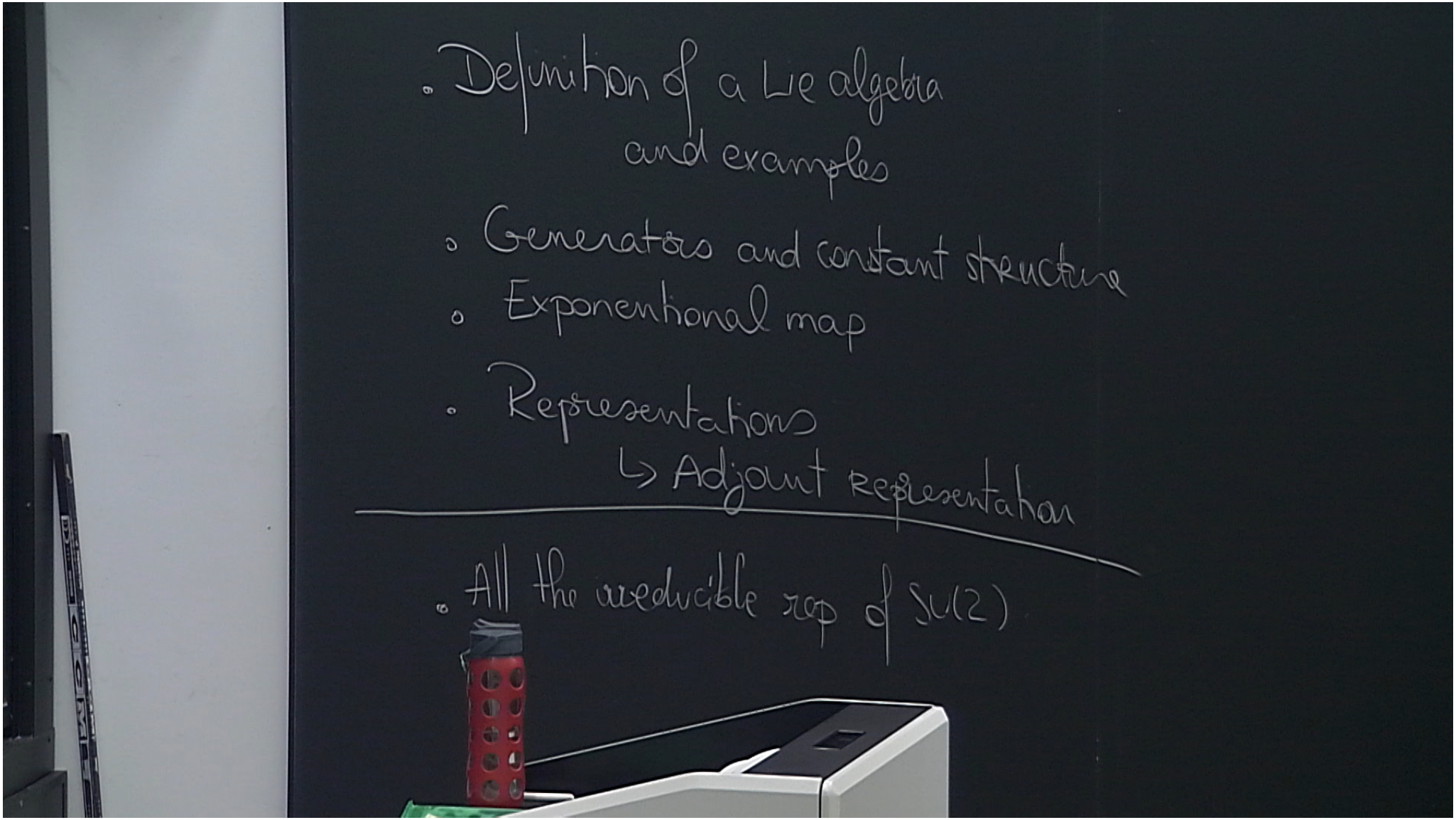
Title: PSI 17/18 Front End - Lie Groups and Lie Algebras (Dupuis)

Date: Aug 15, 2017 10:30 AM

URL: http://pirsa.org/17080054

Abstract:


. Definition of a Lie algebra
and examples
. Generators and constant structure
. Exponential map
. Representations
↳ Adjoint representation
. All the irreducible rep of SU(2)

Lie algebra: Algebra with binary operation, $[,]$

- $[x,y] = -[y,x] \quad \forall x,y$
- $[x,[y,3]] + [y,[3,x]] + [3,[x,y]] = 0$ (Jacobi identity)

- $SO(n) = \{ o(t) \,/\, \langle o(t)x \,|\, o(t)y \rangle = \langle x|y \rangle \}$
- $so(n) \quad \langle o(t)x \,|\, o(t)y \rangle = \langle x|y \rangle$

d. P. $t=0$

Lie algebra: Algebra with binary operation, $[\,,\,]$

- $[x,y] = -[y,x] \quad \forall x,y$
- $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ (Jacobi identity)

- $SO(n) = \{\sigma(t) \,/\, \langle \sigma(t)x \,|\, \sigma(t)y \rangle = \langle x|y \rangle\} \quad \sigma(0) = \mathbb{1}$

d. Parametrate at $t=0$ $\Big\{$

- $so(n) \quad \langle \sigma(t)x \,|\, \sigma(t)y \rangle = \langle x|y \rangle$

Lie algebra: Algebra with binary operation, $[\,,\,]$

- $[x,y] = -[y,x] \quad \forall x,y$
- $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ (Jacobi identity)

d/dt|t=0

- $SO(n) = \{ o(t) \mid \langle o(t)x \mid o(t)y \rangle = \langle x|y \rangle \}$ $\quad o(0) = \mathbb{1}$
- $so(n)$ $\quad \langle o(t)x \mid o(t)y \rangle = \langle x|y \rangle$

$\langle \underbrace{o'(0)}_{T} x \mid y \rangle + \langle x \mid \underbrace{o'(0)}_{T} y \rangle = 0 \Leftrightarrow T_{ij} + T_{ji} = 0 \rightarrow so(n)$: skew-symmetric $n \times n$ matrices

- $U(n) \longrightarrow u(n)$: skew-hermitian $n \times n$ matrices
- $SU(n) \rightarrow su(n)$: ——————————— + $tr = 0$

* Generators and cst structure

$SO(3)$ $\quad g_x(\theta),\ g_y(\theta),\ g_z(\theta)$

$so(3)$ $\quad J_x,\ J_y,\ J_z \longrightarrow$ generators of $so(3)$

$T_{ij} + T_{ji} = 0 \rightarrow$ $so(n)$: skew-symmetric $n \times n$ matrices.

Lie Algebra (finite dimensional) g]

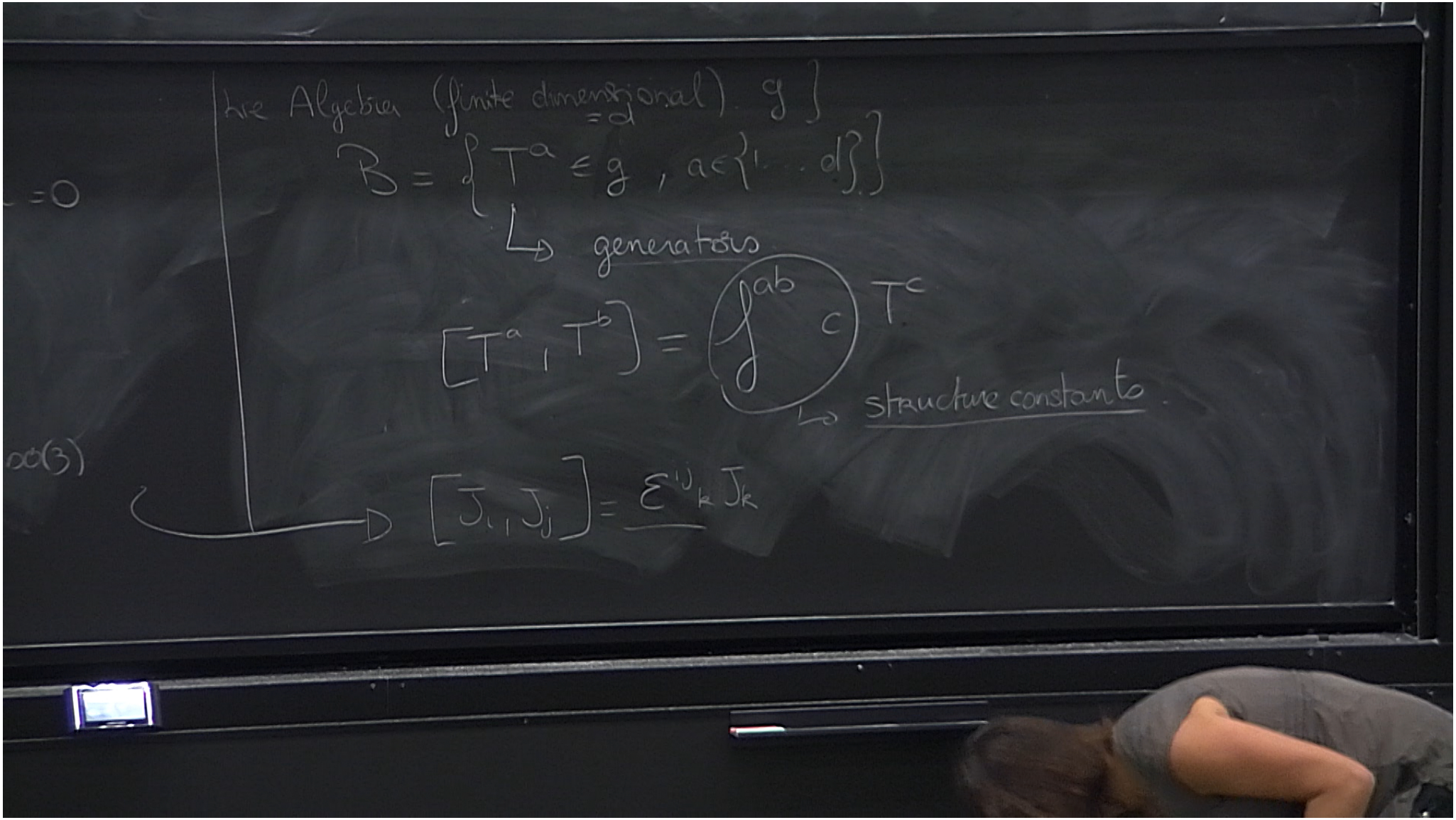
$B = \{ T^a \in g \,,\, a \in \{1 \dots d\} \}$

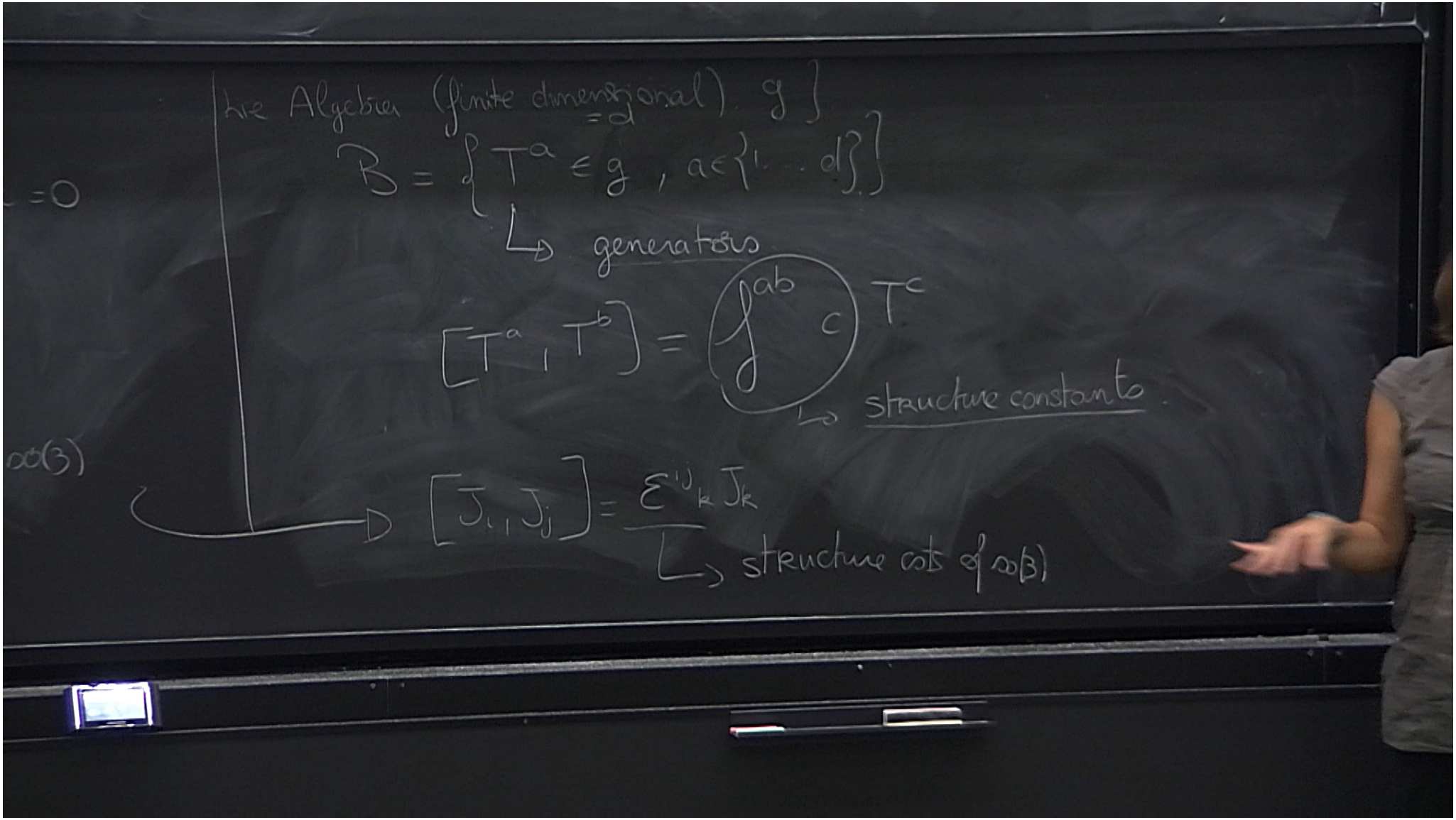
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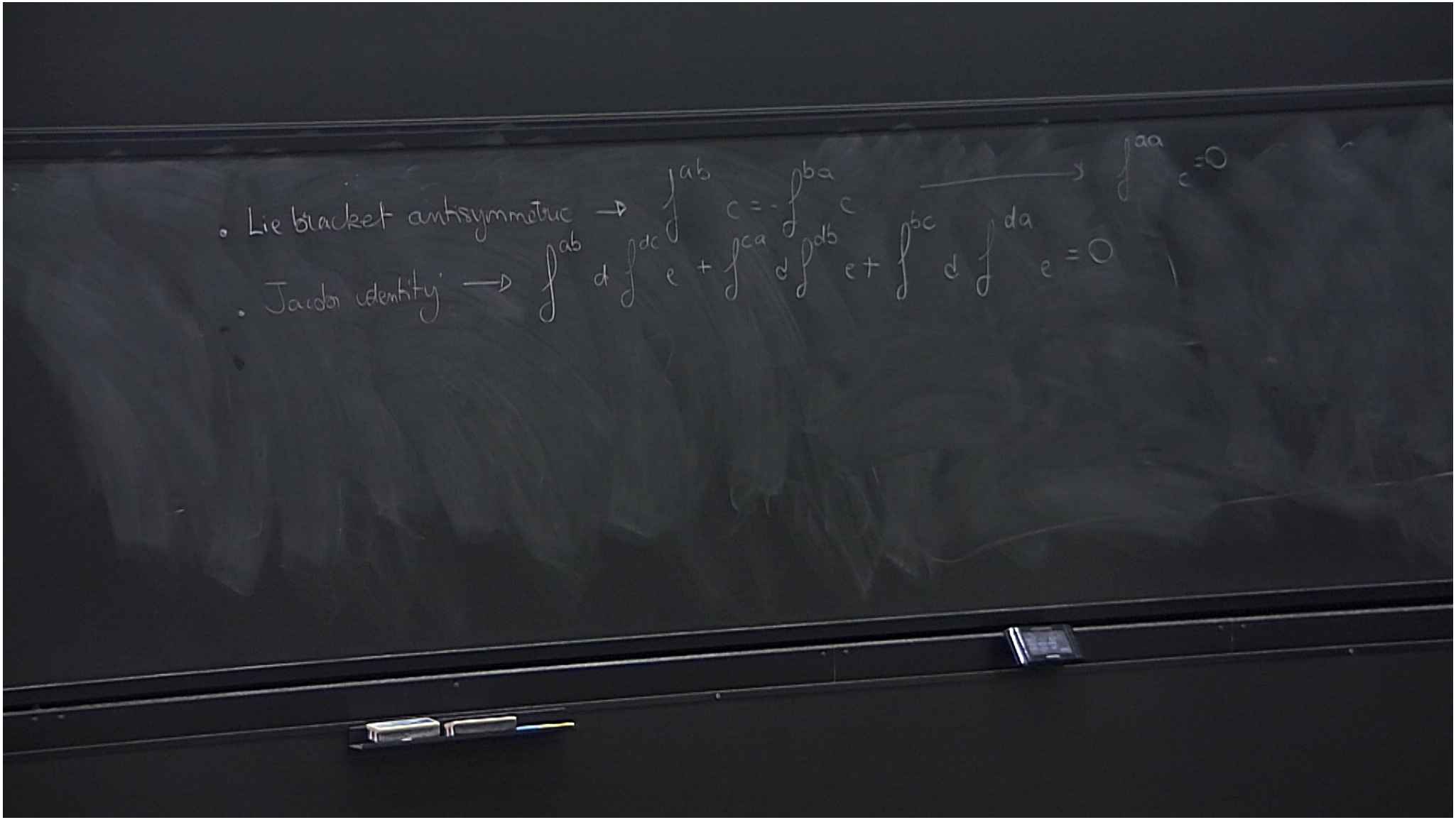
↳ generators

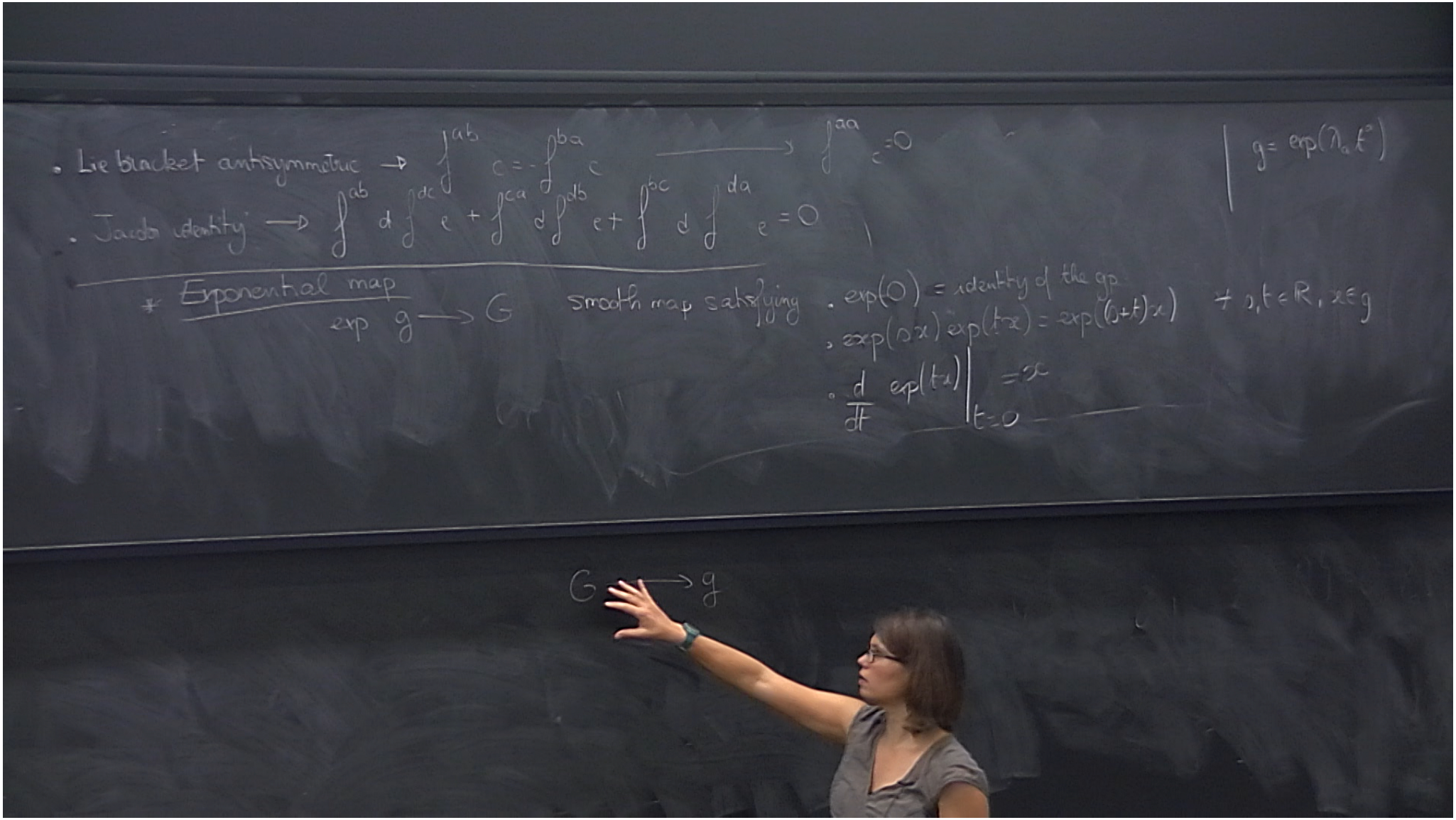
$[T^a, T^b] = f^{ab}{}_c \, T^c$

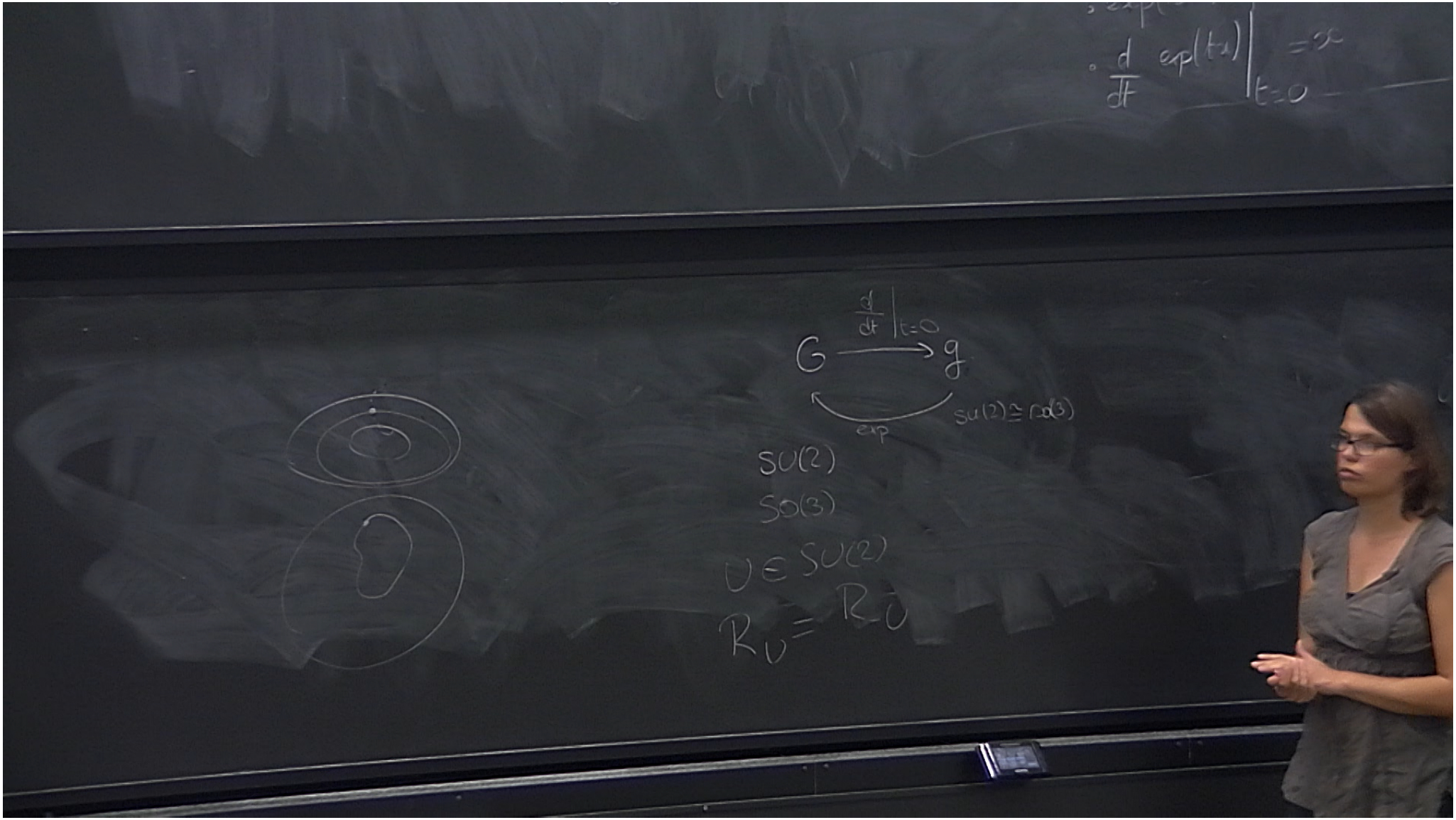
↳ structure constants

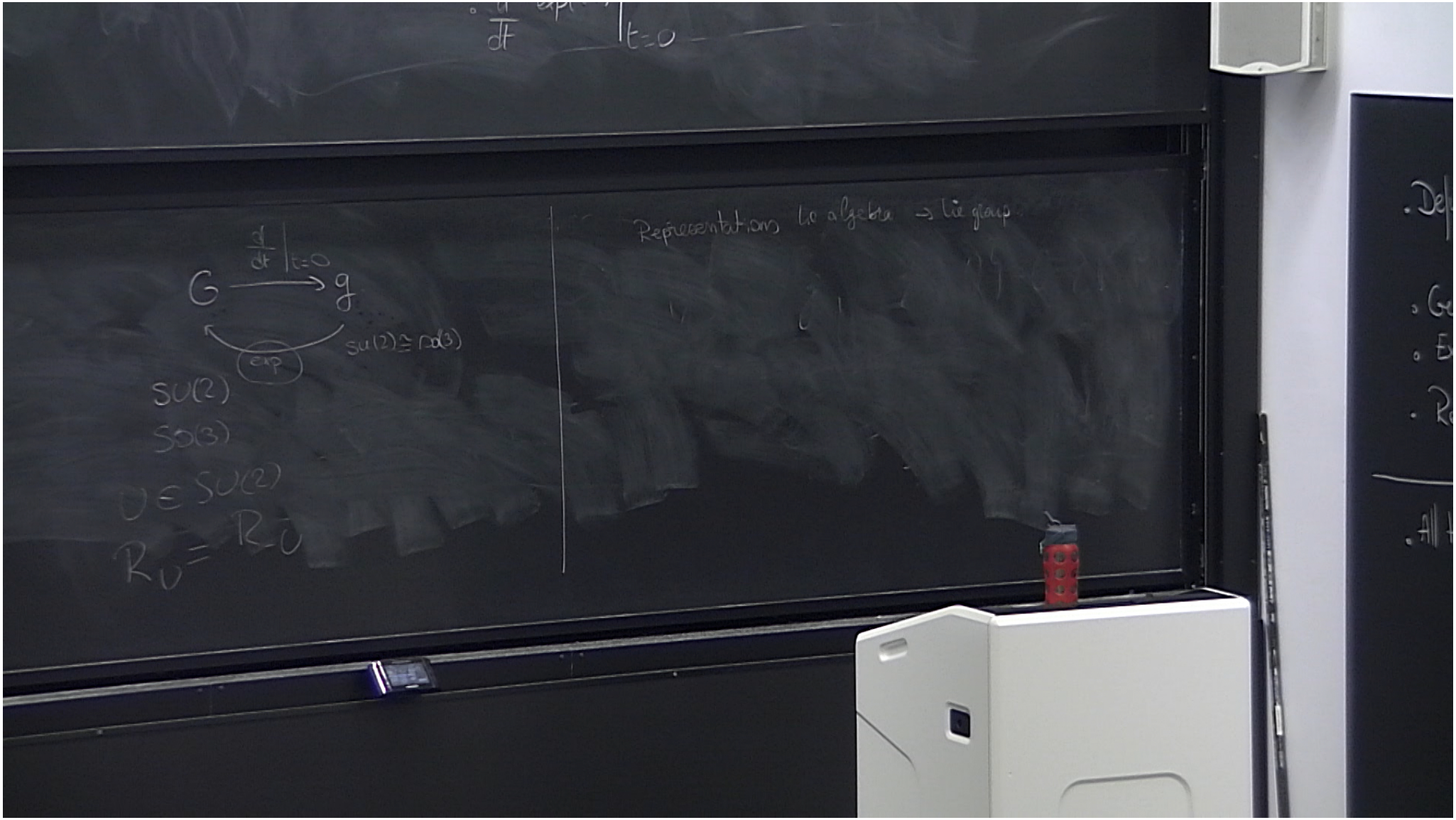

Lie Algebra (finite dimensional) g
$= d$
$B = \{T^a \in g, a \in 1 \dots d\}$
generators
$[T^a, T^b] = f^{ab}{}_c T^c$
structure constants
$[J_i, J_j] = \varepsilon^{ij}{}_k J_k$
=0
so(3)

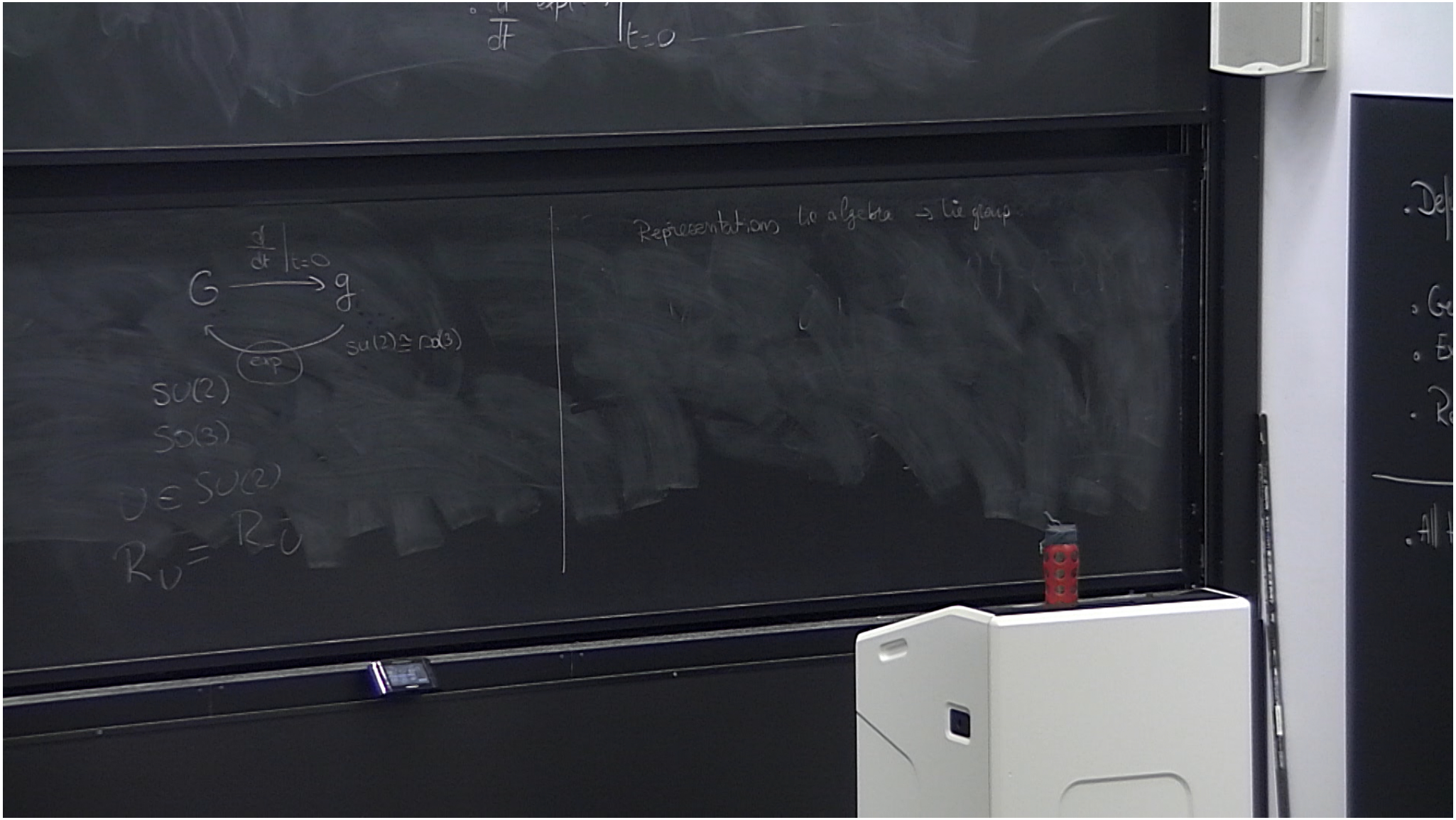

Lie Algebra (finite dimensional) g
$B = \{T^a \in g, a \in \{1 \dots d\}\}$
generators
$[T^a, T^b] = f^{ab}{}_c T^c$
structure constants
$[J_i, J_j] = \varepsilon^{ij}{}_k J_k$
structure cst of so(3)


• Lie bracket antisymmetric → $f^{ab}{}_c = -f^{ba}{}_c \longrightarrow f^{aa}{}_c = 0$
• Jacobi identity → $f^{ab}{}_d f^{dc}{}_e + f^{ca}{}_d f^{db}{}_e + f^{bc}{}_d f^{da}{}_e = 0$

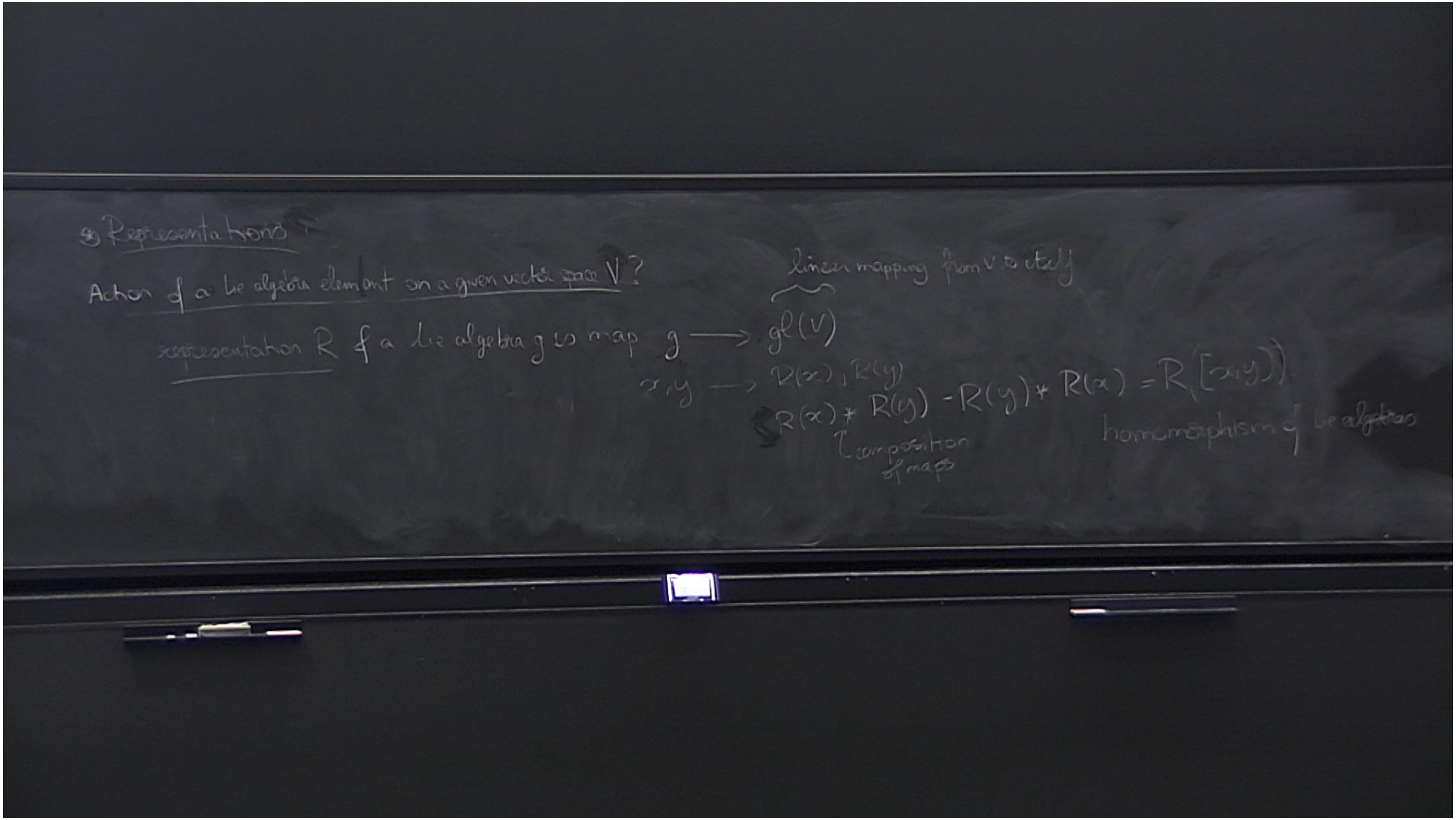

• Lie bracket antisymmetric → $f^{ab}{}_c = -f^{ba}{}_c \longrightarrow f^{aa}{}_c = 0$
• Jacobi identity → $f^{ab}{}_d f^{dc}{}_e + f^{ca}{}_d f^{db}{}_e + f^{bc}{}_d f^{da}{}_e = 0$
$g = \exp(\lambda_a t^a)$
* Exponential map
$\exp\ g \longrightarrow G$ smooth map satisfying
• $\exp(0)$ = identity of the gp
• $\exp(sx)\exp(tx) = \exp((s+t)x) \quad \forall s,t \in \mathbb{R}, x \in g$
• $\frac{d}{dt}\exp(tx)\Big|_{t=0} = x$
$G \longrightarrow g$

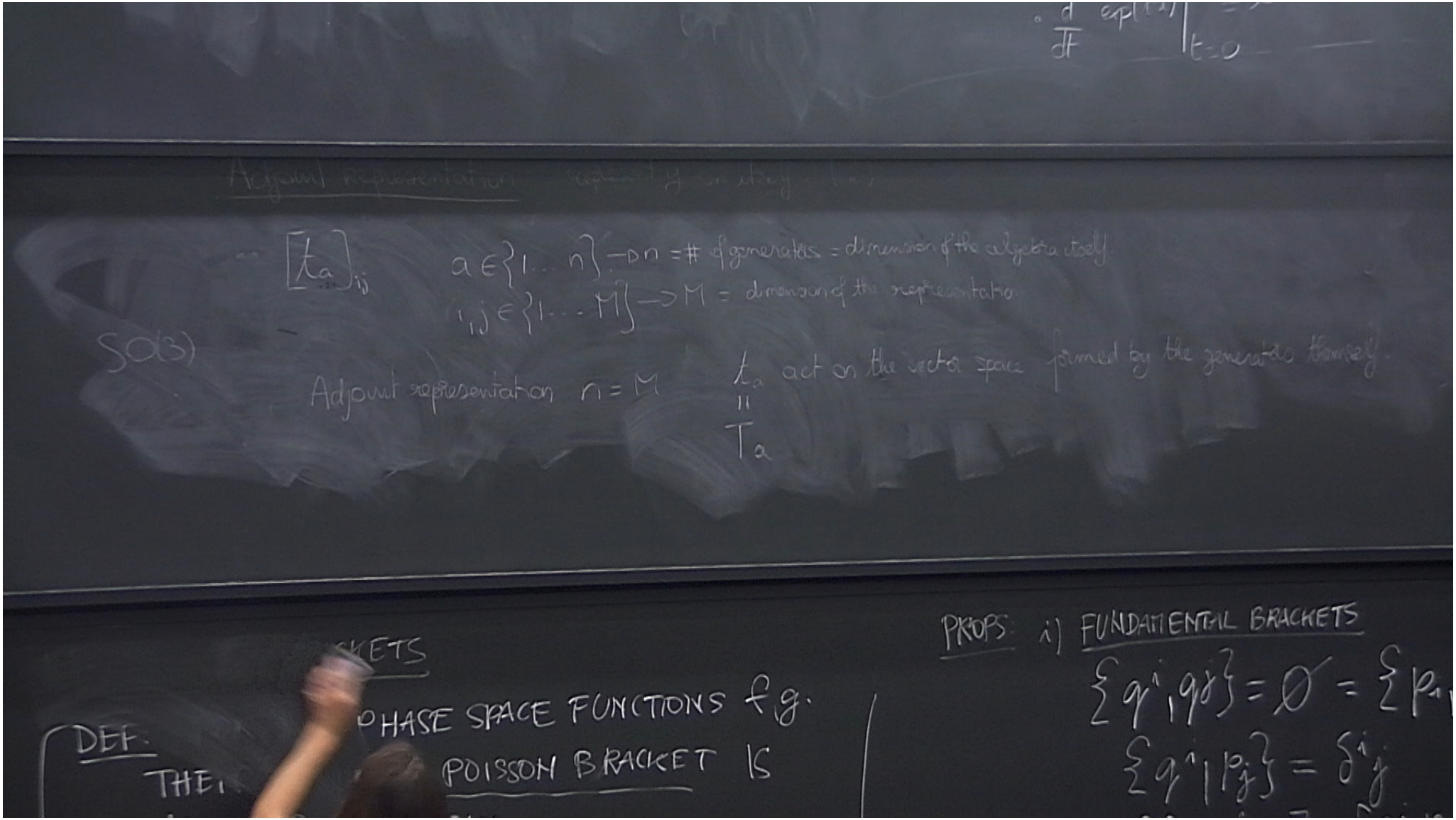

$\frac{d}{dt}\exp(tx)\Big|_{t=0} = x$
$\frac{d}{dt}\Big|_{t=0}$
G
g
exp
$SU(2) \cong \mathfrak{so}(3)$
SU(2)
SO(3)
$U \in SU(2)$
$R_U = R_{-U}$


$\frac{d}{dt}\Big|_{t=0}$
G
g
exp
$SU(2) \cong SO(3)$
$SU(2)$
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$U \in SU(2)$
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Representations Lie algebra → Lie group


$\frac{d}{dt}\Big|_{t=0}$
G
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$SU(2) \cong SO(3)$
SU(2)
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Representations Lie algebra → Lie group


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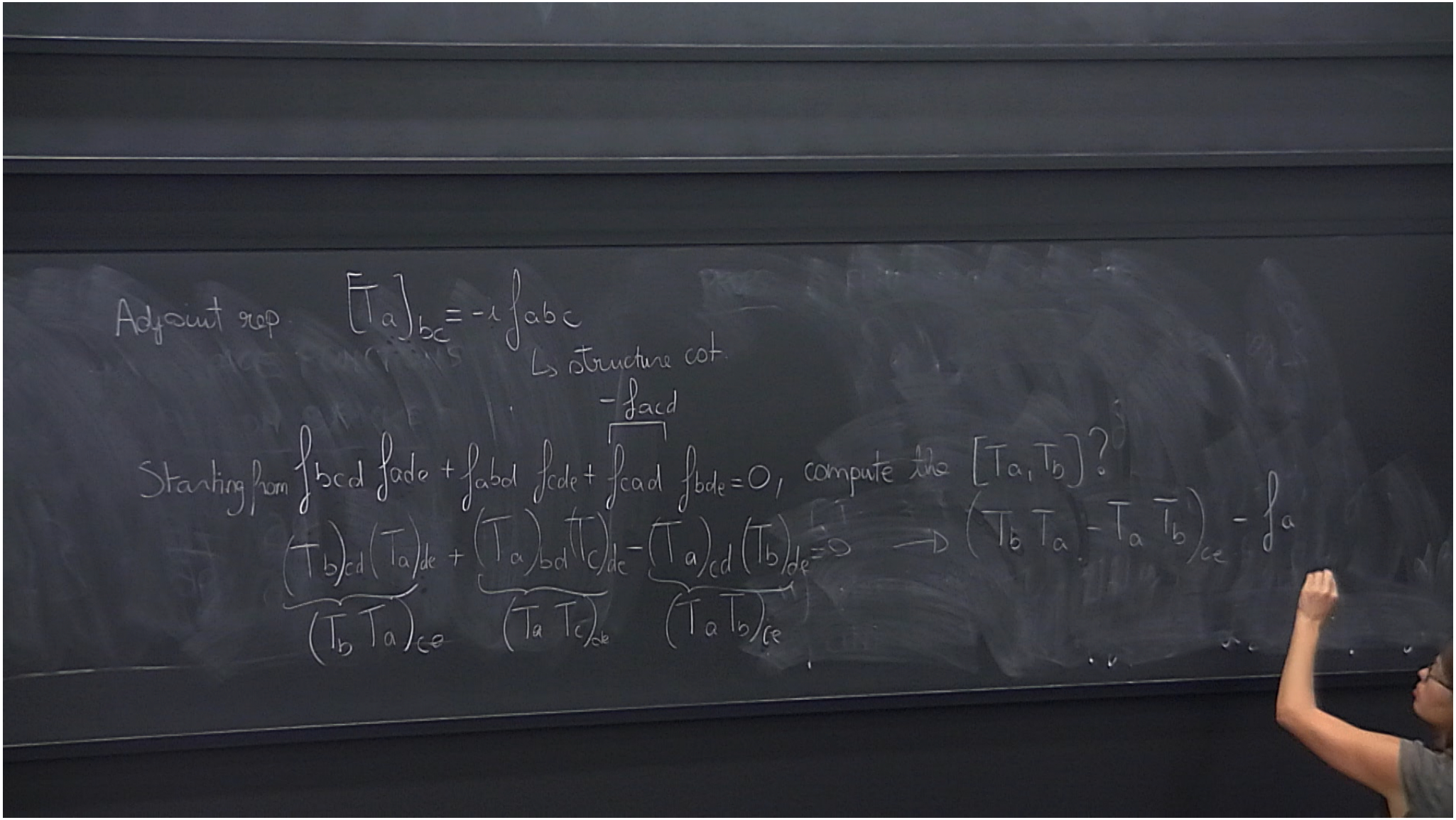

Representations
Action of a Lie algebra element on a given vector space V?
linear mapping from V to itself
representation R of a Lie algebra g is map g ⟶ gl(V)
x, y ⟶ R(x), R(y)
R(x) * R(y) - R(y) * R(x) = R([x,y])
composition of maps
homomorphism of Lie algebras


$\frac{d}{dt}$ exp $|_{t=0}$
Adjoint representation
$[t_a]_{ij}$
$a \in \{1 \dots n\} \to n$ = # of generators = dimension of the algebra itself
$i,j \in \{1 \dots M\} \to M$ = dimension of the representation
SO(3)
Adjoint representation $n = M$
t_a act on the vector space formed by the generators themself
$=$
T_a
PROPS: i) FUNDAMENTAL BRACKETS
$\{q^i, q^j\} = 0 = \{p$
$\{q^i, p_j\} = \delta^i_j$
DEF.
PHASE SPACE FUNCTIONS f, g.
POISSON BRACKET IS

Adjoint rep $[T_a]_{bc} = -i f_{abc}$

↳ structure cst.

Starting from $f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0$, compute the $[T_a, T_b]$?


Adjoint rep $[T_a]_{bc} = -i f_{abc}$
↳ structure cst.
$-f_{acd}$
Starting from $f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0$, compute the $[T_a, T_b]$?
$(T_b)_{cd}(T_a)_{de} + (T_a)_{bd}(T_c)_{de} - (T_a)_{cd}(T_b)_{de} = 0 \rightarrow (T_b T_a - T_a T_b)_{ce} = f_a$
$(T_b T_a)_{ce}$ $(T_a T_c)_{de}$ $(T_a T_b)_{ce}$

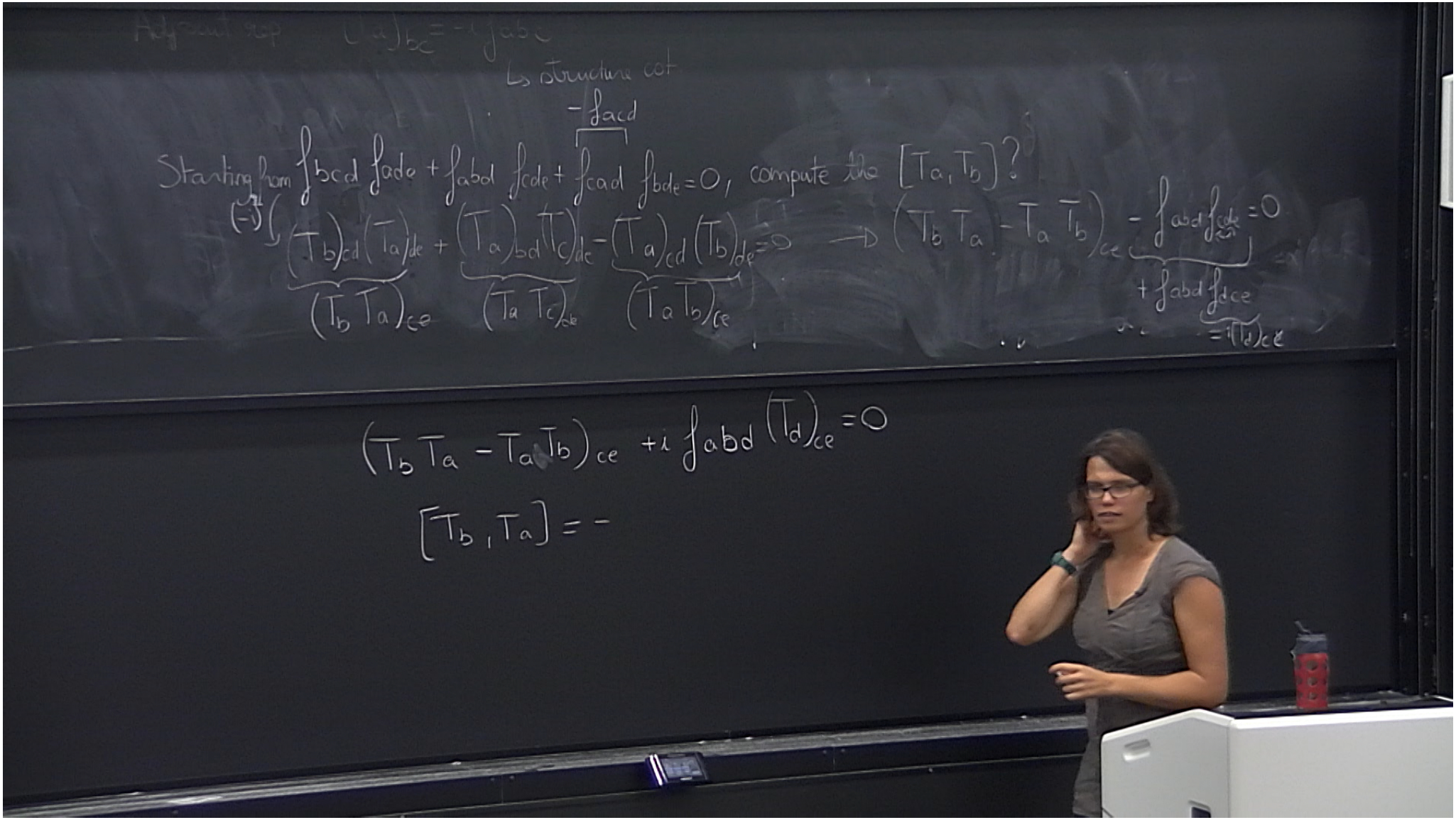
Adjoint rep $[T_a]_{bc} = -i f_{abc}$

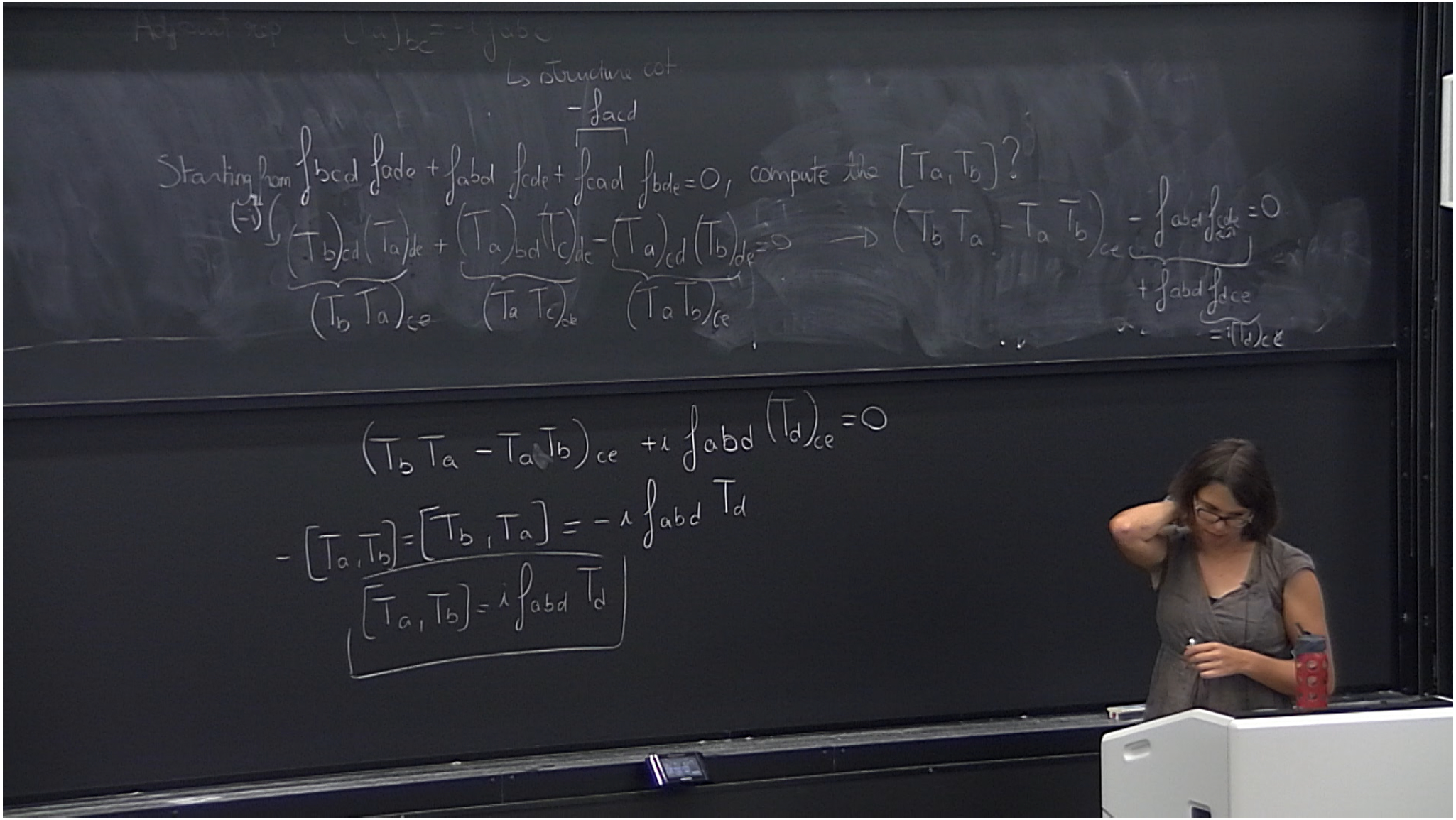
↳ structure cst

Starting from $f_{bcd} f_{ade} + f_{abd} f_{cde} + \underbrace{f_{cad}}_{-f_{acd}} f_{bde} = 0$, compute the $[T_a, T_b]$?

$$\underbrace{(T_b)_{cd}(T_a)_{de}}_{(T_b T_a)_{ce}} + \underbrace{(T_a)_{bd}(T_c)_{de}}_{(T_a T_c)_{ce}} - \underbrace{(T_a)_{cd}(T_b)_{de}}_{(T_a T_b)_{ce}} = 0 \rightarrow (T_b T_a - T_a T_b)_{ce} - f_{abd} f_{cde} = 0$$




Adjoint rep $(T_a)_{bc} = -i f_{abc}$
↳ structure cst
$-f_{acd}$
Starting from $f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0$, compute the $[T_a, T_b]$?
$(-i)^2$
$(T_b)_{cd}(T_a)_{de} + (T_a)_{bd}(T_c)_{de} - (T_a)_{cd}(T_b)_{de} = 0$
$(T_b T_a)_{ce}$ $(T_a T_c)_{ce}$ $(T_a T_b)_{ce}$
$\rightarrow (T_b T_a - T_a T_b)_{ce} - f_{abd} f_{cde} = 0$
$+ f_{abd} f_{dce}$
$= i(T_d)_{ce}$
$(T_b T_a - T_a T_b)_{ce} + i f_{abd} (T_d)_{ce} = 0$
$[T_b, T_a] = -$


Adjoint rep $(T_a)_{bc} = -i f_{abc}$
↳ structure cst
$-f_{acd}$
Starting from $f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0$, compute the $[T_a, T_b]$?
$(-i)^2$
$(T_b)_{cd}(T_a)_{de} + (T_a)_{bd}(T_c)_{de} - (T_a)_{cd}(T_b)_{de} = 0$
$(T_b T_a)_{ce}$ $(T_a T_c)_{ce}$ $(T_a T_b)_{ce}$
$\rightarrow (T_b T_a - T_a T_b)_{ce} - f_{abd} f_{cde} = 0$
$+ f_{abd} f_{dce}$
$= i(T_d)_{ce}$
$(T_b T_a - T_a T_b)_{ce} + i f_{abd} (T_d)_{ce} = 0$
$-[T_a, T_b] = [T_b, T_a] = -i f_{abd} T_d$
$[T_a, T_b] = i f_{abd} T_d$

Inner product on the vector space spanned by the generators

$$\langle T_a | T_b \rangle \equiv T_R [T_a T_b] = (T_a)_{cd} (T_b)_{dc} = (T_a T_b)_{cc}$$

Compact Lie algebra, choose $T_R(T_a T_b) = \lambda \delta_{ab} \quad \lambda > 0$

Inner product on the vector space spanned by the generators

$$\langle T_a | T_b \rangle \equiv \mathrm{Tr}\left[T_a T_b\right] = (T_a)_{cd}\,(T_b)_{dc} = (T_a\,T_b)_{cc}$$

Compact Lie algebra, choose $\mathrm{Tr}(T_a T_b) = \lambda\,\delta_{ab} \quad \lambda > 0$

Let's prove that f_{abc} are completely antisymmetric in all the indices

$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$$\mathrm{Tr}\left(\left[T_a, T_b\right] T_c\right) = \mathrm{Tr}\left(i f_{abd}\, T_d\, T_c\right)$$

$$\mathrm{Tr}\left(T_a T_b T_c - T_b T_a T_c\right) = i f_{abd}\,\lambda\,\delta_{dc}$$

$$\mathrm{Tr}\left(T_b T_c T_a - T_c T_b T_a\right) = i\lambda f_{abc}$$

$$\mathrm{Tr}\left(\left[T_b, T_c\right] T_a\right) = i\lambda f_{abc}$$

$$\mathrm{Tr}\left(i f_{bcd}\, T_d\, T_a\right) = i\lambda f_{abc}$$

$$i\lambda f_{bca} = i\lambda f_{abc}$$

$\langle T_a | T_b \rangle \equiv T_R(T_a T_b) = (T_a)_{cd} (T_b)_{dc}$

Compact lie algebra, choose $T_R(T_a T_b) = \lambda \delta_{ab} \quad \lambda > 0$

Let's prove that f_{abc} are completely anti-symmetric in all the indices

$T_R(AB) = T_R(BA)$

$$T_R\left([T_a, T_b] T_c\right) = T_R\left(i f_{abd} T_d T_c\right)$$

$$T_R(T_a T_b T_c - T_b T_a T_c) = i f_{abd} \lambda \delta_{dc}$$

$$T_R(T_b T_c T_a - T_c T_b T_a) = i \lambda f_{abc}$$

$$\downarrow \quad T_R\left([T_b, T_c] T_a\right) = i \lambda f_{abc}$$

$$T_R\left(i f_{bcd} T_d T_a\right) = i \lambda f_{abc}$$

$$i \lambda f_{bca} = i \lambda f_{abc}$$

$f_{bca} = -f_{cba} = -f_{bac}$ $\Rightarrow$ anti-symmetric in the last two indices

Inner product on the vector space spanned by the generators

$$\langle T_a | T_b \rangle \equiv \mathrm{Tr}\left[T_a T_b\right] = (T_a)_{cd}(T_b)_{dc} = (T_a T_b)_{cc}$$

Compact lie algebra, choose $\mathrm{Tr}(T_a T_b) = \lambda \delta_{ab} \quad \lambda > 0$

Let's prove that f_{abc} are completely anti-symmetric in all the indices

$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$$\mathrm{Tr}\left([T_a, T_b] T_c\right) = \mathrm{Tr}\left(i f_{abd} T_d T_c\right)$$

$$\mathrm{Tr}\left(T_a (T_b T_c) - (T_b T_a) T_c\right) = i f_{abd} \lambda \delta_{dc}$$

$$\mathrm{Tr}\left(T_b T_c T_a - T_c T_b T_a\right) = i \lambda f_{abc}$$

$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ $\Rightarrow$

$$\mathrm{Tr}\left([T_b, T_c] T_a\right) = i \lambda f_{abc}$$

$$\mathrm{Tr}\left(i f_{bcd} T_d T_a\right) = i \lambda f_{abc}$$

$$i \cancel{\lambda} f_{bca} = i \cancel{\lambda} f_{abc}$$

$f_{bca} = -f_{cba} = -f_{bac}$ $\Rightarrow$ anti-symmetric in the last two indices

Adjoint rep $(T_a)_{bc} = -i f_{abc}$

↳ structure cst

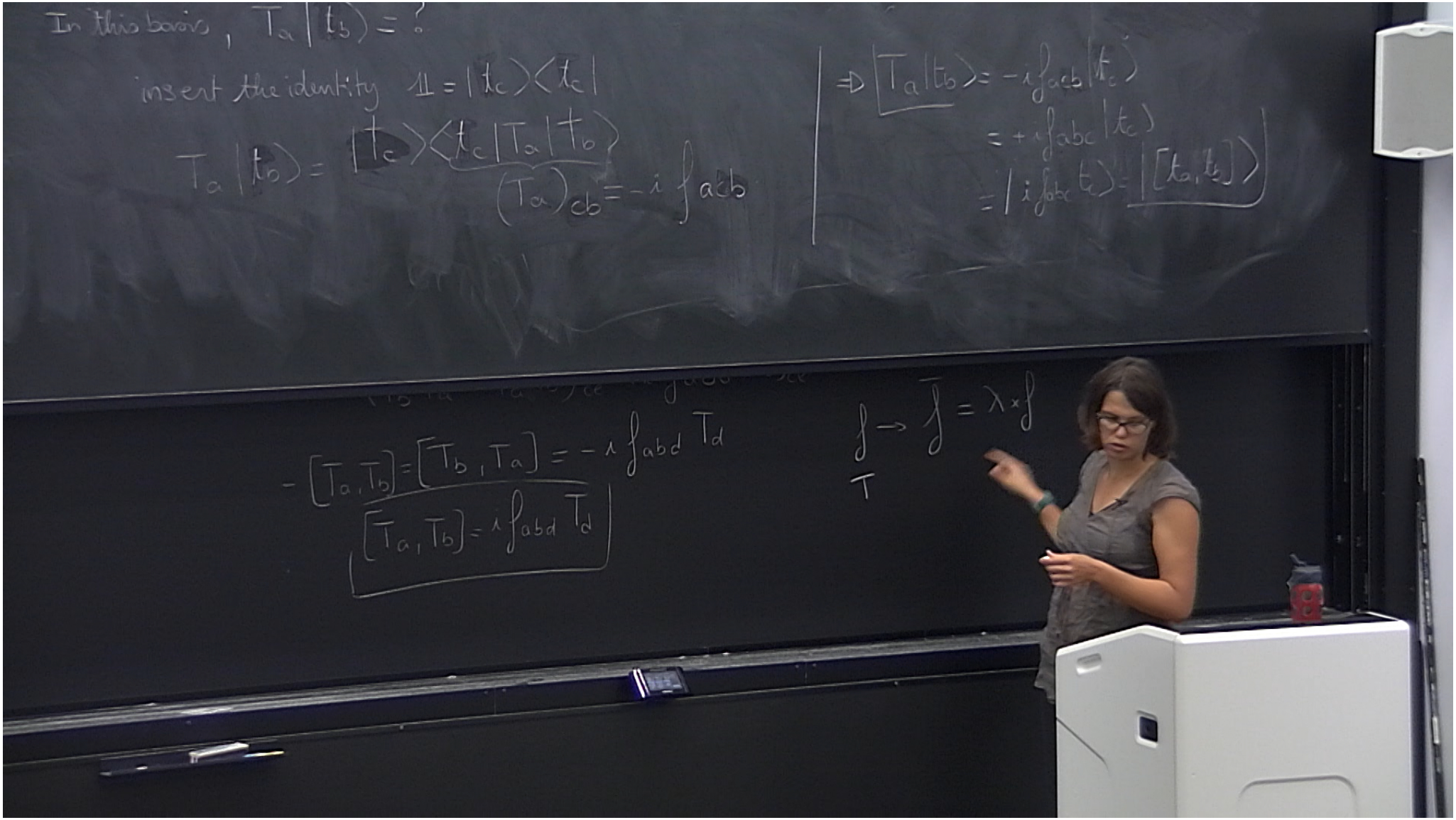
Starting from $f_{bcd} f_{ade} + f_{abd} f_{cde} + \underbrace{f_{cad}}_{-f_{acd}} f_{bde} = 0$, compute the $[T_a, T_b]$?

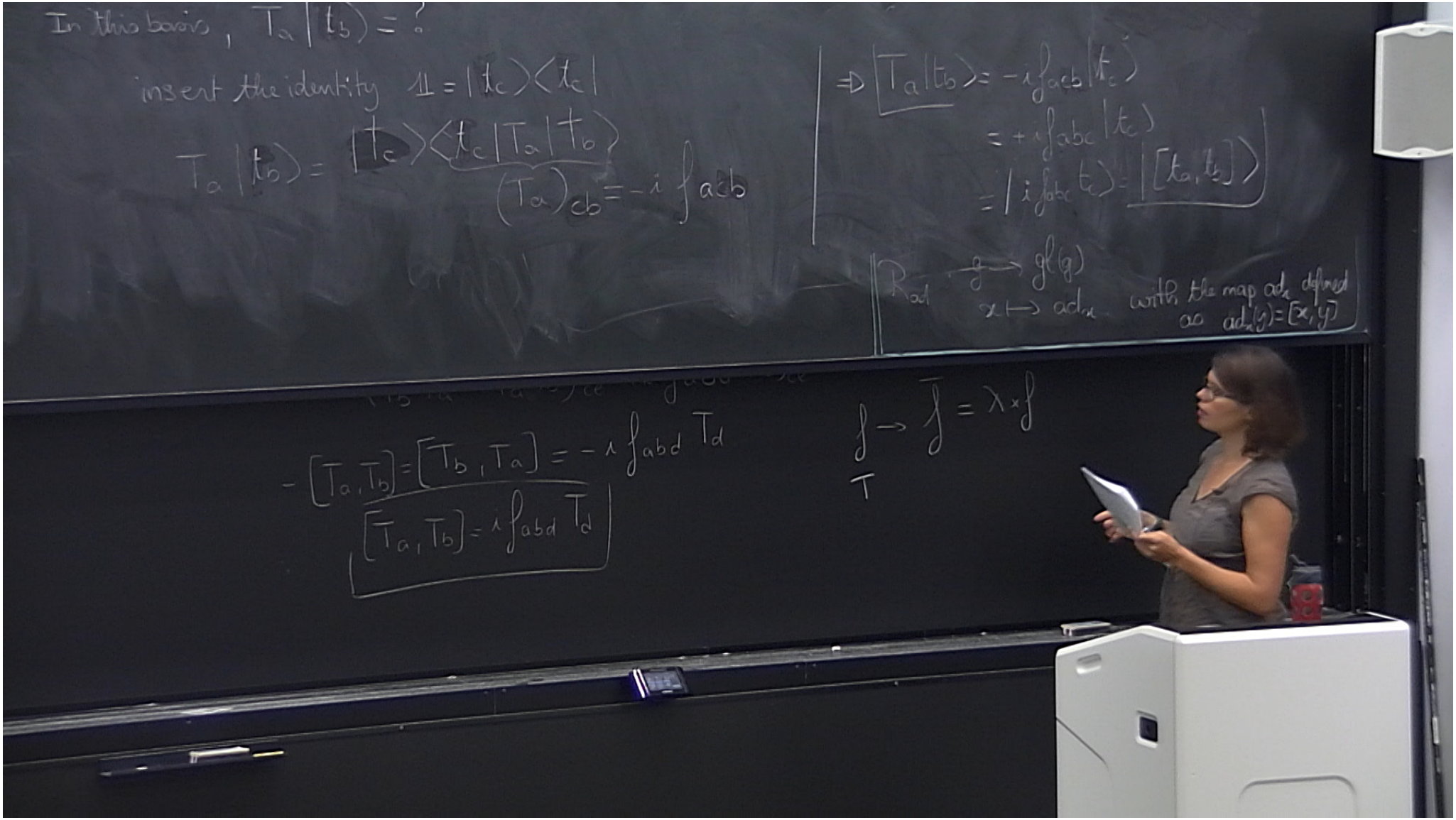
$(-i)^2 \Big(\underbrace{(T_b)_{cd}(T_a)_{de}}_{(T_b T_a)_{ce}} + \underbrace{(T_a)_{bd}(T_c)_{de}}_{(T_a T_c)_{be}} - \underbrace{(T_a)_{cd}(T_b)_{de}}_{(T_a T_b)_{ce}} = 0 \rightarrow (T_b T_a - T_a T_b)_{ce} - \dots$

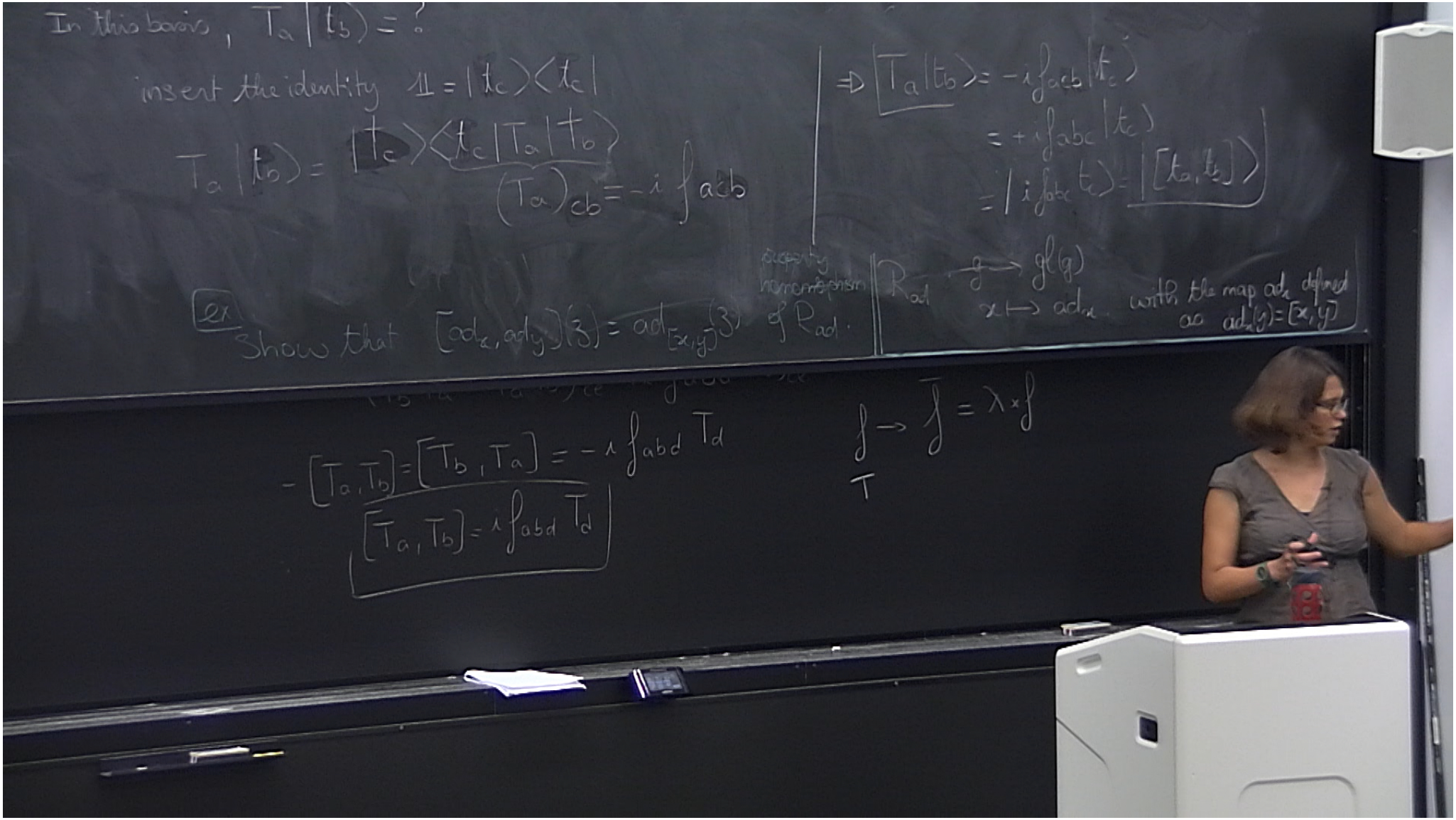
In this basis, $T_a | T_b) = ?$

insert the identity $\mathbb{1} = |T_c)(T_c|$

$T_a | T_b) = |T_c)(T_c | T_a | T_b)$


In this basis, $T_a|t_b\rangle = ?$
insert the identity $\mathbb{1} = |t_c\rangle\langle t_c|$
$T_a|t_b\rangle = |t_c\rangle\langle t_c|T_a|t_b\rangle$
$(T_a)_{cb} = -i f_{acb}$
$\Rightarrow |T_a|t_b\rangle = -i f_{acb}|t_c\rangle$
$= +i f_{abc}|t_c\rangle$
$= |i f_{abc} t_c\rangle = |[t_a, t_b]\rangle$
$-[T_a, T_b] = [T_b, T_a] = -i f_{abd} T_d$
$[T_a, T_b] = i f_{abd} T_d$
$f \to \bar{f} = \lambda \times f$
T


In this basis, $T_a|t_b\rangle = ?$
insert the identity $\mathbb{1} = |t_c\rangle\langle t_c|$
$T_a|t_b\rangle = |t_c\rangle\langle t_c|T_a|t_b\rangle$
$(T_a)_{cb} = -i f_{acb}$
$\Rightarrow |T_a|t_b\rangle = -i f_{acb}|t_c\rangle$
$= +i f_{abc}|t_c\rangle$
$= |i f_{abc} t_c\rangle = |[t_a, t_b]\rangle$
R_{ad} $g \longrightarrow gl(g)$
$x \longmapsto ad_x$ with the map ad_x defined as $ad_x(y) = [x, y]$
$-[T_a, T_b] = [T_b, T_a] = -i f_{abd} T_d$
$[T_a, T_b] = i f_{abd} T_d$
$f \to \bar{f} = \lambda \times f$
T


In this basis, $T_a|t_b\rangle = ?$
insert the identity $\mathbb{1} = |t_c\rangle\langle t_c|$
$T_a|t_b\rangle = |t_c\rangle\langle t_c|T_a|t_b\rangle$
$(T_a)_{cb} = -i f_{acb}$
$\Rightarrow |T_a|t_b\rangle = -i f_{acb}|t_c\rangle$
$= +i f_{abc}|t_c\rangle$
$= |i f_{abc} t_c\rangle = |[t_a, t_b]\rangle$
ex
Show that $[ad_x, ad_y](z) = ad_{[x,y]}(z)$
proving homomorphism of R_{ad}.
R_{ad}
$g \longrightarrow gl(g)$
$x \longmapsto ad_x$ with the map ad_x defined as $ad_x(y) = [x, y]$
$-[T_a, T_b] = [T_b, T_a] = -i f_{abd} T_d$
$[T_a, T_b] = i f_{abd} T_d$
$f \to \bar{f} = \lambda \times f$
T