

Title: PSI Front End - Functions, "Functions", etc.

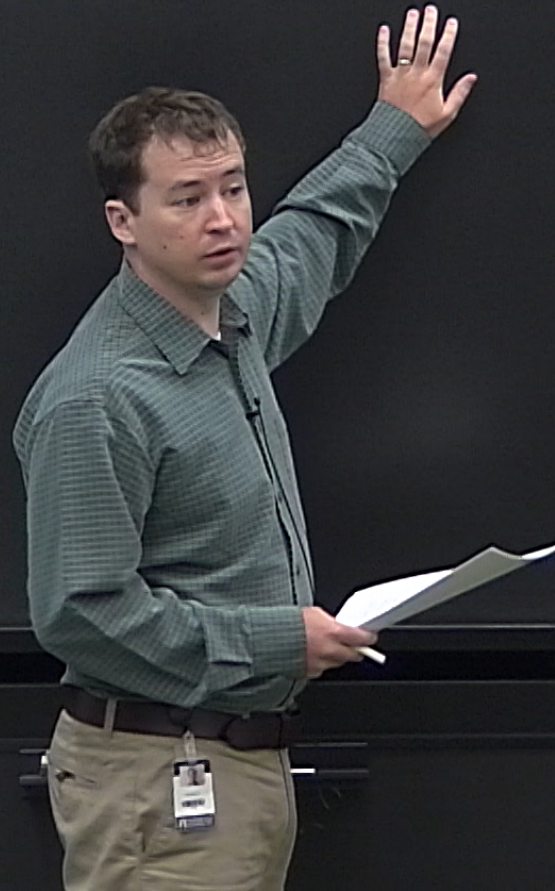
Date: Aug 28, 2017 10:30 AM

URL: <http://pirsa.org/17080067>

Abstract:

Today: Distributions, Gamma, Zeta

Start at 10:48



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Recap: distributions are $\begin{cases} \text{linear} \\ \text{continuous} \\ \text{finite} \end{cases}$ maps from test functions to $\mathbb{R}$

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Today: Multiplication
Composition
Functional Derivatives

10:48

Multiplication

$$\begin{aligned}(\Psi u)(\varphi) &= \int_{-\infty}^{\infty} \Psi(x) u(x) \varphi(x) dx \\ &= \int_{-\infty}^{\infty} u(x) (\Psi(x) \varphi(x)) dx\end{aligned}$$

10:48

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$$(\Psi u)(\varphi) \equiv u(\underbrace{\Psi \varphi})$$

test function if Ψ is a test function

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test function if Ψ is a test function

$\delta_a(\varphi) \equiv \varphi(a)$ shifted delta

Example: $(\psi \delta_a)(\varphi) = \delta_a(\psi \varphi)$
 $= (\psi \varphi)(a)$
 $= \psi(a) \varphi(a)$
 $= \psi(a) \delta_a(\varphi)$

$$\delta_a(\varphi) \equiv \varphi(a) \quad \text{shifted delta}$$

$$\begin{aligned} \text{Example: } (\psi \delta_a)(\varphi) &= \delta_a(\psi \varphi) \\ &= (\psi \varphi)(a) \\ &= \psi(a) \varphi(a) \\ &= \psi(a) \delta_a(\varphi) \\ &\rightarrow \psi \delta_a = 0 \end{aligned}$$

$$\times \delta'(\varphi) =$$

$$\begin{aligned}x \delta'(\varphi) &= -\delta(\partial_x(x\varphi)) \\&= -\delta(\varphi) - x\delta(\varphi') \\&= -\delta(\varphi)\end{aligned}$$

$$x\delta' = -\delta$$

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$$\begin{aligned} \rightarrow x \delta &= 0 \\ \psi &= x \quad a = 0 \end{aligned}$$

$$\begin{aligned} x \delta'(\varphi) &= -\delta(\partial_x(x\varphi)) \\ &= -\delta(\varphi) - x\delta'(\varphi) \\ &= -\delta(\varphi) \end{aligned}$$

$$x\delta' = -\delta$$

$x(x\varphi))$
 $-x\delta(\varphi')$

Composition

$$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(f(x)) \varphi(x) dx$$

$$= \int_{-\infty}^{\infty} u(y) \varphi(g(y)) |g'(y)| dy$$

$x=g(y)$ is the solution to $y=f(x)$

$$(u \circ f)(\varphi) \equiv u((\varphi \circ g)|g'|) \text{ if } (\varphi \circ g)|g'| \text{ is a test function}$$

$$x(x\varphi)) \\ -x\delta(\varphi')$$

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guaranteed if

- f is C^∞
- f has a unique inverse

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$$(\mathcal{J} \circ f)(\varphi) \equiv \sum_i \frac{\delta x_i}{|f'(x_i)|} (\varphi)$$

↑
sum over
roots of f

works if f has only simple roots

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$$x^2(x-1)$$

$$(\mathcal{J} \circ f)(\varphi) \equiv \sum_i \frac{\delta x_i}{|f'(x_i)|}(\varphi) \quad *$$

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$$(\mathcal{J} \circ f)(\varphi) \equiv \sum_i \frac{\delta x_i}{|f'(x_i)|} (\varphi) \quad \text{*important}$$

$\int \delta(f(x)) \varphi(x) dx$

$\uparrow$
 sum over roots of f

works if f has only simple roots

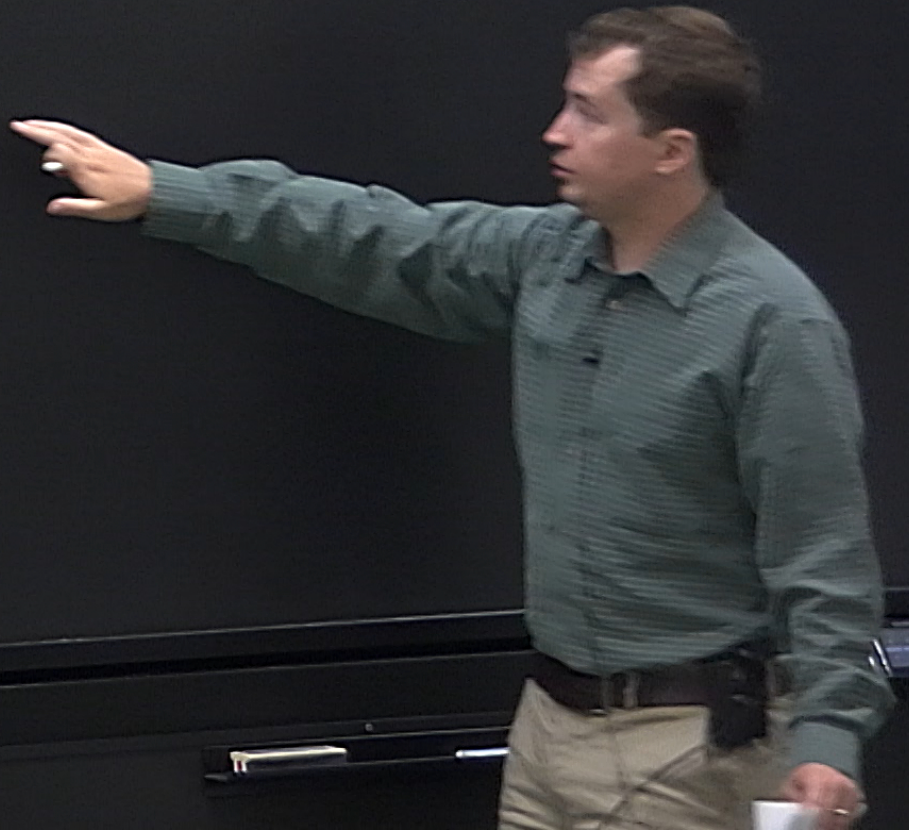
$$x^2(x-1)$$

Functional Derivatives

$$\text{Example: } S[q(t), \dot{q}(t)] = \int dt \underbrace{\left(\frac{1}{2} \dot{q}^2 - V(q) \right)}_{\mathcal{L}}$$

$$\begin{aligned} \delta S &= S[q + \delta q, \dot{q} + \delta \dot{q}] - S[q, \dot{q}] \\ &= \int dt \delta q \underbrace{\left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right]}_{\frac{\delta S}{\delta q(t)} \leftarrow \text{functional derivative}} + \mathcal{O}(\delta q^2) \end{aligned}$$

$$\delta S = \sum_i \frac{\partial S}{\partial q_i} \delta q_i \rightarrow \int dt \frac{\delta S}{\delta q(t)} \delta q(t)$$



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$$\delta q(t) = \epsilon \varphi(t)$$

↑
test function

$$\int \frac{\delta F}{\delta f(x)} \varphi(x) dx = \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \varphi] - F[f]}{\epsilon}$$

Example $F[f] = f^2(x_0)$

$$\int \frac{\delta F}{\delta f(x)} \varphi(x) dx = \lim_{\epsilon \rightarrow 0}$$

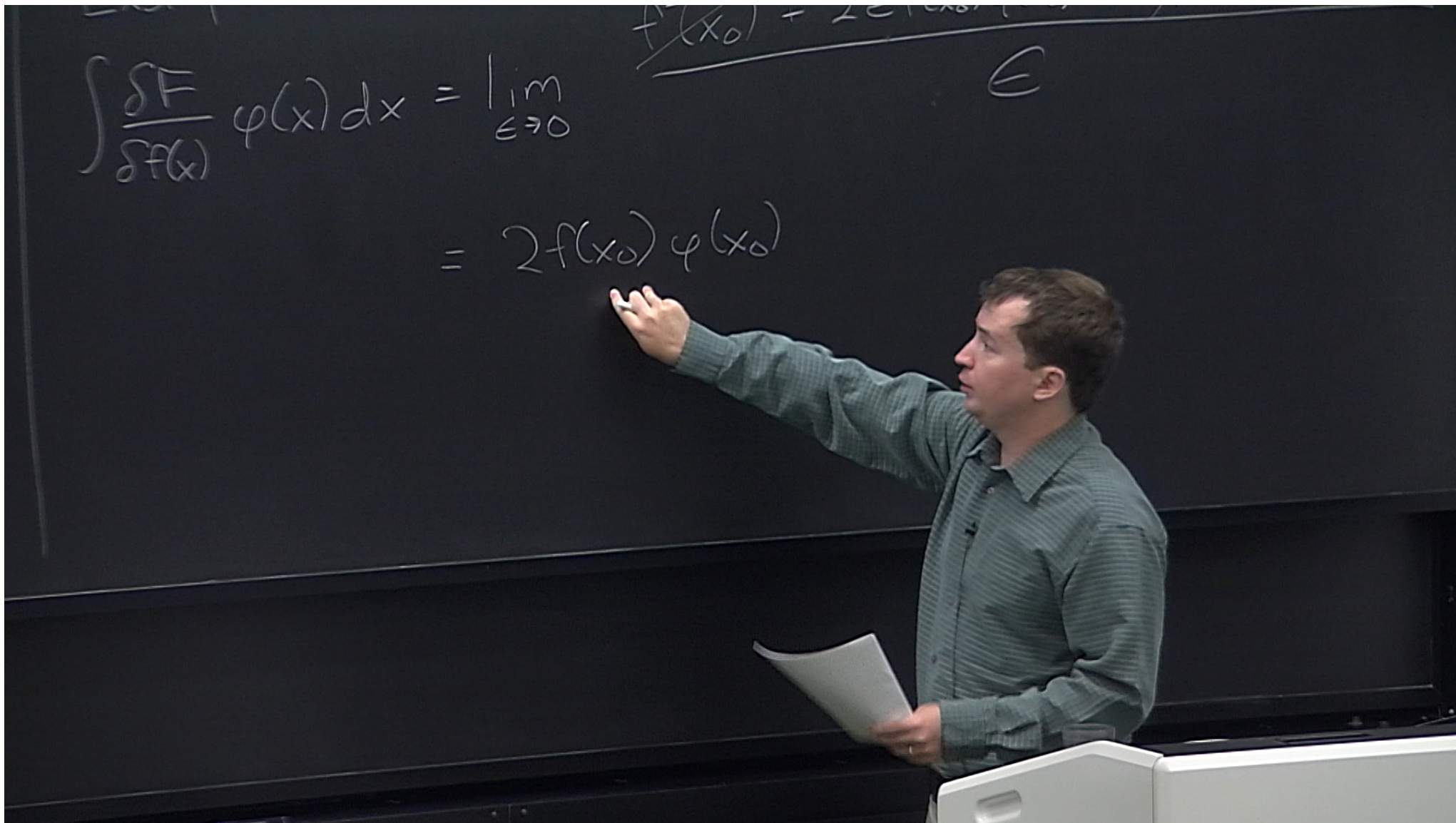
$$\frac{\cancel{f^2(x_0)} + 2\epsilon f(x_0)\varphi(x_0) + \epsilon^2\varphi^2(x_0) - \cancel{f^2(x_0)}}{\epsilon}$$

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$\nearrow 0 \text{ as } \epsilon \rightarrow 0$



Example: $F[f] = f^2(x_0)$

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Example ~~$F[f] = f^2(x_0)$~~

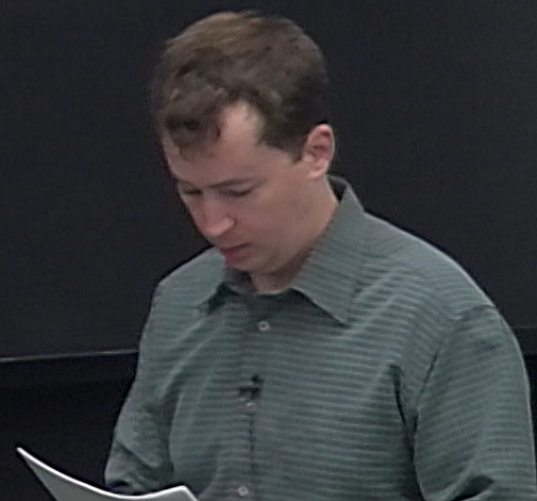
$$\int \frac{\delta F}{\delta f(x)} \varphi(x) dx = \lim_{\epsilon \rightarrow 0}$$

$$\frac{\cancel{f^2(x_0)} + 2\epsilon f(x_0)\varphi(x_0) + \epsilon^2 \varphi^2(x_0) - \cancel{f^2(x_0)}}{\epsilon} \quad \begin{matrix} \nearrow 0 \text{ as } \epsilon \rightarrow 0 \end{matrix}$$

$$= 2f(x_0)\varphi(x_0)$$

$$= \int 2f(x_0)\delta(x-x_0)\varphi(x)dx$$

$$\frac{\delta F}{\delta f(x)} = 2f(x_0)\delta(x-x_0)$$



Properties:

linearity:

$$\frac{\delta(\lambda F + \mu G)}{\delta f(x)} = \lambda \frac{\delta F}{\delta f(x)} + \mu \frac{\delta G}{\delta f(x)}$$

product rule:

$$\frac{\delta(FG)}{\delta f(x)} = \frac{\delta F}{\delta f(x)} G + F \frac{\delta G}{\delta f(x)}$$

Properties:

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chain rules:

$$\frac{\delta F[g(f)]}{\delta f(x)} = \frac{\delta F[g(f)]}{\delta g(f(x))} \frac{dg(f(x))}{df(x)}$$

$$\frac{\delta g(F[f])}{\delta f(x)} = \frac{dg(F[f])}{dF[f]} \frac{\delta F[f]}{\delta f(x)}$$

warnings: $u(\varphi)$ only makes sense if φ is a test function

linear theory

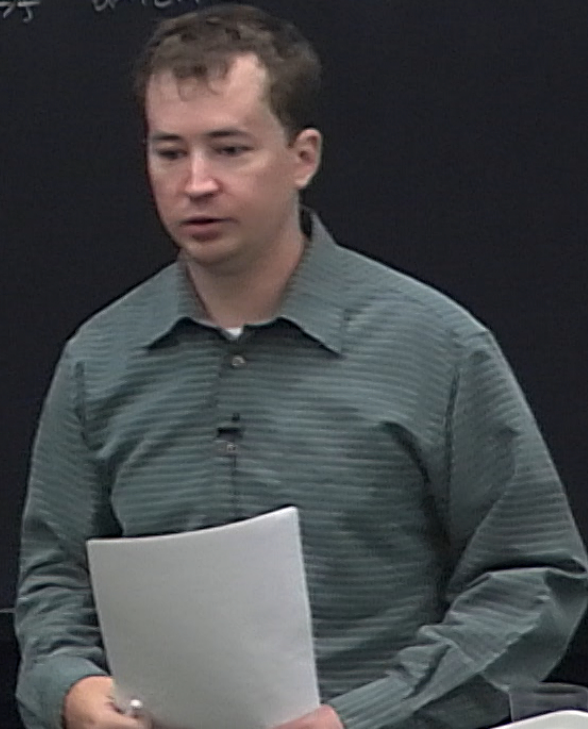
$f_n(x) \rightarrow f(x)$ and $f_n \rightarrow f$ unrelated

warnings: $u(\varphi)$ only makes sense if φ is a test function

linear theory

$f_n(x) \rightarrow f(x)$ and $f_n \rightarrow f$ unrelated

$$(uv)(\varphi) \stackrel{?}{=} \int u(x) v(x) \varphi(x) dx$$



warnings: $u(\varphi)$ only makes sense if φ is a test function

linear theory

$f_n(x) \rightarrow f(x)$ and $f_n \rightarrow f$ unrelated

$$\cancel{(uv)(\varphi) \stackrel{?}{=} \int \cancel{u(x)} \cancel{v(x)} \varphi(x) dx}$$

$$= \int 2f(x_0) \delta(x-x_0) \varphi(x) dx$$

$$\frac{\delta F}{\delta f(x)} = 2f(x_0) \delta(x-x_0)$$

$f_n(x) \rightarrow f(x)$ and $f_n \rightarrow f$ unrelated

~~$$(uv)(\varphi) \stackrel{?}{=} \int u(x) v(x) \varphi(x) dx$$~~

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$$g(x) = x^2$$

$$G[g(f)]$$

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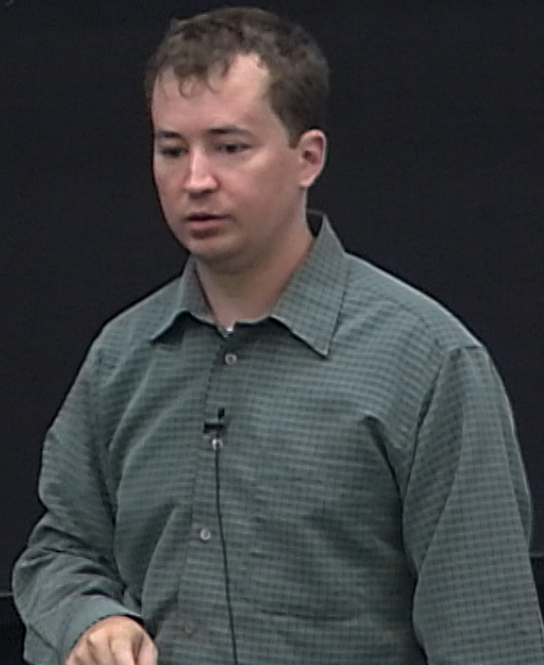
$$(uv)(\varphi) \stackrel{?}{=} \int \cancel{u(x)} v(x) \varphi(x) dx$$

$$F[f] = f^2(x_0) = G[g(f)]$$

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$$G[g(f)]$$

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Gamma + (Zeta)

Gamma

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Functional Equation

$$\begin{aligned} \frac{d}{dt}(t^z e^{-t}) &= z t^{z-1} e^{-t} - t^z e^{-t} \\ t^z e^{-t} \Big|_0^{\infty} &= z \int_0^{\infty} t^{z-1} e^{-t} dt - \int_0^{\infty} t^z e^{-t} dt \\ 0 &= z \Gamma(z) - \Gamma(z+1) \end{aligned}$$

$$\boxed{\Gamma(z+1) = z \Gamma(z)}$$

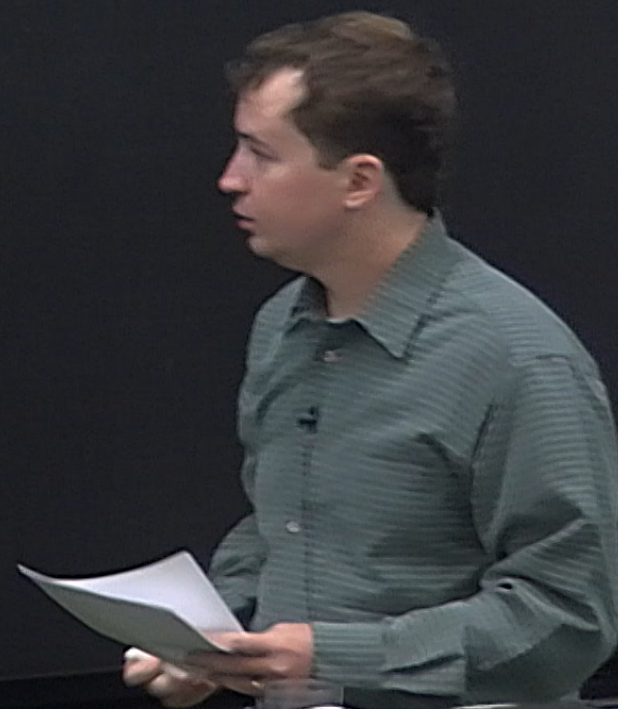
$$n! = n(n-1)!$$

$$\boxed{\Gamma(z+1) = z \Gamma(z)}$$

$$\Gamma(1) = 1$$

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$$n! = n(n-1)!$$

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$$\Gamma(n+1) = n!$$

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$$

$$\operatorname{Re}(z) + n > 0$$

$$z = -n + \epsilon + 1$$

$$\Gamma(z) = \frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)\cdots(\epsilon-n+1)} \approx \frac{\Gamma(1)\epsilon^{n-1}}{(n-1)!\epsilon}$$

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$$\Gamma(z) = \frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)\cdots(\epsilon-n+1)} \approx \frac{\Gamma(1)\epsilon^{n-1}}{(n-1)!\epsilon} \quad \text{as } \epsilon \rightarrow 0$$

simple poles at $z = -n$ with residue $\frac{(-1)^n}{n!}$

Another Definition

$$\Gamma_p(z) = \frac{p! p^z}{z(z+1)\cdots(z+p)}$$

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can show $\Gamma_p(z+1) = \frac{p}{z+p+1} z \Gamma_p(z)$

$$\Gamma(z) = \lim_{p \rightarrow \infty} \Gamma_p(z)$$

Weierstrass Formula

$$\Gamma_p(z) = \frac{p^z}{z(1+\frac{z}{1})(1+\frac{z}{2})\dots(1+\frac{z}{p})}$$

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$$= \frac{e^{z \log p} \cdot e^{-z - z/2 - z/3 \dots - z/p}}{z(1+\frac{z}{1}) \dots (1+\frac{z}{p})} \quad \frac{e^{z + z/2 + \dots + z/p}}{e}$$

$$\gamma = \lim_{p \rightarrow \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{p} - \log p \right) \approx 0.577$$

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$$\boxed{\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{p=1}^{\infty} \left(1 + \frac{z}{p}\right) e^{-z/p}}$$

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$\Gamma(z)$ has no zeros

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$$= \frac{e^{-z(1 + \frac{1}{2} + \dots + \frac{1}{p})}}{(1 + \frac{z}{p})}$$

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$\Gamma(z)$ has no zeros
only poles of $\Gamma(z)$ are at $z = -n$

Complement Formula

$$\frac{1}{\Gamma(z)} \frac{1}{\Gamma(-z)} = -z^2 e^{\gamma z} e^{-\gamma z} \prod_{p=1}^{\infty} \left(1 + \frac{z}{p}\right) e^{-z/p} \left(1 - \frac{z}{p}\right) e^{z/p}$$

at $z = -n$

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$$\frac{1}{\Gamma(z)} \frac{-z}{\Gamma(1-z)} = -z \left(z \prod_{p=1}^{\infty} \left(1 - \frac{z^2}{p^2}\right) \right)$$

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$$\underbrace{\hspace{10em}}_{\frac{\sin(\pi z)}{\pi}}$$

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$$\Gamma(z) \Gamma(1-z) = \frac{\prod_{p=1}^{\infty} \frac{\sin(\pi z)}{p}}{\pi}$$

Complement Formula

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$$\Gamma(z) \Gamma(1-z) = \frac{\hat{1}}{\sin(\pi z)}$$

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$$\Gamma_p(z) = \frac{p! p^z}{z(z+1)\cdots(z+p)}$$

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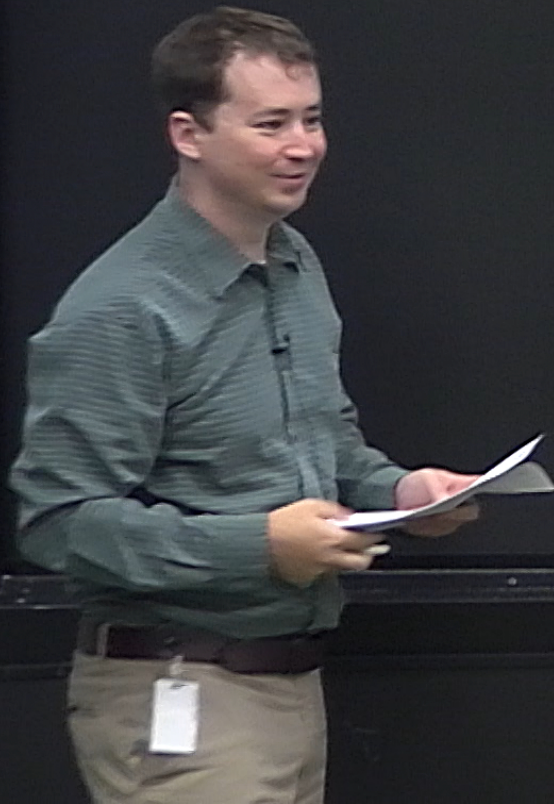
$$\Gamma_p(z+1) = \frac{p}{z+1} \Gamma_p(z)$$

$$\Gamma(z) \Gamma(z + \frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2z-1}} \Gamma(2z)$$



Duplication Formula

$$\Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2z-1}} \Gamma(2z)$$



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