Title: PSI Front End - Functions, "Functions", etc.

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Abstract:

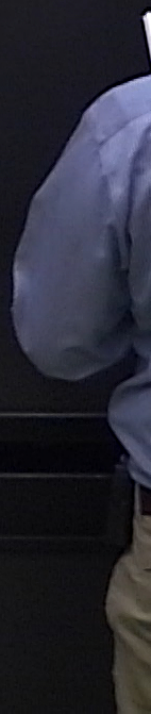
Green's Functions + Orthogonal Functions

$$\Omega u_\lambda(x) + \lambda w(x) u_\lambda(x) = 0 \qquad \text{in interval } [a,b]$$

$$L = \frac{\Omega}{w} \qquad \Omega = \frac{d}{dx}\left[s(x)\frac{d}{dx}\right] + t(x)$$

$$\int_a^b u_\rho \Omega u_\lambda dx = -\lambda \int_a^b u_\rho u_\lambda w\, dx$$

$$\int_a^b (\Omega u_\rho) u_\lambda dx = -\rho \int_a^b u_\rho u_\lambda w\, dx$$



$$\int_a^b u_\rho \Omega u_\lambda \, dx - \int_a^b (\Omega u_\rho) u_\lambda = (\rho - \lambda) \underbrace{\int_a^b u_\rho u_\lambda w \, dx}$$

if Ω is truly self-adjoint $\longrightarrow$ $\langle u_\rho, u_\lambda \rangle_w$

if
$$\left.\begin{aligned} \alpha_a y(a) + \beta_a y'(a) &= 0 \\ \alpha_b y(b) + \beta_b y'(b) &= 0 \end{aligned}\right\} \rightarrow \Omega \text{ is self adjoint}$$

$$\alpha_a^2 + \beta_a^2 > 0$$

$$\alpha_b^2 + \beta_b^2 > 0$$

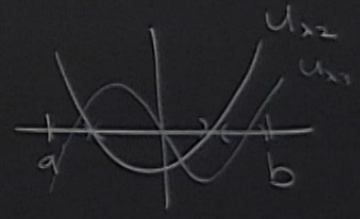
$$\int_a^b (\Omega(u_p))\, u_\lambda \, dx = -\rho \int_a^b u_p u_\lambda w \, dx$$

$\alpha_0^2 + \beta_0^2$

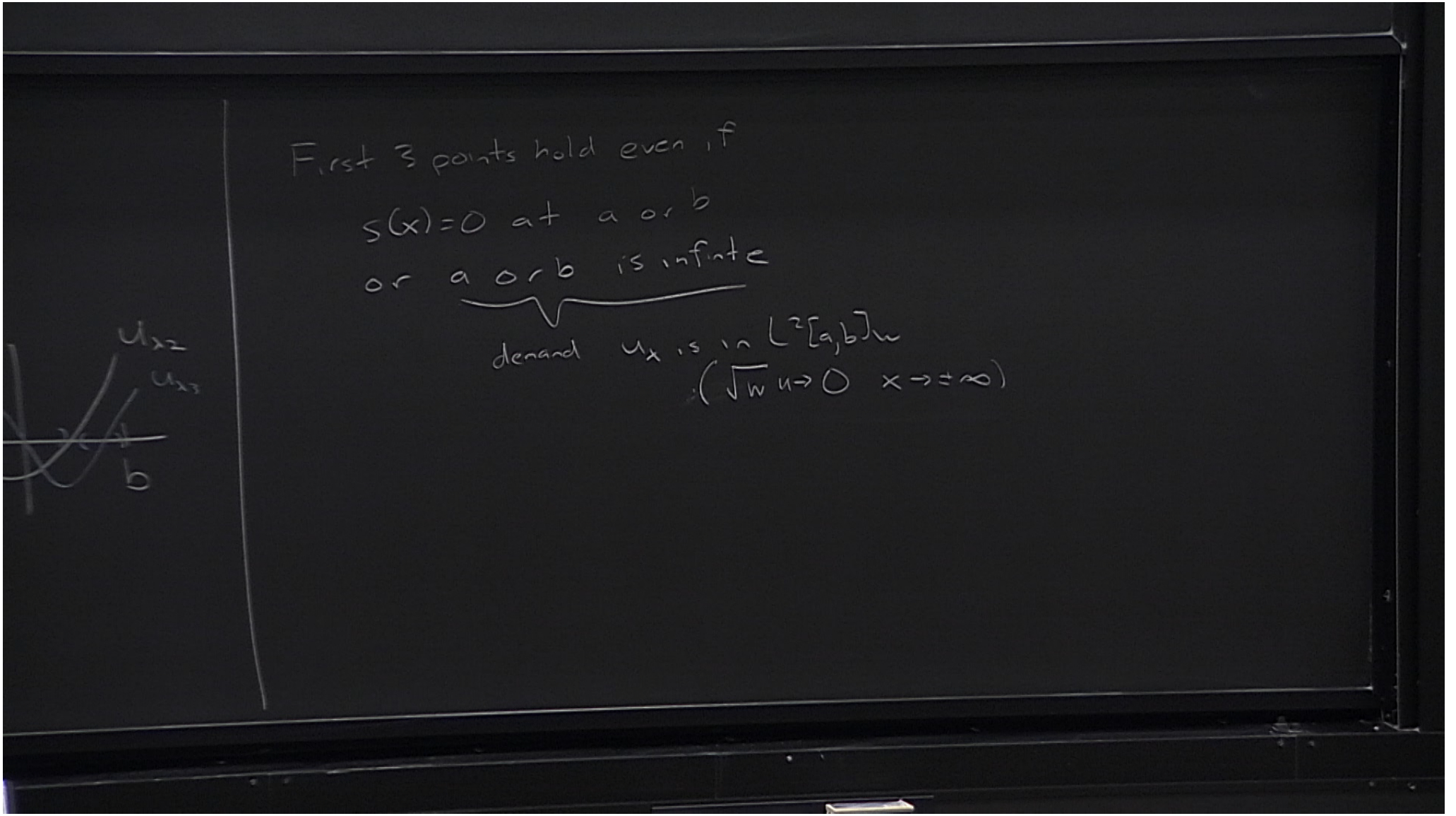
if Ω is self-adjoint

- eigenvalues will be real + distinct $\quad -\infty < \lambda_0 < \lambda_1 < \cdots$
- $u_{\lambda n}$ will form an orthonormal basis of $L^2[a,b]_w$

$$\int_a^b u_{\lambda n}(x)\, u_{\lambda m}(x)\, w(x)\, dx = \delta_{nm}$$



- zeros of $u_{\lambda n+1}$ are in between the zeros of $u_{\lambda n}$
- $u_{\lambda n}$ has exactly n distinct zeros in (a,b)


First 3 points hold even if
s(x)=0 at a or b
or a or b is infinte
demand u_x is in $L^2[a,b]_w$
$(\sqrt{w}\,u \to 0 \quad x \to \pm\infty)$
u_{x2}
u_{x3}
b

Method 3: Eigenfunctions

$$Ly = f$$

L has complete set of eigenfunction ϕ_n with eigenvalues λ_n

$$y = \sum_{n=1}^{\infty} \alpha_n \phi_n \qquad \alpha_n = \langle y, \phi_n \rangle$$

$$f = \sum_{n=1}^{\infty} \beta_n \phi_n \qquad \beta_n = \langle f, \phi_n \rangle$$

$$L \sum_{n=1}^{\infty} \alpha_n \phi_n = \sum_{n=1}^{\infty} \beta_n \phi_n$$

$$\sum_{n=1}^{\infty} \lambda_n \alpha_n \phi_n = \sum_{n=1}^{\infty} \beta_n \phi_n$$

$$\lambda_n \alpha_n = \beta_n$$

if all $\lambda_n \neq 0$

$$y = \sum_{n=1}^{\infty} \frac{\beta_n}{\lambda_n} \phi_n$$

$$y(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n(x) \int \phi_n^*(x') f(x') dx'$$

Orthogonal Polynomials

Polynomials form a complete basis of $L^2[a,b]_w$

Gram-Schmidt:

$$P_0(x) = \frac{1}{\|1\|_w}$$

$$P_{n+1} = \frac{xP_n - \sum_{i=0}^{n} P_i \langle P_i, xP_n \rangle_w}{\alpha_{n+1}}$$

Orthogonal Polynomials

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$$\langle P_i, xP_n \rangle_w = \int_a^b dx\, w(P_i x) P_n$$

$$= \langle xP_i, P_n \rangle_w = 0 \quad \text{for } i < n-1$$

$= \langle x P_i, P_n \rangle_w = 0$ for $i < n-1$

$\int_a^b P_n(x)\, x^p\, w(x)\, dx = 0 \quad p < n$

$n=1 \quad p=0$

$\int_a^b P_1(x)\, w(x)\, dx = 0$

$w(x) P_1(x) \propto \frac{d}{dx} Y_1$

$Y_1(a) = 0 = Y_1(b)$

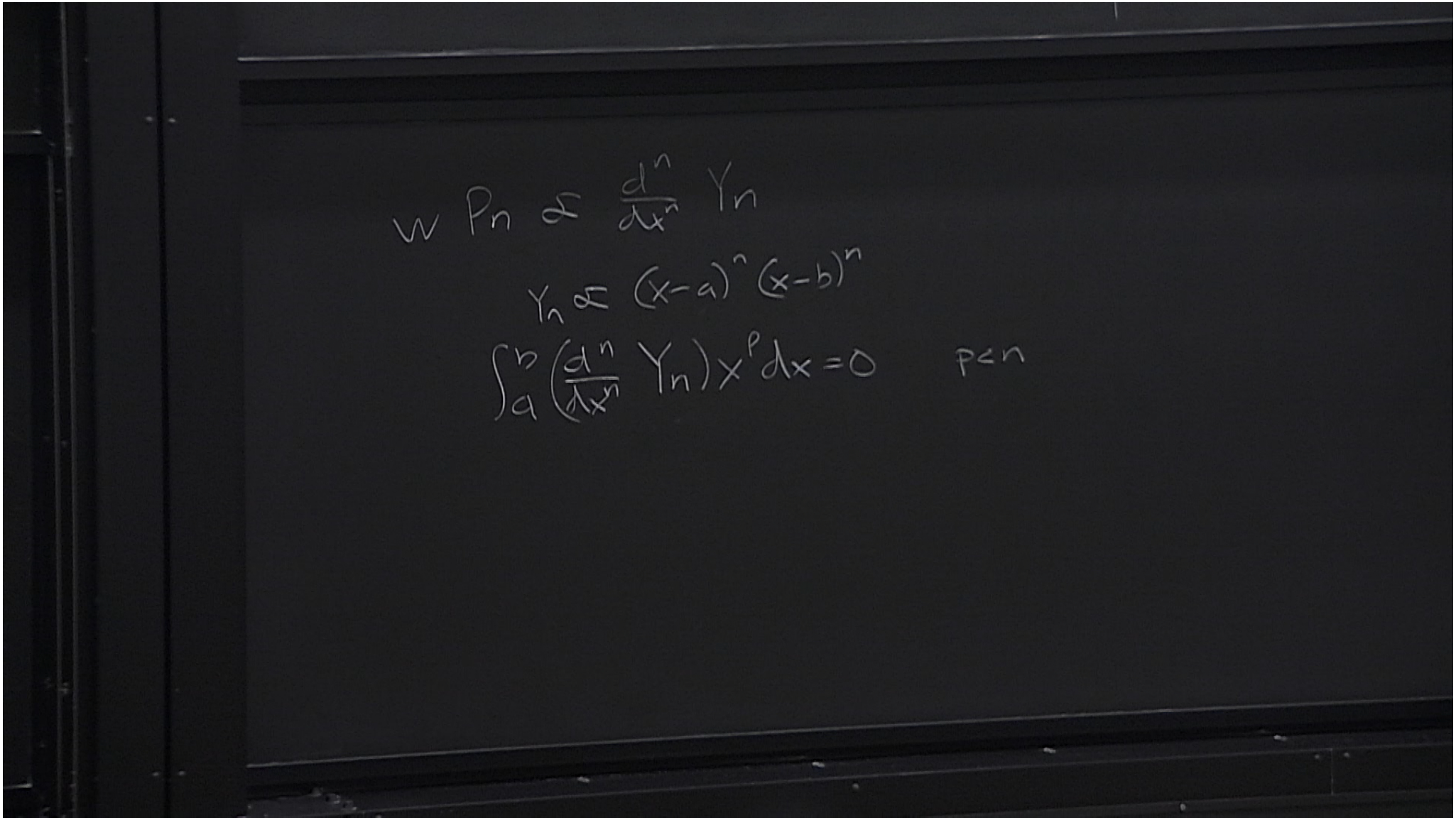
$Y_1(x) = (x-a)(x-b)$

$w P_2 \propto \frac{d^2}{dx^2} Y_2$

$Y_2 \propto (x-a)^2 (x-b)^2$

$\int_a^b w P_2 x\, dx \propto \int_a^b \left(\frac{d^2}{dx^2} Y_2\right) x\, dx$

$= \left(\frac{d}{dx} Y_2\right) x \Big|_a^b - \int_a^b \left(\frac{d}{dx} Y_2\right) dx$


w $P_n \propto \frac{d^n}{dx^n} Y_n$
$Y_n \propto (x-a)^n (x-b)^n$
$\int_a^b \left(\frac{d^n}{dx^n} Y_n\right) x^p dx = 0 \quad p<n$

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$Y_n \propto (x-a)^n (x-b)^n$

$\int_a^b \left(\frac{d^n}{dx^n} Y_n\right) x^p dx = 0 \quad p<n$

$a = -\infty$ b finite

$Y_n = (x-b)^n w$ if $w(x-b)^n \to 0$ as $x \to -\infty$

$$w P_n = \frac{d^n}{dx^n} Y_n$$

$$Y_n \propto s^n(x)\, w(x)$$

$s(x) = (x-a)(x-b)$ — a, b finite

$= x-a$ — a finite, $b=\infty$

$= x-b$ — $a=-\infty$, b finite

$=$ constant — $a=-\infty$, $b=\infty$

$w \to 0$ faster than any polynomial at infinite bds

3 cases:

1) $(a,b) = (-\infty, \infty)$

2) $(a,b) = (0, \infty)$

3) $(a,b) = (-1, 1)$

for any other interval map interval to one of these by linear transformation: $P_n(cx+d)$ will be orthogonal polynomials for some c, d

$w P_n \propto \frac{d^n}{dx^n} Y_n$

$Y_n \propto (x-a)^n (x-b)^n$

$\int_a^b \left(\frac{d^n}{dx^n} Y_n\right) x^p dx = 0 \quad p<n$

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$$s(x) = (x-a)(x-b)$$

$= x-a$ $\quad$ a finite $b=\infty$

$= x-b$ $\quad$ $a=-\infty$ $\quad$ b finite

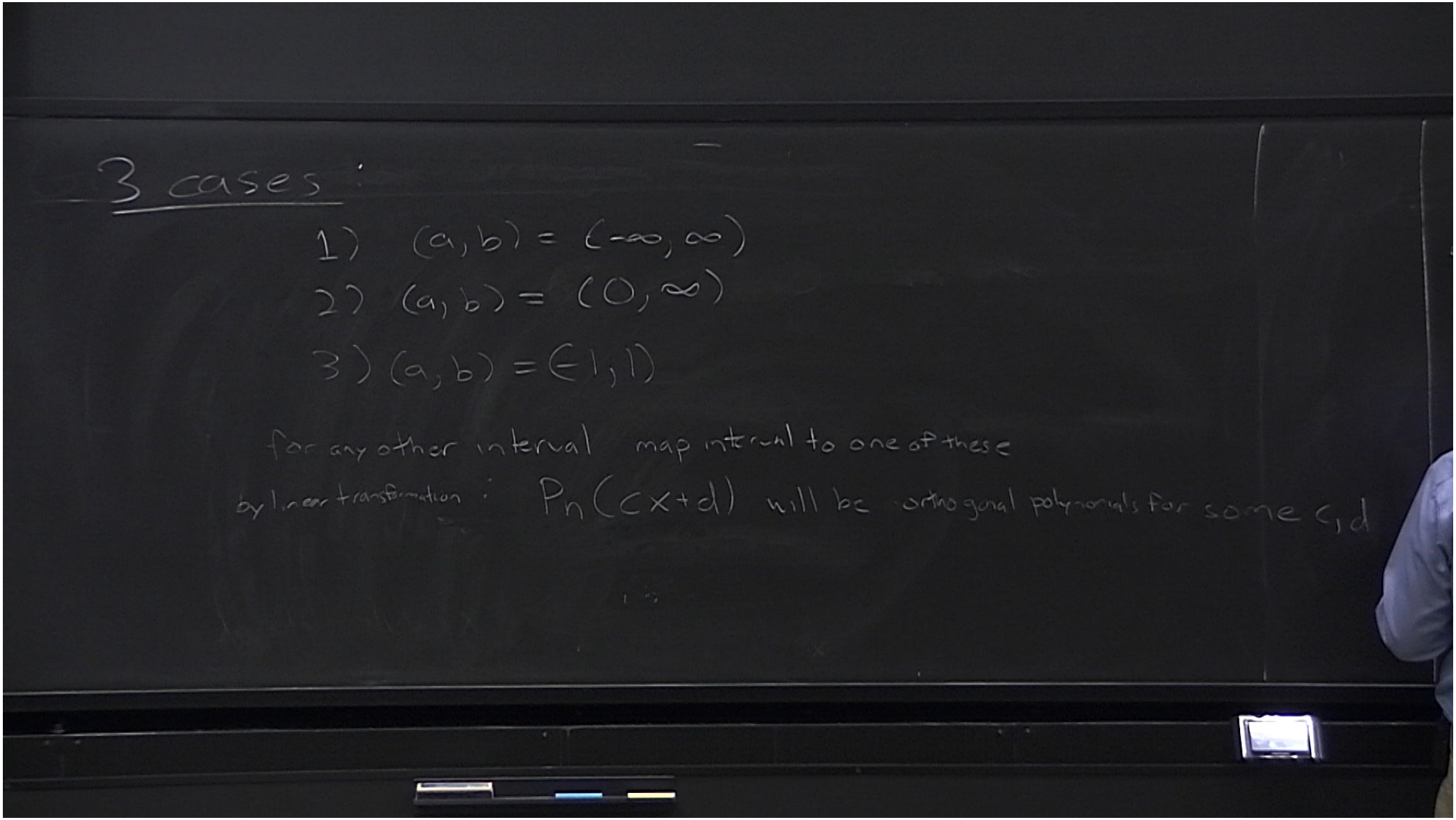
$=$ constant $\quad$ $a=-\infty$ $\quad$ $b=\infty$

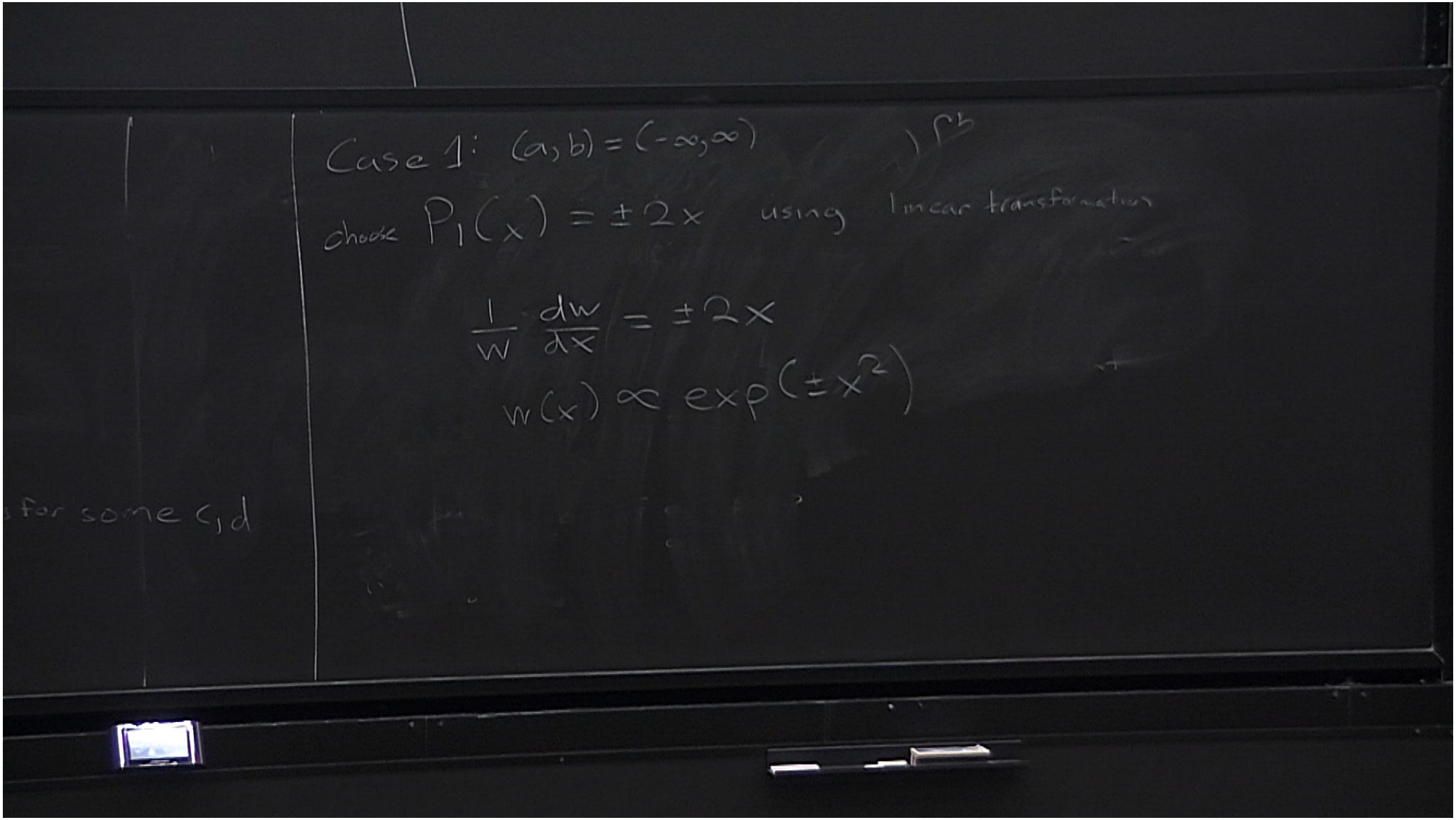
a, b finite

$w \to 0$ faster than any polynomial at infinite bds

$$P_n(x) = \frac{c_n}{w(x)} \left(\frac{d^n}{dx^n} w s^n \right)$$

Rodrigues' Formula


3 cases:
1) $(a,b) = (-\infty, \infty)$
2) $(a,b) = (0, \infty)$
3) $(a,b) = (-1,1)$
for any other interval map interval to one of these
by linear transformation: $P_n(cx+d)$ will be orthogonal polynomials for some c,d


Case 1: $(a,b) = (-\infty,\infty)$
choose $P_1(x) = \pm 2x$ using linear transformation
$\frac{1}{w}\frac{dw}{dx} = \pm 2x$
$w(x) \propto \exp(\pm x^2)$
for some c, d

Case 1: $(a,b) = (-\infty, \infty)$

choose $P_1(x) = \pm 2x$ using linear transformation

$$\frac{1}{w}\frac{dw}{dx} = \pm 2x$$

$$w(x) \propto \exp(\pm x^2)$$

$$P_n(x) = c_n \exp(x^2) \frac{d^n}{dx^n}\left(\exp(-x^2)\right)$$

$$c_n = (-1)^n$$

- Hermite polynomials

Cases 2+3: tutorial

Best approximation

$$f \approx \sum_{n=1}^{N} a_n u_n \quad \text{where } a_n = \langle u_n | f \rangle \quad u_n \text{ are orthonormal}$$

minimizes $\Delta_N = \| f - \sum_{n=1}^{N} a_n u_n \|^2$

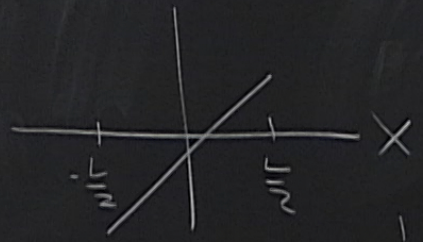
min value is $\Delta_N = \|f\|^2 - \sum_{n=1}^{N} |a_n|^2$

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \, e^{2\pi i n x / L}$$

$$U = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |a_n|^2$$

f

$f(x) = x$

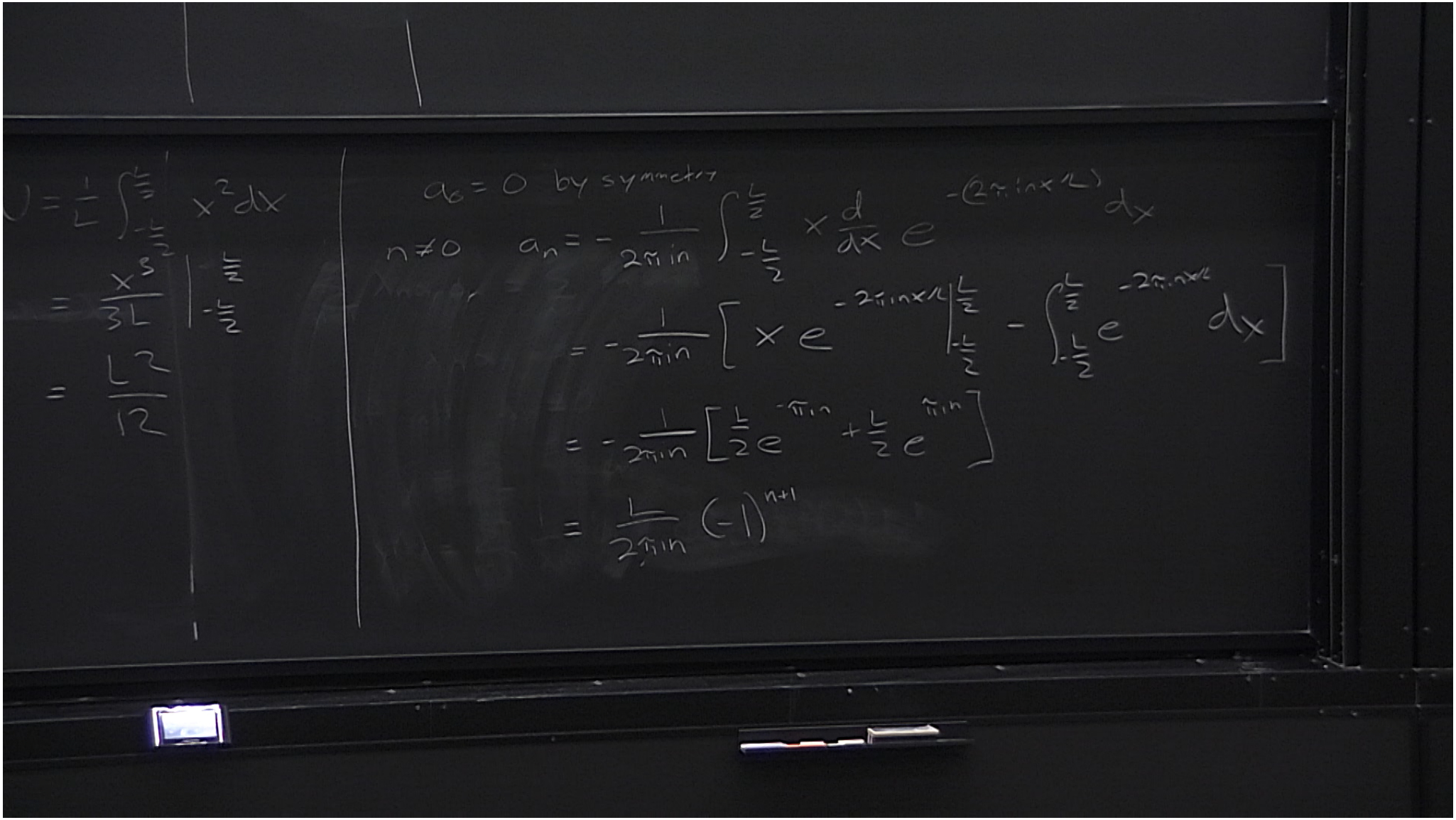


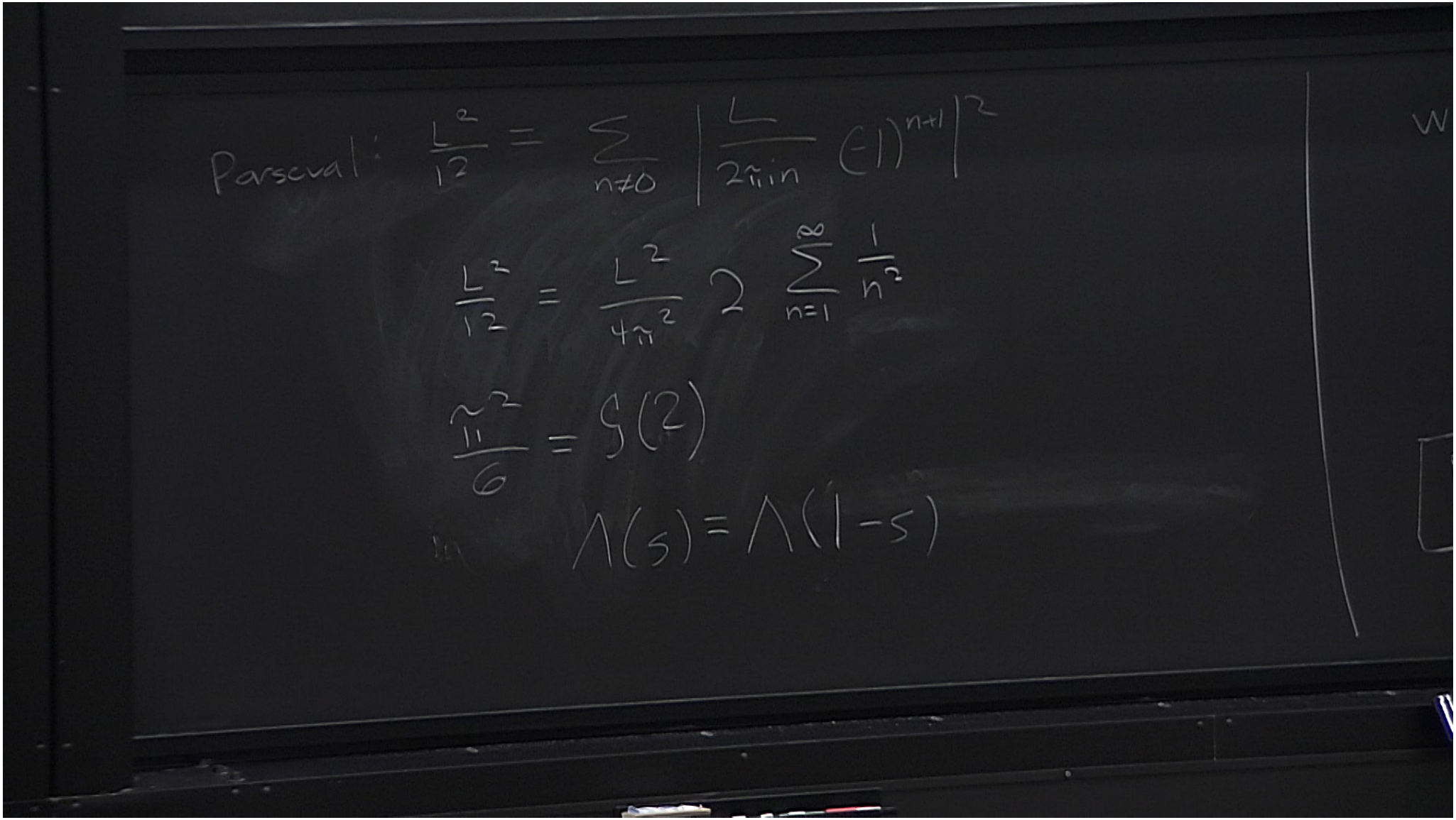
$$a_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x \, e^{-2\pi i n x / L} \, dx$$

$$U = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \, dx$$

$$= \frac{x^3}{3L} \Big|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{L^2}{12}$$


$V = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx$
$= \frac{x^3}{3L}\Big|_{-\frac{L}{2}}^{\frac{L}{2}}$
$= \frac{L^2}{12}$
$a_0 = 0$ by symmetry
$n \neq 0 \quad a_n = -\frac{1}{2\pi i n}\int_{-\frac{L}{2}}^{\frac{L}{2}} x \frac{d}{dx} e^{-(2\pi i n x/L)} dx$
$= -\frac{1}{2\pi i n}\left[x e^{-2\pi i n x/L}\Big|_{-\frac{L}{2}}^{\frac{L}{2}} - \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-2\pi i n x/L} dx \right]$
$= -\frac{1}{2\pi i n}\left[\frac{L}{2} e^{-\pi i n} + \frac{L}{2} e^{\pi i n} \right]$
$= \frac{L}{2\pi i n}(-1)^{n+1}$


Parseval: $\frac{L^2}{12} = \sum_{n\neq 0} \left| \frac{L}{2\pi i n} (-1)^{n+1} \right|^2$
$\frac{L^2}{12} = \frac{L^2}{4\pi^2} 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$
$\frac{\pi^2}{6} = \zeta(2)$
$\Lambda(s) = \Lambda(1-s)$

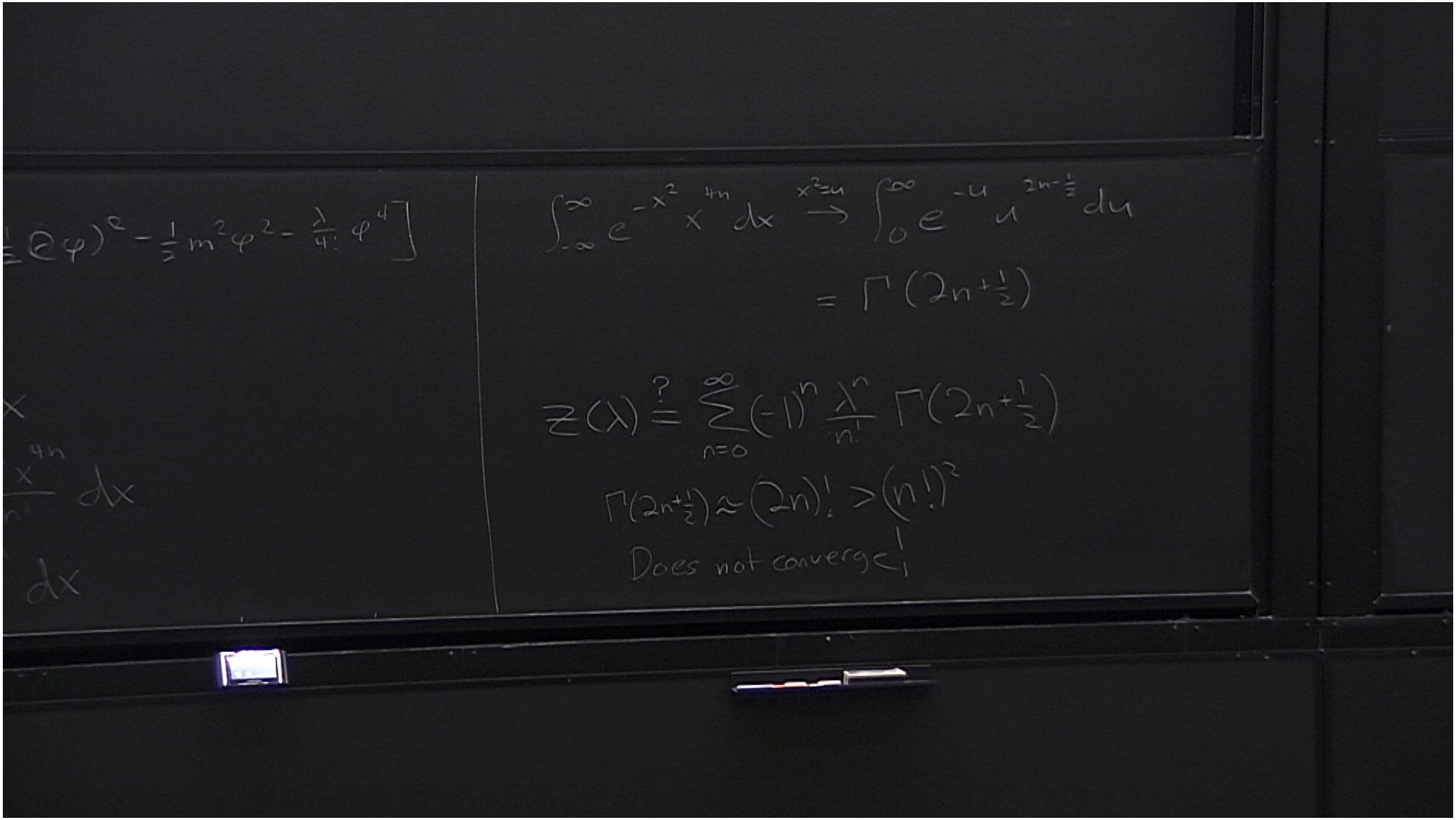
Asymptotics

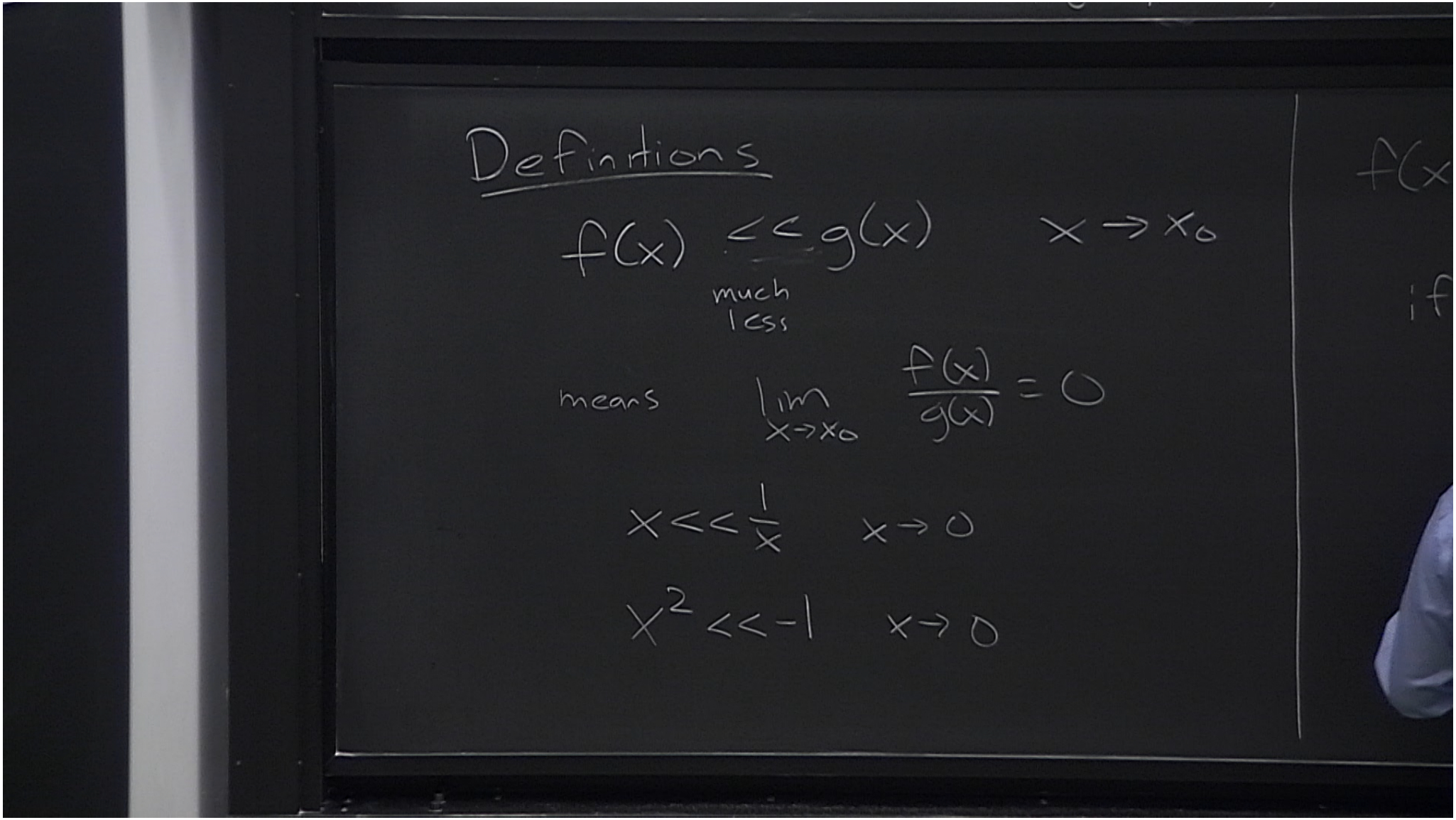
QFT: $Z = \int \mathcal{D}[\varphi]\, e^{\frac{i}{\hbar}\int d^4x \left[-\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4\right]}$

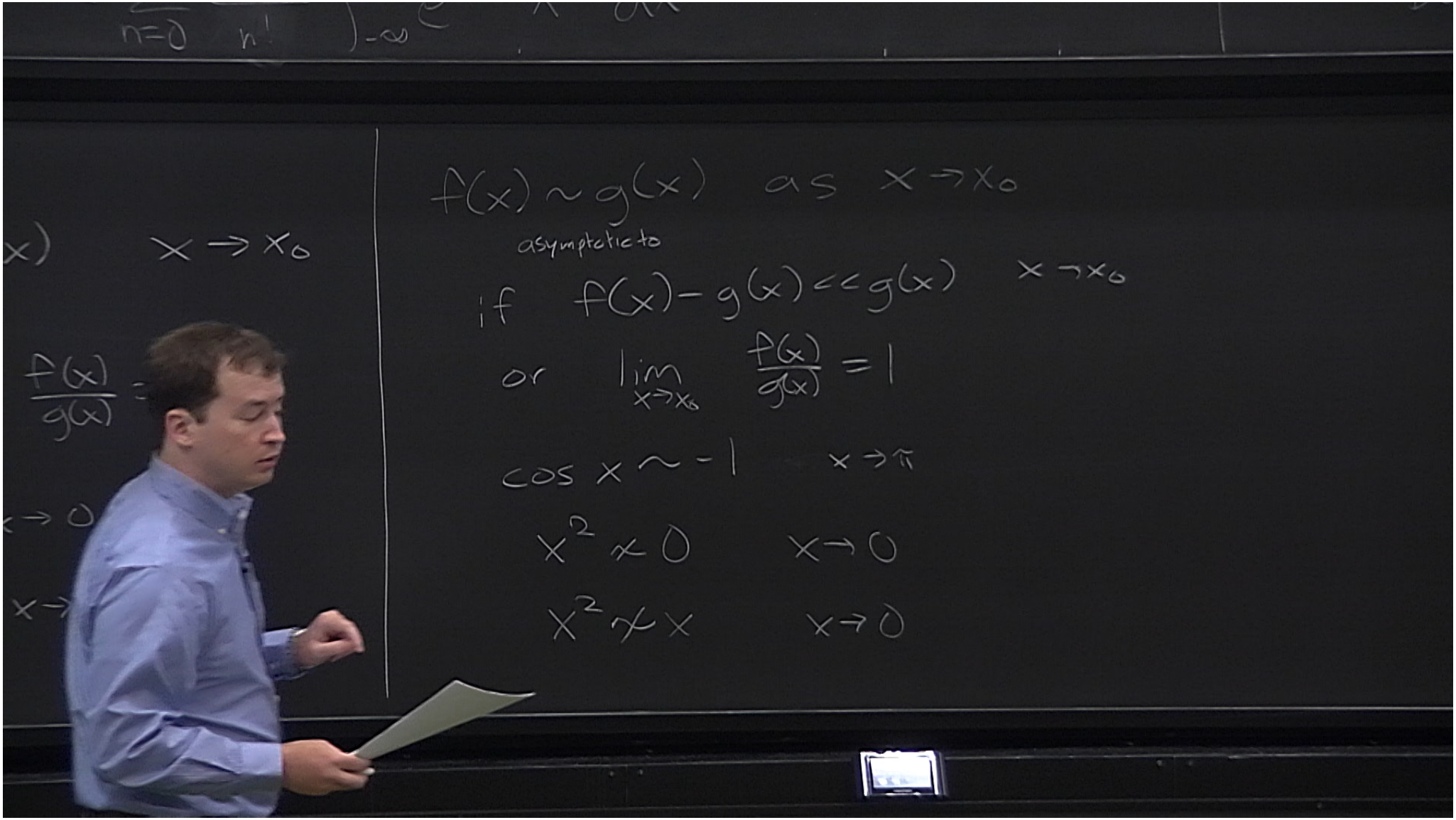
$$Z[\lambda] = \int_{-\infty}^{\infty} e^{-x^2 - \lambda x^4}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2} \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n x^{4n}}{n!}\, dx$$

$$\stackrel{?}{=} \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} \int_{-\infty}^{\infty} e^{-x^2} x^{4n}\, dx$$


$\int_{-\infty}^{\infty} e^{-x^2} x^{4n} dx \xrightarrow{x^2=u} \int_0^{\infty} e^{-u} u^{2n-\frac{1}{2}} du$
$= \Gamma(2n+\frac{1}{2})$
$Z(\lambda) \stackrel{?}{=} \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{n!} \Gamma(2n+\frac{1}{2})$
$\Gamma(2n+\frac{1}{2}) \approx (2n)! > (n!)^2$
Does not converge!


Definitions
$f(x) << g(x)$ $x \to x_0$
much less
means $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$
$x << \frac{1}{x}$ $x \to 0$
$x^2 << -1$ $x \to 0$

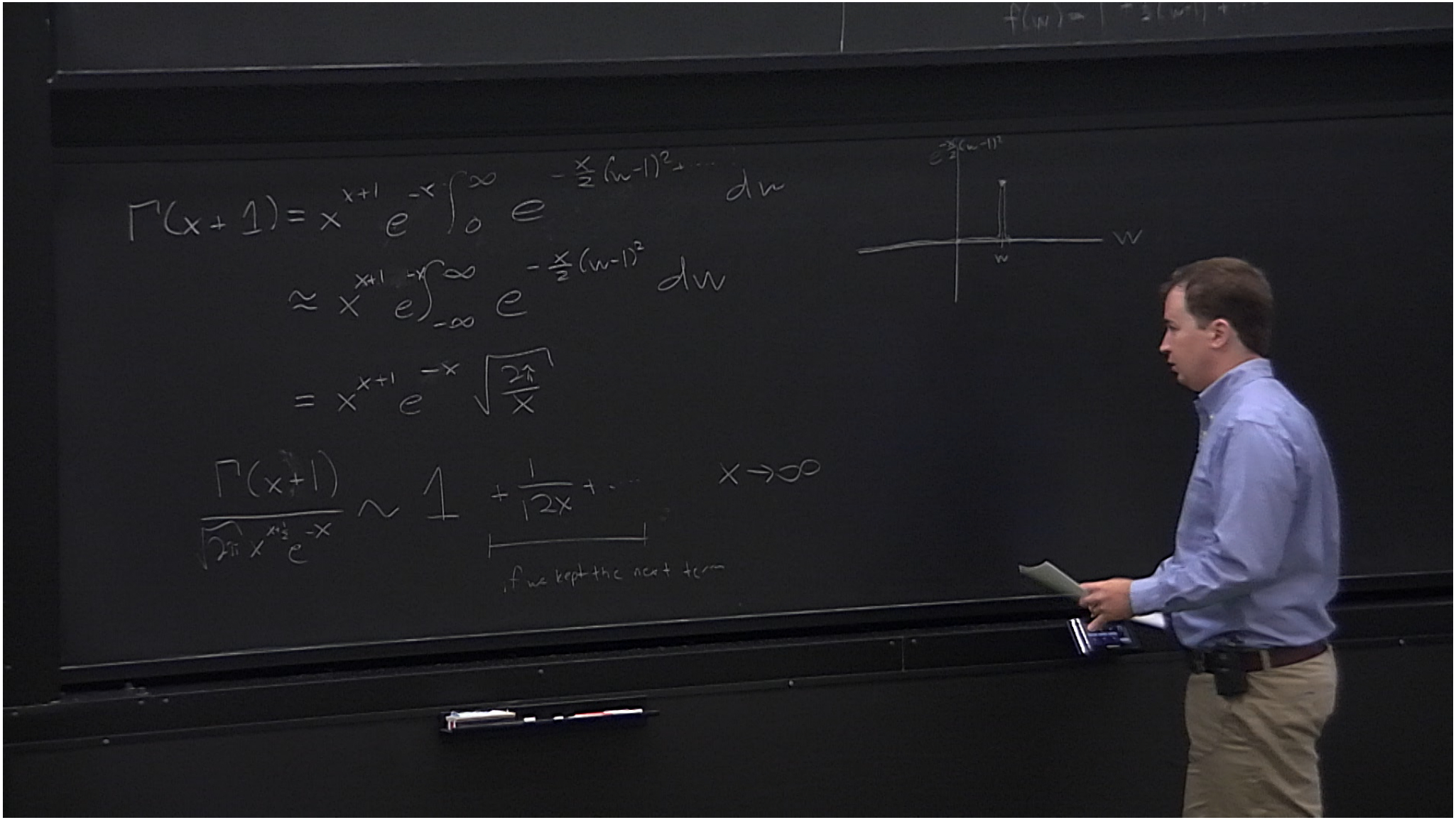

$f(x) \sim g(x)$ as $x \to x_0$
asymptotic to
if $f(x) - g(x) << g(x)$ $x \to x_0$
or $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$
$\cos x \sim -1$ $x \to \pi$
$x^2 \not\sim 0$ $x \to 0$
$x^2 \not\sim x$ $x \to 0$

$$y(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad x \to x_0$$

$x \to x_0$

if $y(x) - \sum_{n=0}^{N} a_n (x - x_0)^n << (x - x_0)^N \quad x \to x_0$

for all N


$\Gamma(x+1) = x^{x+1} e^{-x} \int_0^\infty e^{-\frac{x}{2}(w-1)^2 + \cdots} dw$
$\approx x^{x+1} e^{-x} \int_{-\infty}^{\infty} e^{-\frac{x}{2}(w-1)^2} dw$
$= x^{x+1} e^{-x} \sqrt{\frac{2\pi}{x}}$
$\frac{\Gamma(x+1)}{\sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x}} \sim 1 + \frac{1}{12x} + \cdots \quad x \to \infty$
if we kept the next term
$e^{-\frac{x}{2}(w-1)^2}$
w
w

$\sqrt{2\pi} \times e^{\quad}$

, if we kept the next term

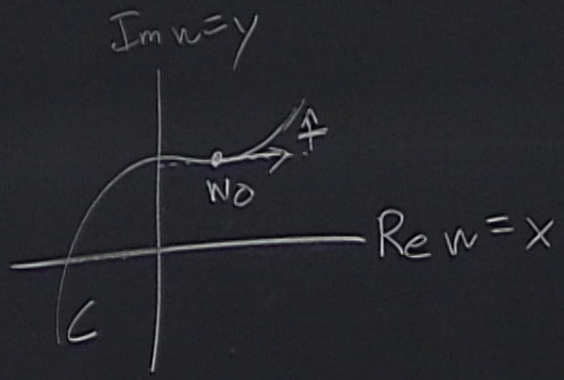
Saddle-Point Method

want $I(z) = \int_C g(w) e^{z f(w)} dw$ for large real z

- dominant contribution will come from near w_0 at which $u \equiv \mathrm{Re} f(w)$ is a local max

as long as phase is constant

phase determined by $v \equiv \mathrm{Im} f(w)$



is a local max

$\nabla v \cdot \hat{f} = 0$ for phase to be stationary

$\nabla u \cdot \hat{f} = 0$ for u to be local max

$\nabla u \cdot \nabla u = 0$ by Cauchy-Riemann

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $e^{-x}\,dx$

must have $\begin{cases} \nabla u = 0 \\ \nabla v = 0 \end{cases}$ at w_0 $\rightarrow f'(w_0) = 0$

w_0 is a critical point or saddle point

contor should pass through w_0 in steepest descent direction

$$0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

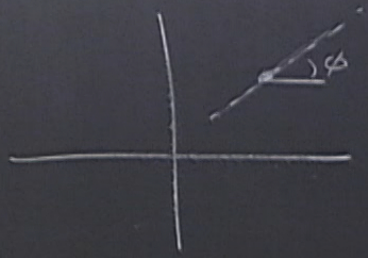
$$0 = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f(w) = f(w_0) + \frac{1}{2}(w - w_0)^2 f''(w_0) + \dots$$

$$= f(w_0) + \frac{1}{2}(w - w_0)^2 e^{i\delta_0} |f''(w_0)| + \dots$$

$w = w_0 + e^{i\phi} t$

t paramctrizes linear approx to C



want $f(w) = f(w_0) - \frac{1}{2} t^2 |f''(w_0)|$

$(w - w_0)^2 e^{i\delta_0} = e^{2i\phi} t^2 e^{i\delta_0} = -t^2$

$2i\phi \quad i\delta_0 \quad \pm i\hat{n}$

$x^2 \quad 4n \quad x^2 = 4 \quad C^\infty$

$$= \sum_{n=0} \frac{(-1)^n \lambda^n}{n!} \int_{-\infty}^{\infty} e^{-x} x^{n} dx$$

Doe

$$I(z) \approx e^{z f(w_0)} \int_{-t_1}^{t_1} g(w_0 + e^{i\phi} t)\, e^{-\frac{z}{2} t^2 |f''(w_0)|} \frac{dw}{dt} dt$$

$-t_1 < t < t_1$ is small region, where we can treat C as linear

$$\approx e^{z f(w_0)} \int_{-\infty}^{\infty} \left[g(w_0) + \frac{g''(w_0)}{2!} e^{2i\phi} t^2 + \cdots \right] e^{-\frac{z}{2} t^2 |f''(w_0)|} e^{i\phi} dt$$

$$\sim \sqrt{\frac{2\pi}{z}} \frac{e^{z f(w_0)}}{\sqrt{-f''(w_0)}}$$

$$\stackrel{?}{=} \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} \int_{-\infty}^{\infty} e^{-x^2} x^{4n}\, dx$$

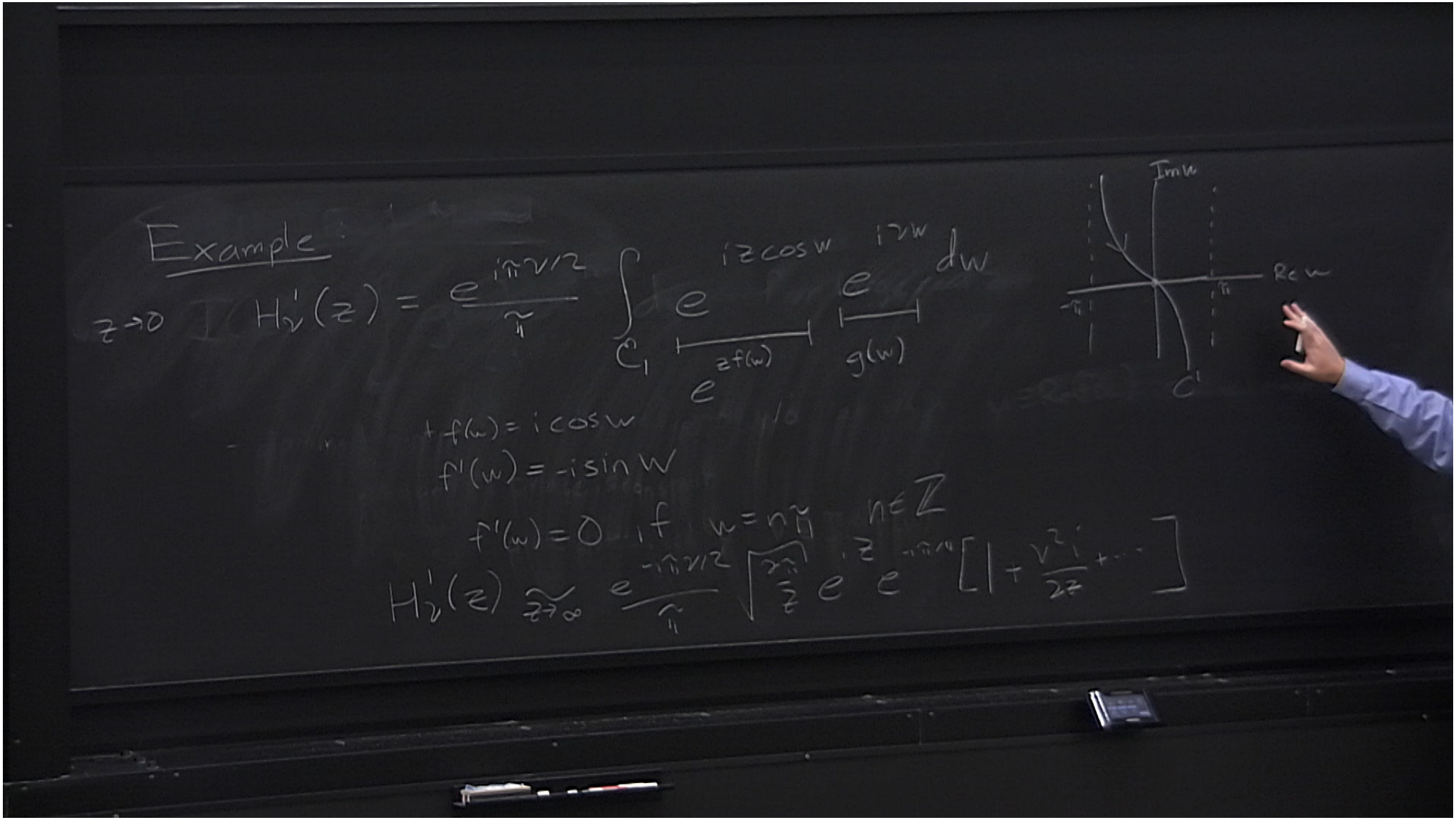
Does no

$$I(z) \approx e^{z f(w_0)} \int_{-t_1}^{+t_1} g(w_0 + e^{i\phi} t)\, e^{-\frac{z}{2} t^2 |f''(w_0)|} \frac{dw}{dt}\, dt$$

$-t_1 < t < t_1$ is small region, where we can treat C as linear

$$\approx e^{z f(w_0)} \int_{-\infty}^{\infty} \left[g(w_0) + \frac{g''(w_0)}{2!} e^{2i\phi} t^2 + \cdots \right] e^{-\frac{z}{2} t^2 |f''(w_0)|} e^{i\phi}\, dt$$

$$\sim \sqrt{\frac{2\pi}{z}}\, \frac{e^{z f(w_0)}}{\sqrt{-f''(w_0)}} \left[g(w_0) - \frac{1}{z} \frac{g''(w_0)}{1! (2 f''(w_0)} + \cdots \right]$$


Example
$z \to 0$
$H_\nu^{(1)}(z) = \frac{e^{i\pi\nu/2}}{\pi} \int_{C_1} e^{iz\cos w} e^{i\nu w} dw$
$e^{zf(w)}$
$g(w)$
$f(w) = i\cos w$
$f'(w) = -i\sin w$
$f'(w) = 0$ if $w = n\pi$ $n \in \mathbb{Z}$
$H_\nu^{(1)}(z) \underset{z\to\infty}{\sim} \frac{e^{-i\pi\nu/2}}{\pi} \sqrt{\frac{2\pi}{z}} e^{iz} e^{-i\pi/4} \left[1 + \frac{\nu^2 i}{2z} + \cdots\right]$
Im w
Re w
$-\pi$
π
C'