Title: PSI 17/18 Front End - Lie Groups and Lie Algebras (Dupuis)

Date: Aug 16, 2017 10:30 AM

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Abstract:

su(2).

Irreducible representation

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Irreducible representation

representation space does not contain any non-trivial invariant subspace

An invariant subspace U of a vector space V for a given representation R is mapped into itself by all $R(x)$

(ie $(R(x))(U) \subseteq U \quad \forall x \in V$)

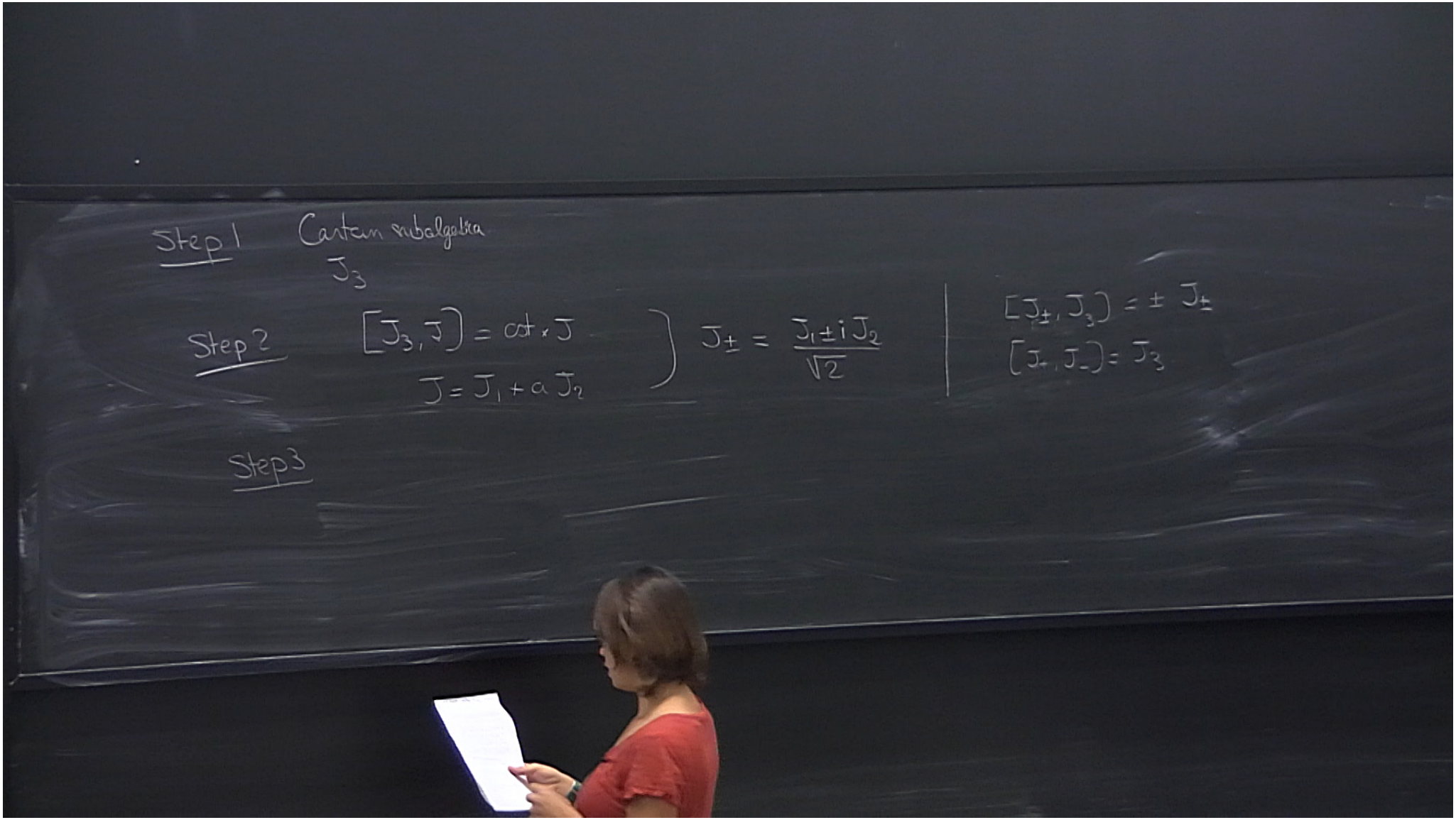
$su(2)$ $\longrightarrow$ $[J_a, J_b] = i\varepsilon_{abc} J_c$ $\quad a,b,c \in \{1,2,3\}$

Irreducible representation

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(ie $(R(x))(U) \subseteq U \quad \forall x \in V$)


Step 1 Cartan subalgebra
J_3
Step 2
$[J_3, J] = \alpha J$
$J = J_1 + a J_2$
$J_\pm = \frac{J_1 \pm i J_2}{\sqrt{2}}$
$[J_\pm, J_3] = \pm J_\pm$
$[J_+, J_-] = J_3$
Step 3

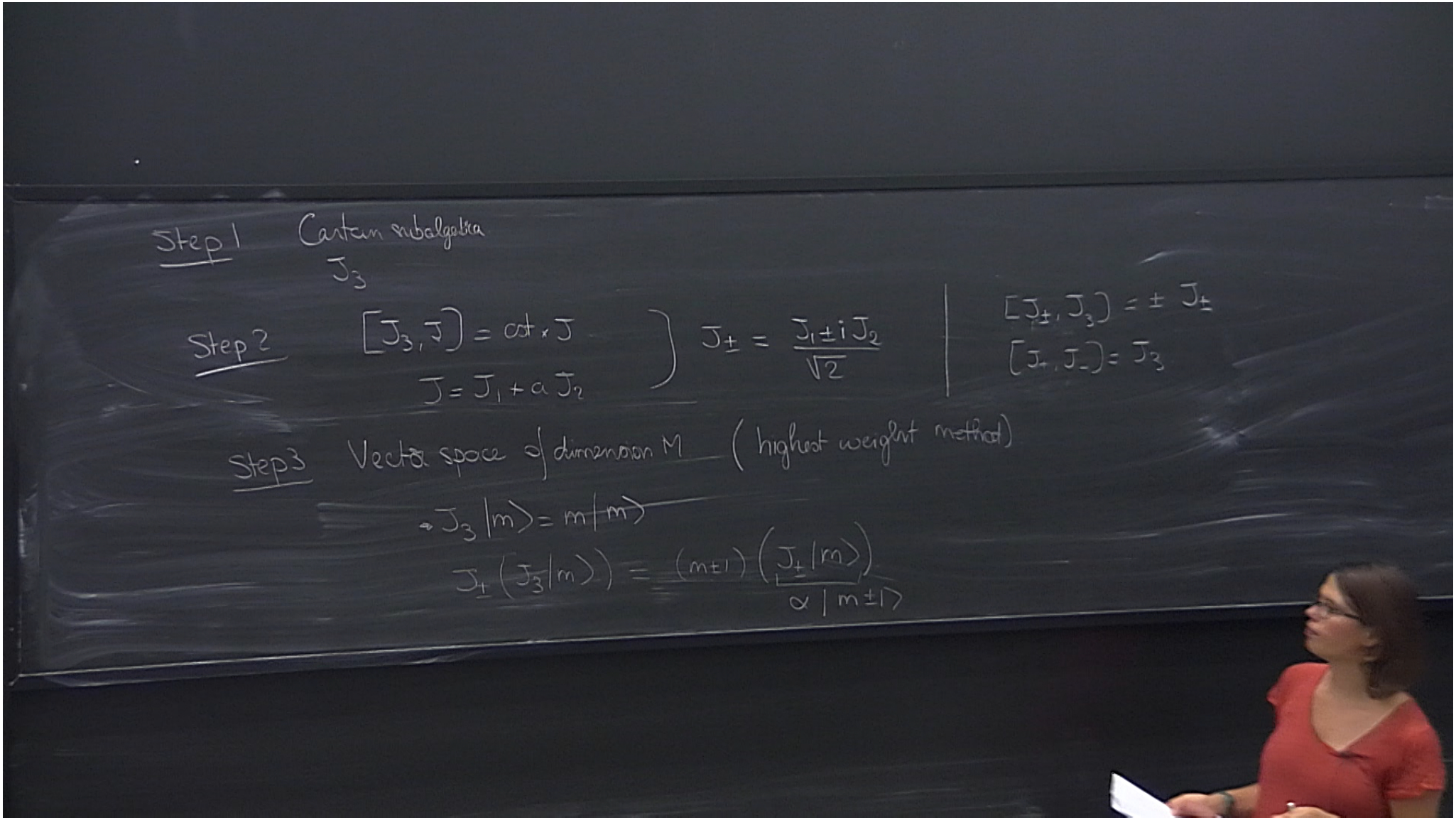
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Step 3 Vector space of dimension M (highest weight method)

$J_3 |m\rangle = m |m\rangle$


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$\alpha |m \pm 1\rangle$

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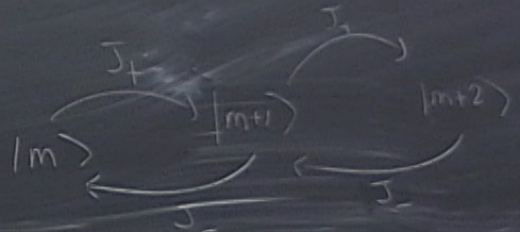
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Finite dimensional representations

highest weight j /

$J_3 \left(J_\pm |m\rangle \right) = (m \pm 1) \underbrace{\left(J_\pm |m\rangle \right)}_{\propto |m \pm 1\rangle}$

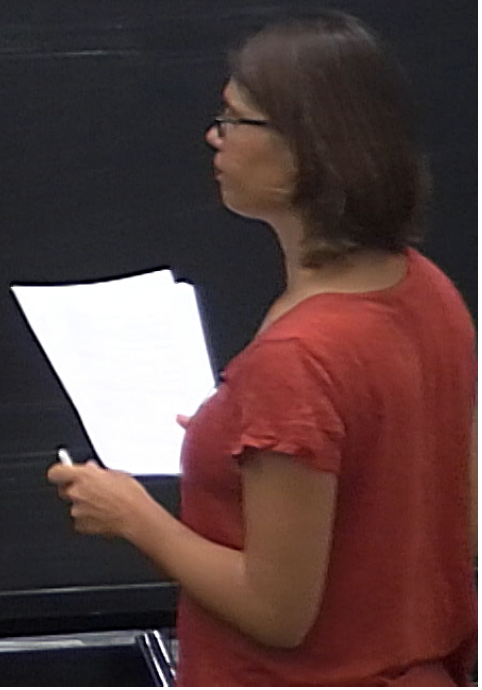
Finite dimensional representations

highest weight j , $J_+ |j\rangle = 0$

$\hookrightarrow J_+ |j\rangle \propto |j+1\rangle$

- $\langle m | m' \rangle = \delta_{mm'}$

$J_- |m\rangle = N(m) |m-1\rangle$



$J_3 (J_\pm |m\rangle) = (m \pm 1)(J_\pm |m\rangle)$

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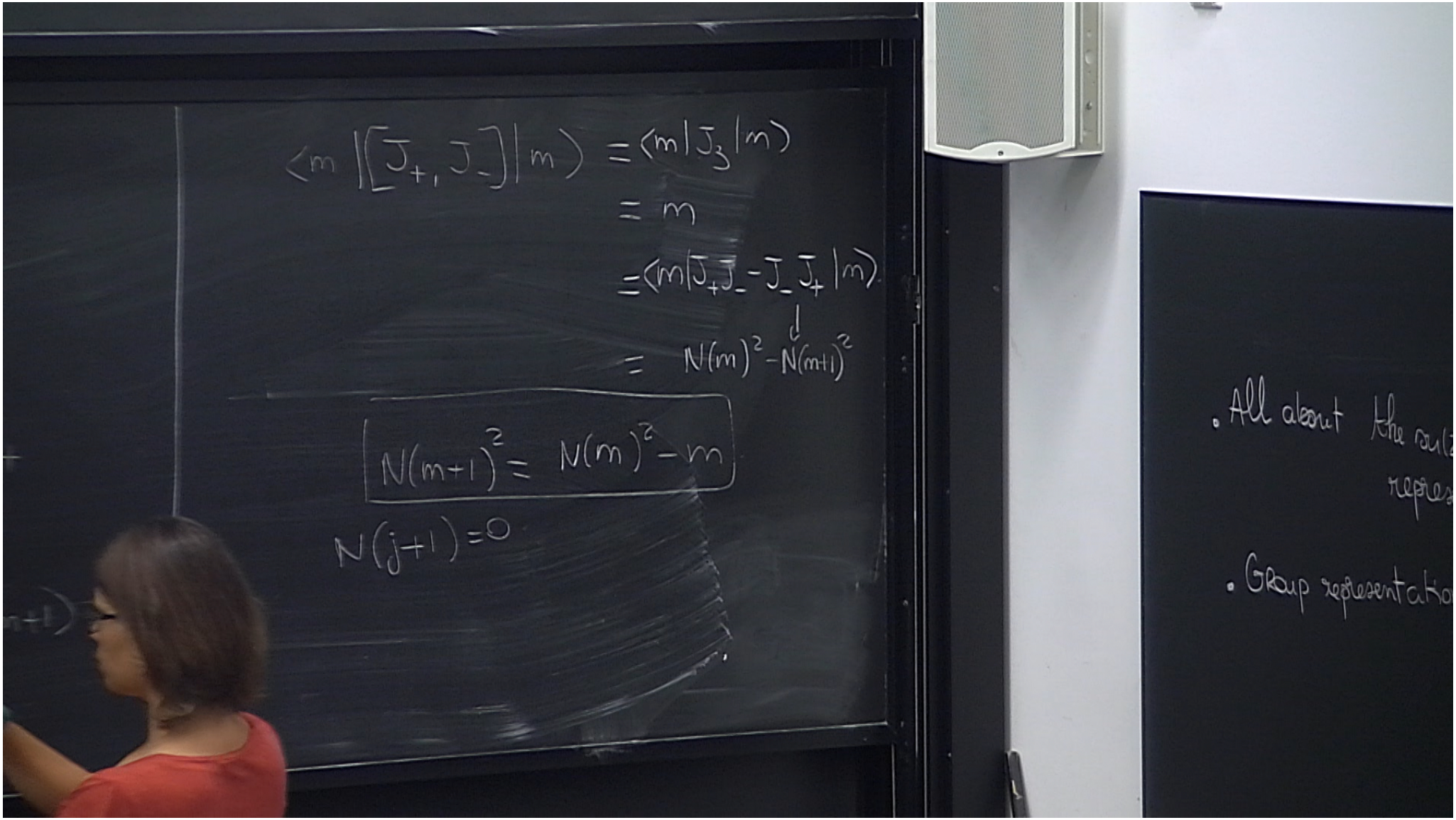
$J_+ |m\rangle = \tilde{N}(m) |m+1\rangle$

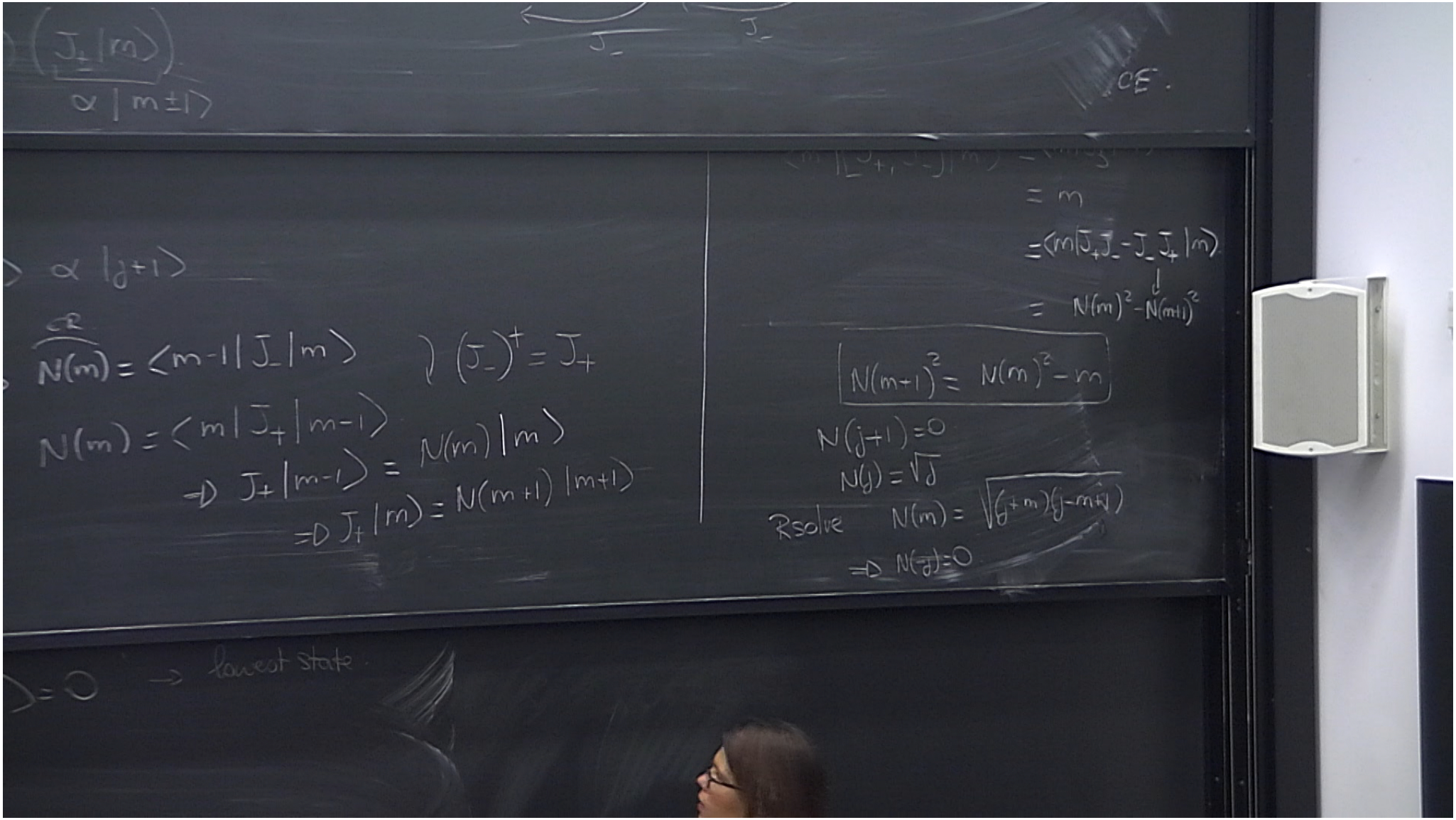
$\xrightarrow{CR}$ $N(m) = \langle m-1 | J_- | m \rangle$ $\Big\} (J_-)^\dagger = J_+$

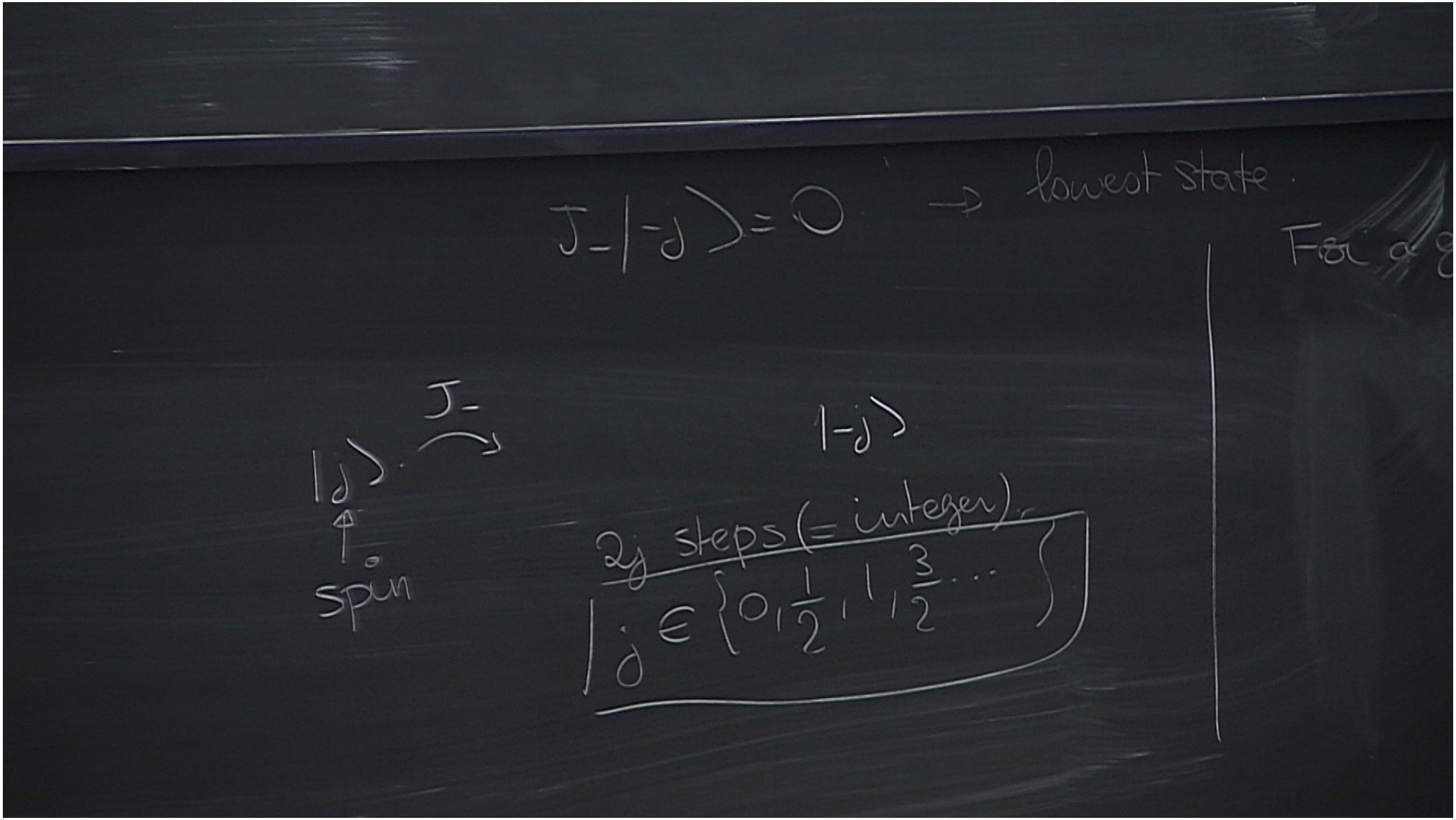
$N(m) = \langle m | J_+ | m-1 \rangle$

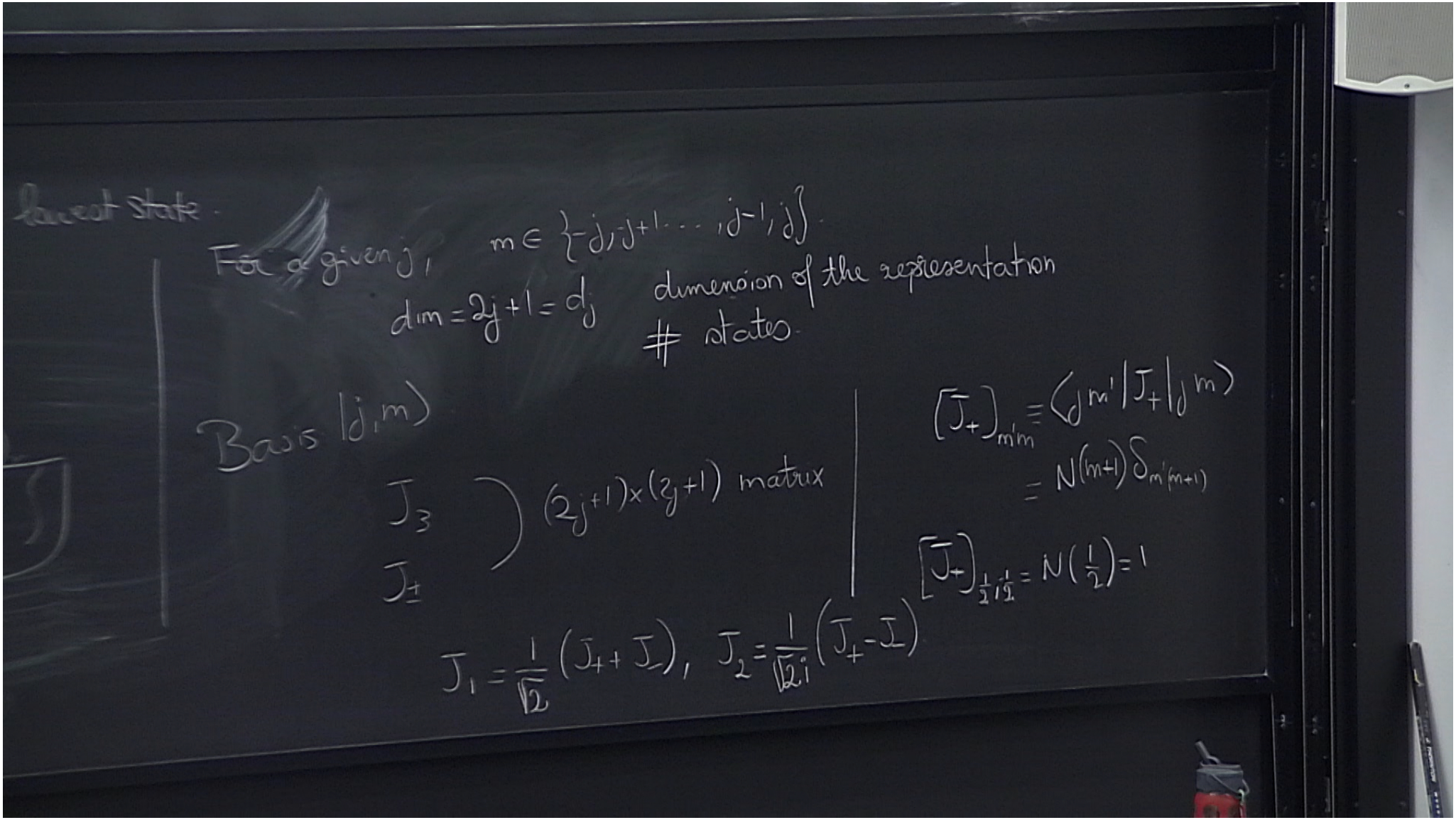
$\Rightarrow J_+ |m-1\rangle = N(m) |m\rangle$

$\Rightarrow J_+ |m\rangle = N(m+1) |m+1\rangle$


$\langle m | [J_+, J_-] | m \rangle = \langle m | J_3 | m \rangle$
$= m$
$= \langle m | J_+ J_- - J_- J_+ | m \rangle$
$= N(m)^2 - N(m+1)^2$
$N(m+1)^2 = N(m)^2 - m$
$N(j+1) = 0$
All about the
Group

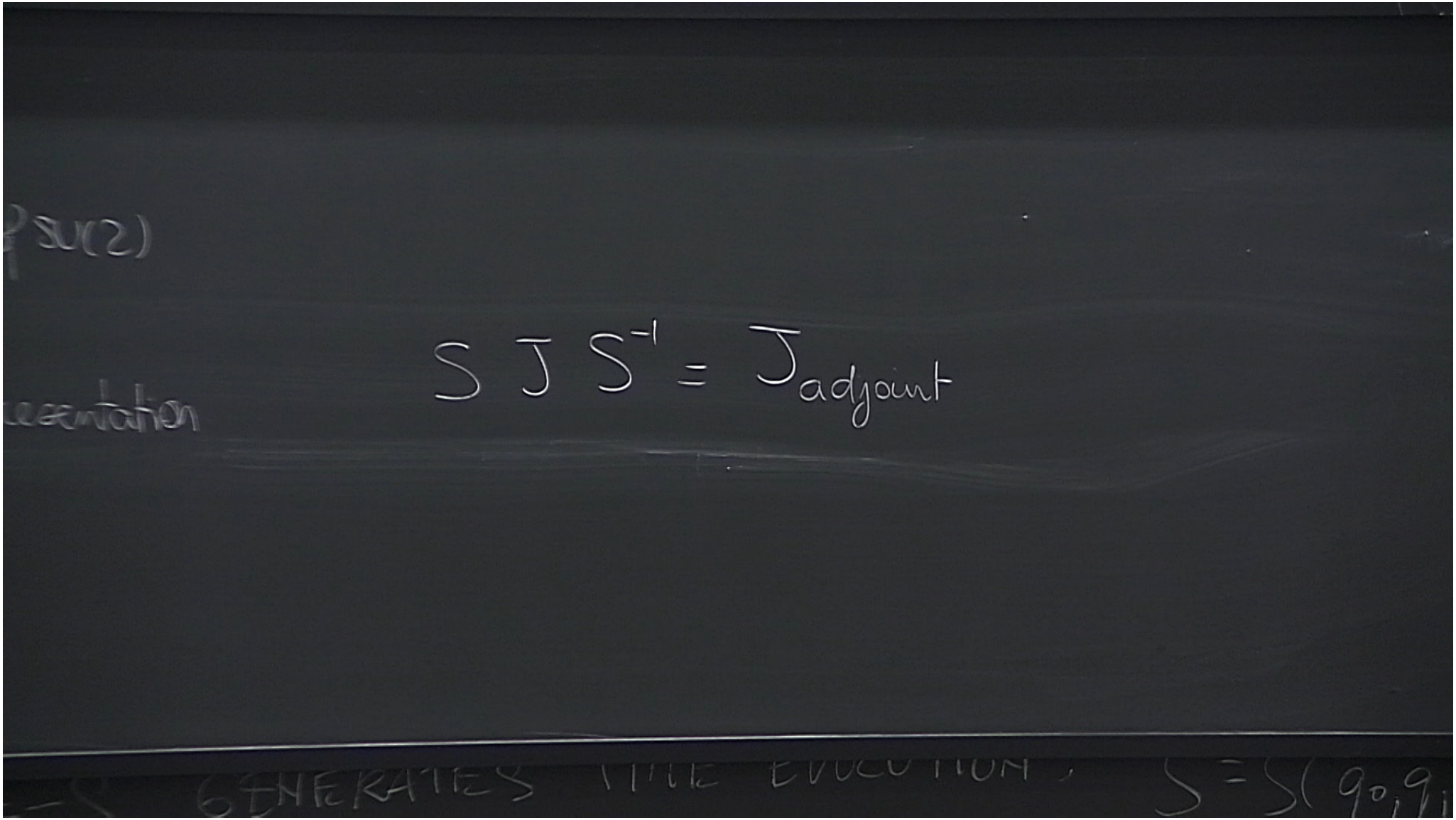

$J_\pm|m\rangle$
$\propto |m\pm1\rangle$
J_-
J_-
CE.
$= m$
$= \langle m|J_+J_- - J_-J_+|m\rangle$
$= N(m)^2 - N(m+1)^2$
$N(m+1)^2 = N(m)^2 - m$
$\propto |j+1\rangle$
CR
$N(m) = \langle m-1|J_-|m\rangle$
$(J_-)^\dagger = J_+$
$N(m) = \langle m|J_+|m-1\rangle$
$\Rightarrow J_+|m-1\rangle = N(m)|m\rangle$
$\Rightarrow J_+|m\rangle = N(m+1)|m+1\rangle$
$N(j+1) = 0$
$N(j) = \sqrt{J}$
Rsolve $N(m) = \sqrt{(j+m)(j-m+1)}$
$\Rightarrow N(-j) = 0$
$= 0$ $\Rightarrow$ lowest state.

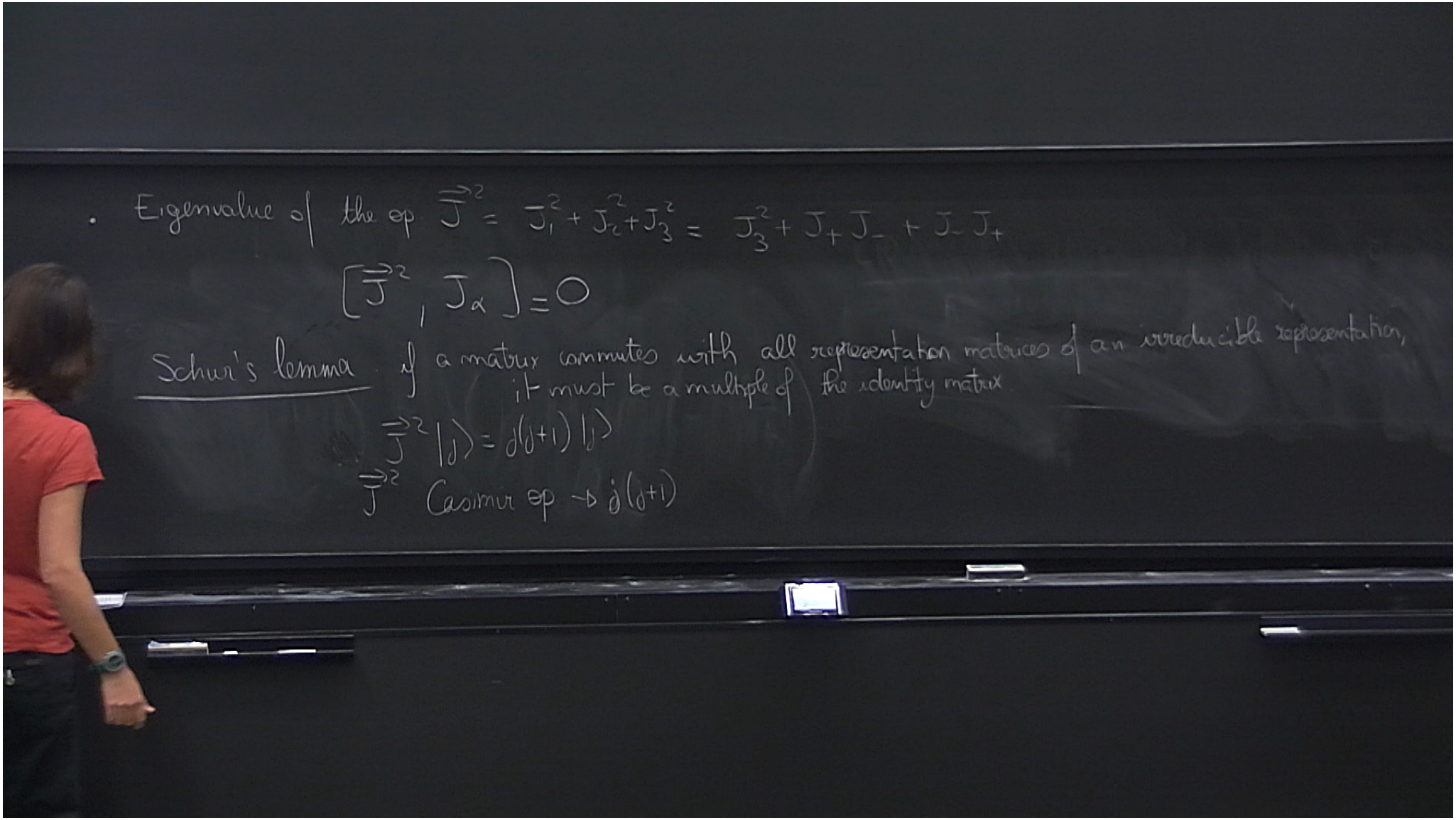

$J_-|-j\rangle = 0$ → lowest state.
$|j\rangle$
J_-
$|-j\rangle$
spin
2j steps (= integer)
$j \in \{0, \frac{1}{2}, 1, \frac{3}{2} \ldots\}$


lowest state.
For a given j, $m \in \{-j, -j+1, \ldots, j-1, j\}$
$dim = 2j+1 = d_j$ dimension of the representation
states
Basis $|j,m\rangle$
J_3
$J_\pm$
$(2j+1)\times(2j+1)$ matrix
$(J_+)_{m'm} \equiv \langle j m' | J_+ | j m \rangle$
$= N(m+1)\,\delta_{m',(m+1)}$
$(J_+)_{\frac{1}{2}\,-\frac{1}{2}} = N(\frac{1}{2}) = 1$
$J_1 = \frac{1}{\sqrt{2}}(J_+ + J_-)$, $J_2 = \frac{1}{\sqrt{2}i}(J_+ - J_-)$

- $j=0$ one state → singlet
- $j=\frac{1}{2}$ two states → doublet (qubit)
  - ↳ the spin $\frac{1}{2}$ representation is the so-called fundamental rep of SU(2)
- $j=1$ 3 states → triplet (qutrit)

Spin 1 representation is equivalent to the adjoint representation


SU(2)
$S J S^{-1} = J_{adjoint}$


Eigenvalue of the op $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2 = J_3^2 + J_+ J_- + J_- J_+$
$[\vec{J}^2, J_\alpha] = 0$
Schur's lemma if a matrix commutes with all representation matrices of an irreducible representation, it must be a multiple of the identity matrix
$\vec{J}^2 |j\rangle = j(j+1)|j\rangle$
$\vec{J}^2$ Casimir op $\rightarrow j(j+1)$

• $j=1$ 3 states → triplet (qutrit)

Spin 1 representation is equivalent to the adjoint representation

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$\vec{J}^2, J^3$

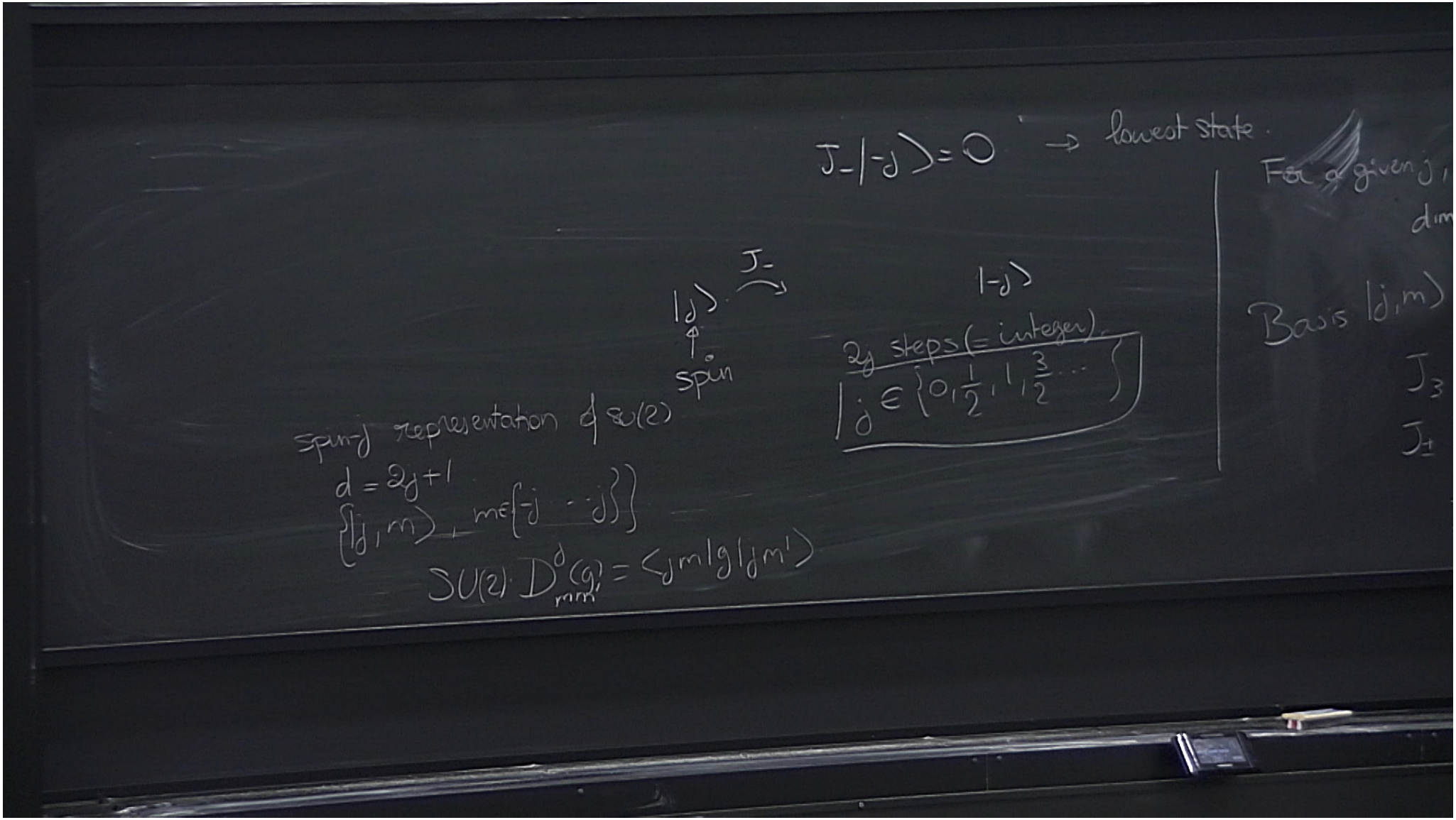
$|j\ m\rangle$

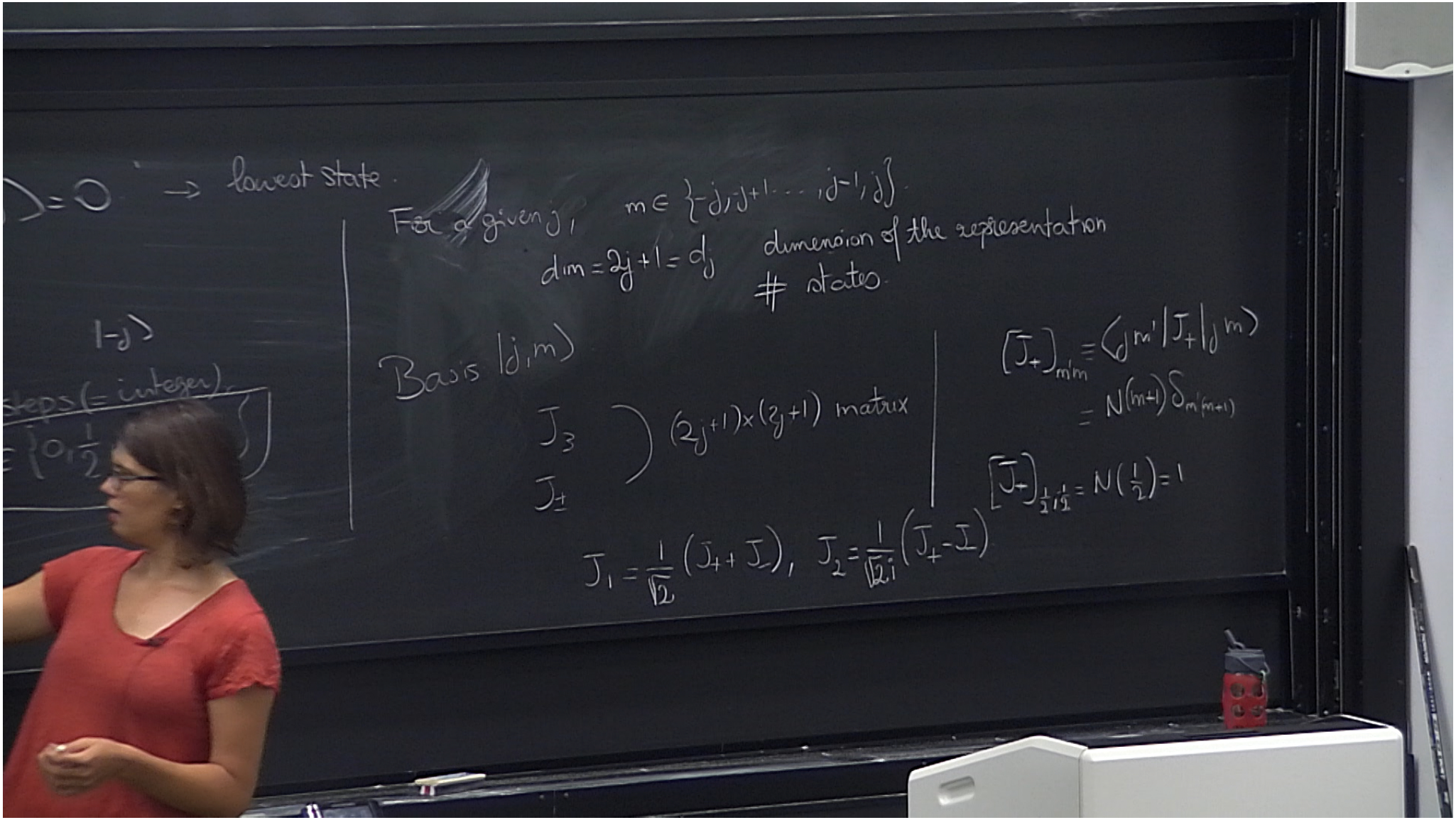
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For a given j
dim
$|j\rangle$ J_- $|-j\rangle$
spin
2j steps (= integer)
$j \in \{0, \frac{1}{2}, 1, \frac{3}{2} \cdots\}$
Basis $|j,m\rangle$
J_3
$J_\pm$
spin-j representation of SU(2)
$d = 2j+1$
$\{|j,m\rangle, m \in \{-j \cdots j\}\}$
$SU(2)\ D^d_{mm'}(g) = \langle jm|g|jm'\rangle$


= 0 → lowest state
|-j⟩
steps (= integer)
For a given j, m ∈ {-j, j+1, ..., j-1, j}
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J_±
(2j+1)×(2j+1) matrix
[J_+]_{m'm} ≡ ⟨j m'|J_+|j m⟩
= N(m±1) δ_{m',m+1}
[J_+]_{1/2 -1/2} = N(1/2) = 1
J_1 = 1/√2 (J_+ + J_-), J_2 = 1/√2 i (J_+ - J_-)

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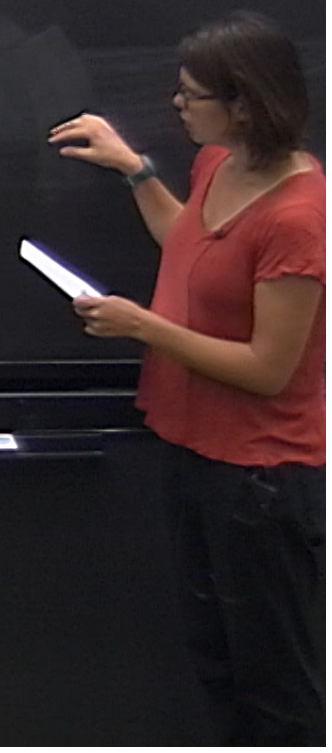
Group representation theory again

A rep. of a group G on a vector space V is a D from G to linear transformations of V such that

$$D(gh)v = D(g)D(h)v \quad \forall v \in V$$

Lie groups: $D: G \to GL(V)$ V finite dimension

Lie group rep $\rho: G \to GL(V)$ · differentiate it to obtain Lie algebra rep. $d\rho: g \to gl(V)$



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Group representation theory again

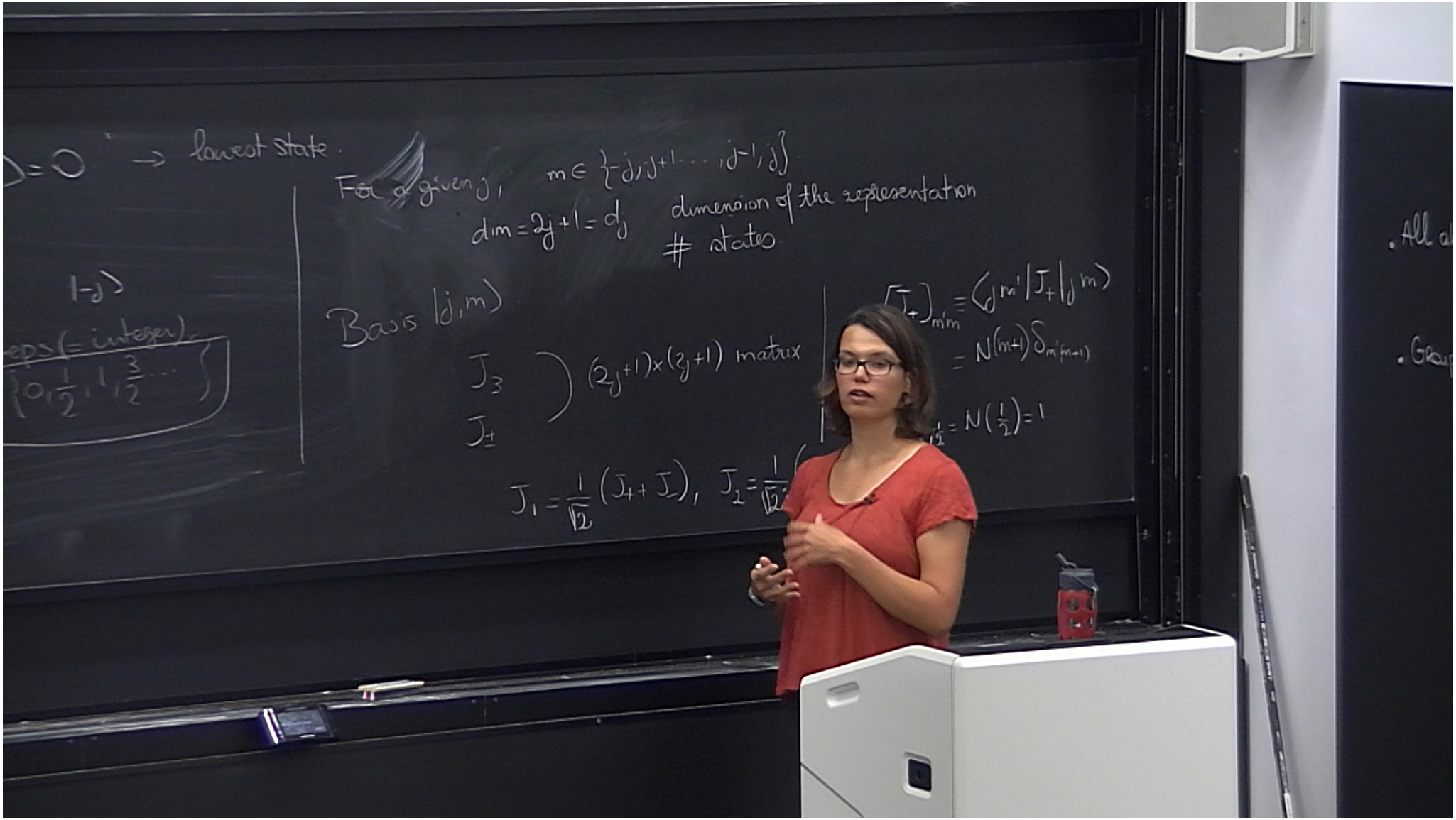
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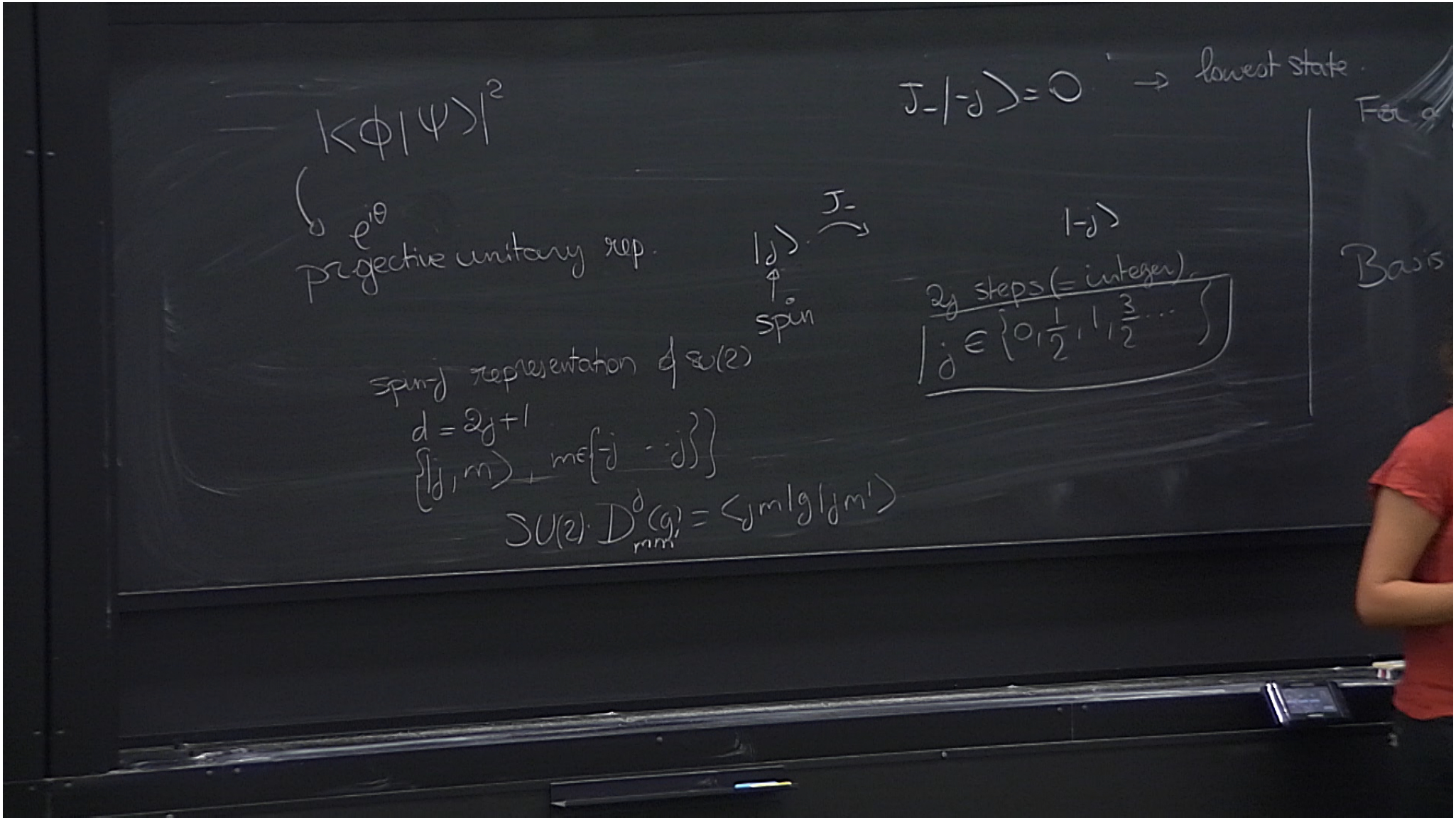
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exponentiation (simply connected Lie group)


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= N(m+1) δ_{m',m+1}
N(1/2) = 1


$|\langle\phi|\psi\rangle|^2$
projective unitary rep.
$e^{i\theta}$
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