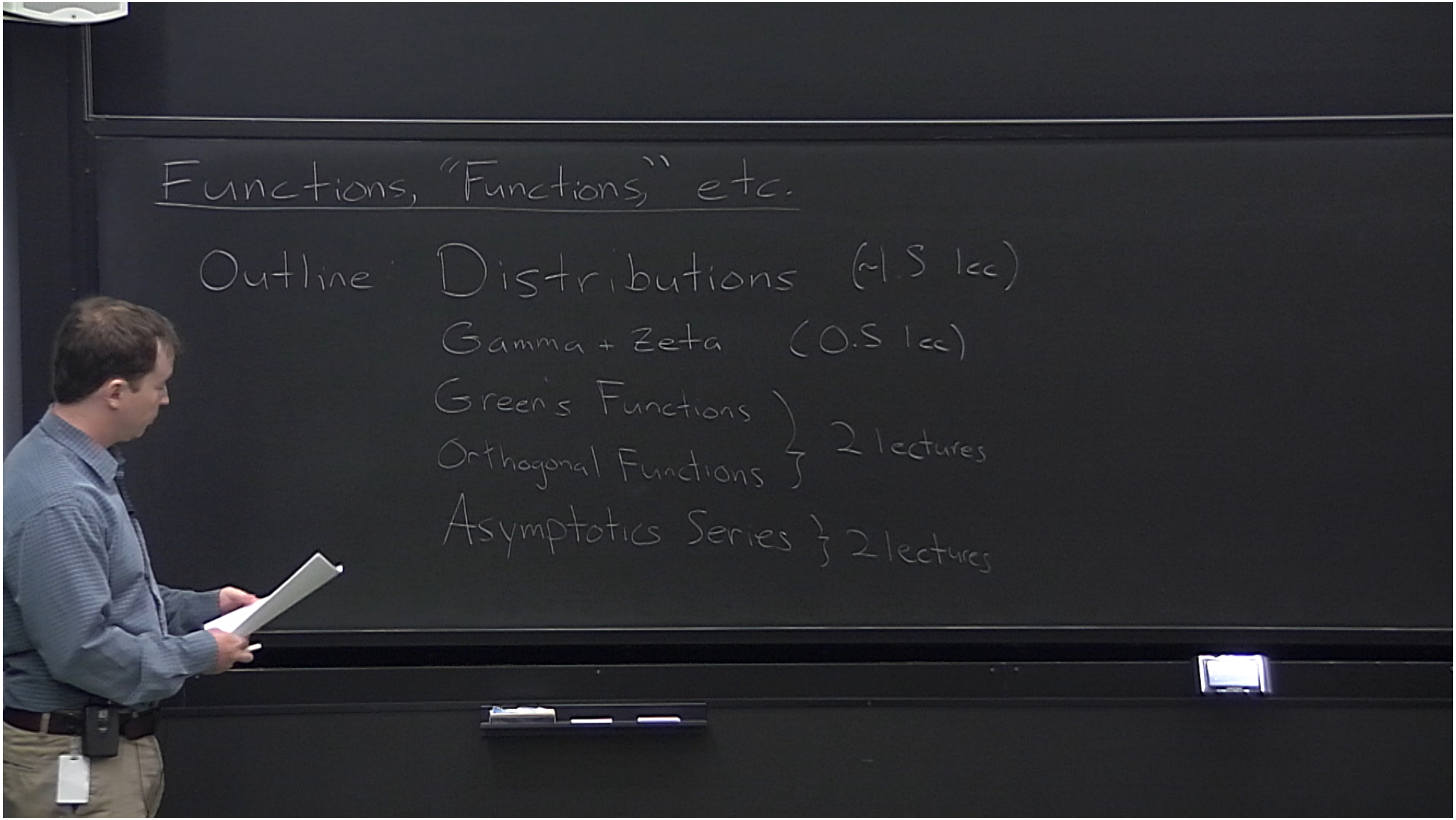
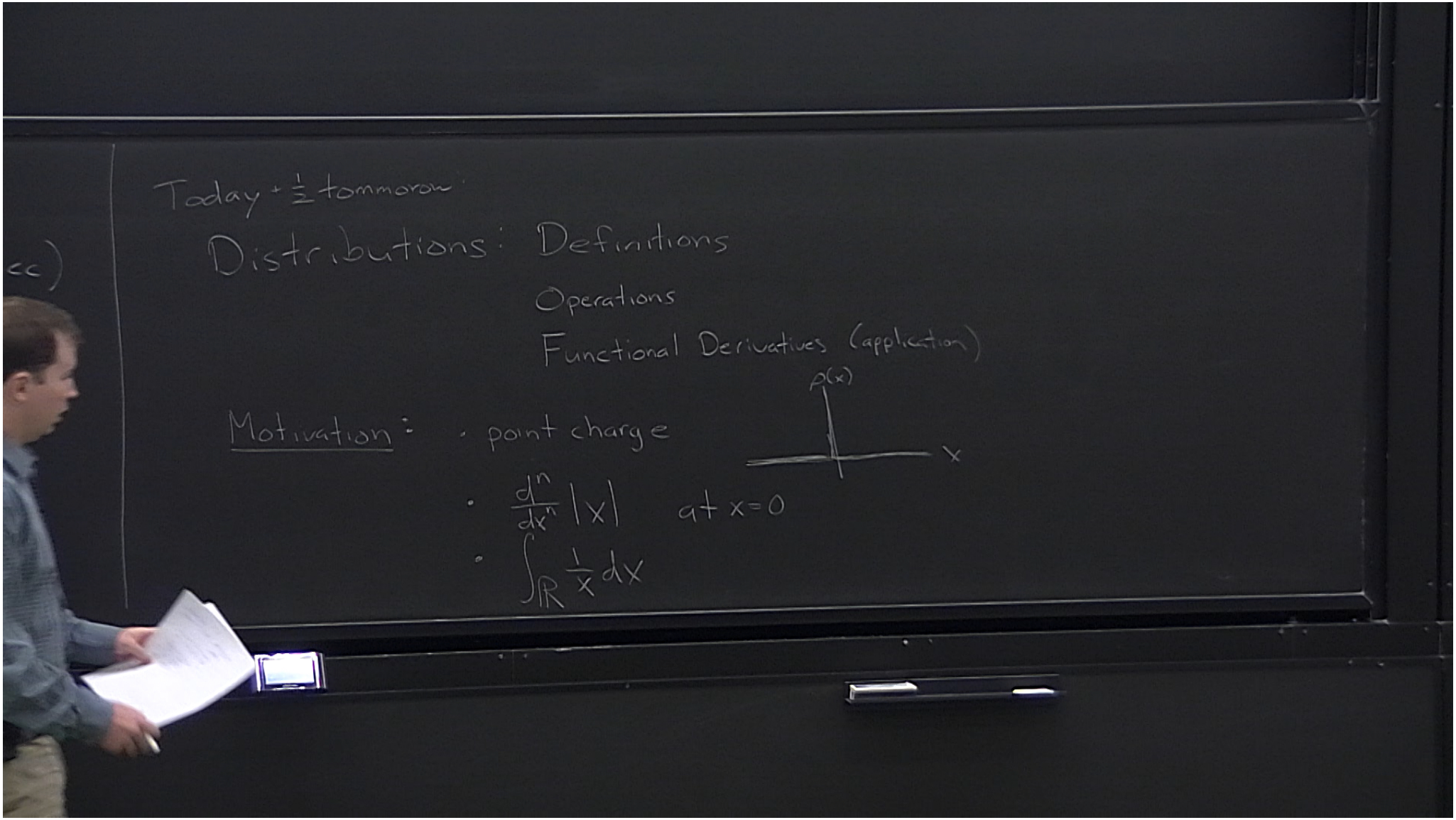
Title: PSI Front End - Functions, "Functions", etc.

Date: Aug 25, 2017 10:30 AM

URL: http://pirsa.org/17080066

Abstract:


Functions, "Functions" etc.
Outline: Distributions (~1.5 lec)
Gamma + Zeta (0.5 lec)
Green's Functions
Orthogonal Functions
2 lectures
Asymptotics Series
2 lectures

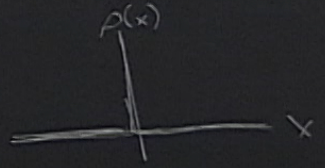

Today + ½ tommorow
Distributions: Definitions
Operations
Functional Derivatives (application)
Motivation:
· point charge
ρ(x)
x
· $\frac{d^n}{dx^n}|x|$ at x=0
· $\int_R \frac{1}{x} dx$

Today + $\frac{1}{2}$ tommorow

Distributions: Definitions

Operations

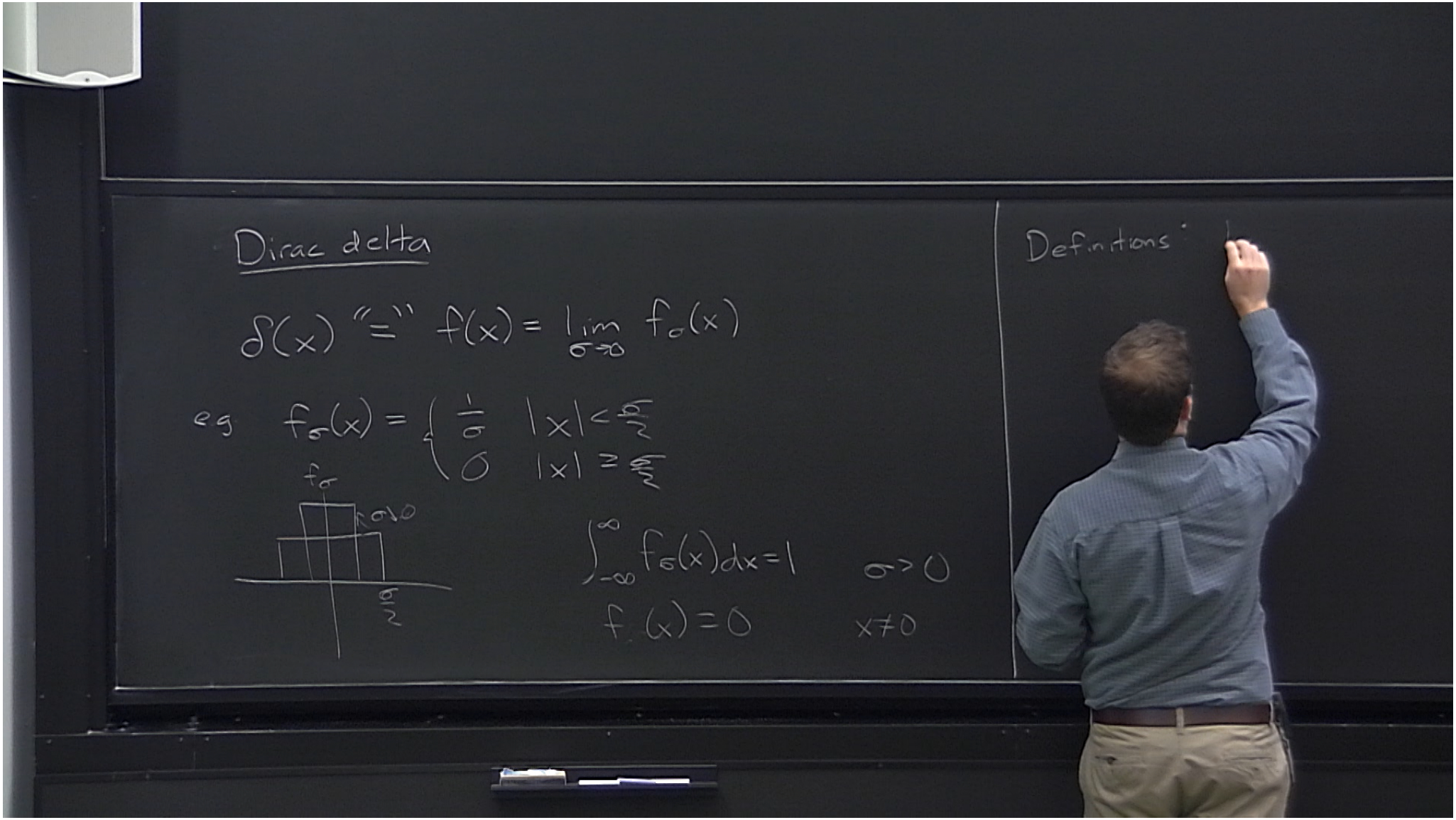
Functional Derivatives (application)



Motivation:

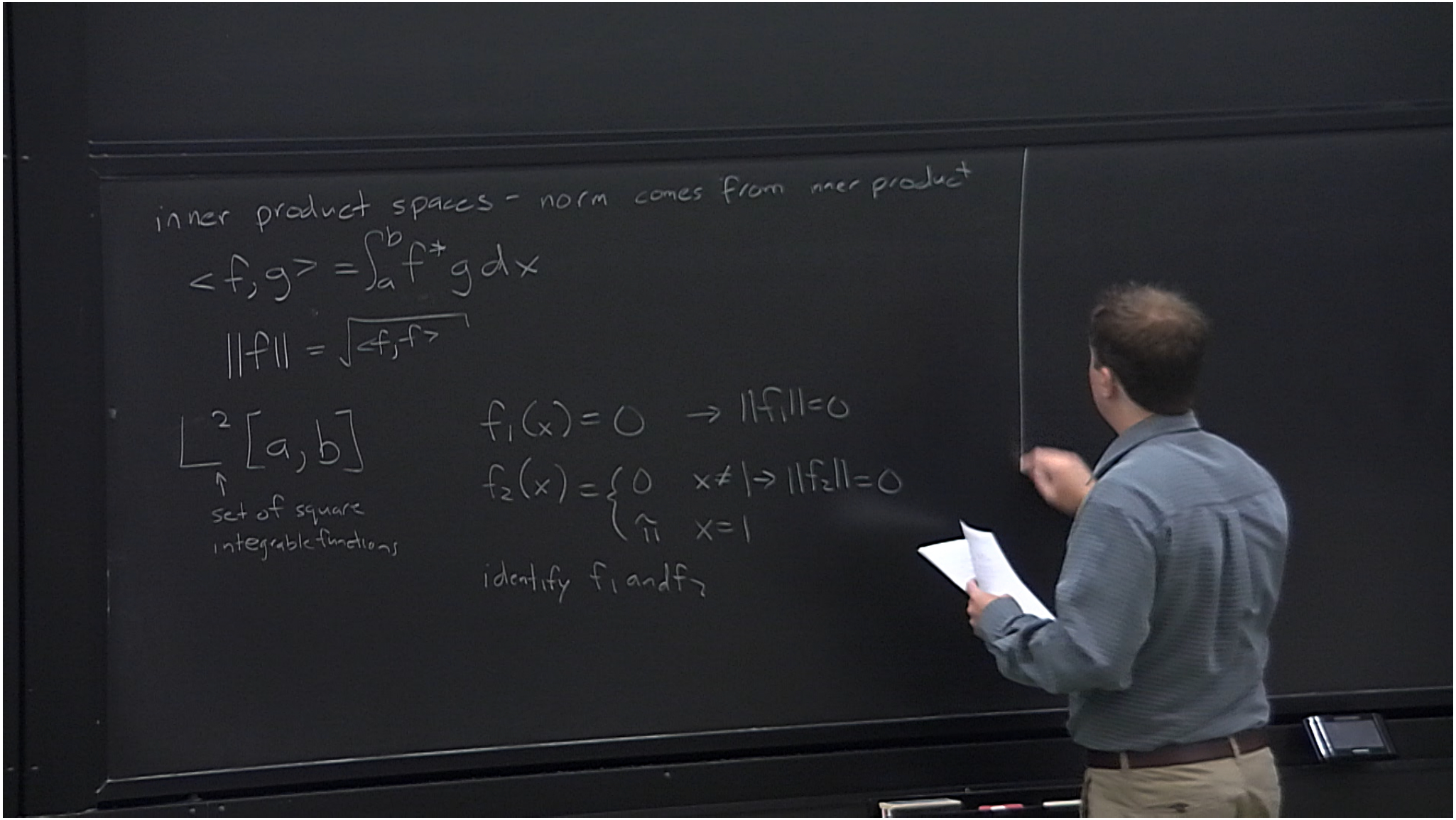
- point charge
- $\frac{d^n}{dx^n}|x|$ at $x=0$
- $\int_R \frac{1}{x} dx$

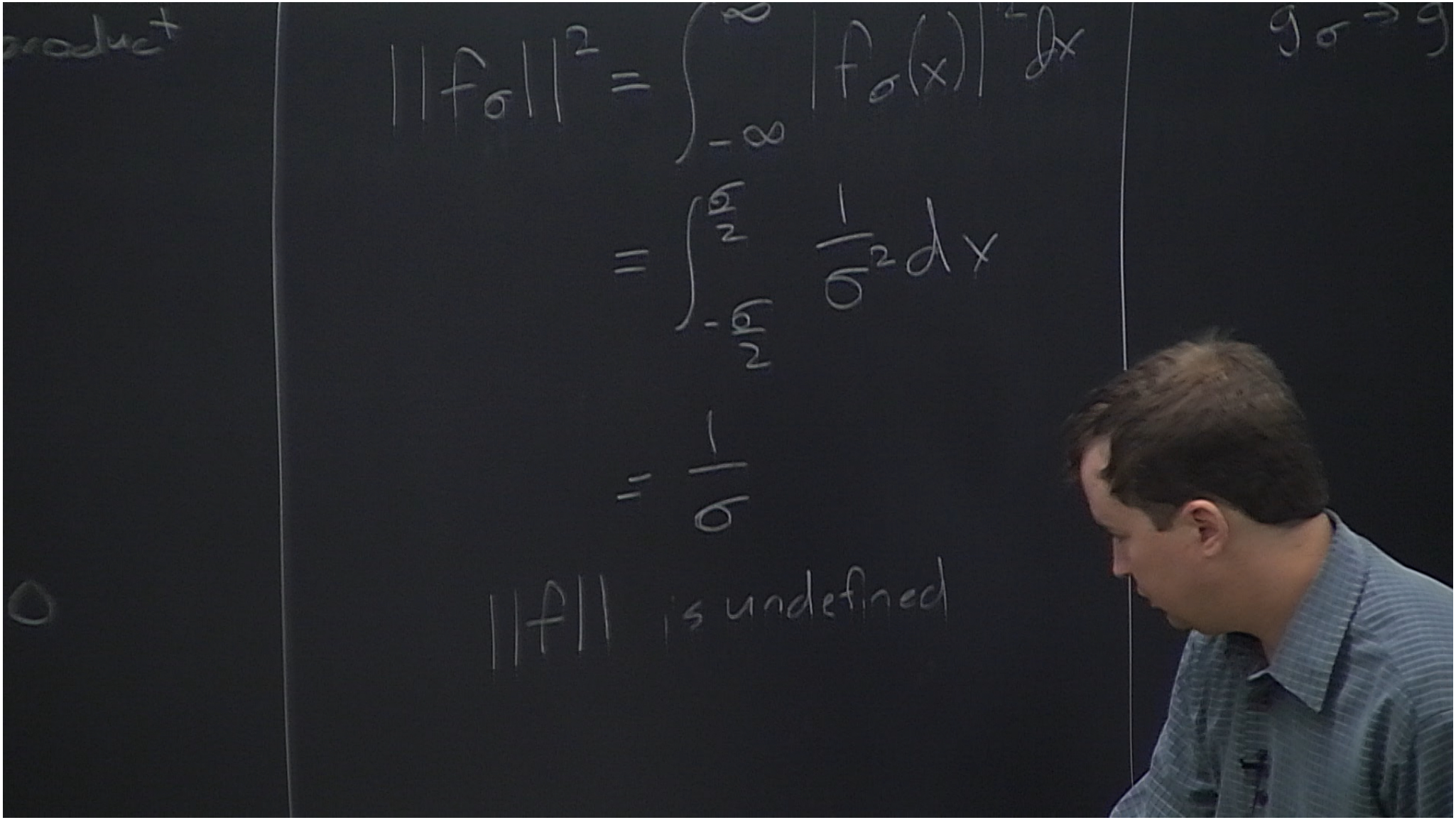
- variational problems
- limit of sequences of functions

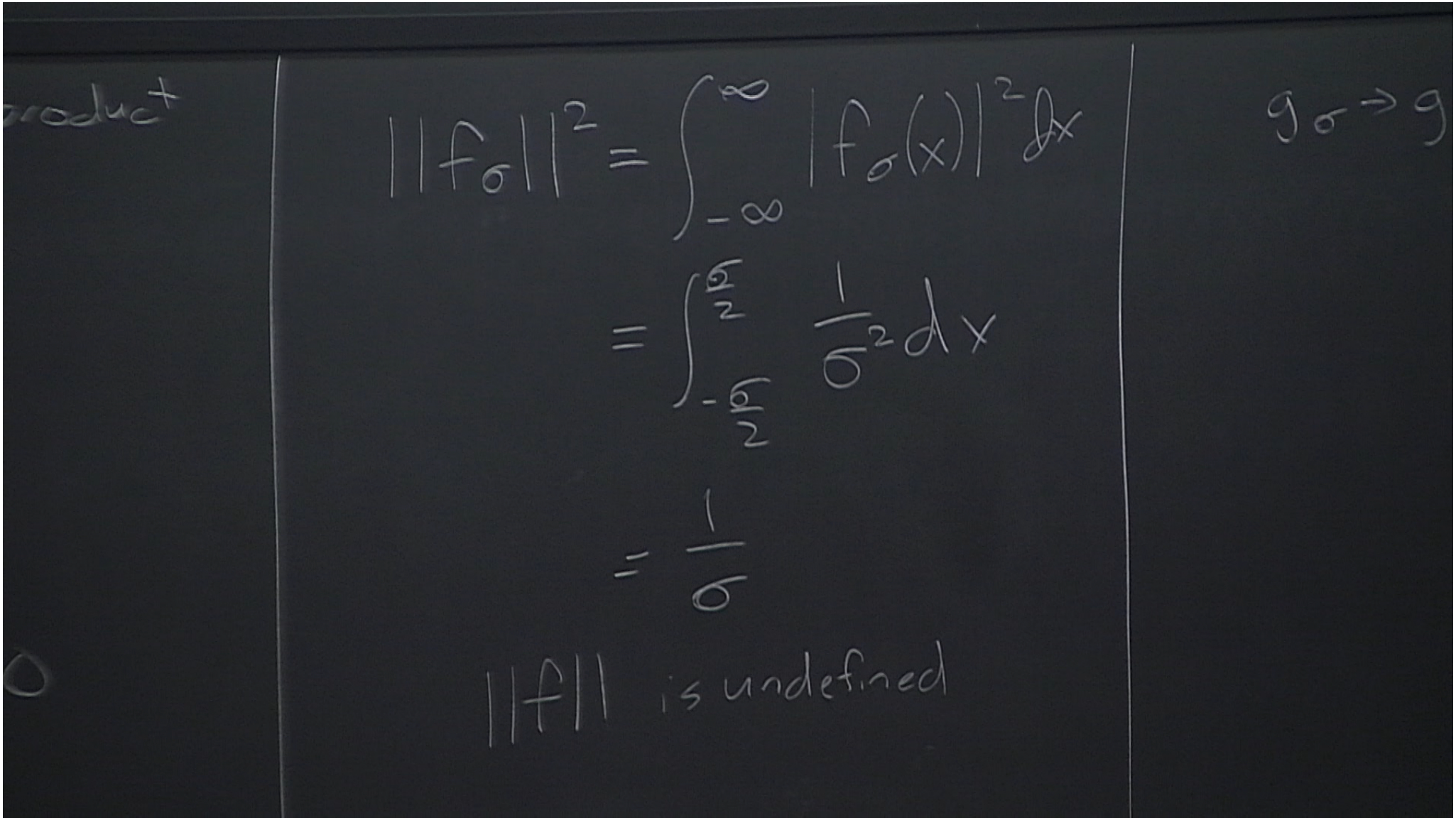

Dirac delta
$\delta(x)$ "=" $f(x) = \lim_{\sigma \to 0} f_\sigma(x)$
eg $f_\sigma(x) = \begin{cases} \frac{1}{\sigma} & |x| < \frac{\sigma}{2} \\ 0 & |x| \geq \frac{\sigma}{2} \end{cases}$
f_σ
$\sigma \to 0$
$\frac{\sigma}{2}$
$\int_{-\infty}^{\infty} f_\sigma(x)\,dx = 1$ $\sigma \to 0$
$f_\sigma(x) = 0$ $x \neq 0$
Definitions:

Definitions: Function Space: Vector space whose elements are _____ functions and has a norm which satisfies

- positivity $\|f\| \geq 0$
- unique identity $\|f\| = 0$ iff $f = 0$
- triangle inequality $\|f+g\| \leq \|f\| + \|g\|$
- linear homogeneity $\|\lambda f\| = |\lambda| \|f\|$

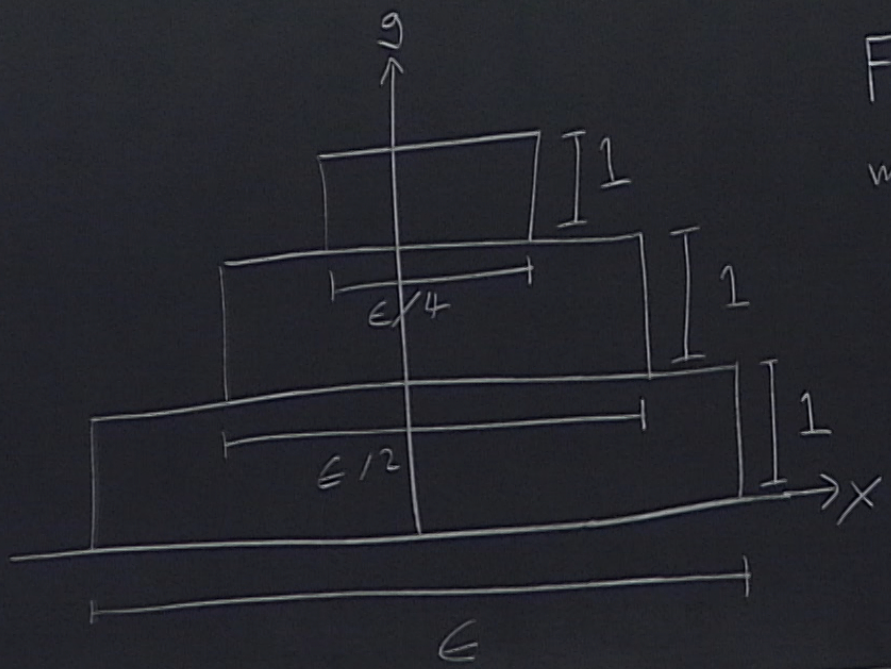

inner product spaces - norm comes from inner product
$\langle f, g \rangle = \int_a^b f^* g \, dx$
$\|f\| = \sqrt{\langle f, f \rangle}$
$L^2[a, b]$
set of square integrable functions
$f_1(x) = 0 \rightarrow \|f_1\| = 0$
$f_2(x) = \begin{cases} 0 & x \neq 1 \\ \pi & x = 1 \end{cases} \rightarrow \|f_2\| = 0$
identify f_1 and f_2


product
$\|f_\sigma\|^2 = \int_{-\infty}^{\infty} |f_\sigma(x)|^2 dx$
$= \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{\sigma^2} dx$
$= \frac{1}{\sigma}$
$\|f\|$ is undefined
$g_\sigma \to g$


product
$\|f_\sigma\|^2 = \int_{-\infty}^{\infty} |f_\sigma(x)|^2 dx$
$= \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{\sigma^2} dx$
$= \frac{1}{\sigma}$
$\|f\|$ is undefined
$g_\sigma \to g$

$g_\sigma \to g$ "=" δ for which $\lim_{\sigma \to 0} g_\sigma(x) = 0 \quad x \neq 0$

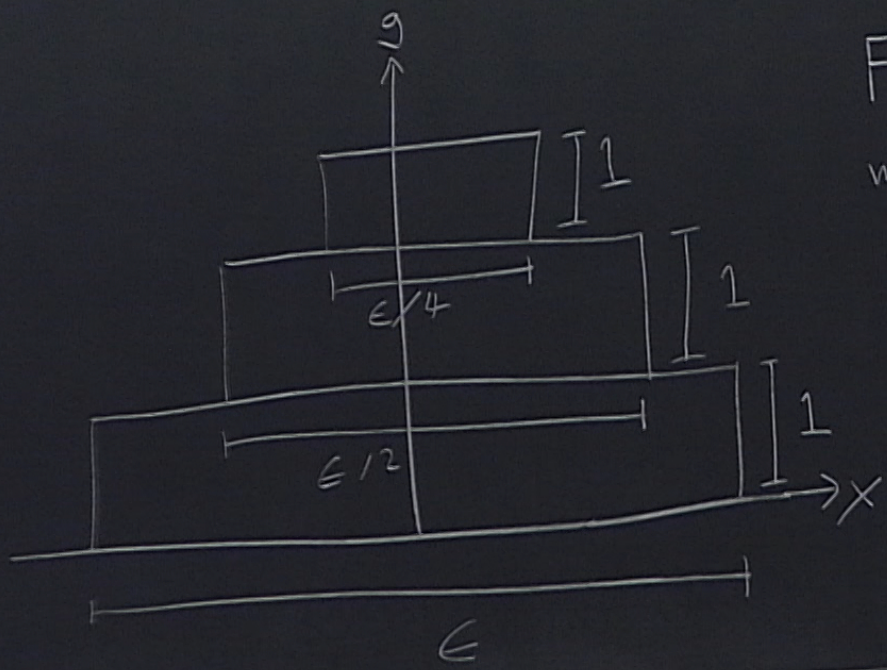
$\int g_\sigma(x)\,dx = 1 \quad \sigma \to 0$



For each ϵ and each x we can find σ_0 so that $g_\sigma(x)$ lies inside boxes for all $\sigma < \sigma_0$

$g_\sigma \to g$ "=" δ for which $\lim_{\sigma \to 0} g_\sigma(x) = 0 \quad x \neq 0$

$\int g_\sigma(x)\,dx = 1 \quad \sigma \to 0$



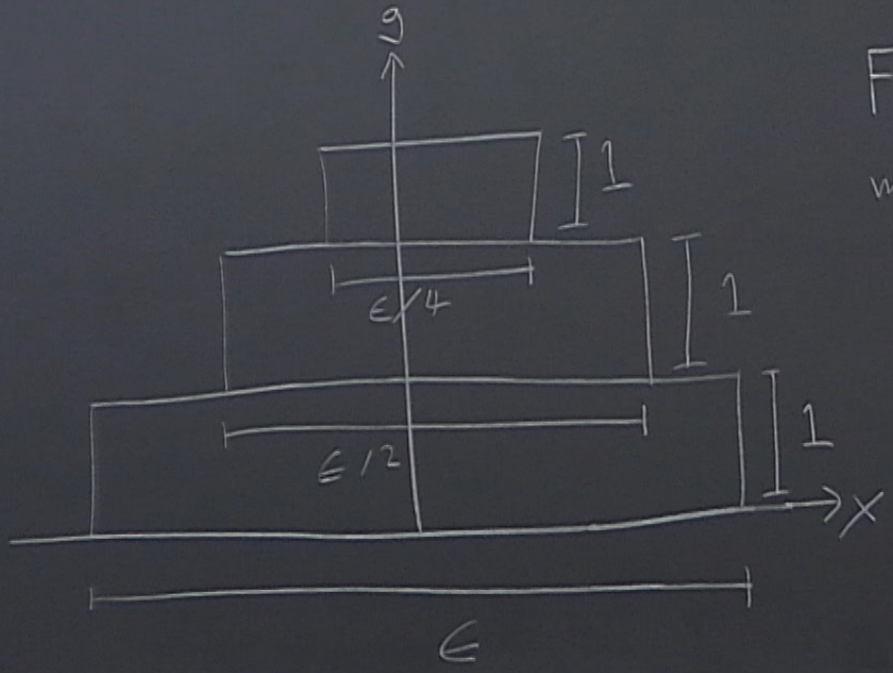
For each ϵ and each x we can find σ_0 so that $g_\sigma(x)$ lies inside boxes for all $\sigma < \sigma_0$

For f_σ choose $\sigma_0 = x$

$\sigma < x$

$g_\sigma \to g$ "$=$" δ for which $\lim_{\sigma \to 0} g_\sigma(x) = 0 \quad x \neq 0$

$\int g_\sigma(x)\,dx = 1 \quad \sigma \to 0$



For each ϵ and each x we can find σ_0 so that $g_\sigma(x)$ lies inside boxes for all $\sigma < \sigma_0$

For f_σ choose $\sigma_0 = x$

$\sigma < x \Rightarrow f_\sigma(x) = 0 < \epsilon$

Then g lies inside boxes

$$\int_{-\infty}^{\infty} g(x)dx < A = \epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \cdots = 2\epsilon \quad \text{for any } \epsilon$$

$$\int_{-\infty}^{\infty} g(x)dx = 0 \neq 1 = \lim_{\sigma \to 0} \int_{-\infty}^{\infty} g_\sigma(x)dx$$

order matters!

ϵ

$\sigma_0 = X$

$\sigma < X \Rightarrow f_\sigma(x) = 0 < \epsilon$

Test functions:

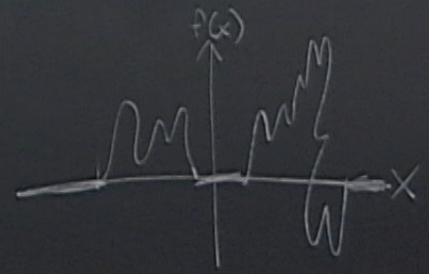
$\mathcal{D}$: set of C^∞ functions with compact support

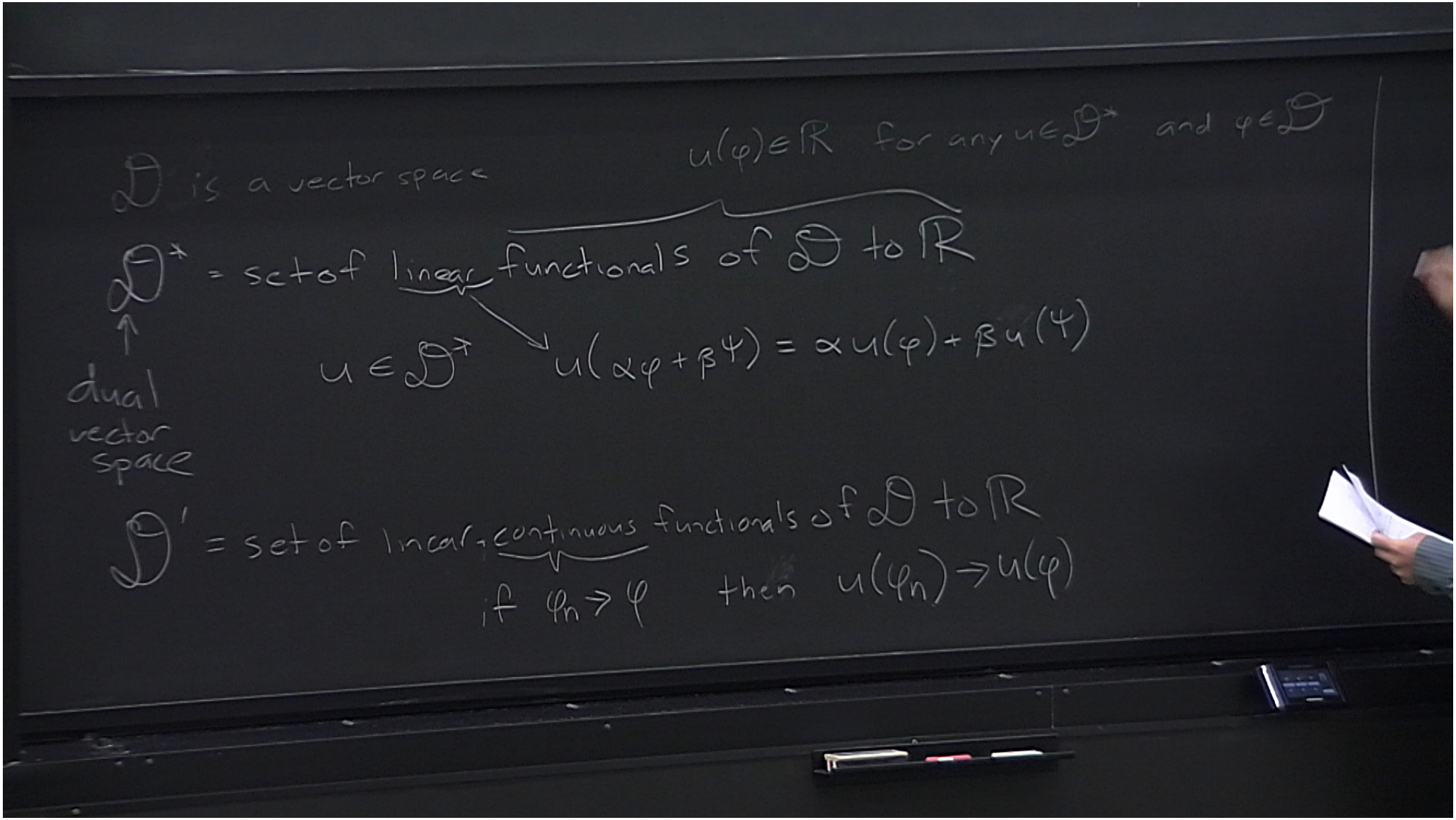
C^∞: continuous w/ ∞ many derivatives

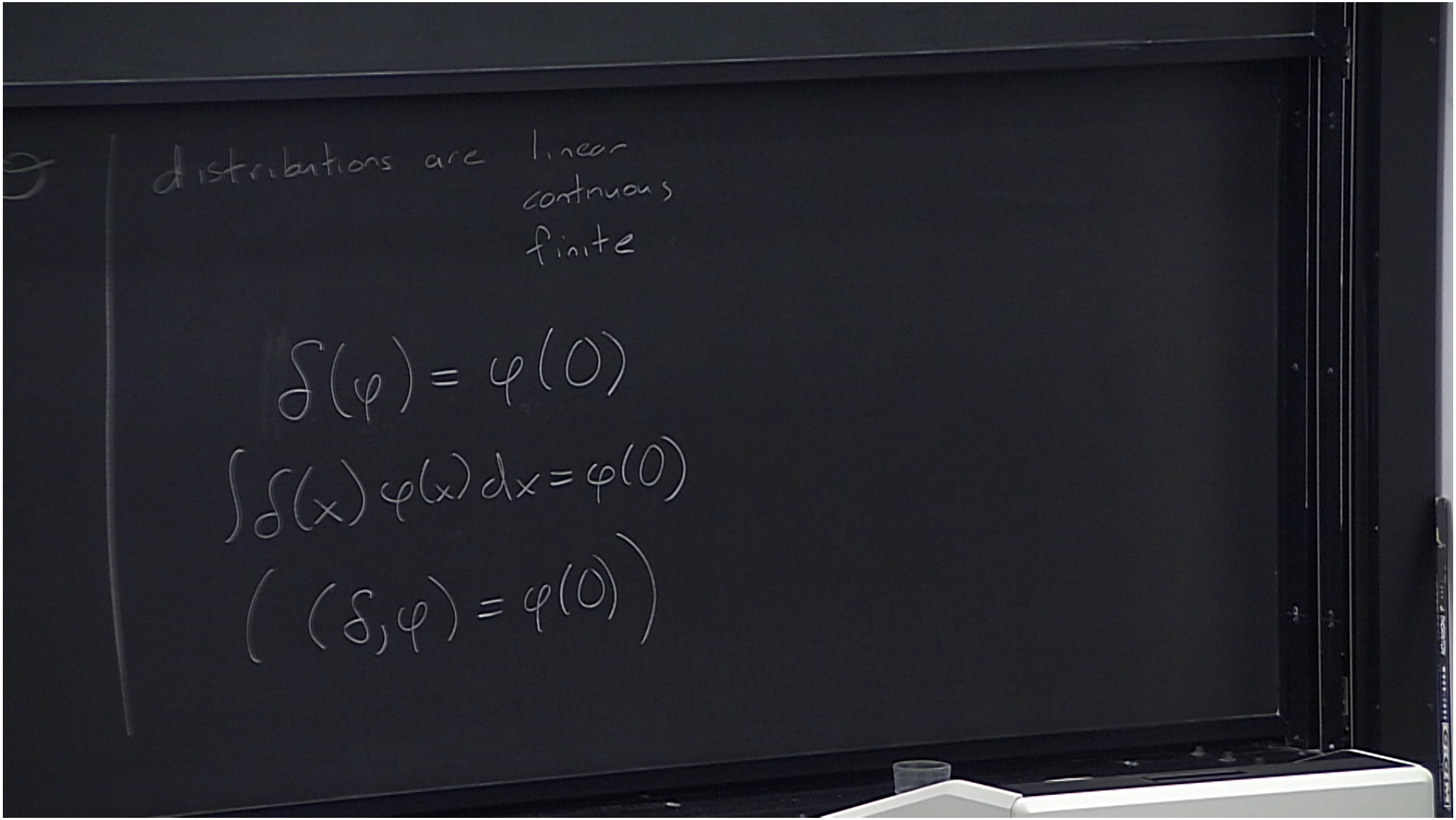
compact support: vanishes outside of a finite set of bounded intervals

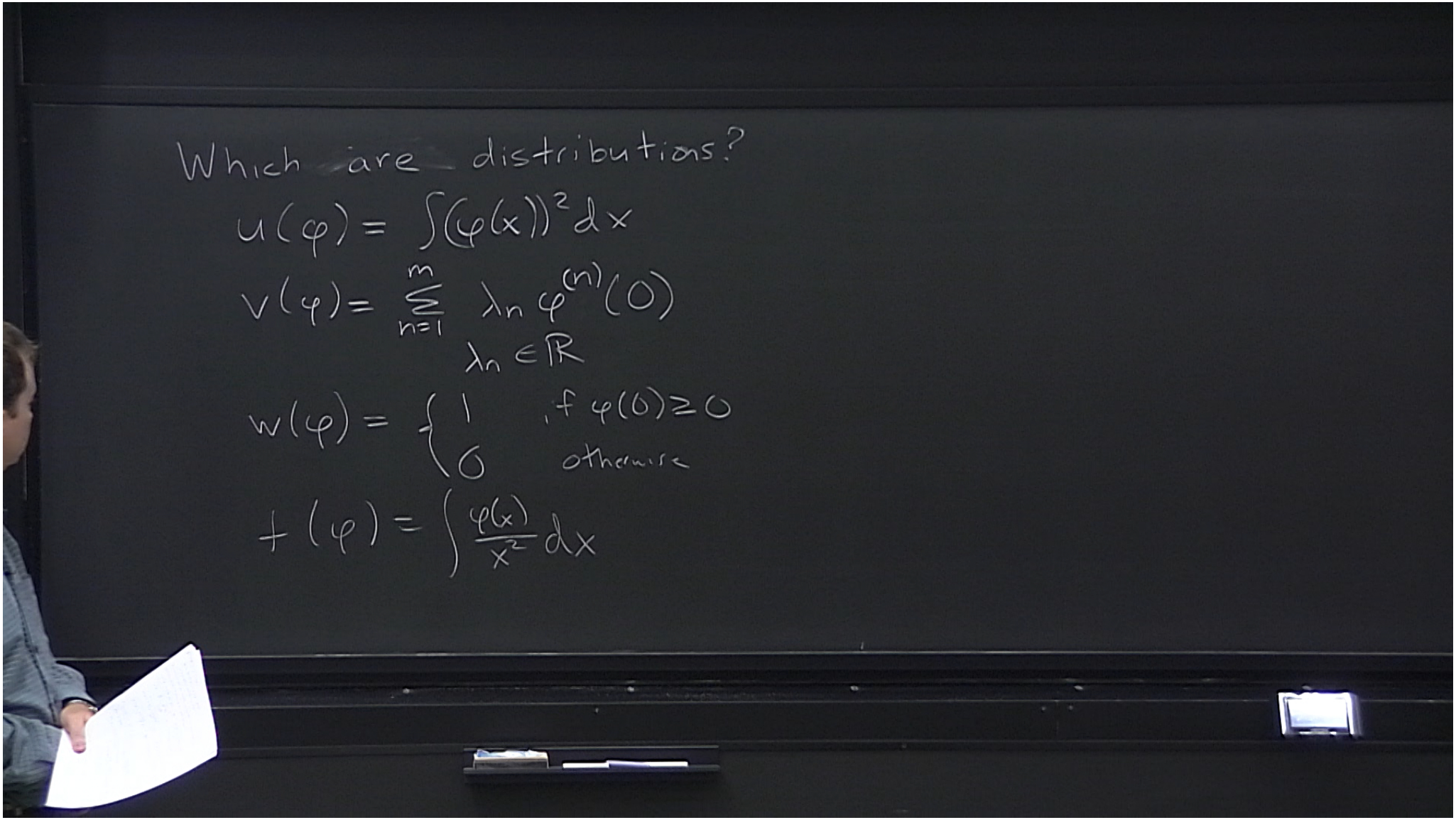
$f(x) = 0$

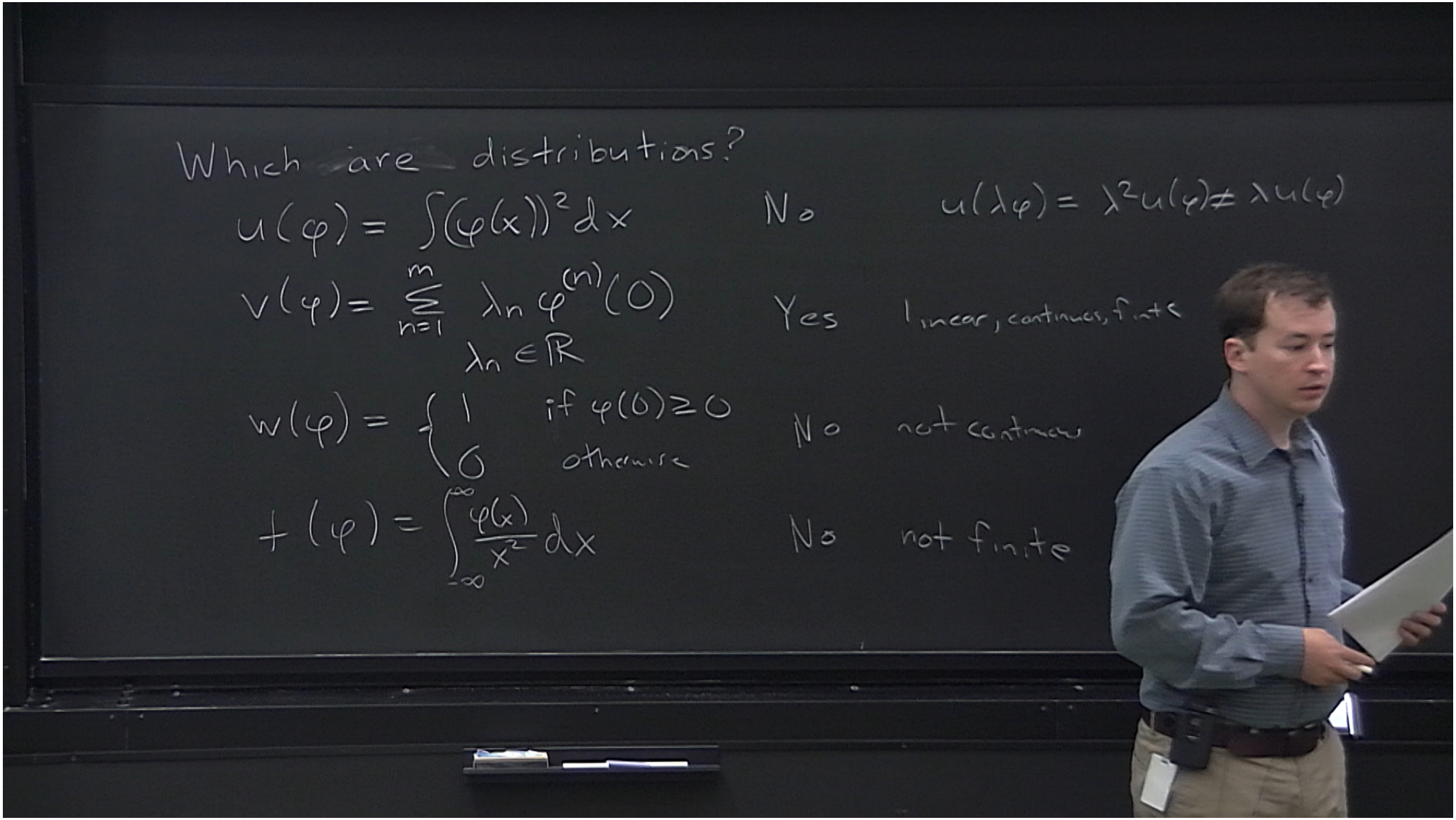
$$g(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$




$\mathcal{D}$ is a vector space
$u(\varphi) \in \mathbb{R}$ for any $u \in \mathcal{D}^*$ and $\varphi \in \mathcal{D}$
$\mathcal{D}^*$ = set of linear functionals of $\mathcal{D}$ to $\mathbb{R}$
$u \in \mathcal{D}^*$ $u(\alpha\varphi + \beta\psi) = \alpha u(\varphi) + \beta u(\psi)$
dual vector space
$\mathcal{D}'$ = set of linear, continuous functionals of $\mathcal{D}$ to $\mathbb{R}$
if $\varphi_n \to \varphi$ then $u(\varphi_n) \to u(\varphi)$


distributions are linear
continuous
finite
$\delta(\varphi) = \varphi(0)$
$\int \delta(x)\varphi(x)\,dx = \varphi(0)$
$((\delta, \varphi) = \varphi(0))$


Which are distributions?
$u(\varphi) = \int (\varphi(x))^2 dx$
$v(\varphi) = \sum_{n=1}^{m} \lambda_n \varphi^{(n)}(0)$
$\lambda_n \in \mathbb{R}$
$w(\varphi) = \begin{cases} 1 & \text{if } \varphi(0) \geq 0 \\ 0 & \text{otherwise} \end{cases}$
$t(\varphi) = \int \frac{\varphi(x)}{x^2} dx$

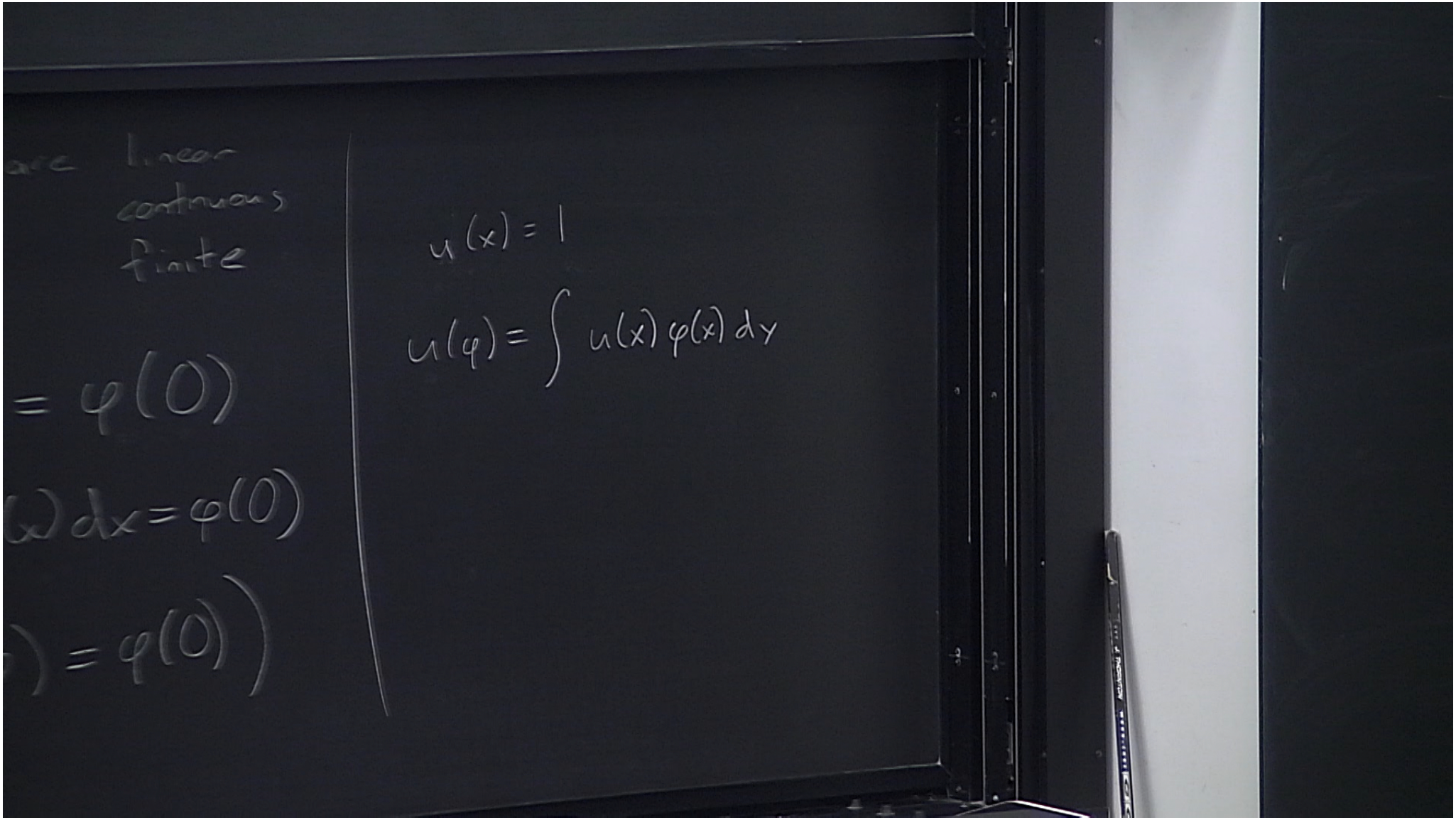

Which are distributions?
$u(\varphi) = \int (\varphi(x))^2 dx$ No $u(\lambda\varphi) = \lambda^2 u(\varphi) \neq \lambda u(\varphi)$
$v(\varphi) = \sum_{n=1}^{m} \lambda_n \varphi^{(n)}(0)$ Yes linear, continuous, finite
$\lambda_n \in \mathbb{R}$
$w(\varphi) = \begin{cases} 1 & \text{if } \varphi(0) \geq 0 \\ 0 & \text{otherwise} \end{cases}$ No not continuous
$t(\varphi) = \int_{-\infty}^{\infty} \frac{\varphi(x)}{x^2} dx$ No not finite

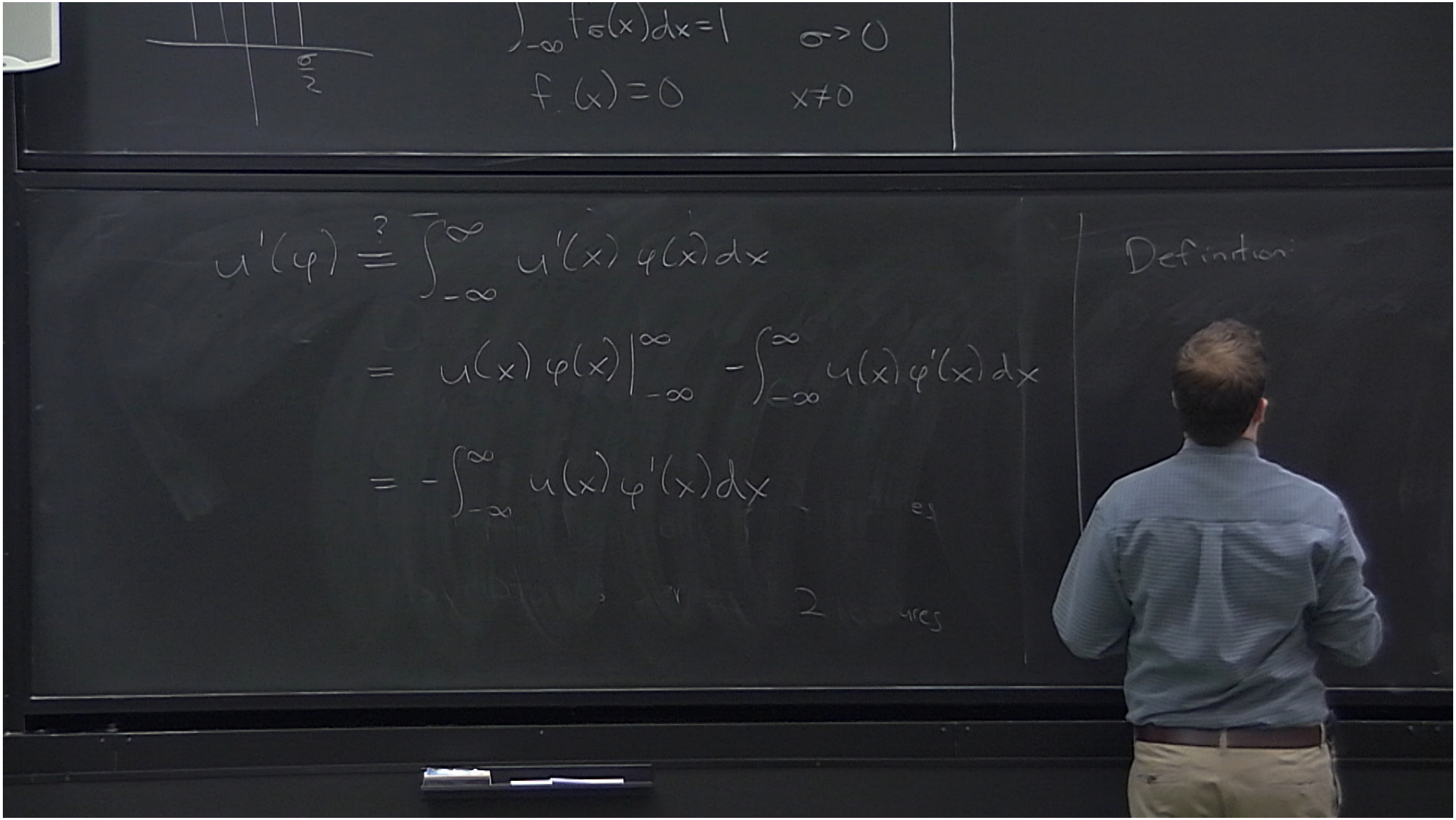
$\neq \lambda u(\varphi)$

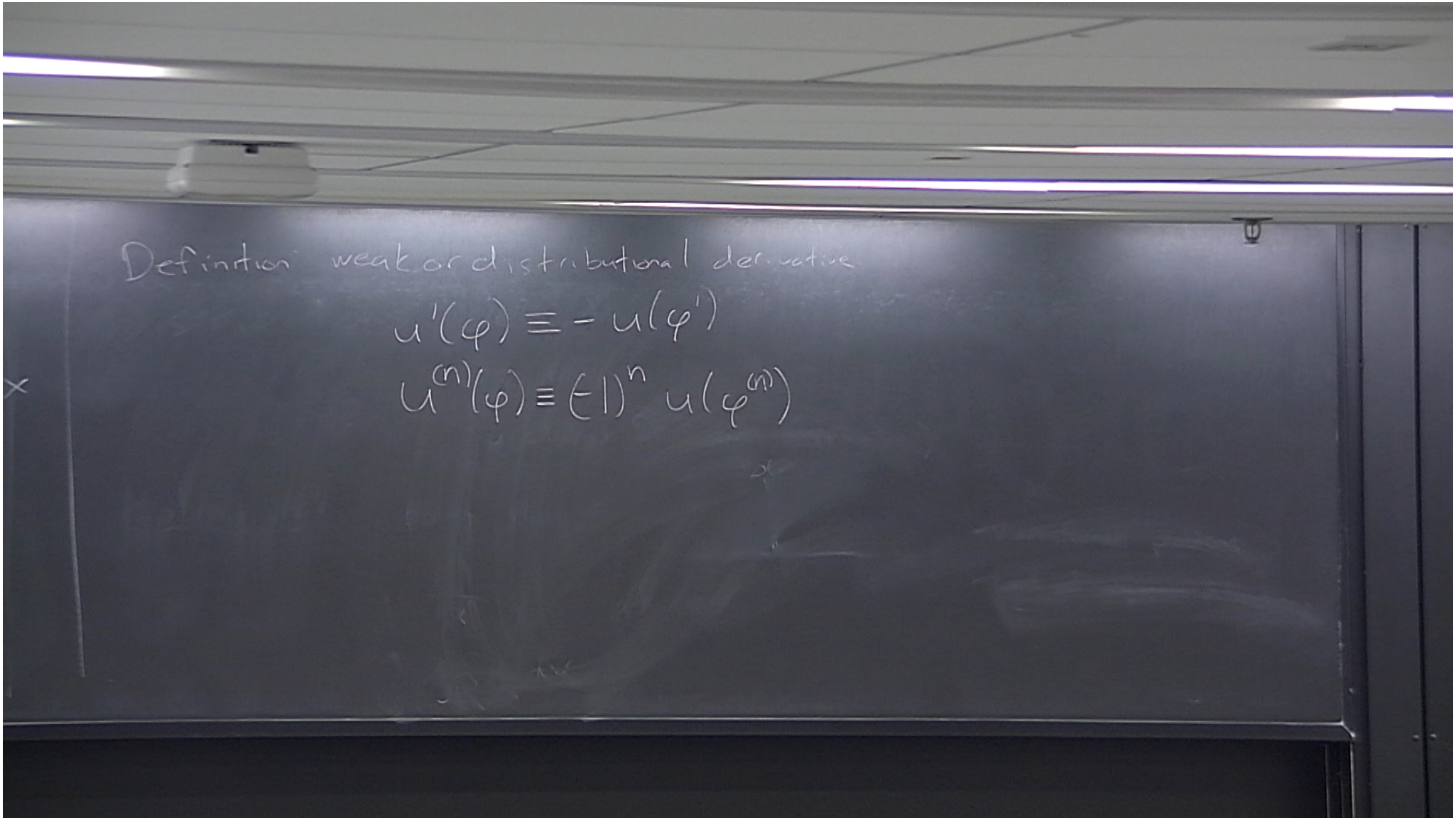
Operations: Derivation
Multiplication
Composition

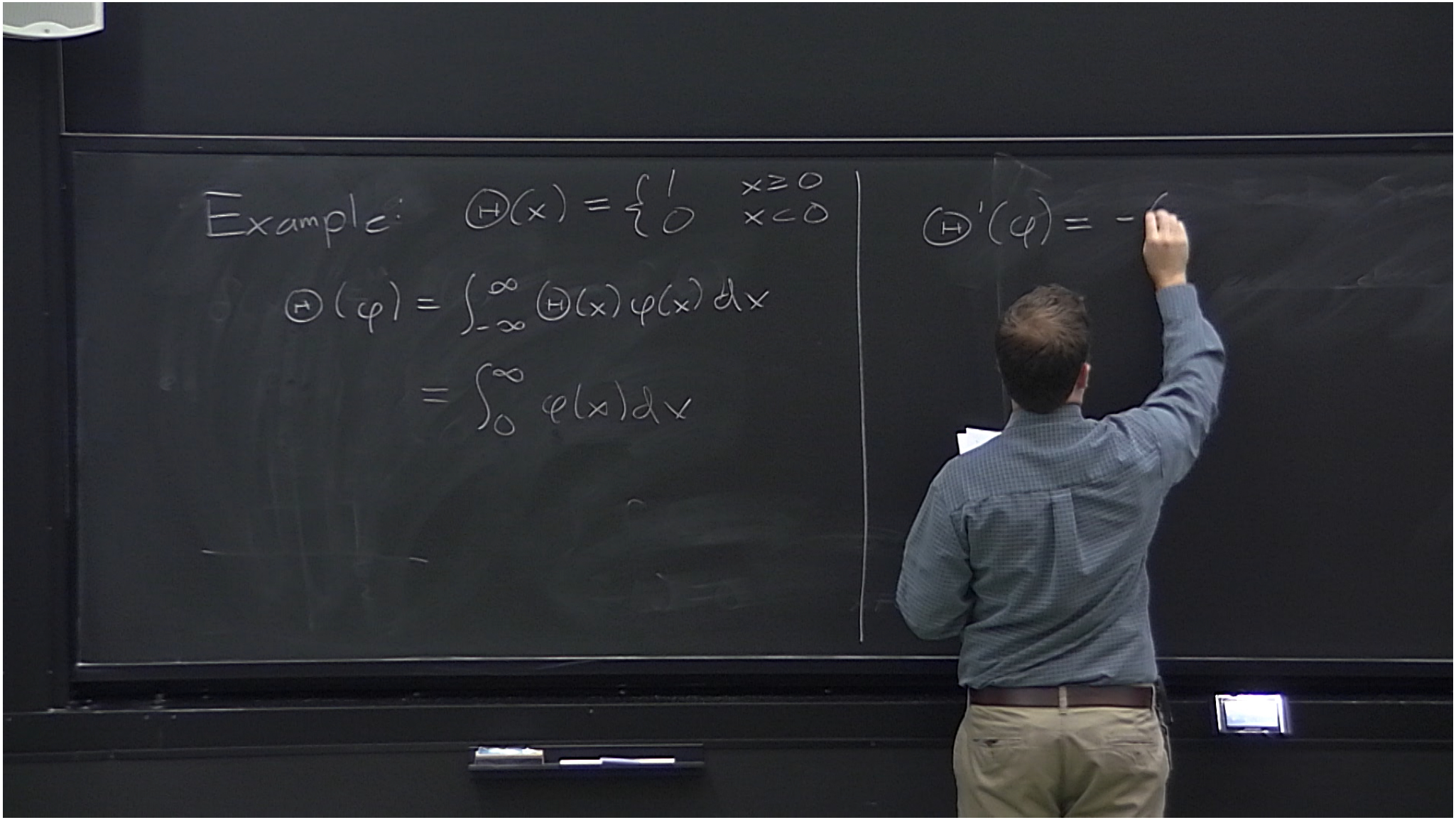
Derivatives of Distributions:

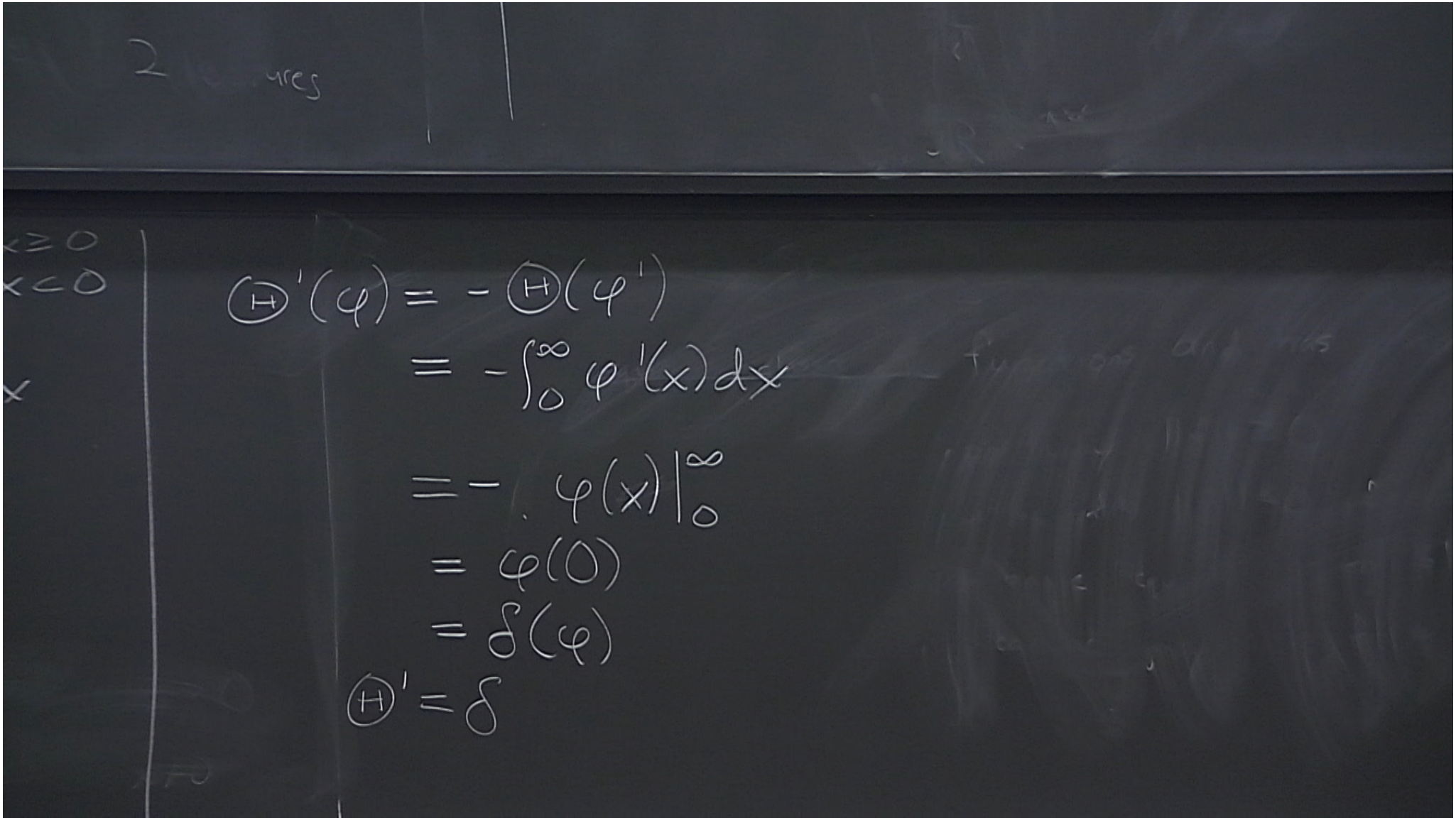
$$u'(x) \overset{?}{=} \lim_{\epsilon \to 0} \frac{u(x+\epsilon) - u(x)}{\epsilon}$$


are linear
continuous
finite
$= \varphi(0)$
$\omega)\,dx = \varphi(0)$
$) = \varphi(0)$
$u(x) = 1$
$u(\varphi) = \int u(x)\,\varphi(x)\,dx$


$\int_{-\infty}^{\infty} f_\sigma(x)\,dx = 1 \qquad \sigma > 0$
$f_\sigma(x) = 0 \qquad x \neq 0$
$\frac{\sigma}{2}$
$u'(\varphi) \stackrel{?}{=} \int_{-\infty}^{\infty} u'(x)\,\varphi(x)\,dx$
$= u(x)\,\varphi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)\,\varphi'(x)\,dx$
$= -\int_{-\infty}^{\infty} u(x)\,\varphi'(x)\,dx$
Defintion:


Definition weak or distributional derivative
$u'(\varphi) \equiv -u(\varphi')$
$u^{(n)}(\varphi) \equiv (-1)^n u(\varphi^{(n)})$

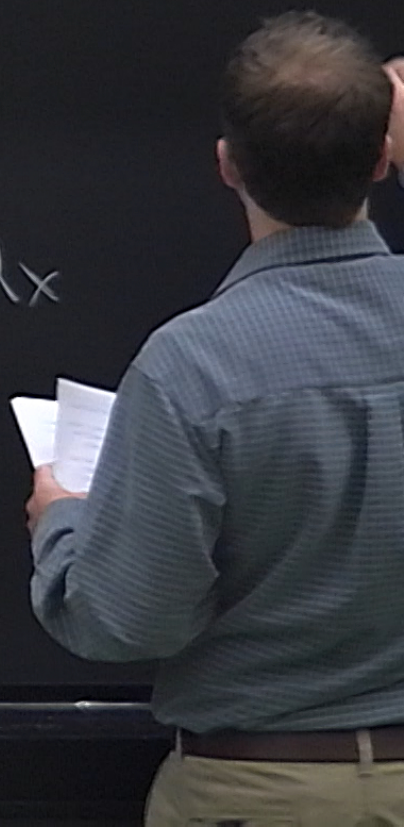

Example: $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$
$H(\varphi) = \int_{-\infty}^{\infty} H(x)\varphi(x)\,dx$
$= \int_{0}^{\infty} \varphi(x)\,dx$
$H'(\varphi) = -$


2
$x \geq 0$
$x < 0$
x
$\Theta'(\varphi) = -\Theta(\varphi')$
$= -\int_0^\infty \varphi'(x)\,dx$
$= -\varphi(x)\Big|_0^\infty$
$= \varphi(0)$
$= \delta(\varphi)$
$\Theta' = \delta$

Example: $(\ln|x|)'$

$$(\ln|x|)'(\varphi) = -\int_{-\infty}^{\infty} \varphi'(x) \ln|x|\, dx$$

$$= -\lim_{\epsilon \to 0} \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \varphi'(x) \ln|x|\, dx$$



cut out part is

$$\left| \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \ln|x|\, \varphi(x)\, dx \right|$$

$$\leq \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} |\ln|x||\, |\varphi(x)|\, dx$$

$$\leq \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} |\ln|x||\, \max|\varphi(x)|\, dx$$

$$= \max|\varphi(x)| \lim_{\epsilon \to 0} 2 \int_{0}^{\epsilon} -\ln x\, dx$$

$$= \max|\varphi(x)| \lim_{\epsilon \to 0} -2x(\ln x - 1)\Big|_{0}^{\epsilon} dx = 0$$

cut out part is

$$\left| \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \ln|x|\, \varphi'(x)\, dx \right|$$

$$\leq \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} |\ln|x||\, |\varphi'(x)|\, dx$$

$$\leq \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} |\ln|x||\, \max|\varphi'(x)|\, dx$$

$$= \max|\varphi(x)| \lim_{\epsilon \to 0} 2 \int_{0}^{\epsilon} -\ln x\, dx$$

$$= \max|\varphi(x)| \lim_{\epsilon \to 0} -2x(\ln x - 1)\Big|_{0}^{\epsilon} dx = 0$$

$\ln|x|)'$

$$\varphi) = -\int_{\infty}^{\infty} \varphi'(x) \ln|x|\, dx$$

$$= -\lim_{\epsilon\to 0} \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \varphi'(x) \ln|x|\, dx$$

$$= \lim_{\epsilon\to 0} \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \frac{1}{x} \varphi(x)\, dx + \underbrace{\varphi(\epsilon)\ln|\epsilon| - \varphi(-\epsilon)\ln|\epsilon|}_{(\varphi(\epsilon)-\varphi(-\epsilon))\ln|\epsilon|}$$

$$\approx 2\varphi'(0)\epsilon \ln|\epsilon| \to 0 \text{ as } \epsilon \to 0$$

cut out part is

$$\left| \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} \ln|\right.$$

$$\leq \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} |\ln$$

$$\leq \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} |\ln$$

$$= \max|\varphi(x)| \lim_{\epsilon\to 0} 2$$

$$= \max|\varphi(x)| \lim_{\epsilon\to 0} -2$$

$\ln|x|)'$

$$\varphi) = -\int_{\infty}^{\infty} \varphi'(x) \ln|x|\, dx$$

$$= -\lim_{\epsilon\to 0} \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \varphi'(x) \ln|x|\, dx$$

$$= \underbrace{\lim_{\epsilon\to 0} \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \frac{1}{x} \varphi(x)\, dx}_{\equiv \left(PV \frac{1}{x}\right)(\varphi)} + \underbrace{\varphi(\epsilon)\ln|\epsilon| - \varphi(-\epsilon)\ln|\epsilon|}_{(\varphi(\epsilon)-\varphi(-\epsilon))\ln|\epsilon|}$$

$$\approx 2\varphi'(0)\epsilon \ln|\epsilon| \to 0 \text{ as } \epsilon \to 0$$

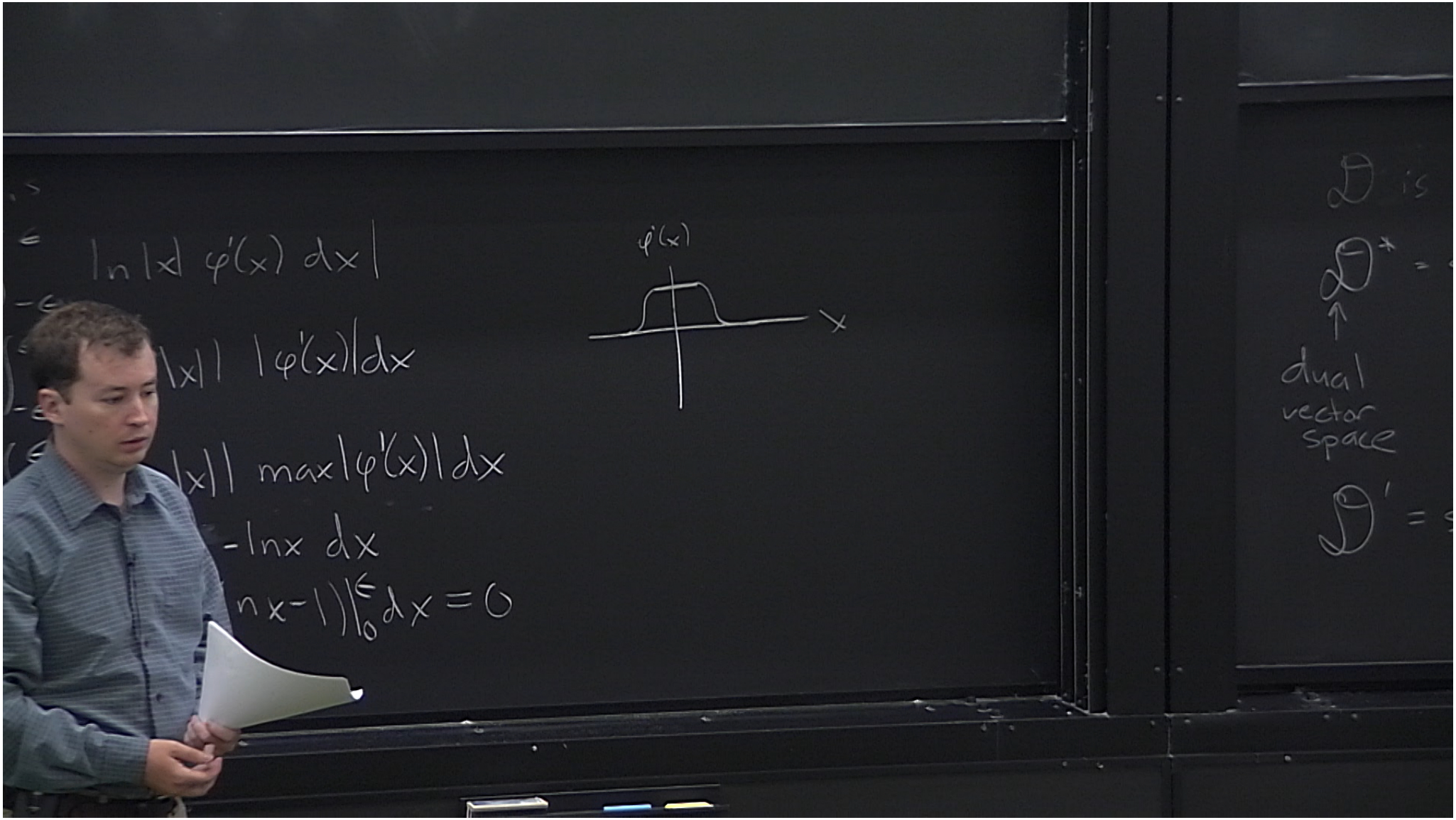
cut out part is

$$\left| \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} \ln|x| \right|$$

$$\leq \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} |\ln|x||$$

$$\leq \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} |\ln|x|| = \max|\varphi(x)| \lim_{\epsilon\to 0} 2$$

$$= \max|\varphi(x)| \lim_{\epsilon\to 0} -2$$


$\int_\epsilon^1 \ln|x|\,\varphi'(x)\,dx$
$\varphi(x)$
x
$\mathcal{D}$ is
$\mathcal{D}^*$
dual vector space
$\mathcal{D}'$


$\int_{-\epsilon}^{\epsilon} \ln|x|\,\varphi'(x)\,dx$
$\varphi'(x)$
x
$f(x)$ $\varphi(x)$
$\int f(x)\,\varphi(x)\,dx$
$\lim_{\epsilon \to 0} 2\int_0^{\epsilon} -\ln$
$\lim_{\epsilon \to 0} -2\epsilon(\ln$
$\mathcal{D}$ is
$\mathcal{D}^*$ =
dual vector space
$\mathcal{D}'$ =