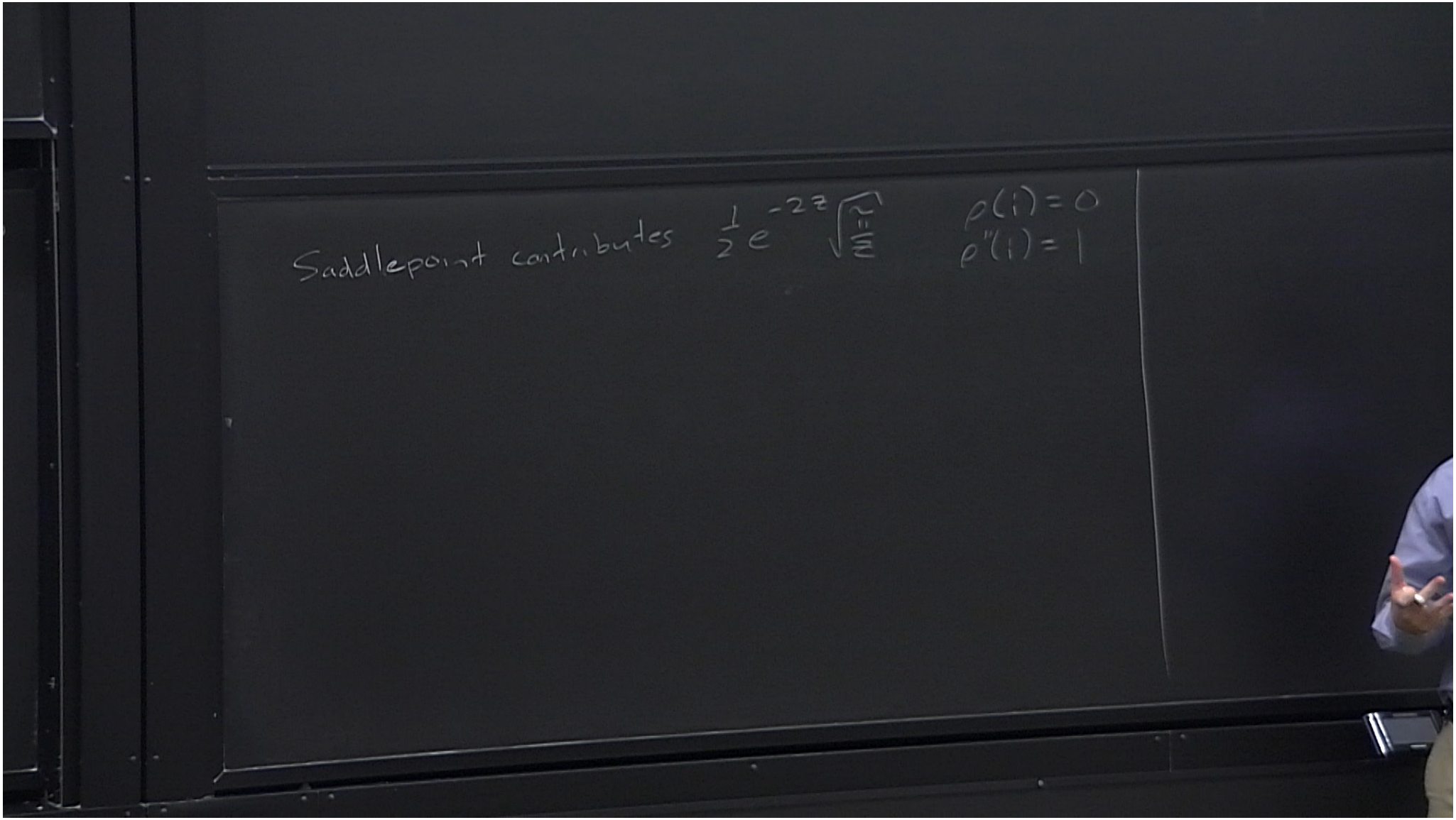


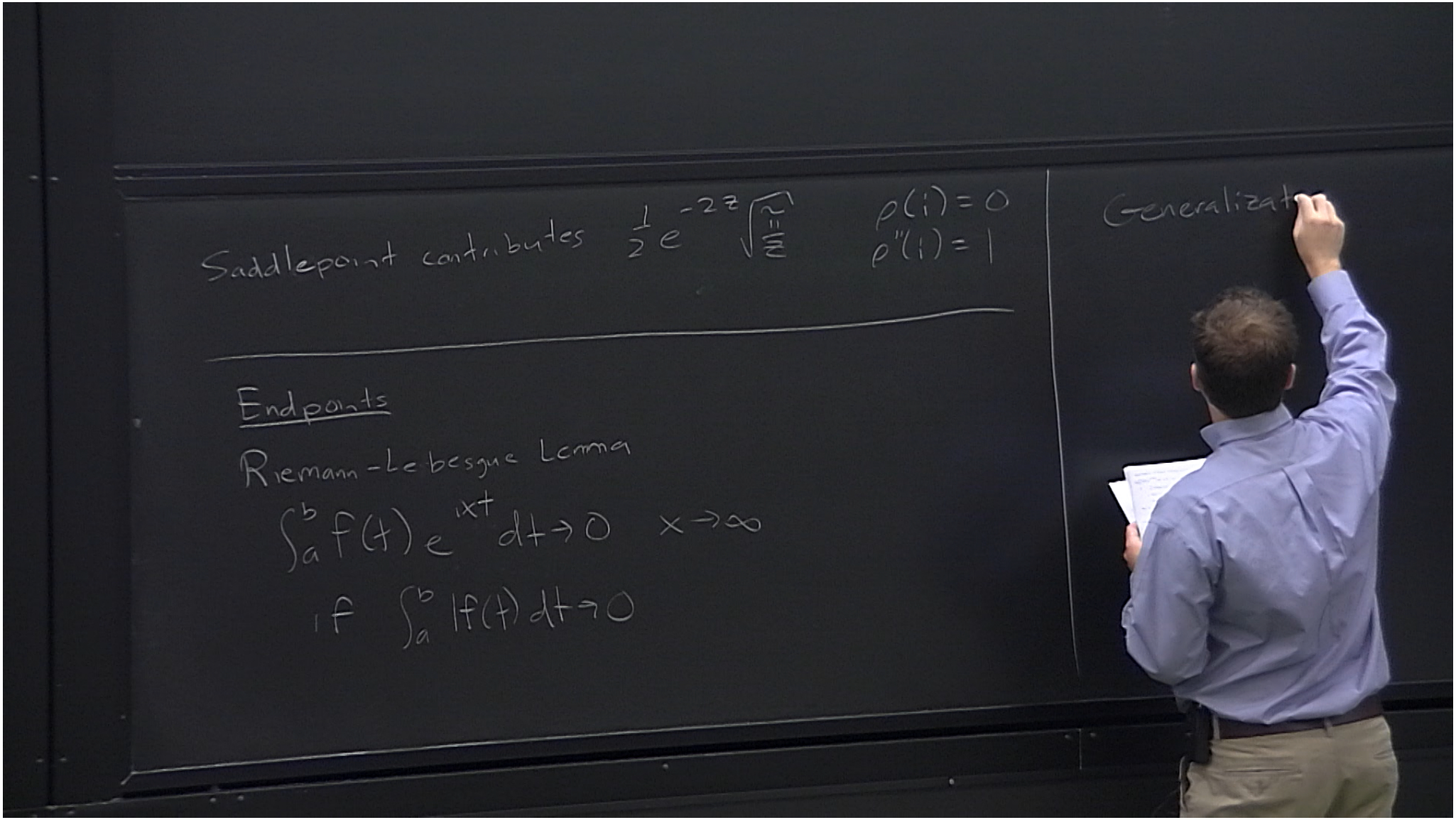
constant phas

$\mathrm{Im}\rho(+) =$

$\mathrm{Im}\rho(-1)$

$\mathrm{Im}\rho(1)$


Saddlepoint contributes $\frac{1}{2} e^{-2z} \sqrt{\frac{\pi}{z}}$
$\rho(1) = 0$
$\rho''(1) = 1$


Saddlepoint contributes $\frac{1}{2} e^{-2z} \sqrt{\frac{\pi}{z}}$
$\rho(i) = 0$
$\rho''(i) = 1$
Generalizat
Endpoints
Riemann-Lebesgue Lemma
$\int_a^b f(t) e^{ixt} dt \to 0 \quad x \to \infty$
if $\int_a^b |f(t)| dt \to 0$

Saddlepoint contributes $\frac{1}{2} e^{-2z} \sqrt{\frac{2\pi}{\rho''(i)}}$ $\rho(i) = 0$ $\rho''(i) = 1$

Endpoints

Riemann-Lebesgue Lemma

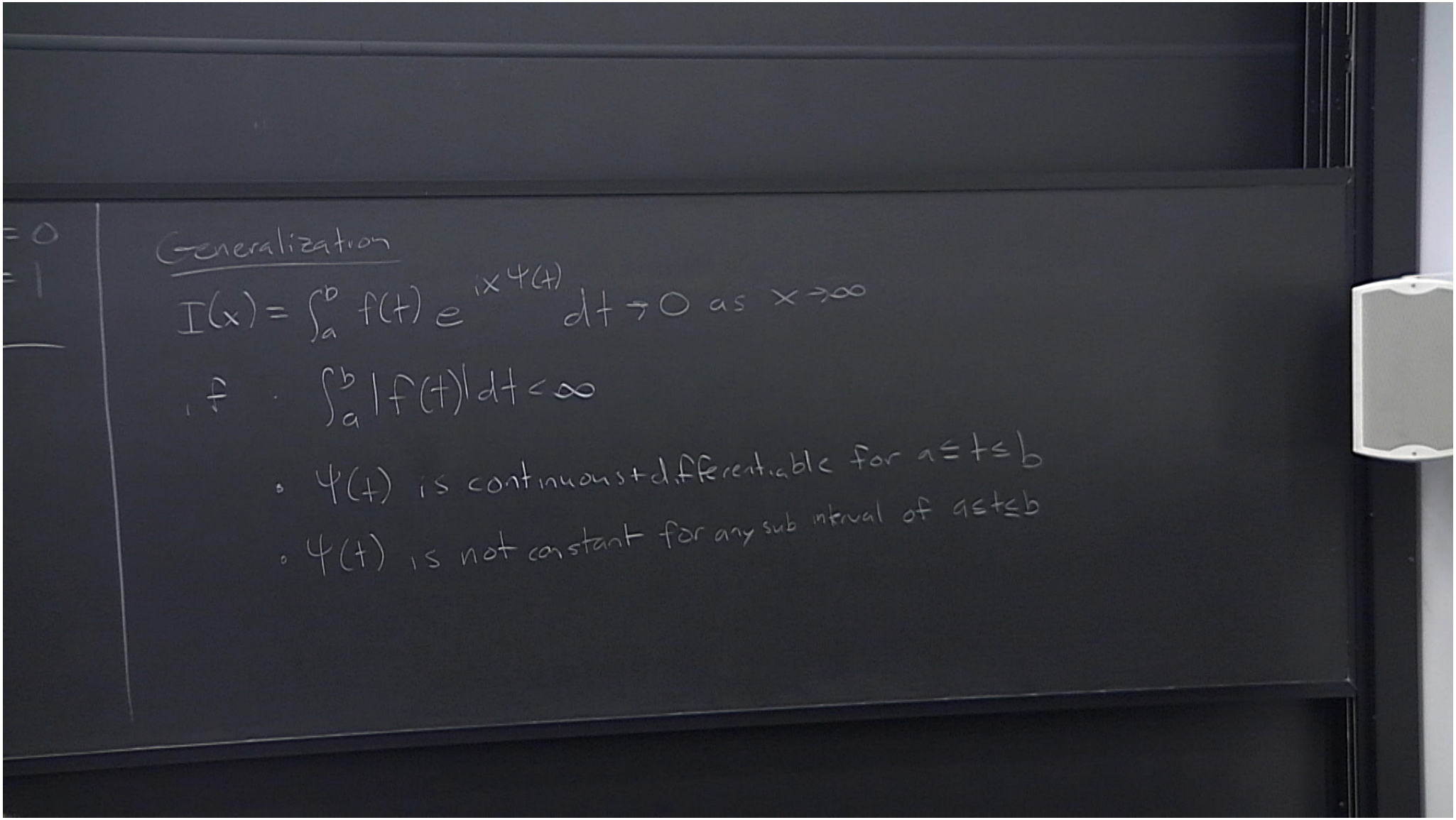
$\int_a^b f(t) e^{ixt} dt \to 0 \quad x \to \infty$

if $\int_a^b |f(t)| dt < \infty$

Generalization

$I(x) = \int_a^b f(t) e^{ix}$

if $\int_a^b |f(t)| d$


Generalization
$I(x) = \int_a^b f(t)\, e^{ix\psi(t)}\, dt \to 0$ as $x \to \infty$
if $\int_a^b |f(t)|\, dt < \infty$
• $\psi(t)$ is continuous + differentiable for $a \le t \le b$
• $\psi(t)$ is not constant for any sub interval of $a \le t \le b$

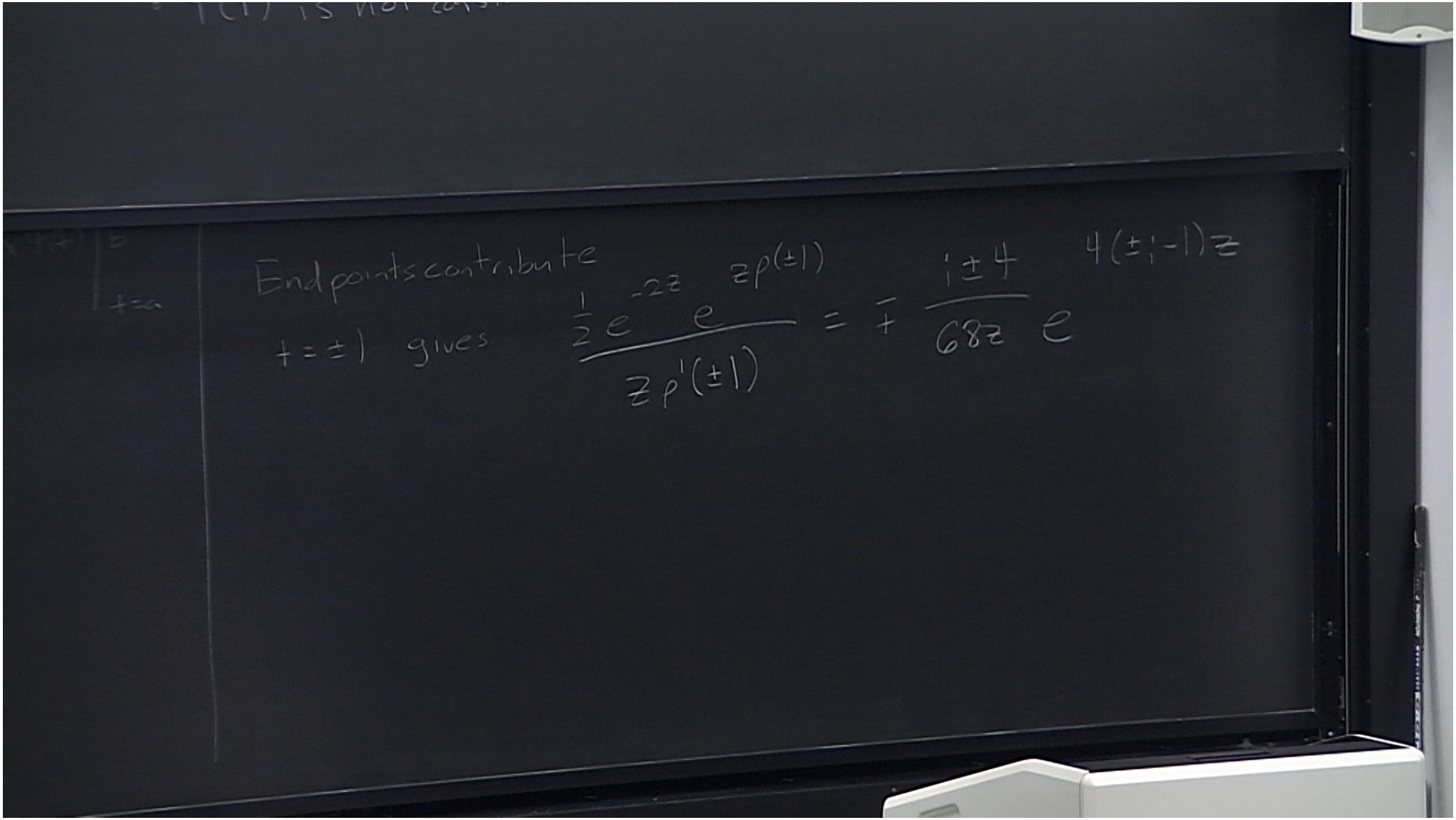
if $\int_a^b |f(t)|\,dt < \infty$

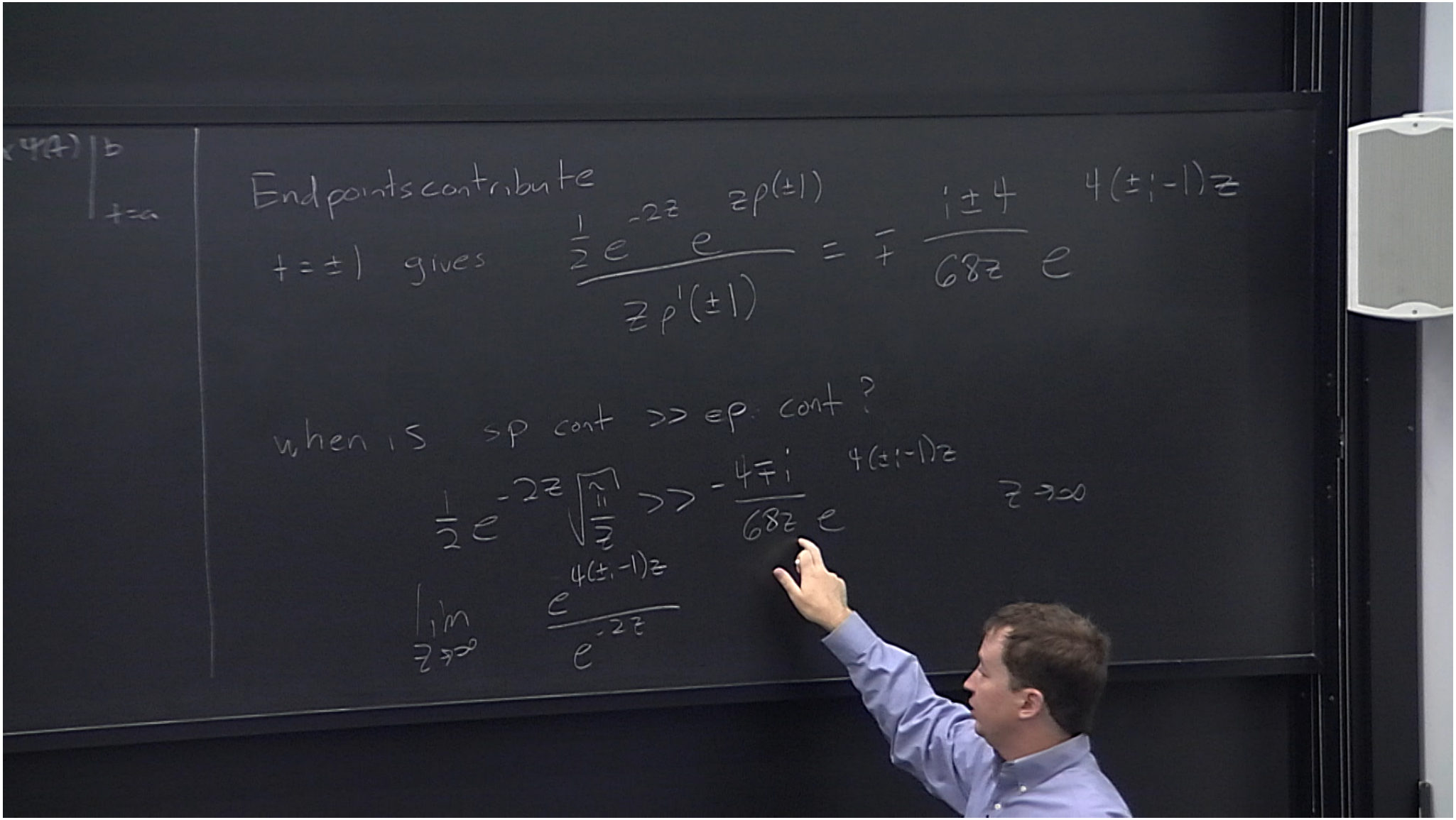
$$I(x) = -\frac{1}{ix}\int_a^b \frac{d}{dt}\frac{f(t)}{\psi'(t)} e^{ix\psi(t)}\,dt - \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)}\Big|_{t=a}^{t=b}$$

$$I(x) \sim \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)}\Big|_{t=a}^{t=b} \qquad x \to \infty$$

if above hold and

- $\frac{f(t)}{\psi'(t)}$ is cont. + diff. for $a \le x \le b$
- at least one of $\frac{f(a)}{\psi'(a)}, \frac{f(b)}{\psi'(b)} \neq 0$


Endpoints contribute
$t = \pm 1$ gives $\frac{\frac{1}{2} e^{-2z} e^{zp(\pm 1)}}{zp'(\pm 1)} = \mp \frac{i \pm 4}{68z} e^{4(\pm i - 1)z}$


Endpoints contribute
t = ±1 gives
when is sp cont >> ep cont ?
lim
z→∞

$\times 4(t) \Big|_a^b$

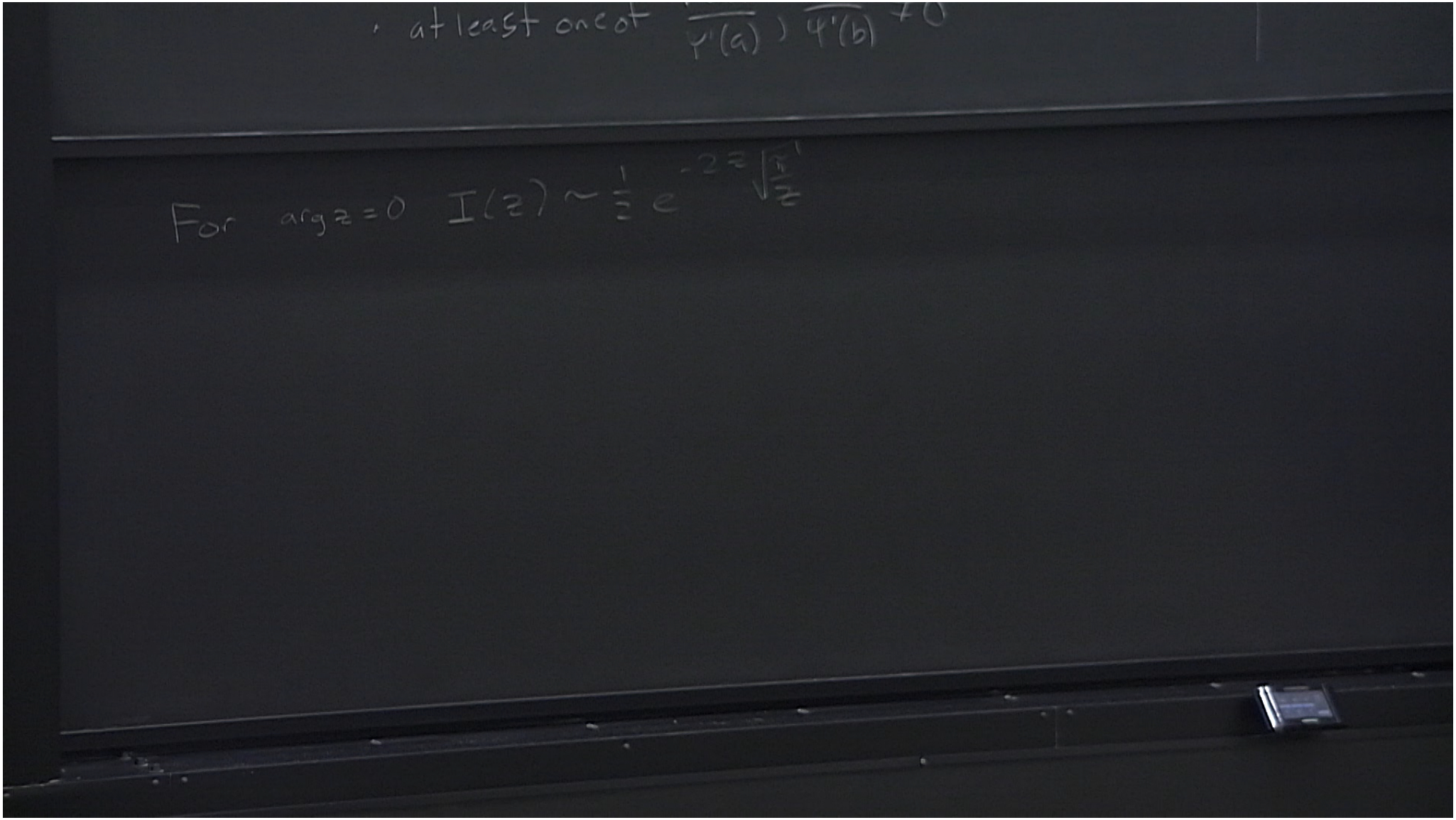
Endpoints contribute

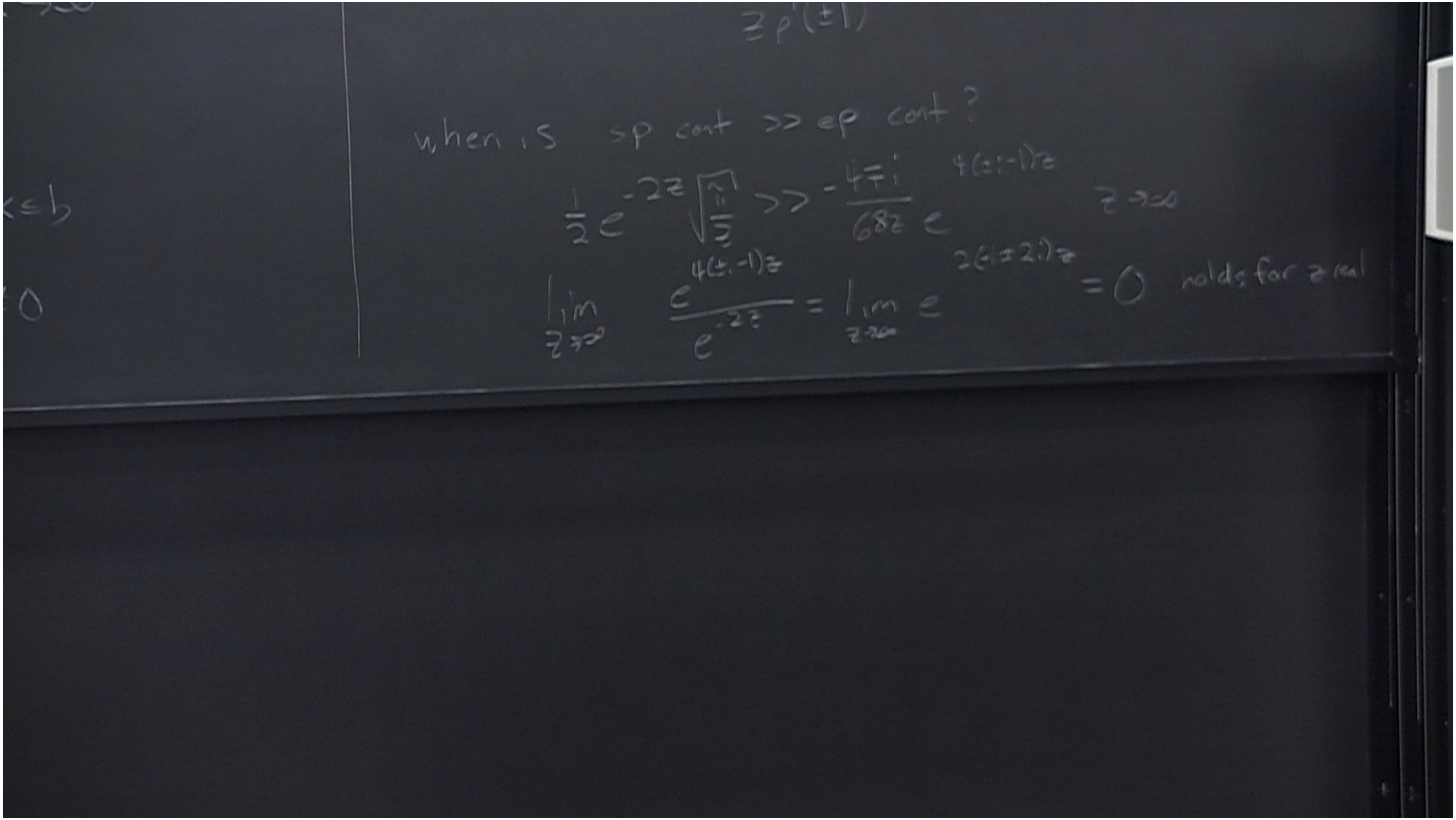
$t = \pm 1$ gives $\dfrac{\frac{1}{2} e^{-2z} e^{zp(\pm 1)}}{zp'(\pm 1)} = \mp \dfrac{i \pm 4}{68z} e^{4(\pm i, -1)z}$

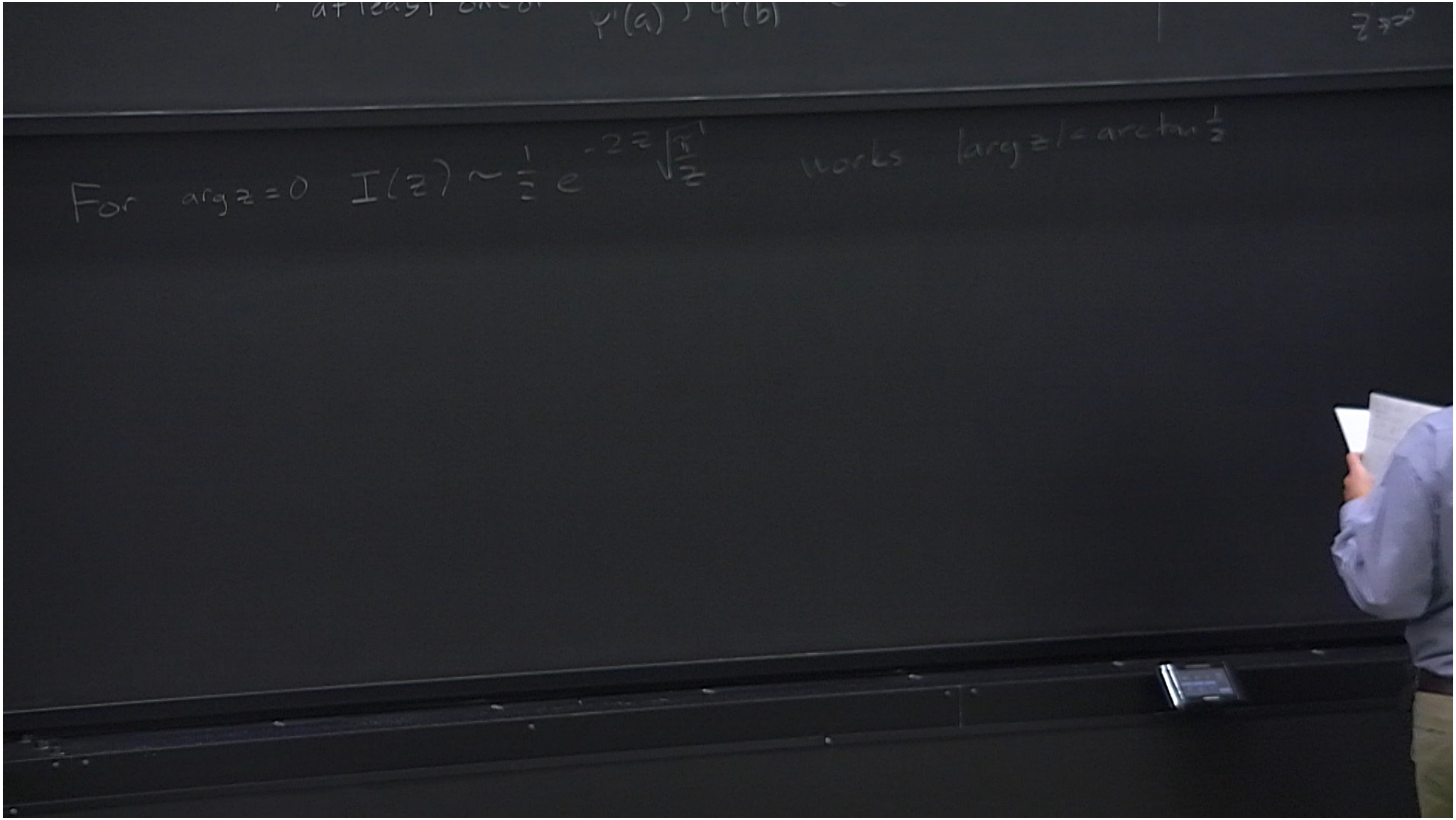
when is sp cont >> ep cont?

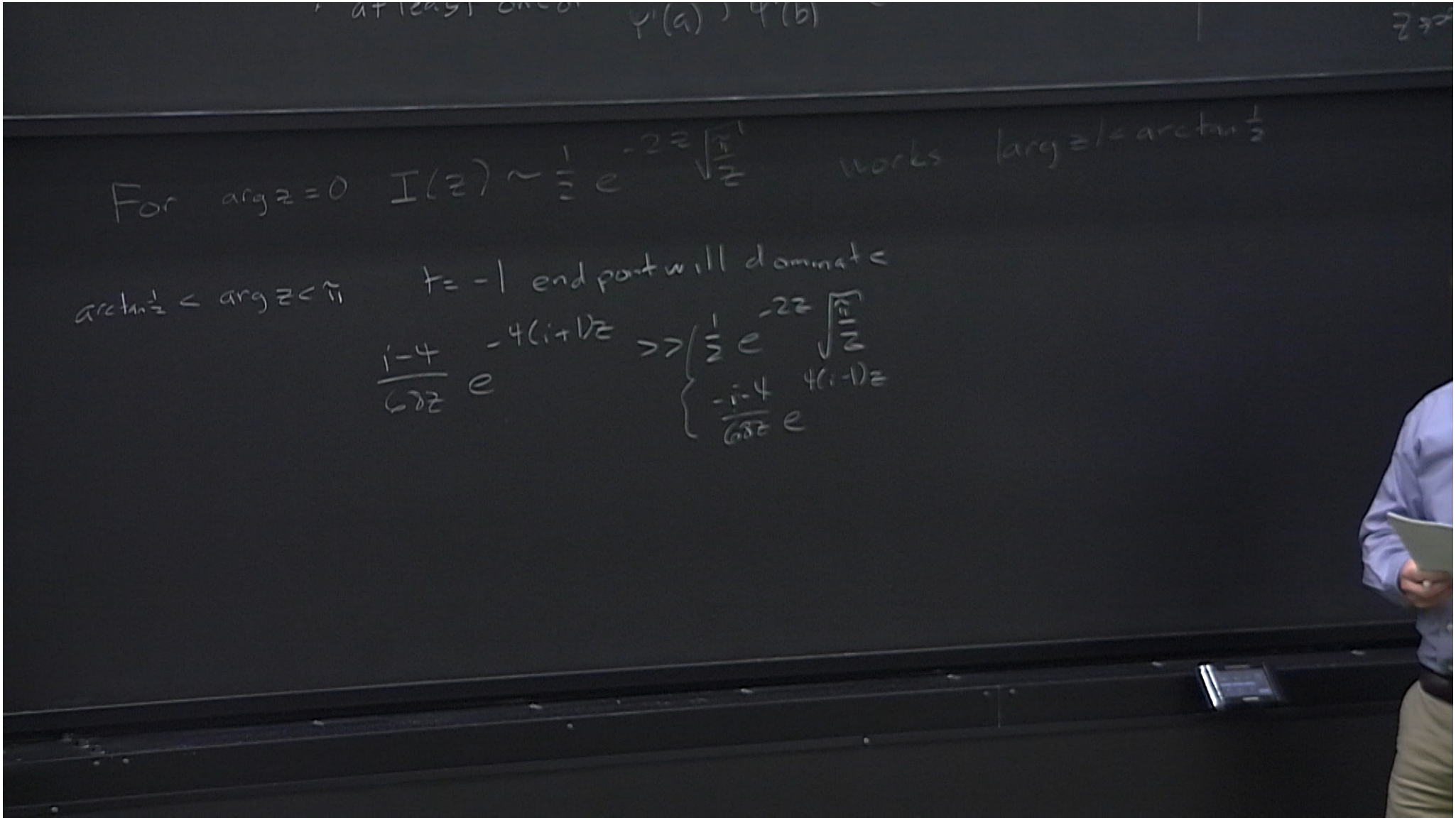
$\frac{1}{2} e^{-2z} \sqrt{\frac{\pi}{z}} >> -\frac{4 \mp i}{68z} e^{4(\pm i, -1)z}$ $\quad z \to \infty$

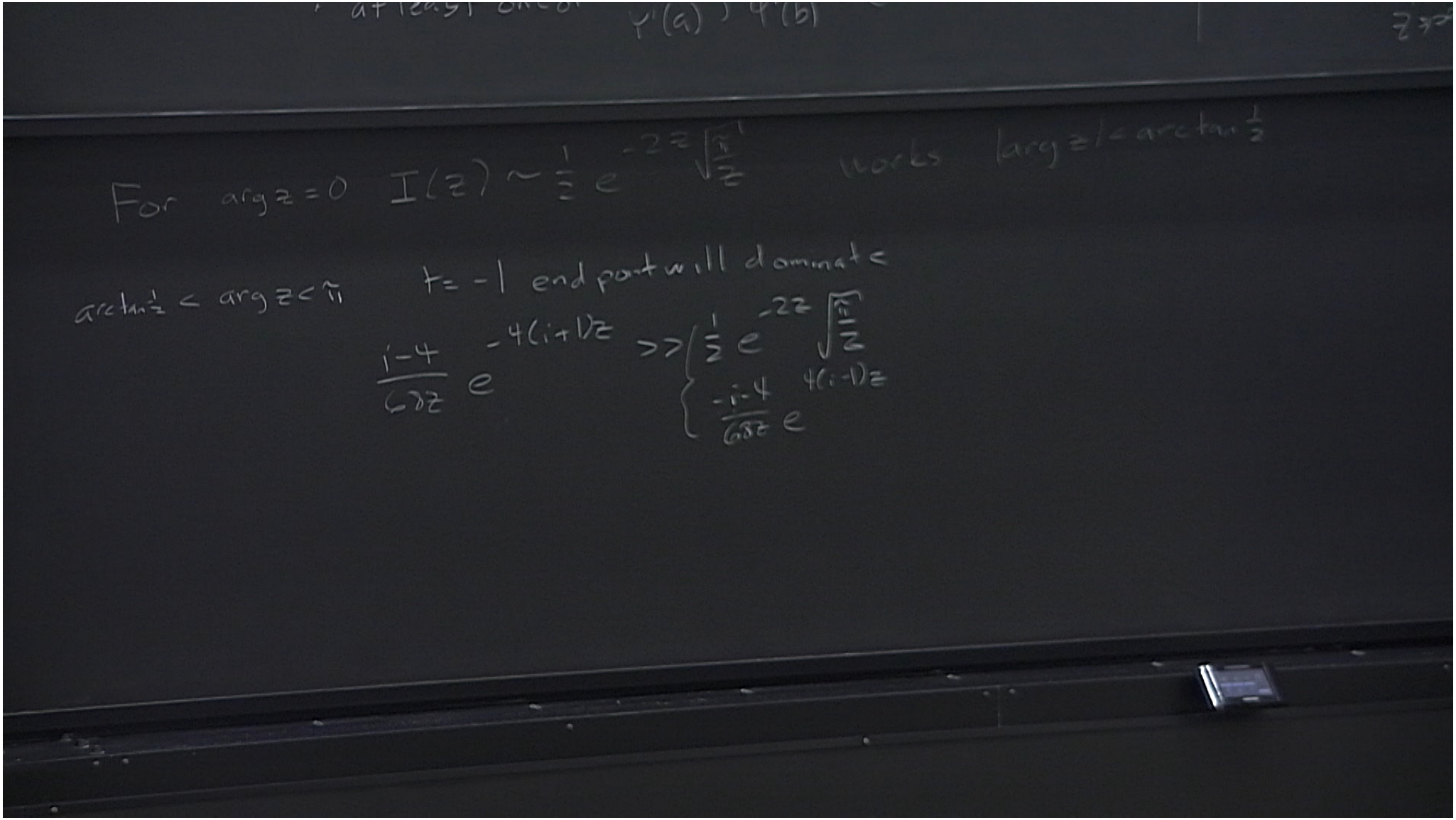
$\lim_{z \to \infty} \dfrac{e^{4(\pm i, -1)z}}{e^{-2z}} = \lim_{z \to \infty} e^{2(-1 \pm 2i)z} = 0$ holds for z real

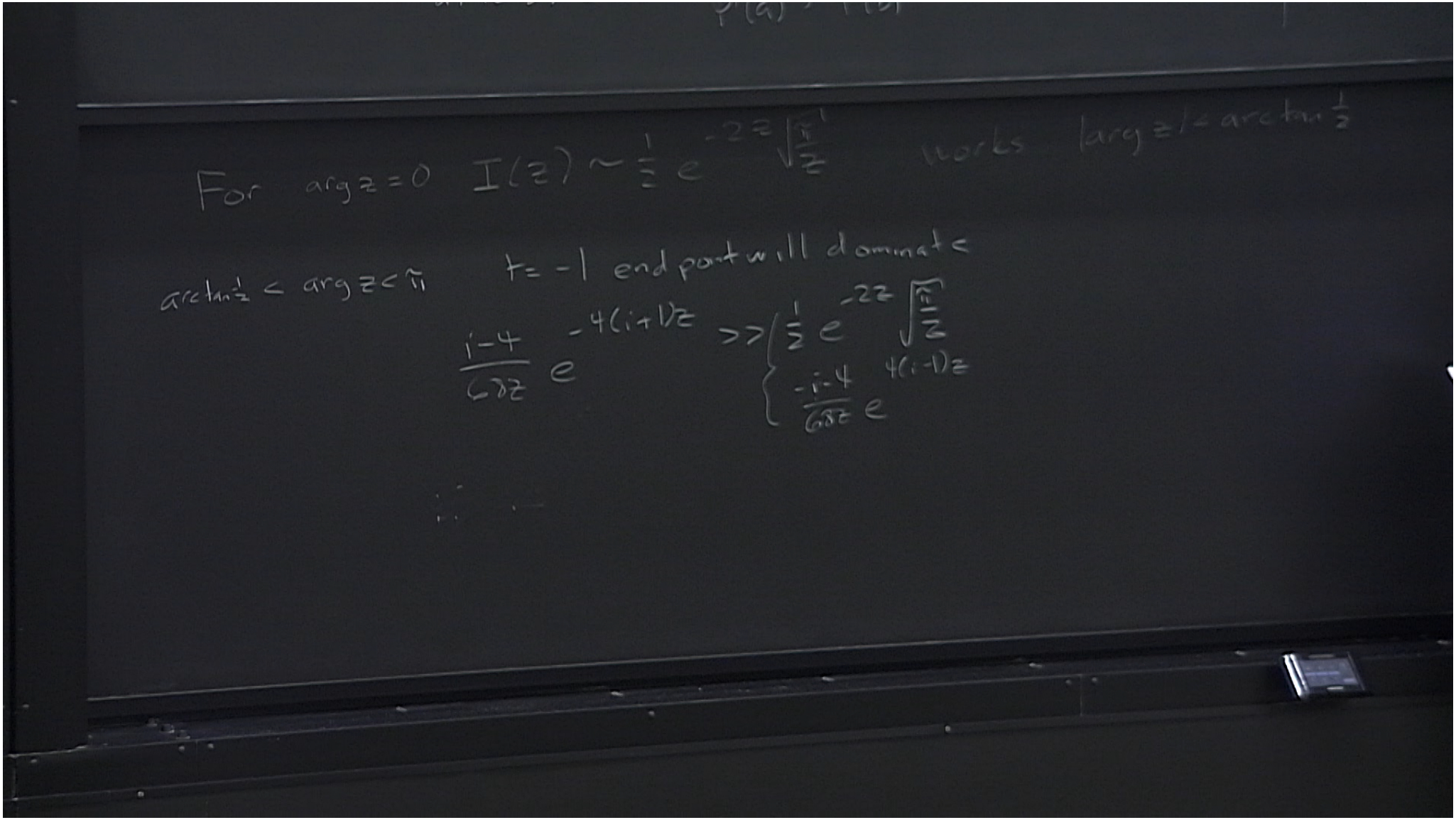

· at least one of
For arg z = 0

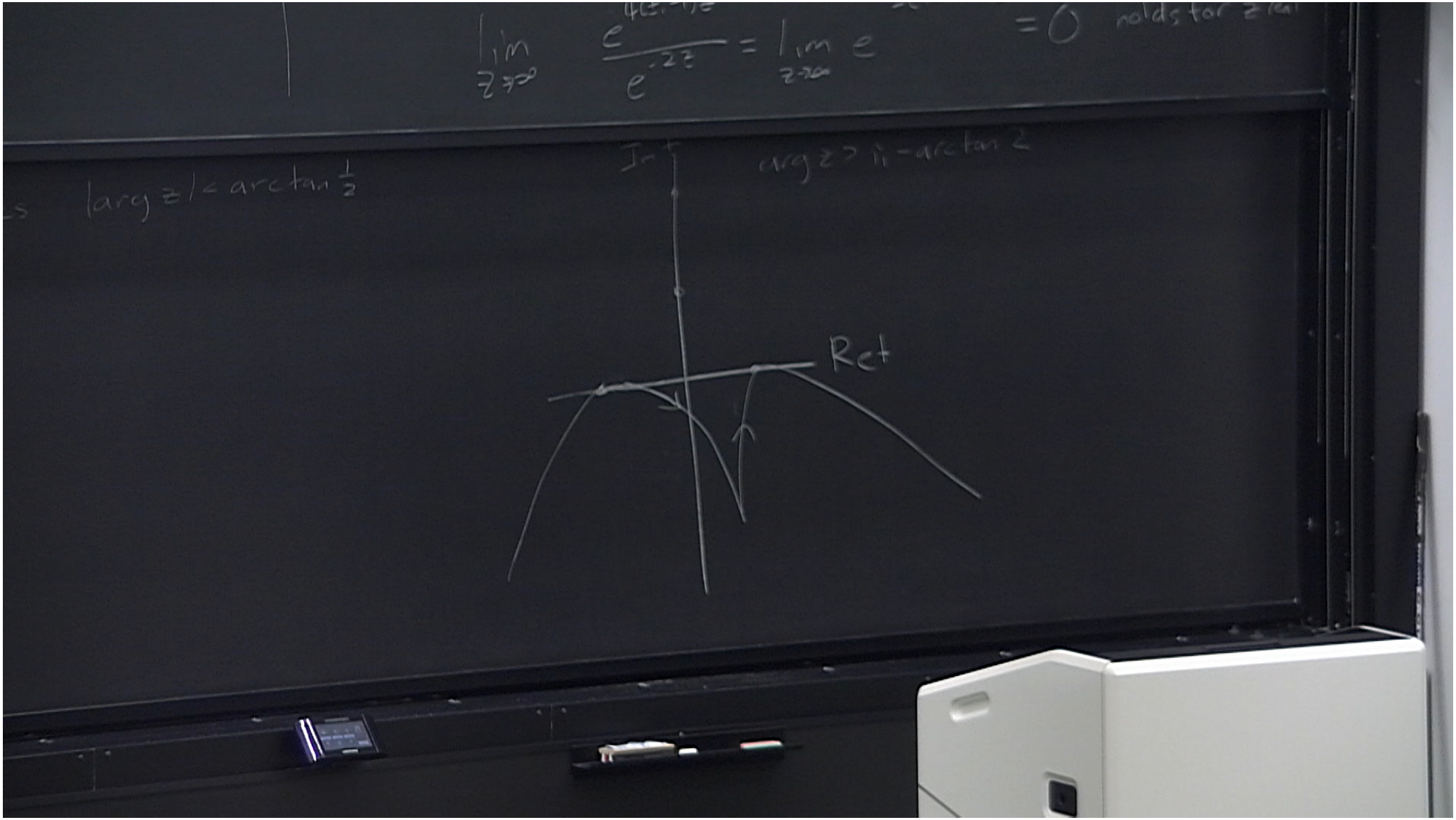

$z\rho'(\pm 1)$
when is sp cont >> ep cont?
$\frac{1}{2}e^{-2z}\sqrt{\frac{\pi}{z}} >> -\frac{4\mp i}{68z}e^{4(\pm i-1)z}$ $z\to\infty$
$\lim_{z\to\infty}\frac{e^{4(\pm i-1)z}}{e^{-2z}} = \lim_{z\to\infty} e^{2(\pm i+2)z} = 0$ holds for z real


For arg z = 0 $I(z) \sim \frac{1}{z} e^{-2z} \sqrt{\frac{\pi}{2z}}$ works $|\arg z| < \arctan \frac{1}{2}$


For arg z = 0 I(z) ~ $\frac{1}{2} e^{-2z} \sqrt{\frac{\pi}{z}}$ works arg z < arctan $\frac{1}{2}$
arctan $\frac{1}{2}$ < arg z < π t = -1 endpoint will dominate
$\frac{i-4}{6\pi z} e^{-4(i+1)z}$ >> $\frac{1}{2} e^{-2z} \sqrt{\frac{\pi}{z}}$
$\frac{-i-4}{6\pi z} e^{4(i-1)z}$


For arg z = 0 I(z) ~ $\frac{1}{z} e^{-2z}\sqrt{\frac{\pi}{z}}$ works arg z < arctan $\frac{1}{2}$
arctan $\frac{1}{2}$ < arg z < π
t = -1 end part will dominate
$\frac{i-4}{68z} e^{-4(i+1)z}$ >> $\begin{cases} \frac{1}{2} e^{-2z}\sqrt{\frac{\pi}{z}} \\ \frac{-i-4}{68z} e^{4(i-1)z} \end{cases}$


For arg z = 0 I(z) ~ $\frac{1}{z} e^{-2z}\sqrt{\frac{\pi}{z}}$ works for arg z < arctan $\frac{1}{2}$
arctan $\frac{1}{2}$ < arg z < $\tilde{\pi}$ t = -1 end part will dominate
$\frac{i-4}{6\sqrt{2}} e^{-4(i+1)z}$ >> $\frac{1}{2} e^{-2z}\sqrt{\frac{\pi}{z}}$
$\frac{-i-4}{6\sqrt{2}} e^{4(i-1)z}$


Im
Re
arg z > π - arctan 2
arg z < arctan 1/2

$p'(t)=0$ at $t=i$ and $t=\frac{5i}{3}$

smaller Rep

const

"Summation" of Divergent Series

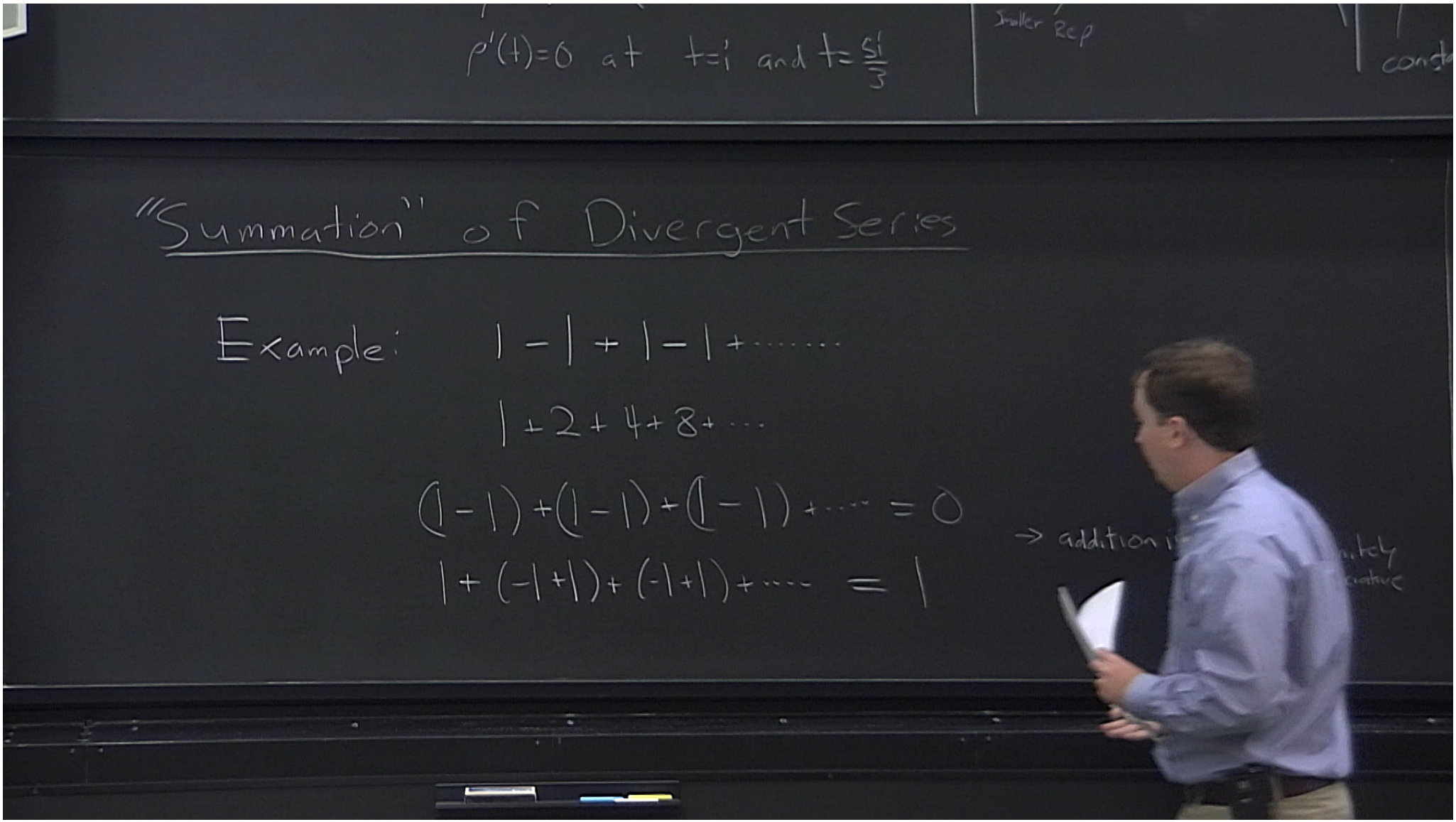
Example: $1-1+1-1+\cdots\cdots$

$1+2+4+8+\cdots$

$(1-1)+(1-1)+(1-1)+\cdots = 0$

$1+(-1+1)+(-1+1)+\cdots = 1$

→ addition i… …tely …iative



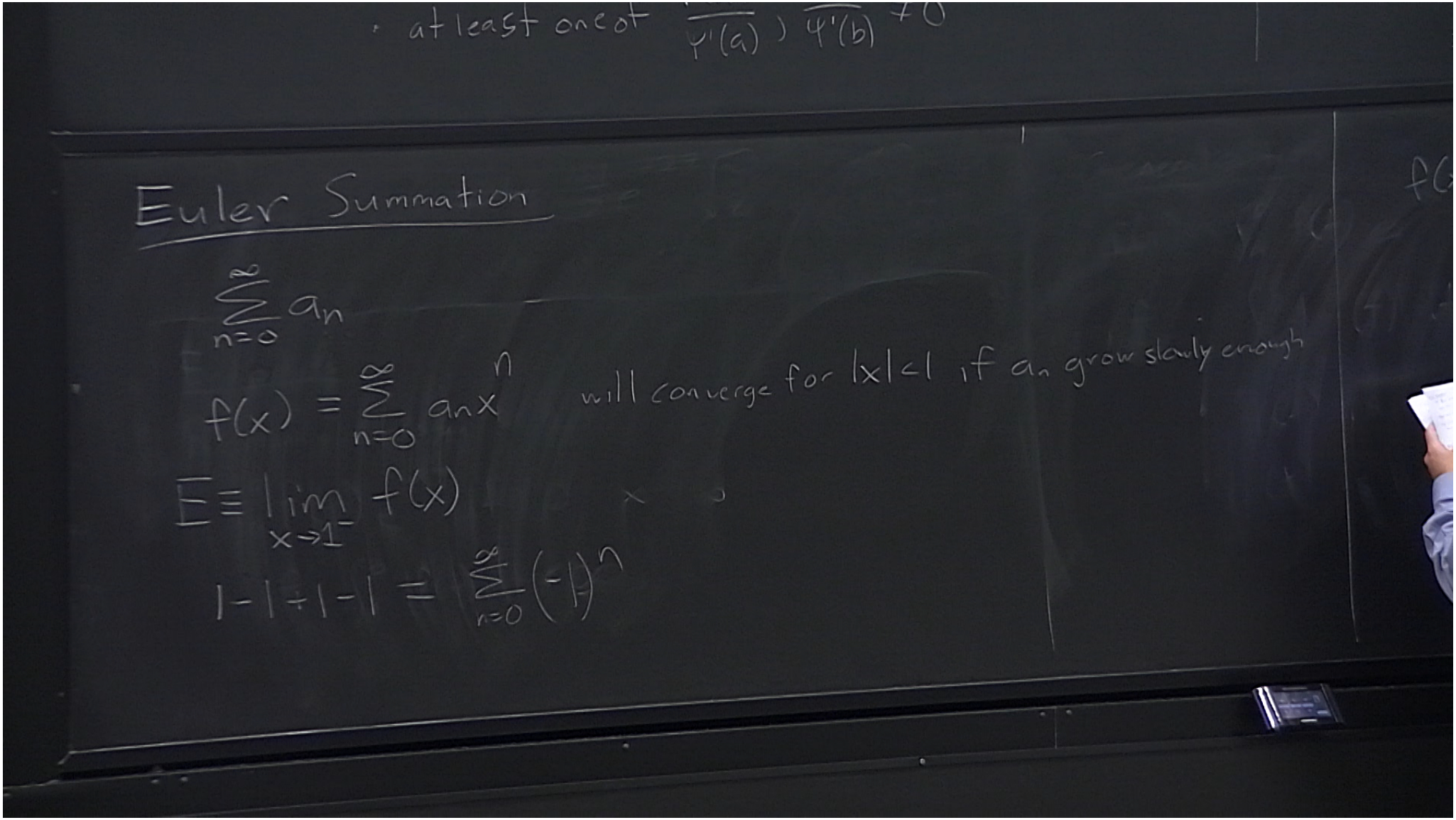
$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

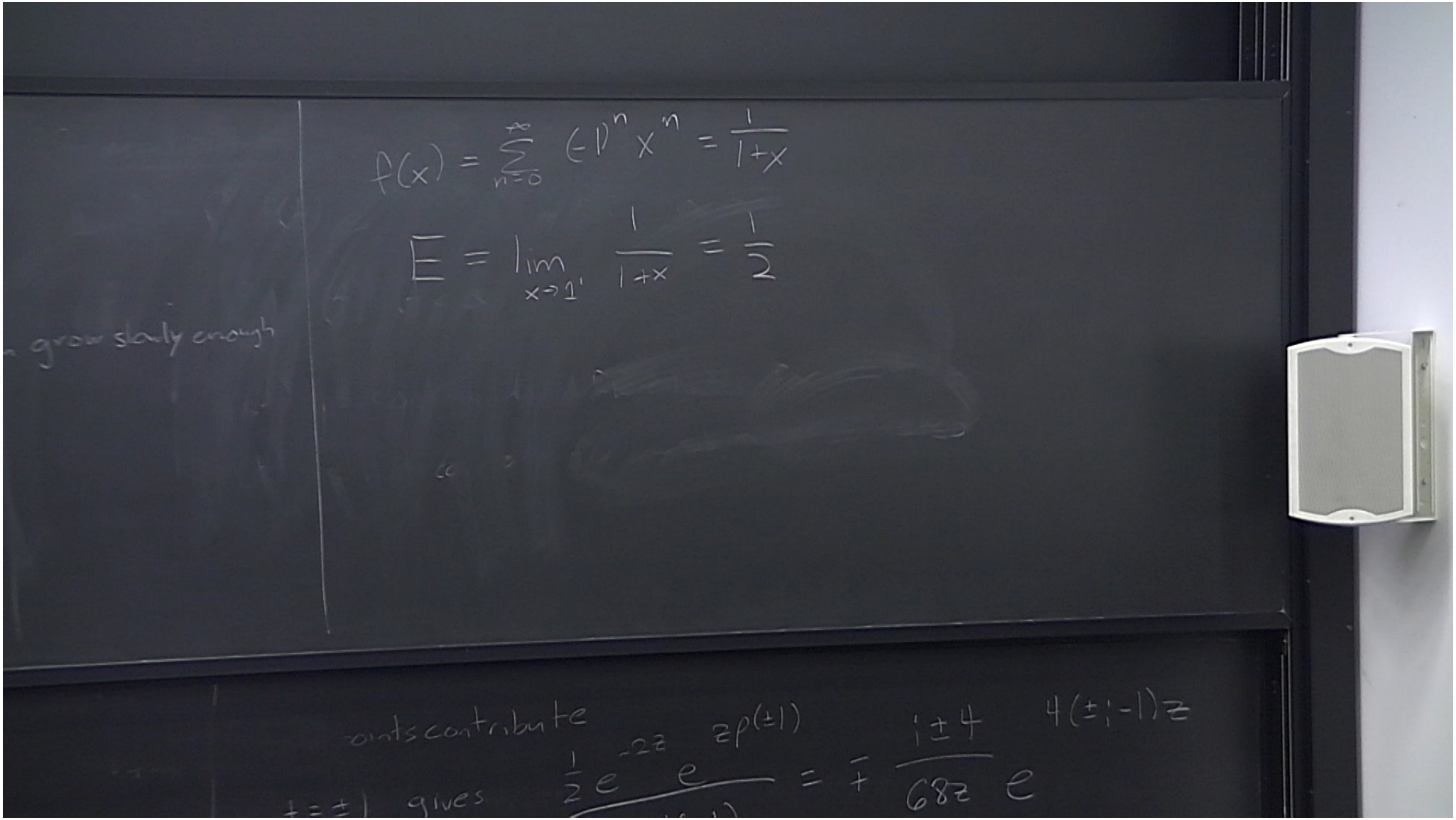
$1 + \frac{1}{3} + \frac{1}{5} + \cdots$ diverges

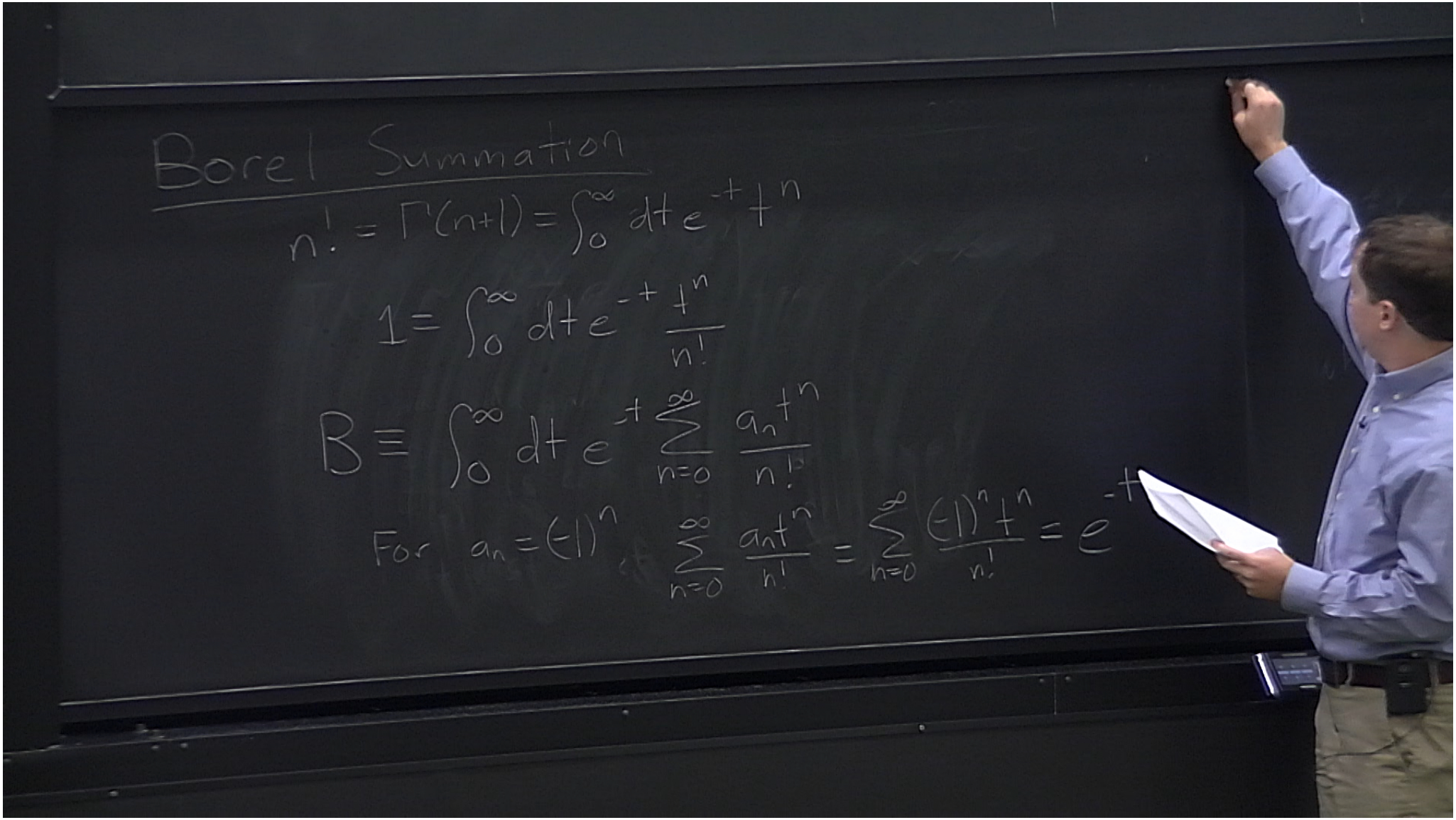
$$\tilde{\pi} = \underbrace{1 + \frac{1}{3} + \frac{1}{5} + \cdots}_{> \tilde{\pi}_1} \underbrace{- \frac{1}{2} - \frac{1}{4} - \cdots}_{< \tilde{\pi}_1} + \cdots$$

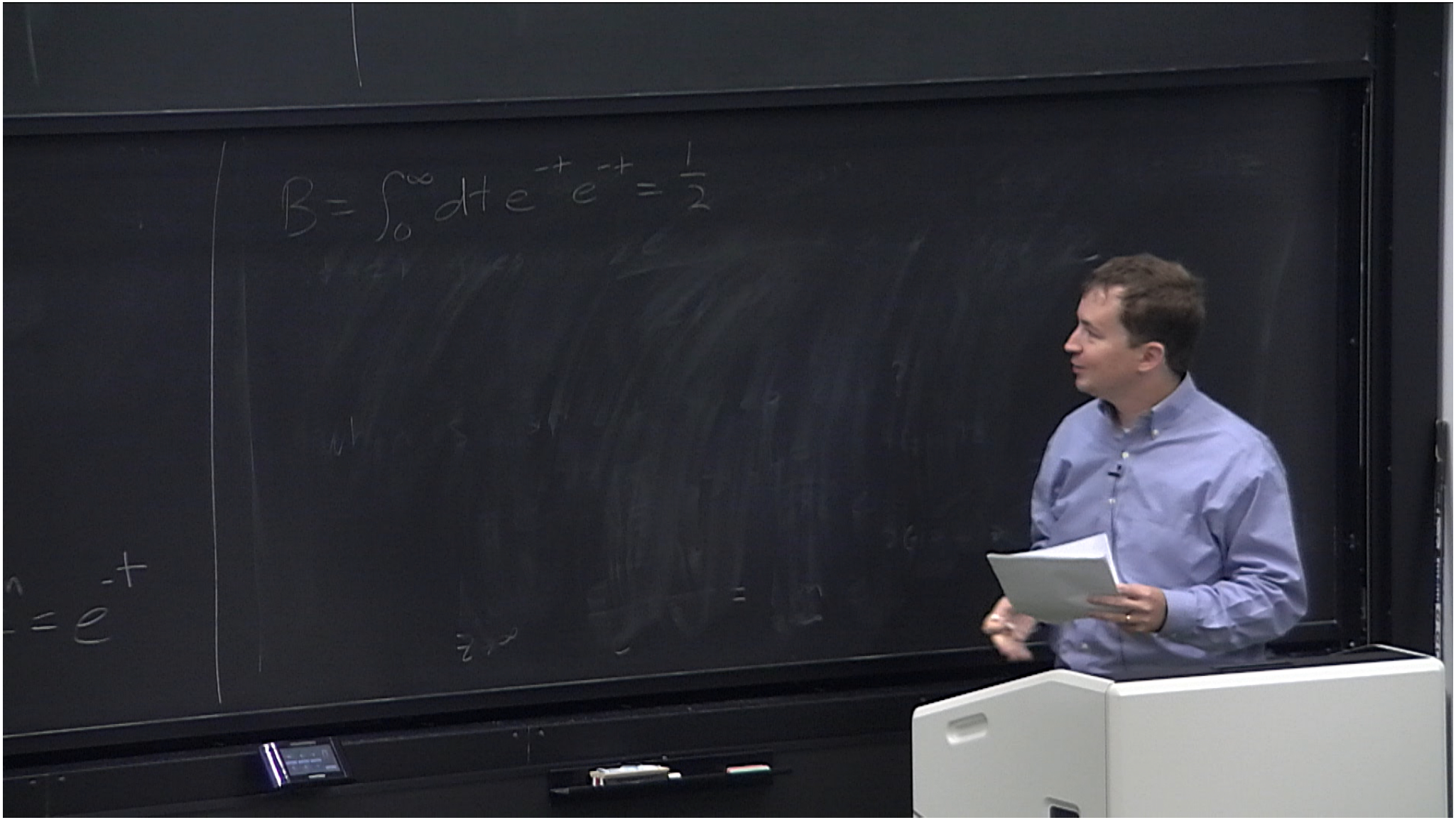
→ addition is not infinitely commutative

infinitely associative


· at least one of $\frac{1}{\varphi'(a)}$, $\frac{1}{\varphi'(b)} \neq 0$
Euler Summation
$\sum_{n=0}^{\infty} a_n$
$f(x) = \sum_{n=0}^{\infty} a_n x^n$ will converge for $|x|<1$ if a_n grow slowly enough
$E \equiv \lim_{x \to 1^-} f(x)$
$1-1+1-1 = \sum_{n=0}^{\infty} (-1)^n$


$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$
$E = \lim_{x \to 1^-} \frac{1}{1+x} = \frac{1}{2}$
grow slowly enough
oints contribute
$t = \pm 1$ gives
$\frac{1}{2} e^{-2z} \exp(\pm 1)$
$= \mp \frac{i \pm 4}{68z} e^{4(\pm i - 1)z}$


Borel Summation
$n! = \Gamma(n+1) = \int_0^\infty dt\, e^{-t} t^n$
$1 = \int_0^\infty dt\, e^{-t} \frac{t^n}{n!}$
$B \equiv \int_0^\infty dt\, e^{-t} \sum_{n=0}^{\infty} \frac{a_n t^n}{n!}$
For $a_n = (-1)^n$, $\sum_{n=0}^{\infty} \frac{a_n t^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} = e^{-t}$

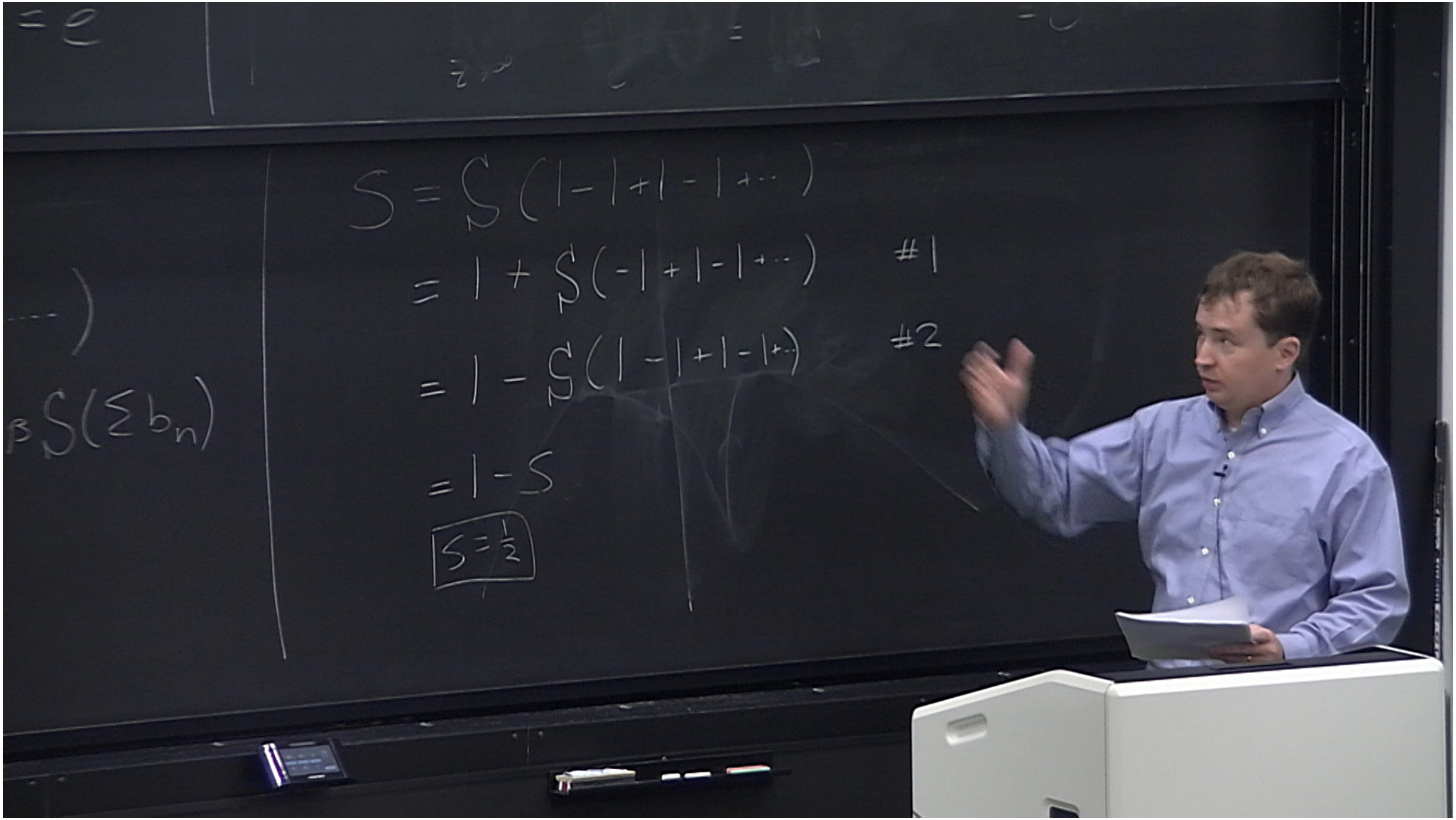

$B = \int_0^\infty dt\, e^{-t} e^{-t} = \frac{1}{2}$

$\sum_{n=0} \frac{}{n!}$ $\sum_{n=0} \frac{}{n!}$

Generic Summation $\sum$

1. $\sum(a_0 + a_1 + a_2 + \cdots) = a_0 + \sum(a_1 + a_2 + \cdots)$

2. $\sum\left(\sum_n \alpha a_n + \beta b_n\right) = \alpha \sum(\Sigma a_n) + \beta \sum(\Sigma b_n)$


$S = S(1-1+1-1+\cdots)$
$= 1 + S(-1+1-1+\cdots)$ #1
$= 1 - S(1-1+1-1+\cdots)$ #2
$= 1 - S$
$S = \frac{1}{2}$
$\beta S(\Sigma b_n)$