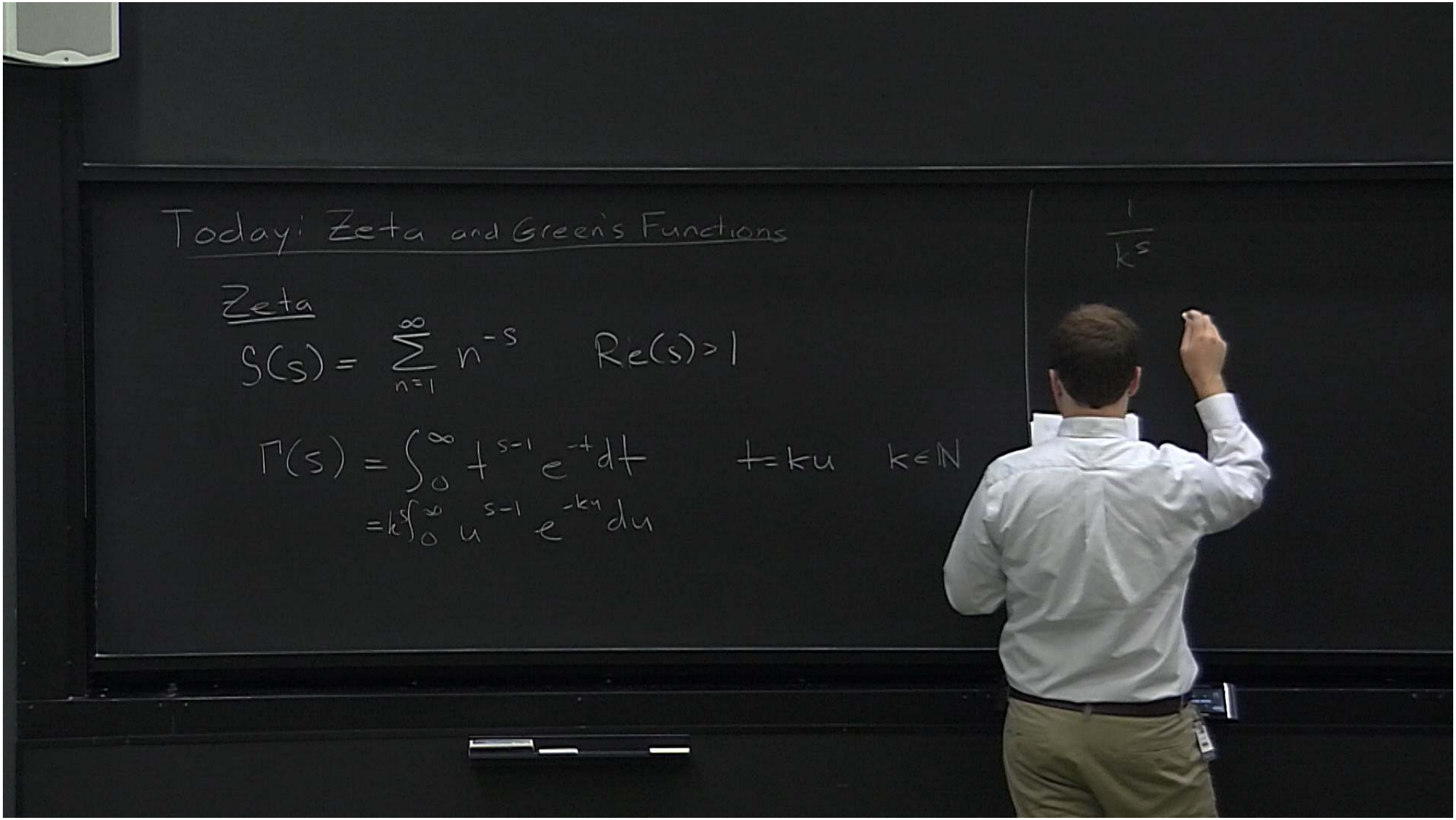
Title: PSI Front End - Functions, "Functions", etc.

Date: Aug 29, 2017 10:30 AM

URL: http://pirsa.org/17080068

Abstract:

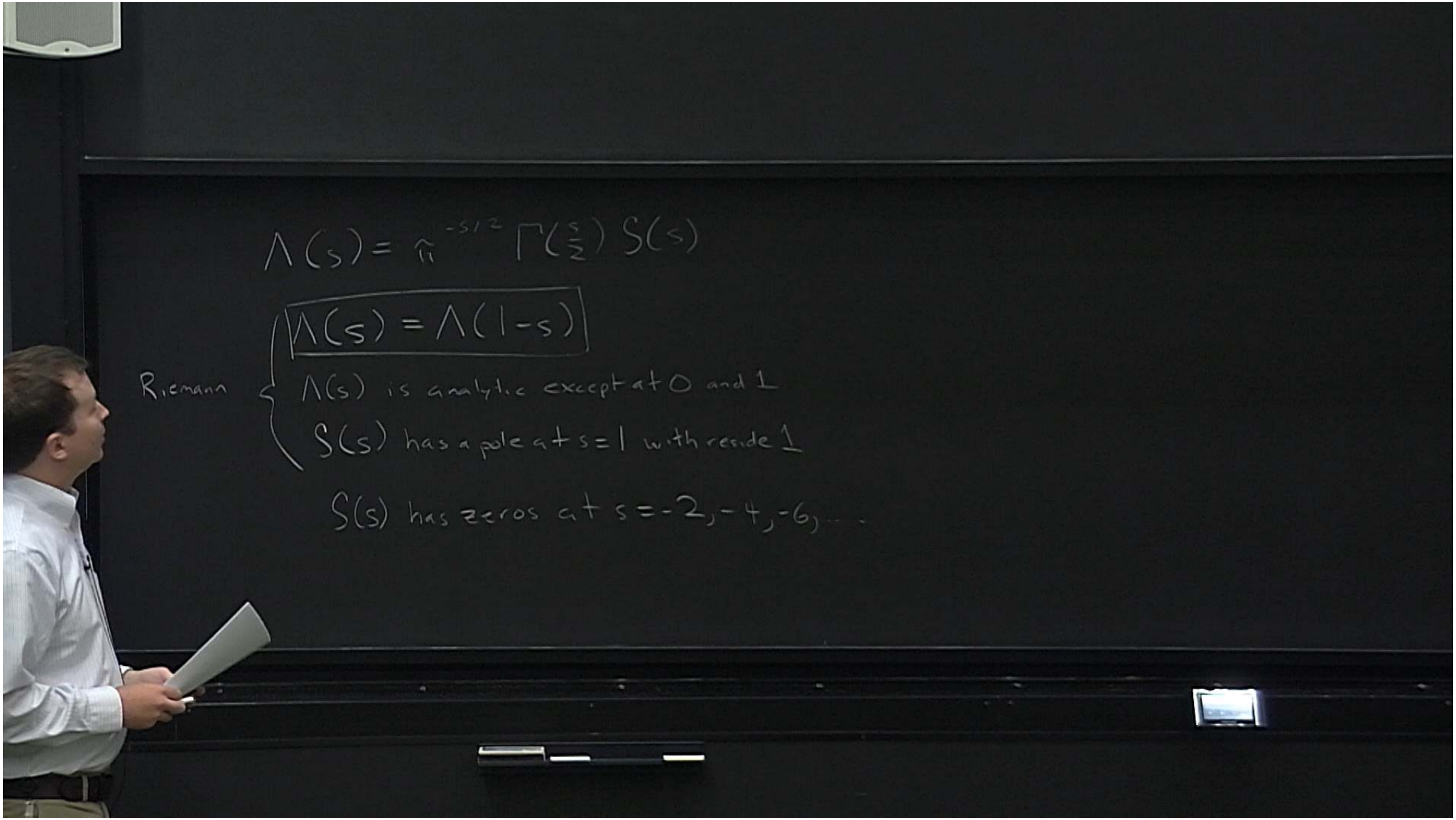

Today: Zeta and Green's Functions
Zeta
$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ $\mathrm{Re}(s) > 1$
$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ $t = ku$ $k \in \mathbb{N}$
$= k^s \int_0^\infty u^{s-1} e^{-ku} du$
$\frac{1}{k^s}$

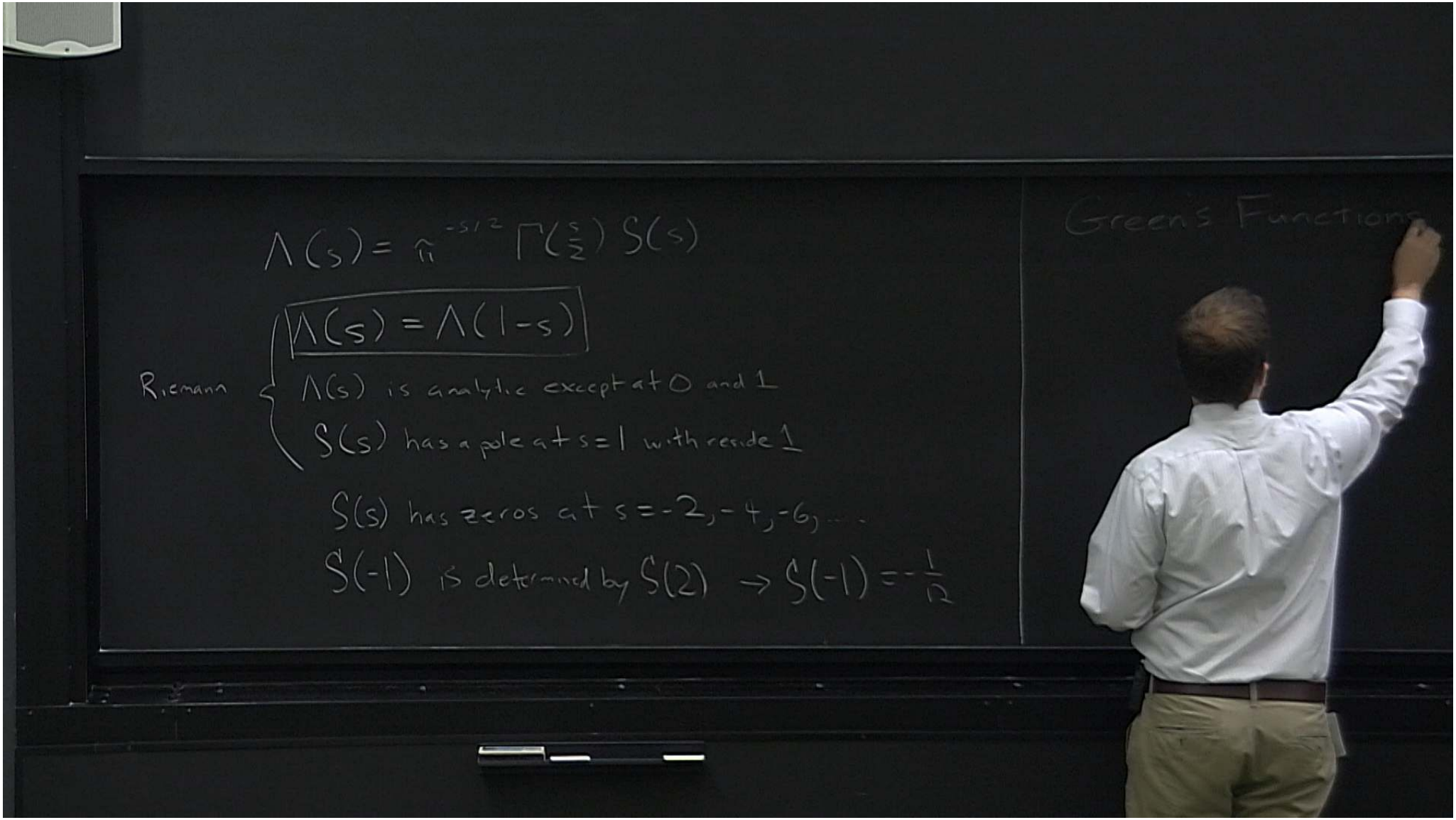
$$\sum_{k=1}^{\infty} \frac{1}{k^s} = \sum_{k=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} e^{-ku} \, du$$

$$S(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \sum_{k=1}^{\infty} e^{-ku} \, du$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \left(\frac{e^{-u}}{1-e^{-u}} \right) du$$

$$\boxed{S(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{u^{s-1}}{e^u - 1} \, du}$$


$\Lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$
$\Lambda(s) = \Lambda(1-s)$
Riemann
$\Lambda(s)$ is analytic except at 0 and 1
$\zeta(s)$ has a pole at $s=1$ with reside 1
$\zeta(s)$ has zeros at $s=-2,-4,-6,\ldots$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$\zeta(s)$ has zeros at $s=-2,-4,-6,\ldots$
$\zeta(-1)$ is determined by $\zeta(2) \rightarrow \zeta(-1) = -\frac{1}{12}$
Green's Functions

Green's Functions

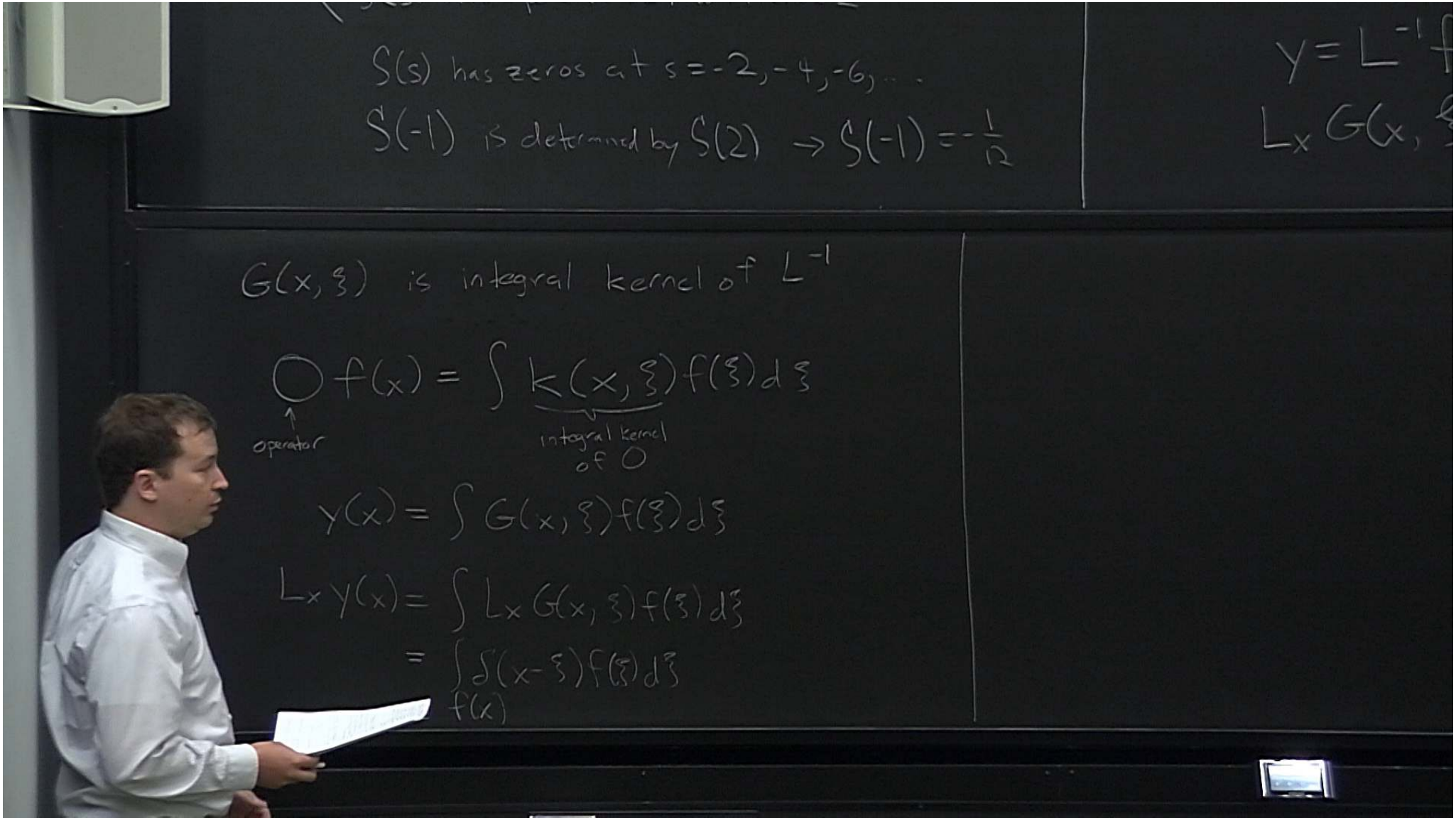
$$L\,y(x) = f(x)$$

L: diff op; $f(x)$: function

$$L = P_0(x)\frac{d^n}{dx^n} + P_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + P_n(x)$$

$$y = L^{-1} f$$

$$L_x\, G(x, \xi) = \delta(x - \xi)$$

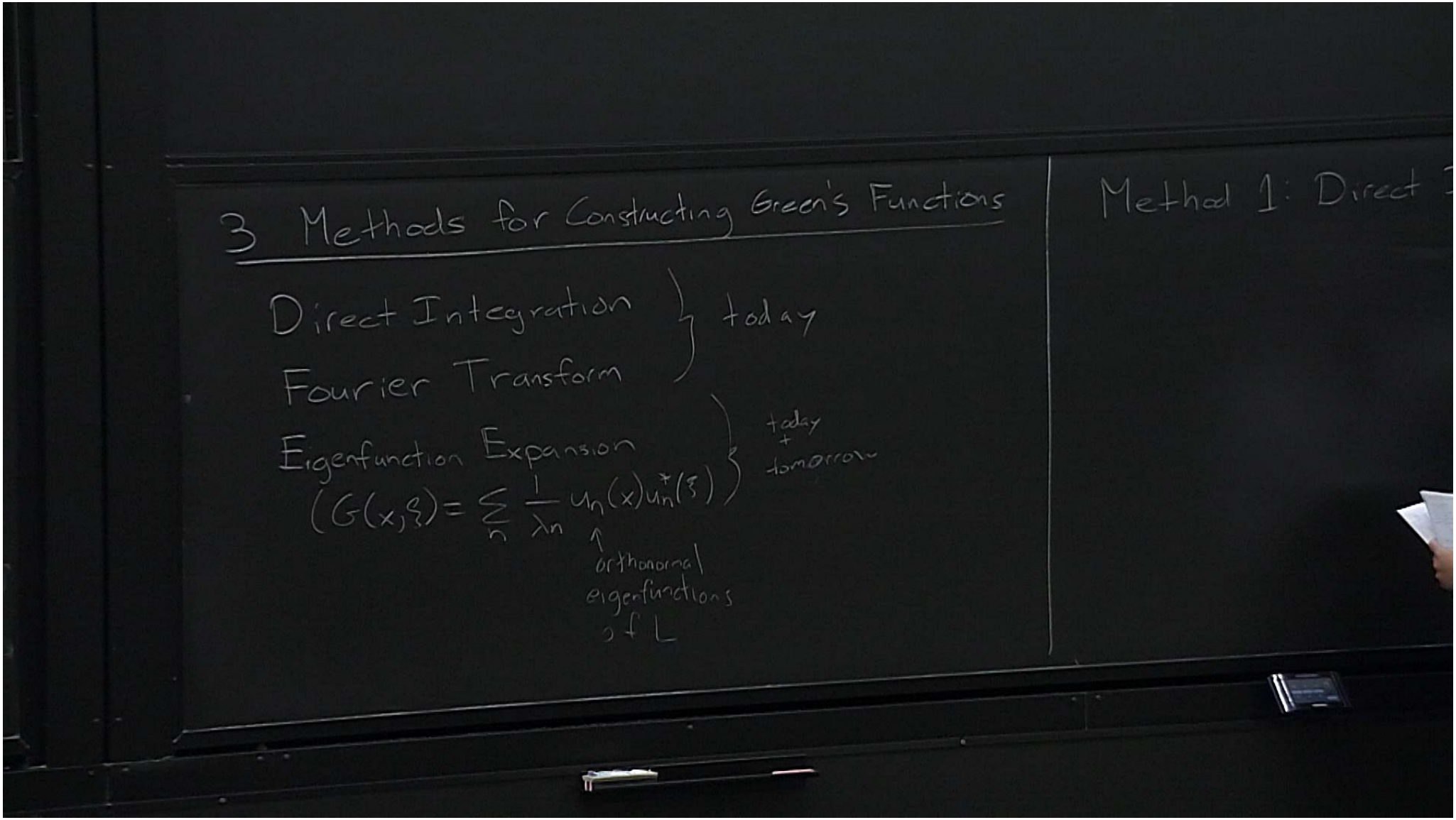

$\zeta(s)$ has zeros at $s=-2,-4,-6,\ldots$
$\zeta(-1)$ is determined by $\zeta(2) \rightarrow \zeta(-1) = -\frac{1}{12}$
$y = L^{-1} f$
$L_x G(x,$
$G(x,\xi)$ is integral kernel of L^{-1}
$O f(x) = \int k(x,\xi) f(\xi) d\xi$
operator
integral kernel of O
$y(x) = \int G(x,\xi) f(\xi) d\xi$
$L_x y(x) = \int L_x G(x,\xi) f(\xi) d\xi$
$= \int \delta(x-\xi) f(\xi) d\xi$
$f(x)$

$$y = L^{-1} f$$

$$L_x G(x, \xi) = \delta(x - \xi)$$

Analogy

Finite Dim	Infinite Dim
vector $\vec{v}$	function f
component v_i	value at a point $f(x)$
matrix M	operator O
matrix element M_{ij}	integral kernel $K(x,y)$
identity matrix $\mathbb{1}$	Dirac delta δ


3 Methods for Constructing Green's Functions
Direct Integration
Fourier Transform
today
Eigenfunction Expansion
$G(x,\xi) = \sum_n \frac{1}{\lambda_n} u_n(x) u_n^*(\xi)$
today + tomorrow
orthonormal eigenfunctions of L
Method 1: Direct

nctions

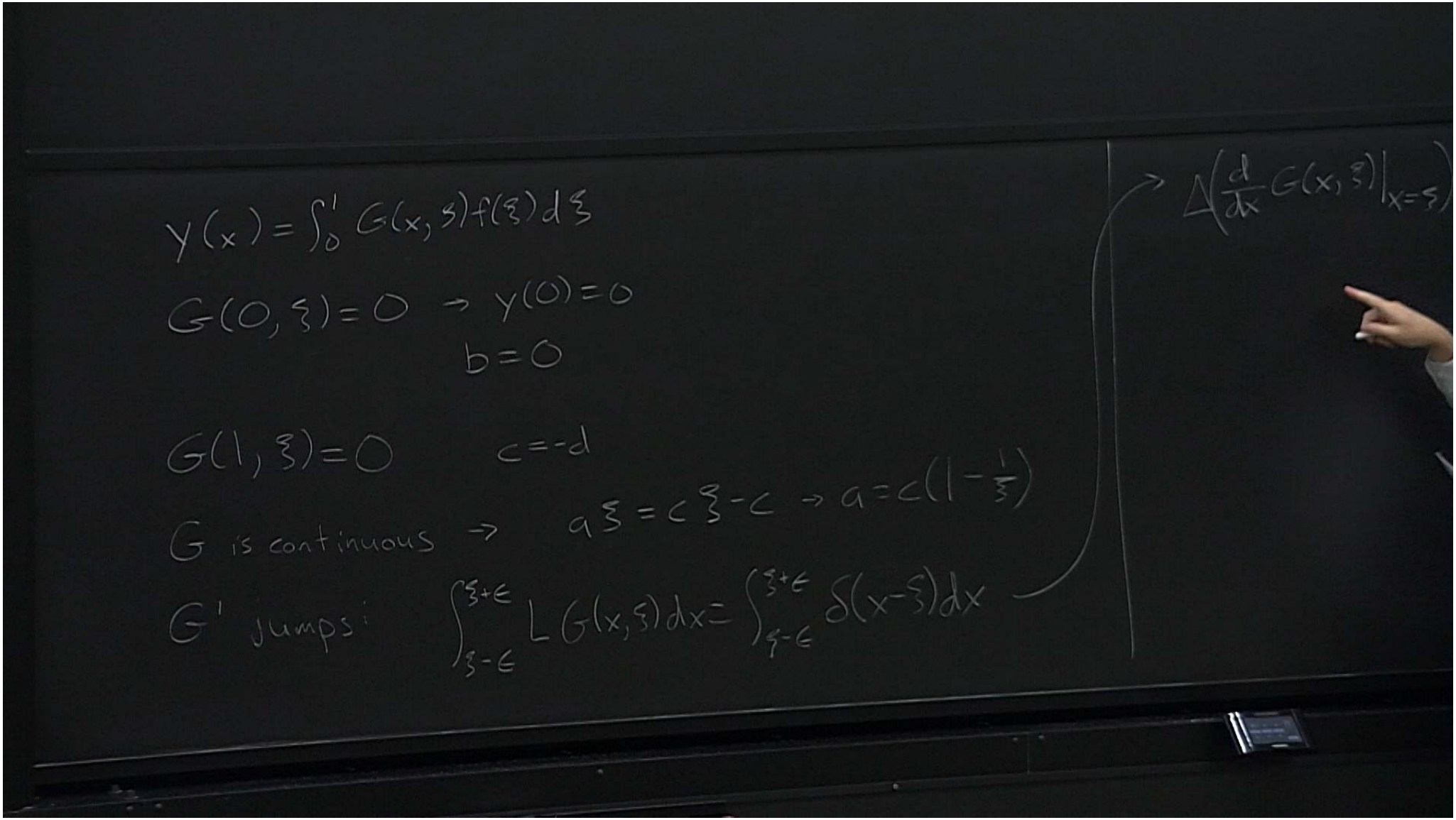
Method 1: Direct Integration

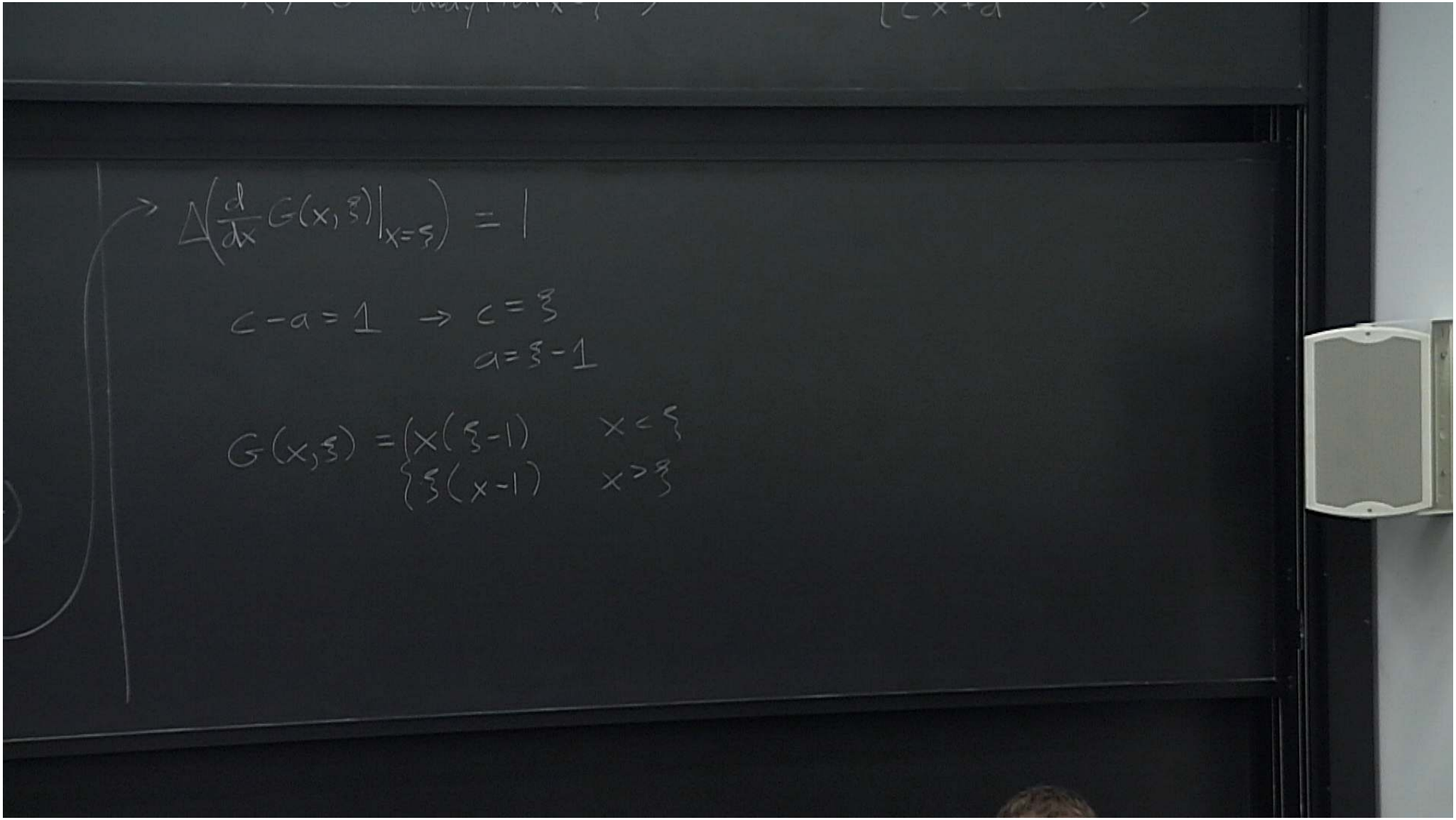
$\frac{d^2y}{dx^2} = f(x)$ $\quad L = \frac{d^2}{dx^2}$

$y(0)=0$
$y(1)=0$
} homogeneous if y_1 and y_2 satisfy b.cs and cons $\alpha y_1 + \beta y_2$

$L\,G(x,\xi) = \delta(x-\xi)$

$L\,G(x,\xi) = 0$ away from $x=\xi$ $\Rightarrow$ $G(x,\xi) = \begin{cases} ax+b & x<\xi \\ cx+d & x>\xi \end{cases}$


$y(x) = \int_0^1 G(x,\xi) f(\xi)\, d\xi$
$G(0,\xi) = 0 \rightarrow y(0) = 0$
$b = 0$
$G(1,\xi) = 0 \qquad c = -d$
G is continuous $\rightarrow \quad a\xi = c\xi - c \rightarrow a = c\left(1 - \frac{1}{\xi}\right)$
G' jumps: $\int_{\xi-\epsilon}^{\xi+\epsilon} L\, G(x,\xi)\, dx = \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)\, dx$
$\rightarrow \Delta\left(\frac{d}{dx} G(x,\xi)\Big|_{x=\xi}\right)$


$\Delta\left(\frac{d}{dx}G(x,\xi)\Big|_{x=\xi}\right) = 1$
$c - a = 1 \rightarrow c = \xi$
$a = \xi - 1$
$G(x,\xi) = \begin{cases} x(\xi-1) & x<\xi \\ \xi(x-1) & x>\xi \end{cases}$

G is continuous

G' jumps: $\int_{\xi-\epsilon}^{\xi+\epsilon} L\,G(x,\xi)\,dx = \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)\,dx$

$\frac{d^2y}{dx^2} = f(x)$

$y(0)=0 \rightarrow b=0$

$y'(0)=0 \rightarrow a=0$

$\Rightarrow 0 = c\xi + d$

continuity

jump $\rightarrow c=1 \quad d=-\xi$

$$G(x,\xi) = \begin{cases} 0 & x<\xi \\ x-\xi & x>\xi \end{cases}$$

$\zeta(-1)$ is determined by $\zeta(2) \rightarrow \zeta(-1) = -\frac{1}{12}$

$L_x G(x, \xi)$

Inhomogeneous b.cs

$y = y_1 + y_2$

y_1 solves $Ly = 0$ with inhomogeneous bcs

y_2 solves $Ly = f$ with homogeneous bcs $\quad y_2 = \int d\xi\, G(x, \xi) f(\xi)$

$$\frac{d^2 y_2}{dx} = f(x)$$

$y(0) = y_0$

$y'(0) = v_0$

$S(-1)$ is determined by $S(2) \rightarrow S(-1) = -\frac{1}{2}$

L_x GG

Method 2: Fourier Transforms

$$\ddot{x}(t) + 2\gamma \dot{x}(t) + \omega_0^2 x(t) = F(t)$$

FT: $$(-k^2 - 2i\gamma k + \omega_0^2)\tilde{x}(k) = \tilde{F}(k)$$

$$\tilde{x}(k) = \frac{\tilde{F}(k)}{-k^2 - 2i\gamma k + \omega_0^2}$$

$x(t) = \frac{1}{2\pi}$

$$L_x G(x,\xi) = \delta(x-\xi)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\tilde{F}(k)}{-k^2 - 2i\gamma k + \omega_0^2} e^{-ikt}\, dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dk\, \frac{F(t')e^{ikt'}}{-k^2 - 2i\gamma k + \omega_0^2} e^{-ikt}$$

$$G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{e^{-ik(t-t')}}{-k^2 - 2i\gamma k + \omega_0^2}$$

$$= \frac{1}{\Omega} e^{-\gamma(t-t')} \sin(\Omega(t-t'))\,\Theta(t-t')$$

$$\Omega = \sqrt{\omega_0^2 - \gamma^2}$$

$\zeta(-1)$ is determined by $\zeta(2) \rightarrow \zeta(-1) = -\frac{1}{12}$

$L_x G(x, \xi) =$

Method 2: Fourier Transforms

$$\ddot{x}(t) + 2\gamma \dot{x}(t) + \omega_0^2 x(t) = F(t)$$

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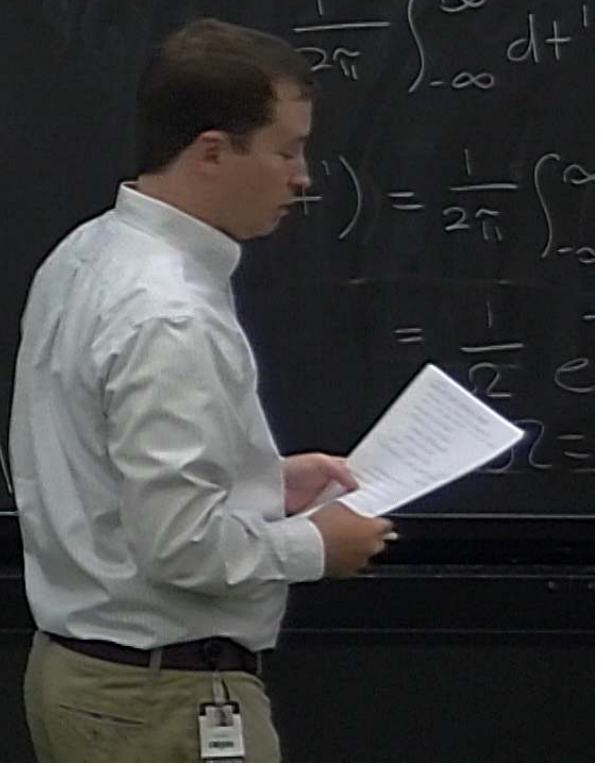
$$\tilde{x}(k) = \frac{\tilde{F}(k)}{-k^2 - 2i\gamma k + \omega_0^2}$$

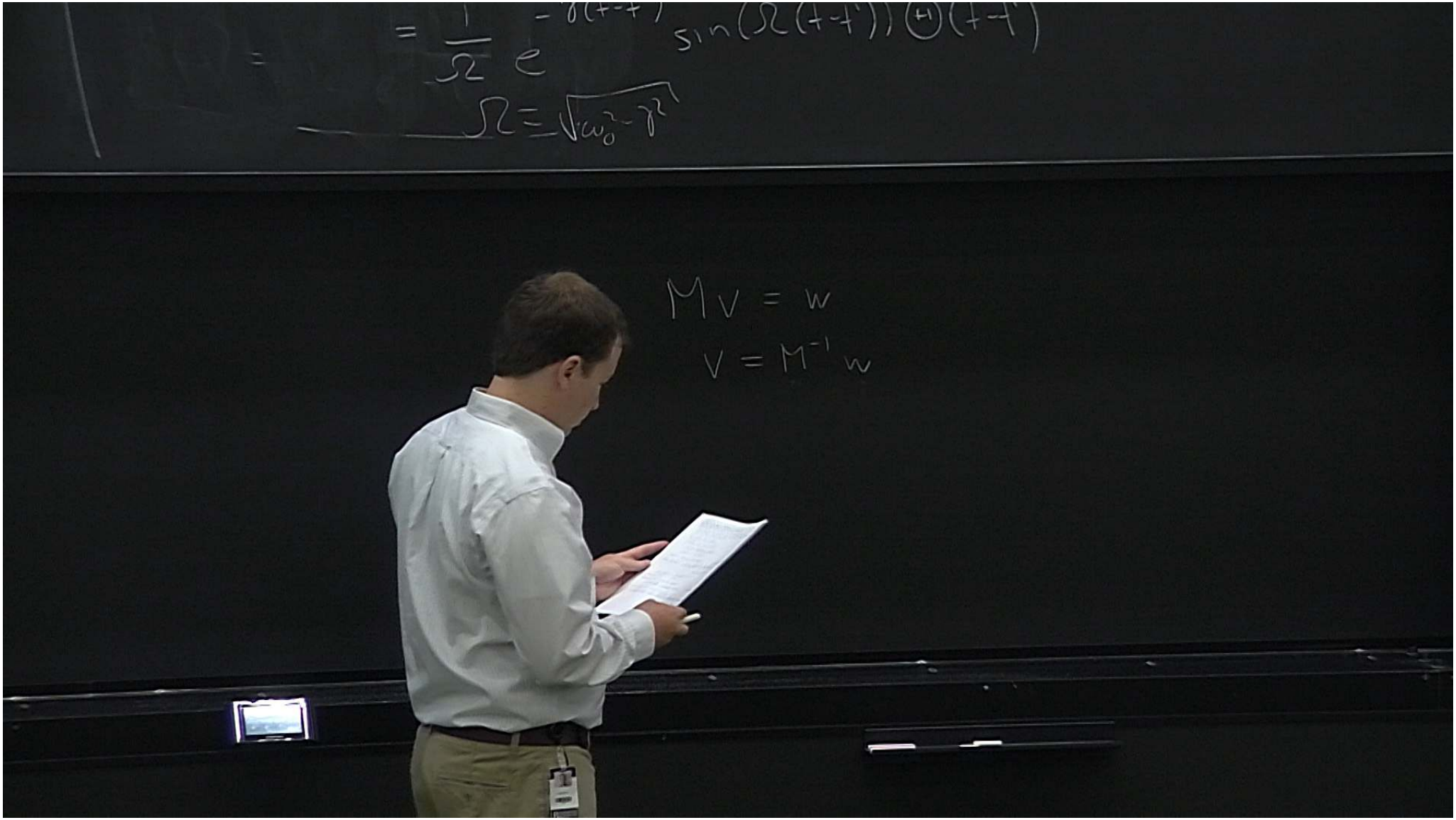
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{F}}{-k^2}$$

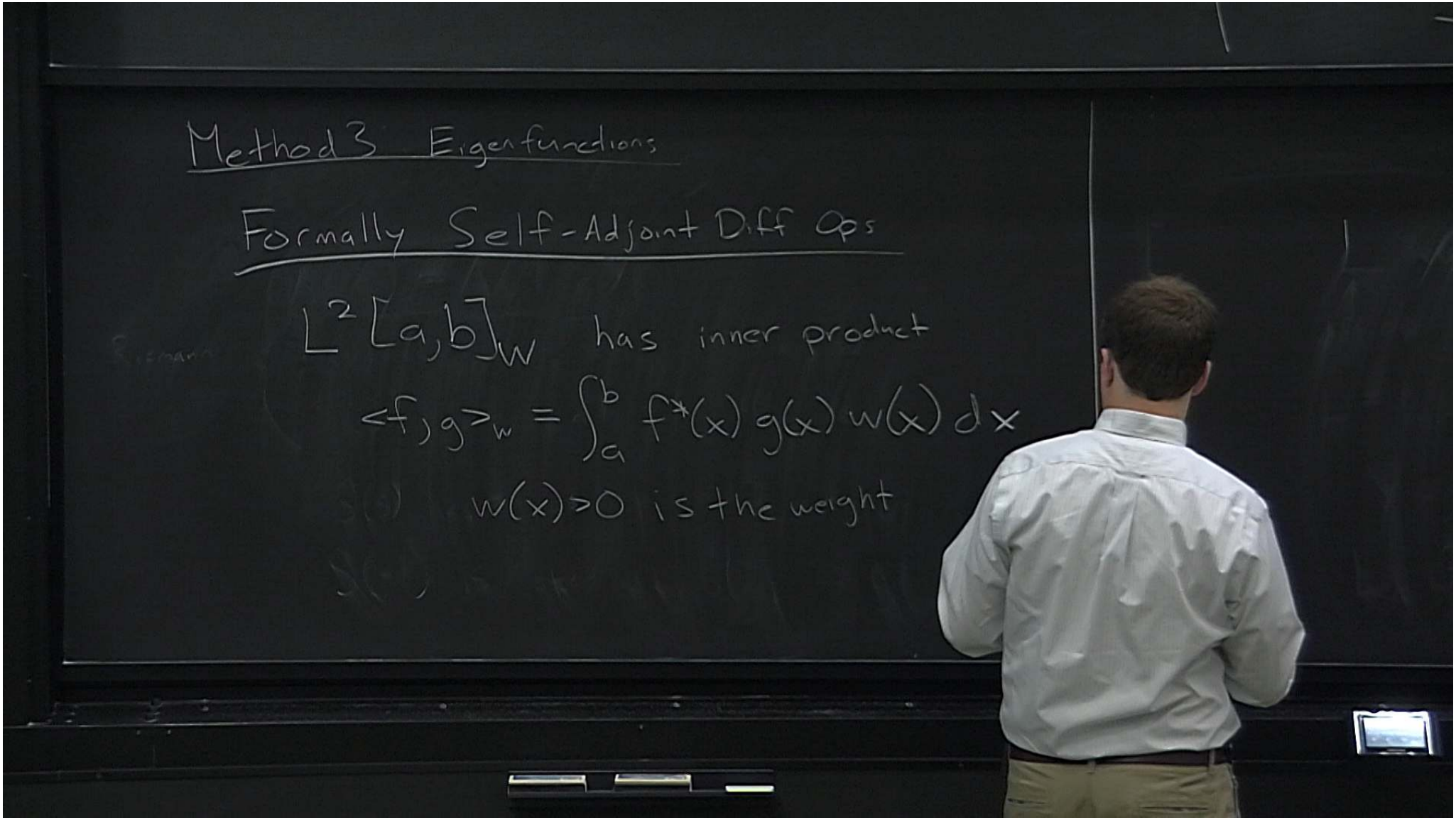
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dt'$$

$$t') = \frac{1}{2\pi} \int_{-\infty}^{\infty}$$

$$= \frac{1}{2} e$$




$= \frac{1}{\Omega} e^{-\gamma(t-t')} \sin(\Omega(t-t'))\,\Theta(t-t')$
$\Omega = \sqrt{\omega_0^2 - \gamma^2}$
$Mv = w$
$v = M^{-1} w$


Method 3 Eigenfunctions
Formally Self-Adjoint Diff Ops
$L^2[a,b]_w$ has inner product
$\langle f, g \rangle_w = \int_a^b f^*(x)\, g(x)\, w(x)\, dx$
$w(x) > 0$ is the weight

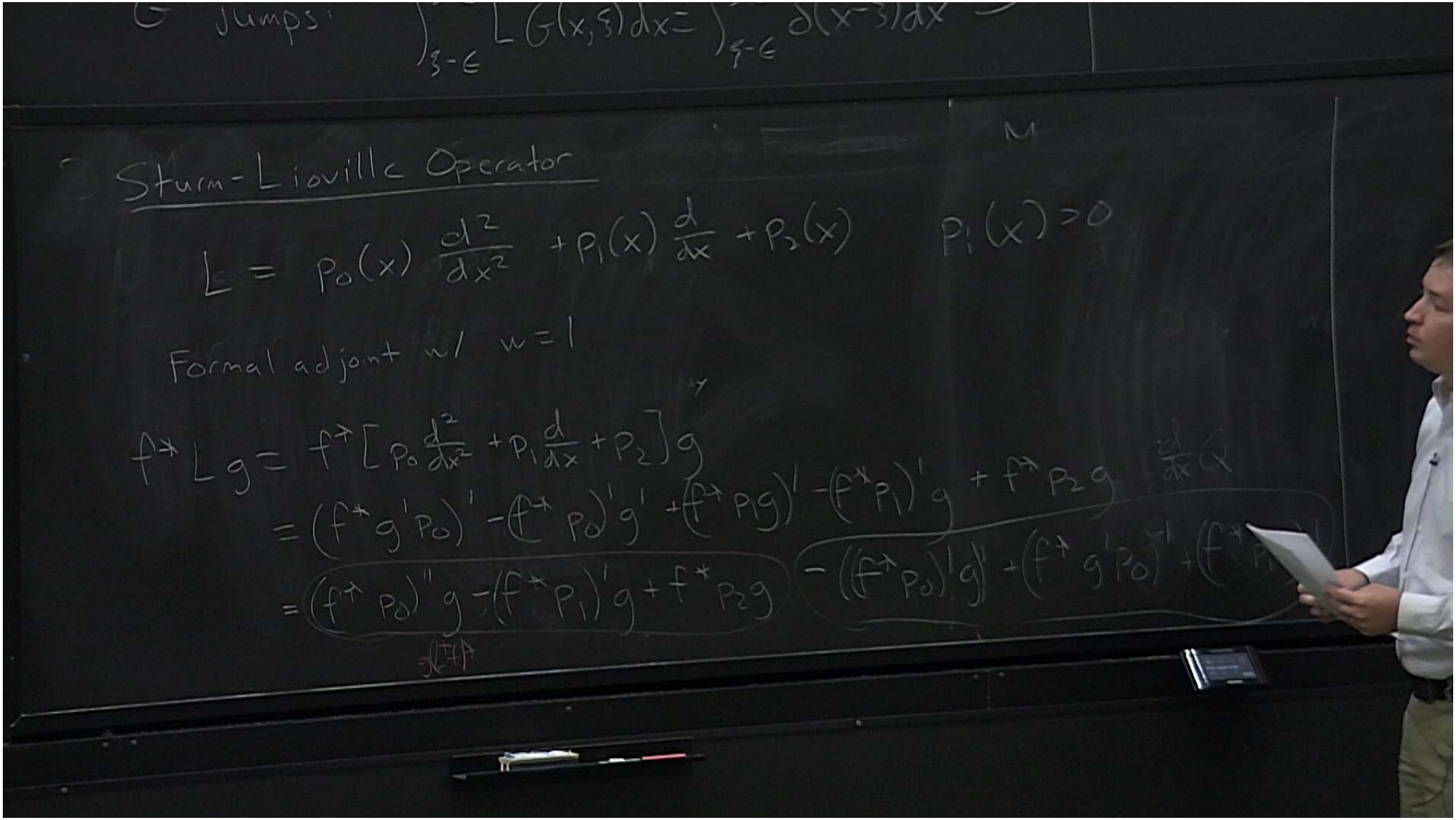
$\Omega = \sqrt{\omega_0^2 - \gamma^2}$

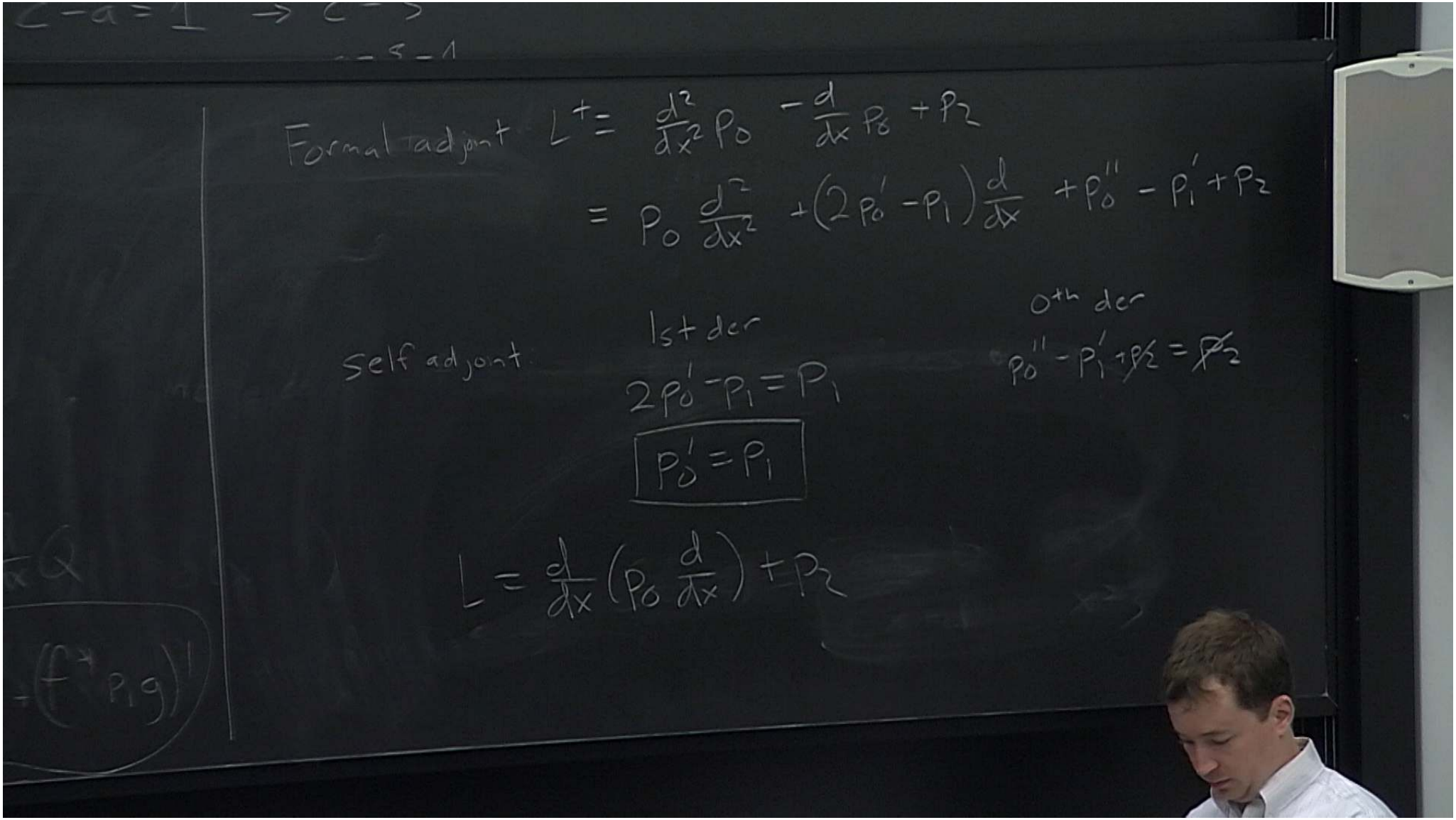
$\langle f, Lg \rangle_w = \langle L^+ f, g \rangle_w$

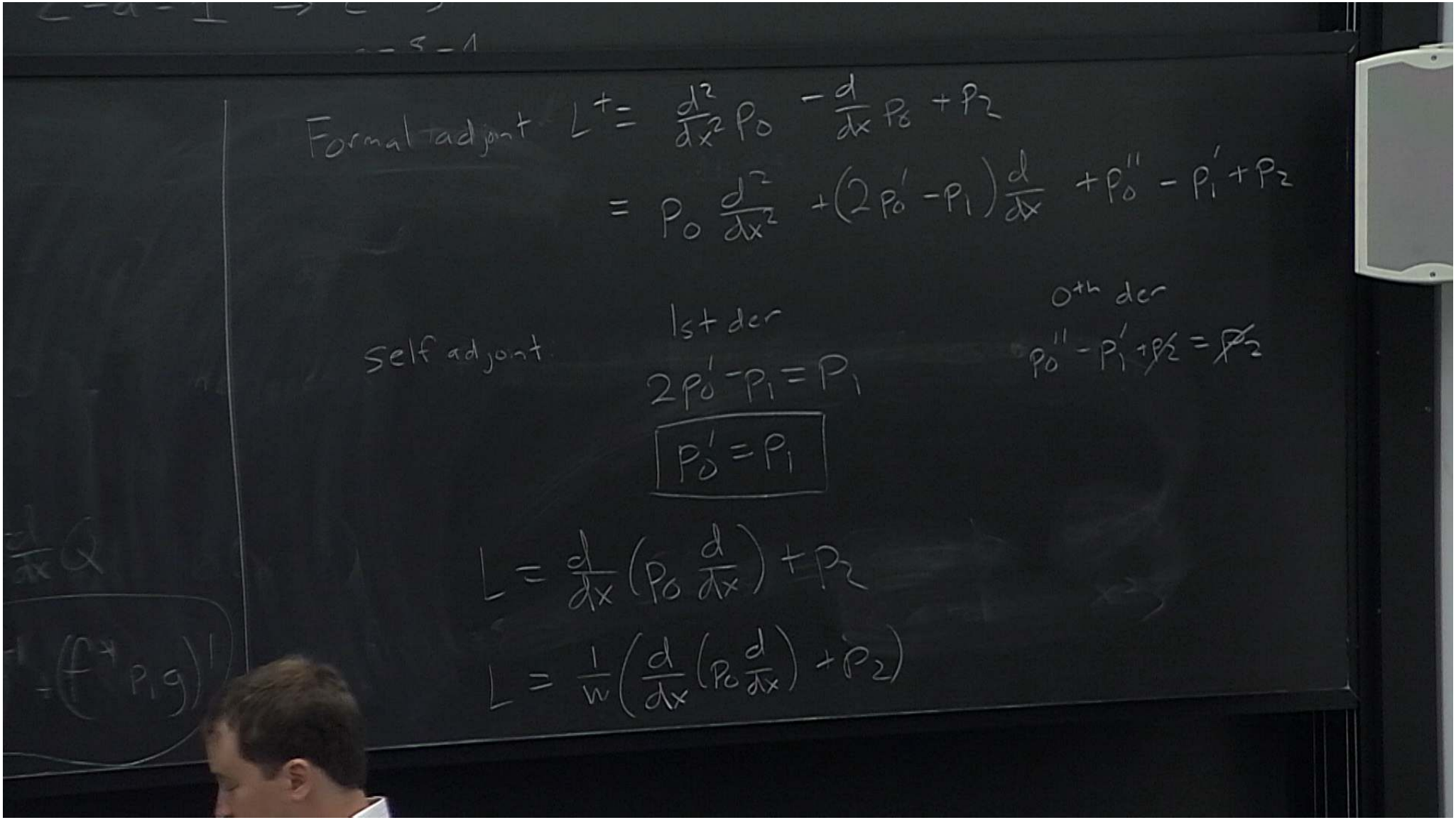
Formal adjoint L^+: $w(f^* Lg - g(L^+ f)^*) = \frac{d}{dx} Q[f, g]$

no specification of domain of L^+

Formal self-adjoint: if $L = L^+$


Jumps:
$\int_{\xi-\epsilon}^{} L\,G(x,\xi)\,dx = \int_{\xi-\epsilon}^{} \delta(x-\xi)\,dx$
Sturm-Liouville Operator
$L = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$
$p_1(x) > 0$
Formal adjoint w/ $w = 1$
$f^* L g = f^*\left[p_0\frac{d^2}{dx^2} + p_1\frac{d}{dx} + p_2\right] g$
$= (f^* g' p_0)' - (f^* p_0)' g' + (f^* p_1 g)' - (f^* p_1)' g + f^* p_2 g$
$= (f^* p_0)'' g - (f^* p_1)' g + f^* p_2 g$
$- (f^* p_0)' g' + (f^* g' p_0)' + (f^*$


Formal adjoint $L^+ = \frac{d^2}{dx^2} p_0 - \frac{d}{dx} p_1 + p_2$
$= p_0 \frac{d^2}{dx^2} + (2p_0' - p_1)\frac{d}{dx} + p_0'' - p_1' + p_2$
Self adjoint
1st der
$2p_0' - p_1 = p_1$
$p_0' = p_1$
0th der
$p_0'' - p_1' + p_2 = p_2$
$L = \frac{d}{dx}(p_0 \frac{d}{dx}) + p_2$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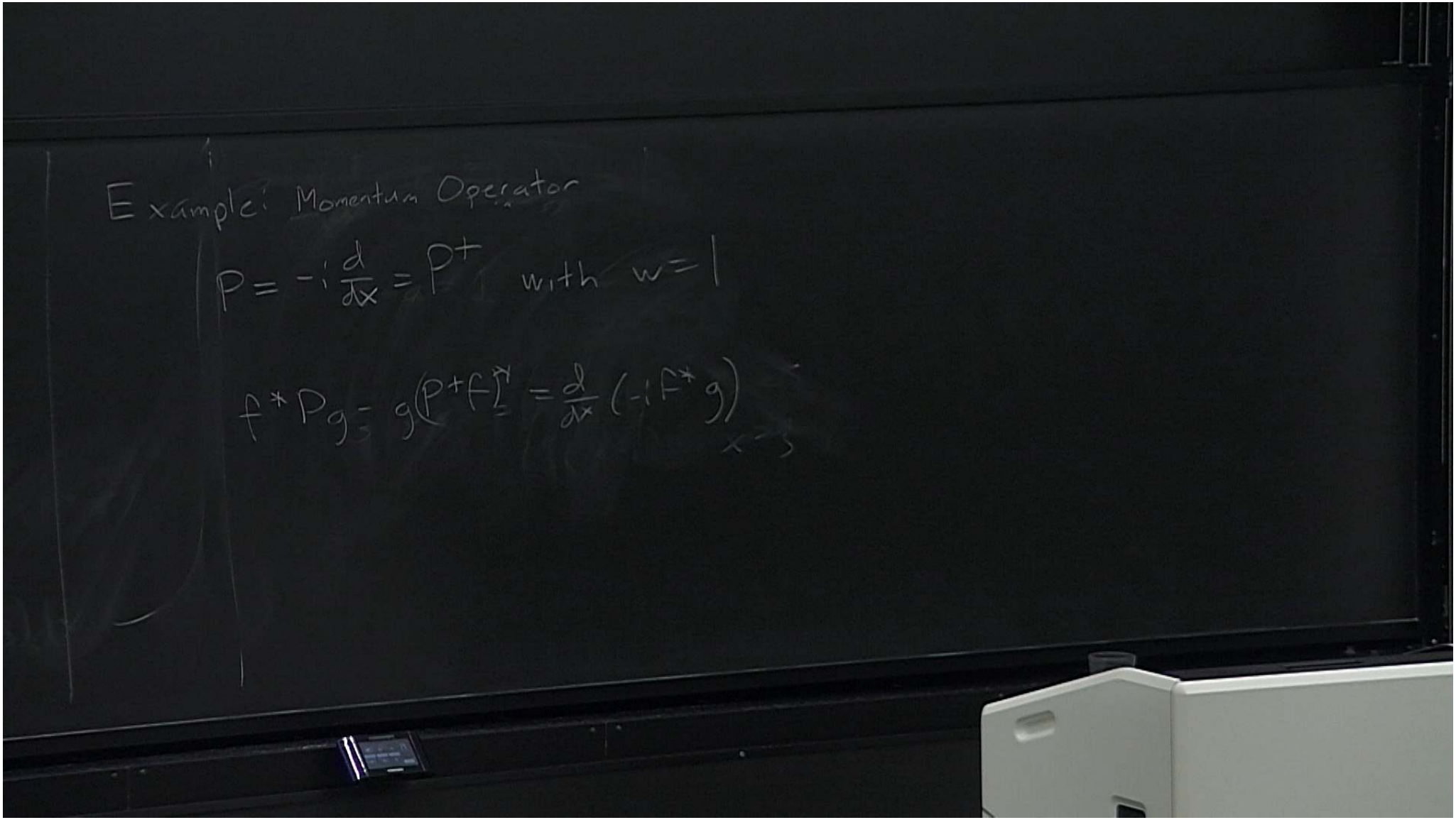
0th der
$p_0'' - p_1' + p_2 = p_2$
$L = \frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_2$
$L = \frac{1}{w}\left(\frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_2\right)$

Truly Self-Adjoint

L is truly self-adjoint

if
- formally self adjoint
- $\frac{d}{dx} Q = 0$ when bcs are satisfied
- domain of L and L^+ to be same


Example: Momentum Operator
$P = -i\frac{d}{dx} = P^+$ with $w = 1$
$f^* P g = g(P^+ f)^* = \frac{d}{dx}(-i f^* g)$
$x \to \infty$

Truly Self-Adjoint

L is truly self-adjoint

if
- formally self adjoint
- $\int_a^b \frac{d}{dx} Q = 0$ when bcs are satisfied
- domain of L and $L^\dagger$ to be same

Example: Mom

$P = -i$

$f^* P g$

$$\ldots g)' + (f^* g' p_0)' + (f^* p_1 g)'$$

$$L = \frac{1}{w}\left(\frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_2\right)$$

Example: Momentum Operator

$$P = -i\frac{d}{dx} = P^\dagger \quad \text{with } w = 1 \quad \text{in } L^2[a,b]$$

$$f^* P g - g(P^\dagger f)^* = \frac{d}{dx}(-i f^* g)$$

integrate: $\langle f, Pg\rangle - \langle P^\dagger f, g\rangle = -i f^* g \Big|_a^b$

Need is $f^*(b)g(b) - f^*(a)g(a) = 0$

1 $g(a) = 0$ $f(b) = 0$ → not self adjoint

2 $g(a) = e^{i\delta} g(b)$ and $f(a) = e^{i\delta} f(b)$

Fredholm Alternative

L of order n with n homogeneous bcs

→ either: $Ly = f$ has a unique solution

or

2. $Ly = 0$ has a nontrivial solution

If $Ly = 0$ has n indep solutions so does $L^+y = 0$

If 2 holds $Ly = f$ has no solutions unless F is orthogonal to all solutions of $L^+y = 0$

Need is $f^*(b)\,g(b) - f^*(a)\,g(a) = 0$

1. $g(a) = 0$ $\quad f(b) = 0$ $\quad \Rightarrow$ not self adjoint

2. $g(a) = e^{i\delta}\, g(b)$ and $f(a) = e^{i\delta} f(b)$

L acts on g

L^+ acts on f

Fre

L

→ e

If

If