Title: PSI 17/18 Front End - Lie Groups and Lie Algebras (Dupuis)

Date: Aug 18, 2017 10:30 AM

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Abstract:

Classification of semi-simple Lie algebras

↳ g semi-simple algebra

we want to build commutation relations similar to the ones we have for SU(2)

$$[J_3, J_\pm] = \pm J_\pm$$

$$[J_+, J_-] = J_3$$

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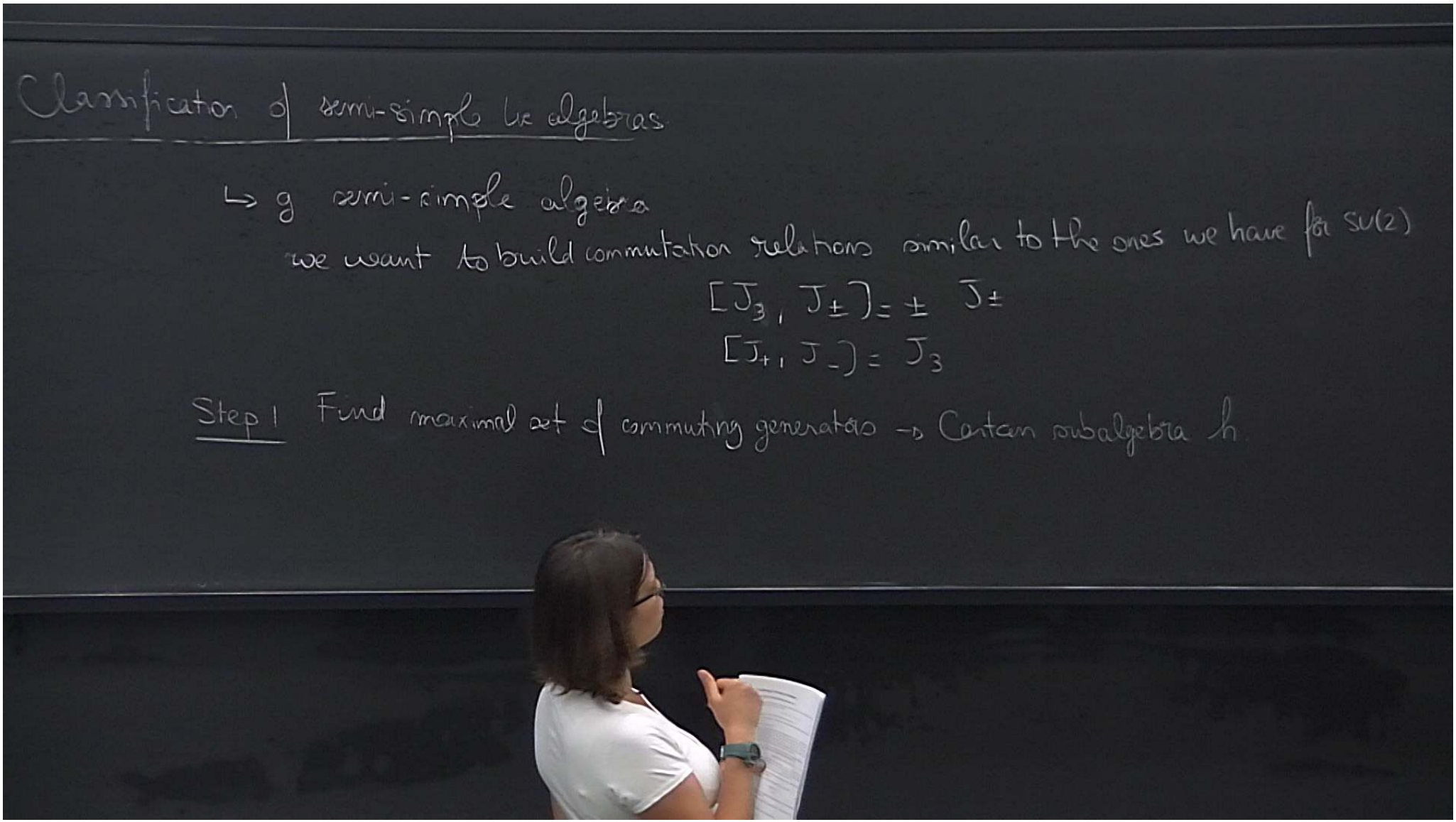
Step 1 Find maximal set of commuting generators → Cartan subalgebra h



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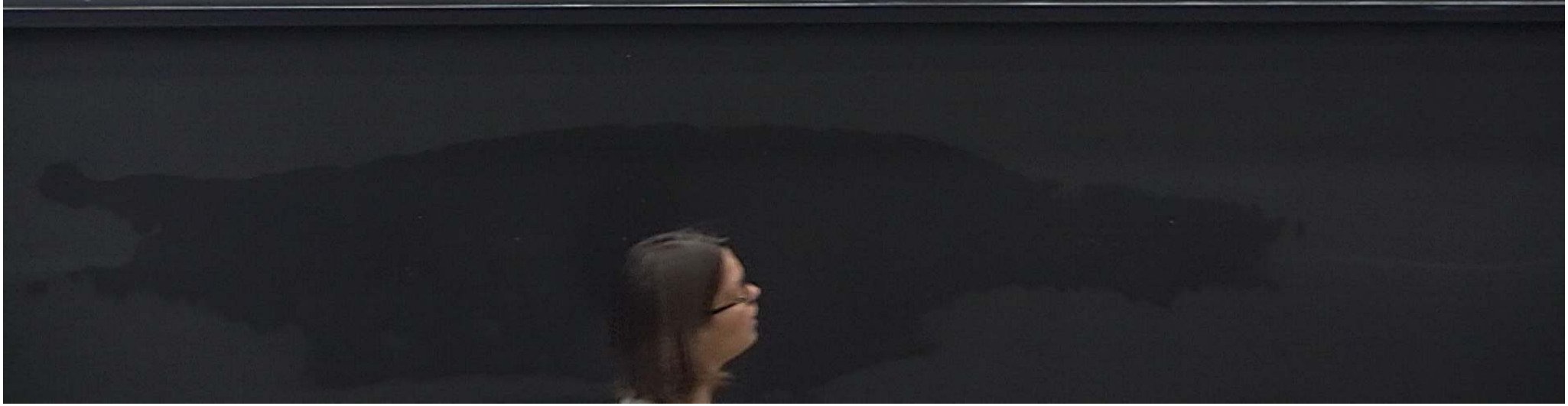
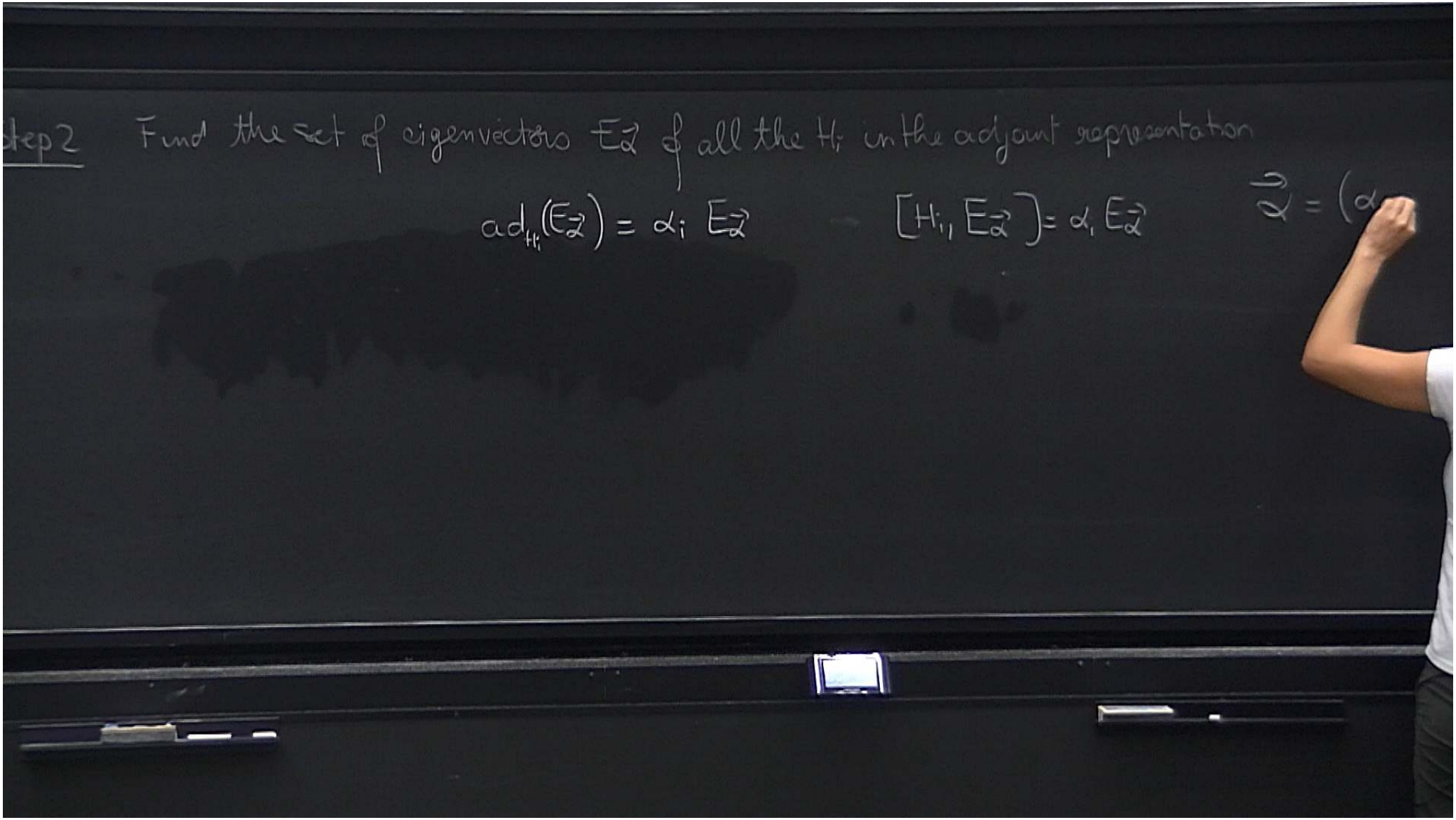
Rank = $\dim h$ Rank(su(2)) = 1 = A_1

in this lecture: rank 2.




Step 2 Find the set of eigenvectors $E_{\vec{\alpha}}$ of all the H_i in the adjoint representation
$ad_{H_i}(E_{\vec{\alpha}}) = \alpha_i E_{\vec{\alpha}}$
$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$
$\vec{\alpha} = (\alpha$

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$$ad_{H_i}(E_{\vec{\alpha}}) = \alpha_i E_{\vec{\alpha}} \qquad [H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$$

$\vec{\alpha} = (\alpha_1 \dots \alpha_r) \quad \alpha_i \in \mathbb{R}$

Roots of the Lie algebras

Step 2 Find the set of eigenvectors $E_{\vec{\alpha}}$ of all the H_i in the adjoint representation

$$ad_{H_i}(E_{\vec{\alpha}}) = \alpha_i E_{\vec{\alpha}} \qquad [H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$$

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Roots

Case of rank 2: roots are +1 and -1

Rank 2 Lie algebra.

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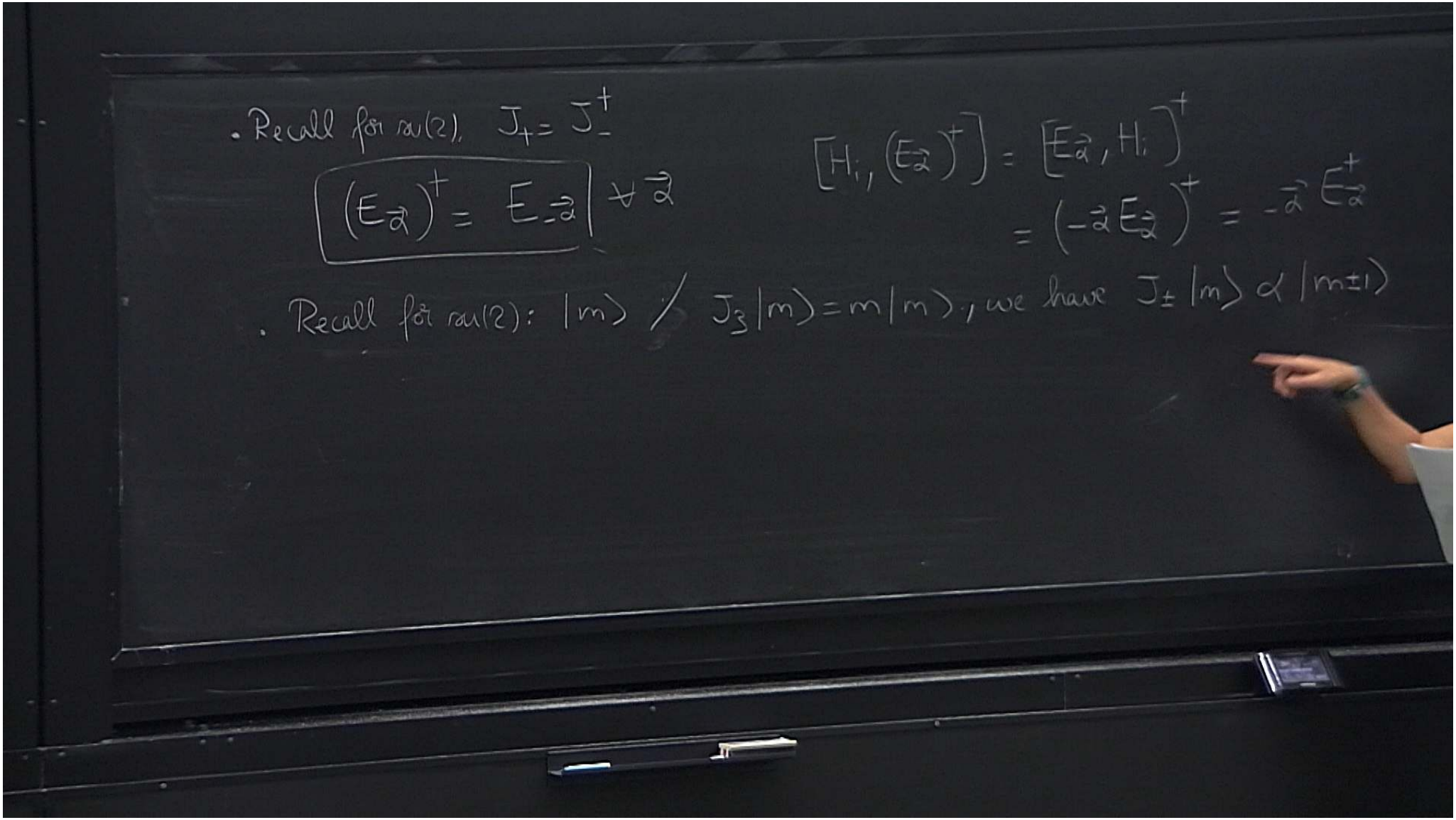
Roots

(Case of su(2): roots are +1 and -1

Rank 2 Lie algebra: roots are vectors in the 2D plane (tutorial: roots of su(3))

Step 3 Miracle! (geometry) $R=2$


• Recall for su(2), $J_+ = J_-^\dagger$
$(E_{\vec{\alpha}})^\dagger = E_{-\vec{\alpha}}$ $\forall \vec{\alpha}$
$[H_i, (E_{\vec{\alpha}})^\dagger] = [E_{\vec{\alpha}}, H_i]^\dagger$
$= (-\vec{\alpha} E_{\vec{\alpha}})^\dagger = -\vec{\alpha} E^\dagger_{\vec{\alpha}}$
• Recall for su(2): $|m\rangle$ / $J_3|m\rangle = m|m\rangle$, we have $J_\pm |m\rangle \propto |m\pm 1\rangle$

su(2): $J_+ = J_-^\dagger$

$(E_{\vec{\alpha}})^\dagger = E_{-\vec{\alpha}} \quad \forall \vec{\alpha}$

$$[H_i, (E_{\vec{\alpha}})^\dagger] = [E_{\vec{\alpha}}, H_i]^\dagger$$
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for su(2): $|m\rangle$ / $J_3|m\rangle = m|m\rangle$, we have $J_\pm |m\rangle \propto |m \pm 1\rangle$: analogues of raising and lowe

$E_{\vec{\alpha}}|\vec{\mu}\rangle$, $|\vec{\mu}\rangle$ is some eigenstate for all H_i with eigenvalue μ_i $\quad H_i|\vec{\mu}\rangle = \mu_i|\vec{\mu}\rangle$

$$H_i(E_{\vec{\alpha}}|\vec{\mu}\rangle) = ([H_i, E_{\vec{\alpha}}] + E_{\vec{\alpha}} H_i)|\vec{\mu}\rangle$$
$$(\alpha_i E_{\vec{\alpha}}|\vec{\mu}\rangle + \mu_i E_{\vec{\alpha}}|\vec{\mu}\rangle) = (\alpha_i + \mu_i)(E_{\vec{\alpha}}|\vec{\mu}\rangle) \Rightarrow E_{\vec{\alpha}}|\vec{\mu}\rangle \propto |\vec{\mu} + \vec{\alpha}\rangle$$

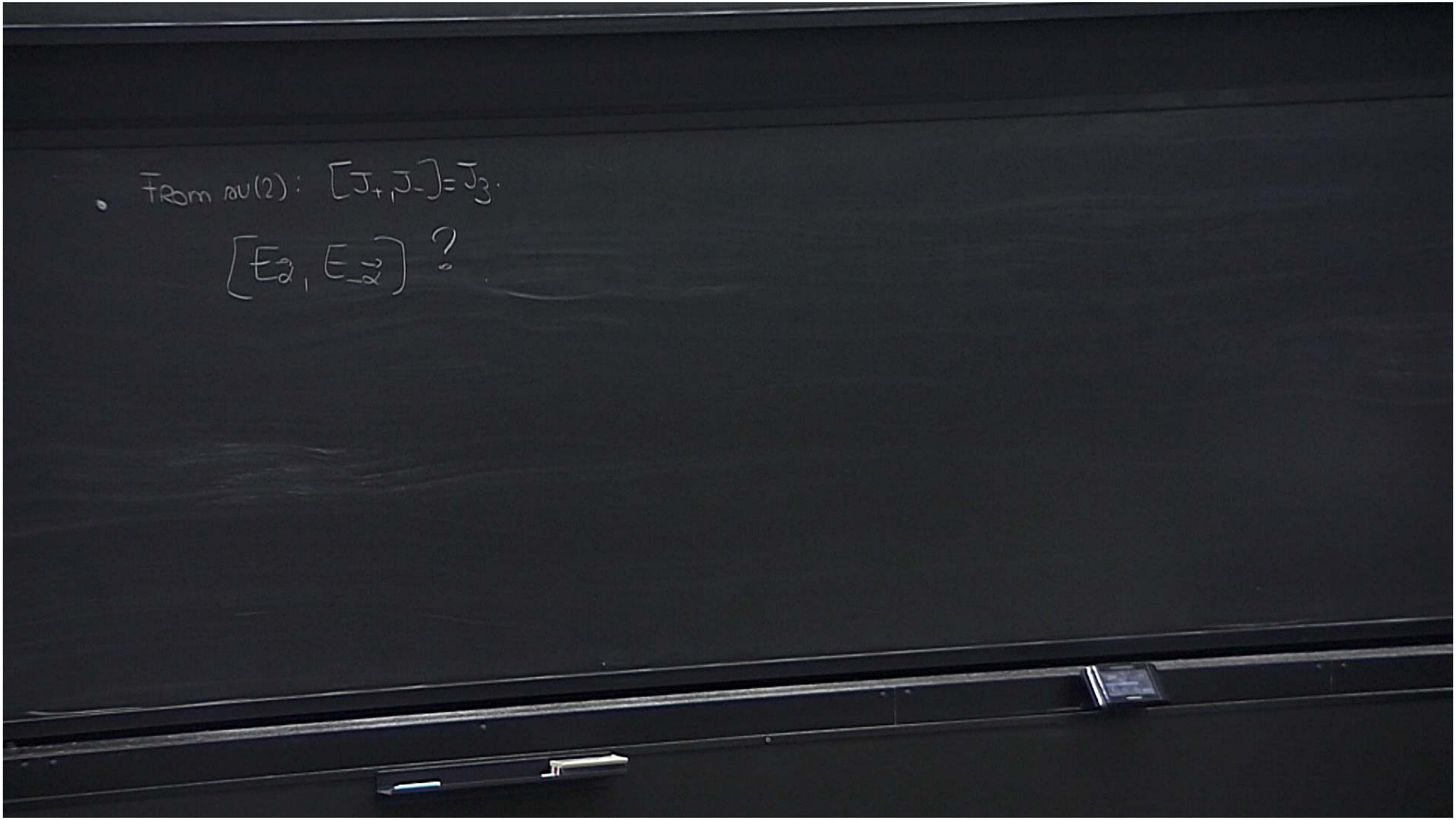
$J_z|m\rangle = m|m\rangle$, we have $J_\pm|m\rangle \propto |m\pm1\rangle$: analogues of raising and lowering operators?

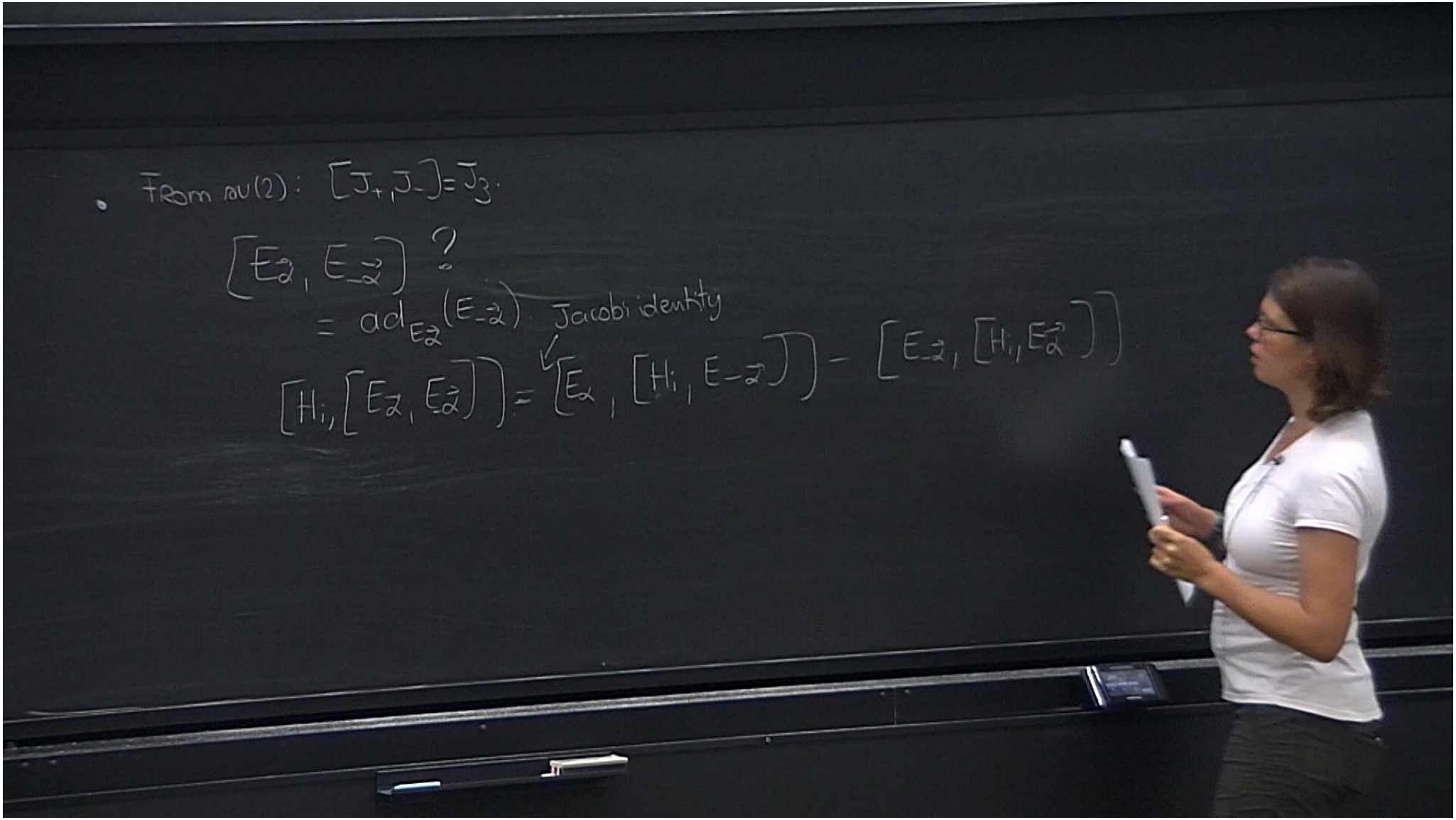
$|\vec{\mu}\rangle$ eigenstate for all H_i with eigenvalue μ_i $\quad H_i|\vec{\mu}\rangle = \mu_i|\vec{\mu}\rangle$

$$\left([H_i, E_{\vec{\alpha}}] + E_{\vec{\alpha}} H_i\right)|\vec{\mu}\rangle$$

$$\alpha_i E_{\vec{\alpha}}|\vec{\mu}\rangle + \mu_i E_{\vec{\alpha}}|\vec{\mu}\rangle = (\alpha_i + \mu_i)\left(E_{\vec{\alpha}}|\vec{\mu}\rangle\right) \Rightarrow E_{\vec{\alpha}}|\vec{\mu}\rangle \propto |\vec{\mu}+\vec{\alpha}\rangle$$

$E_{\vec{\alpha}}$ Raising operator!


From su(2): $[J_+, J_-] = J_3$.
$[E_2, E_{-2}]$?


From su(2): $[J_+, J_-] = J_3$.
$[E_\alpha, E_{-\alpha}]$?
$= ad_{E_\alpha}(E_{-\alpha})$
Jacobi identity
$[H_i, [E_\alpha, E_{-\alpha}]] = [E_\alpha, [H_i, E_{-\alpha}]] - [E_{-\alpha}, [H_i, E_\alpha]]$

- From su(2): $[J_+, J_-] = J_3$.

$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}]$?

$= ad_{E_{\vec{\alpha}}}(E_{-\vec{\alpha}})$

Jacobi identity

$[H_i, [E_{\vec{\alpha}}, E_{-\vec{\alpha}}]] = [E_{\vec{\alpha}}, \underbrace{[H_i, E_{-\vec{\alpha}}]}_{-\alpha_i E_{-\vec{\alpha}}}] - [E_{-\vec{\alpha}}, \underbrace{[H_i, E_{\vec{\alpha}}]}_{\alpha_i E_{\vec{\alpha}}}] = 0$

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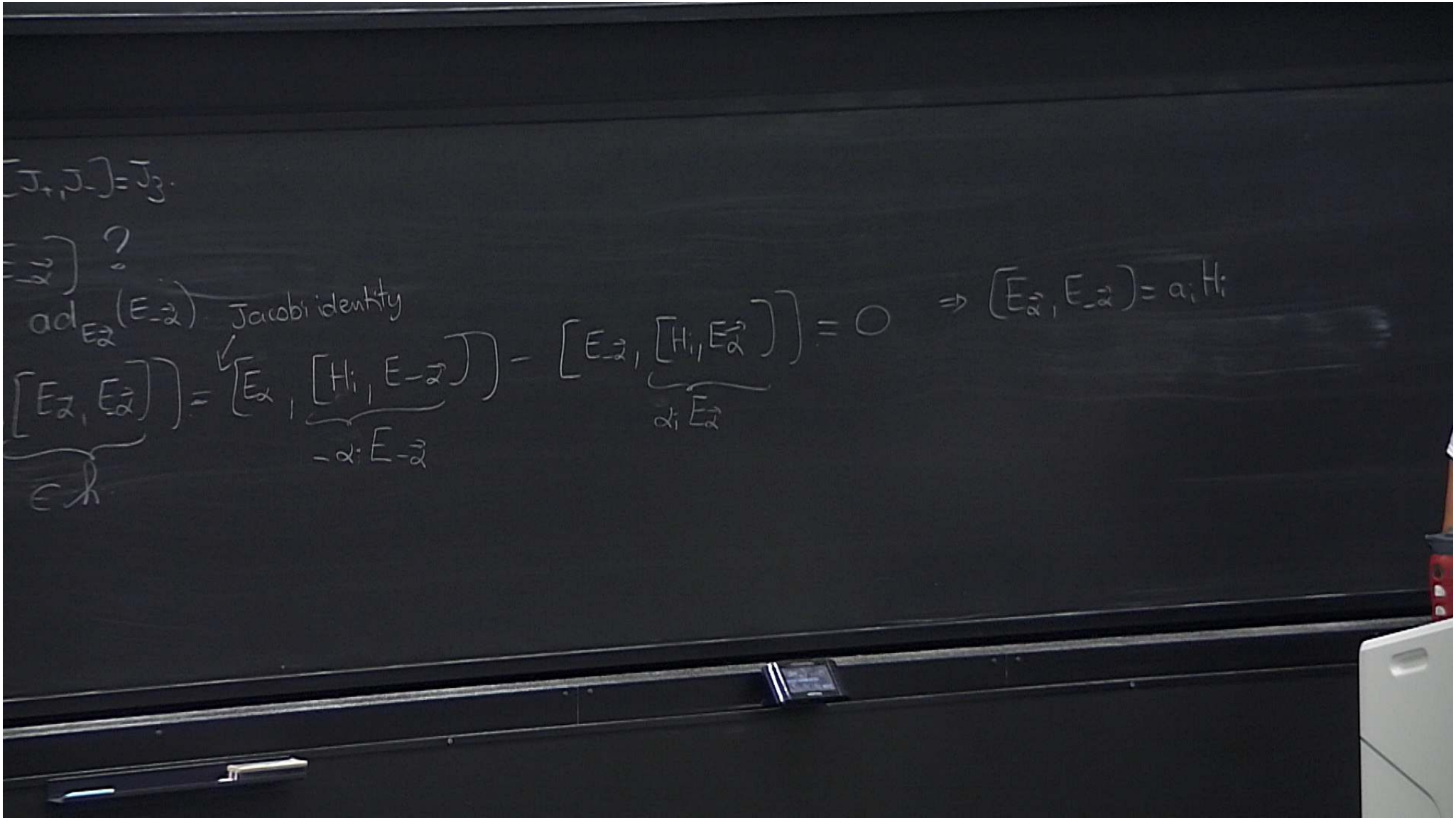
$[J_+, J_-] = J_3$
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$[E_{\vec{\alpha}}, [E_{-\vec{\alpha}}, E_{\vec{\alpha}}]] = [E_{\vec{\alpha}}, [H_i, E_{-\vec{\alpha}}]] - [E_{-\vec{\alpha}}, [H_i, E_{\vec{\alpha}}]] = 0 \Rightarrow [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = a_i H_i$
$\in \mathfrak{h}$
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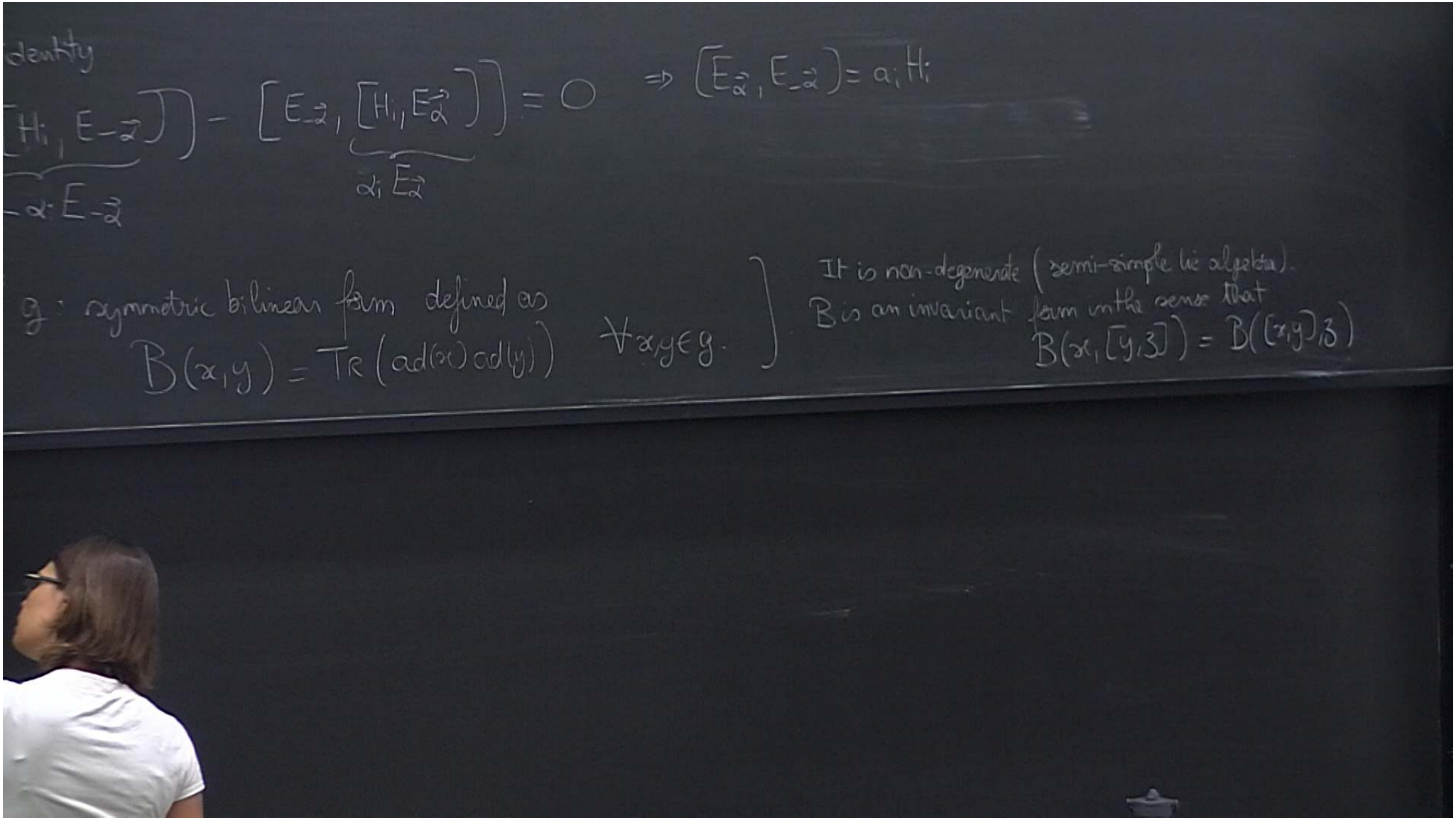
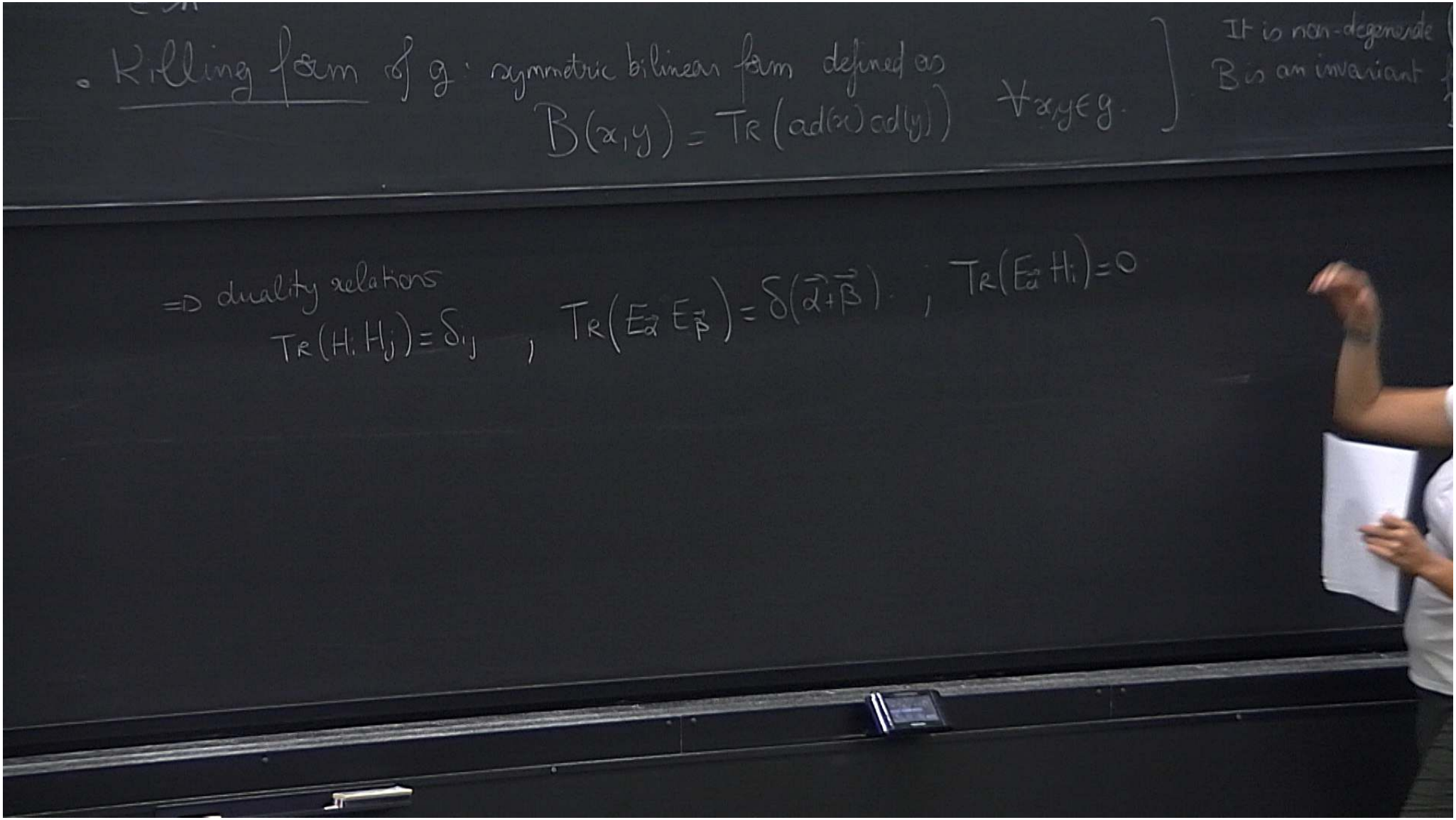
- Killing form of g: symmetric bilinear form defined as

$$B(x,y) = \mathrm{Tr}(ad(x)\,ad(y)) \quad \forall x,y \in g.$$


dentity
$[H_i, E_{-\vec{\alpha}}]]$ - $[E_{-\vec{\alpha}}, [H_i, E_{\vec{\alpha}}]] = 0$ $\Rightarrow$ $[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = a_i H_i$
$-\alpha_i E_{-\vec{\alpha}}$
$\alpha_i E_{\vec{\alpha}}$
g: symmetric bilinear form defined as
$B(x,y) = Tr(ad(x)\,ad(y))$ $\forall x,y \in g.$
It is non-degenerate (semi-simple lie algebra).
B is an invariant form in the sense that
$B(x,[y,z]) = B([x,y],z)$


• Killing form of g: symmetric bilinear form defined as
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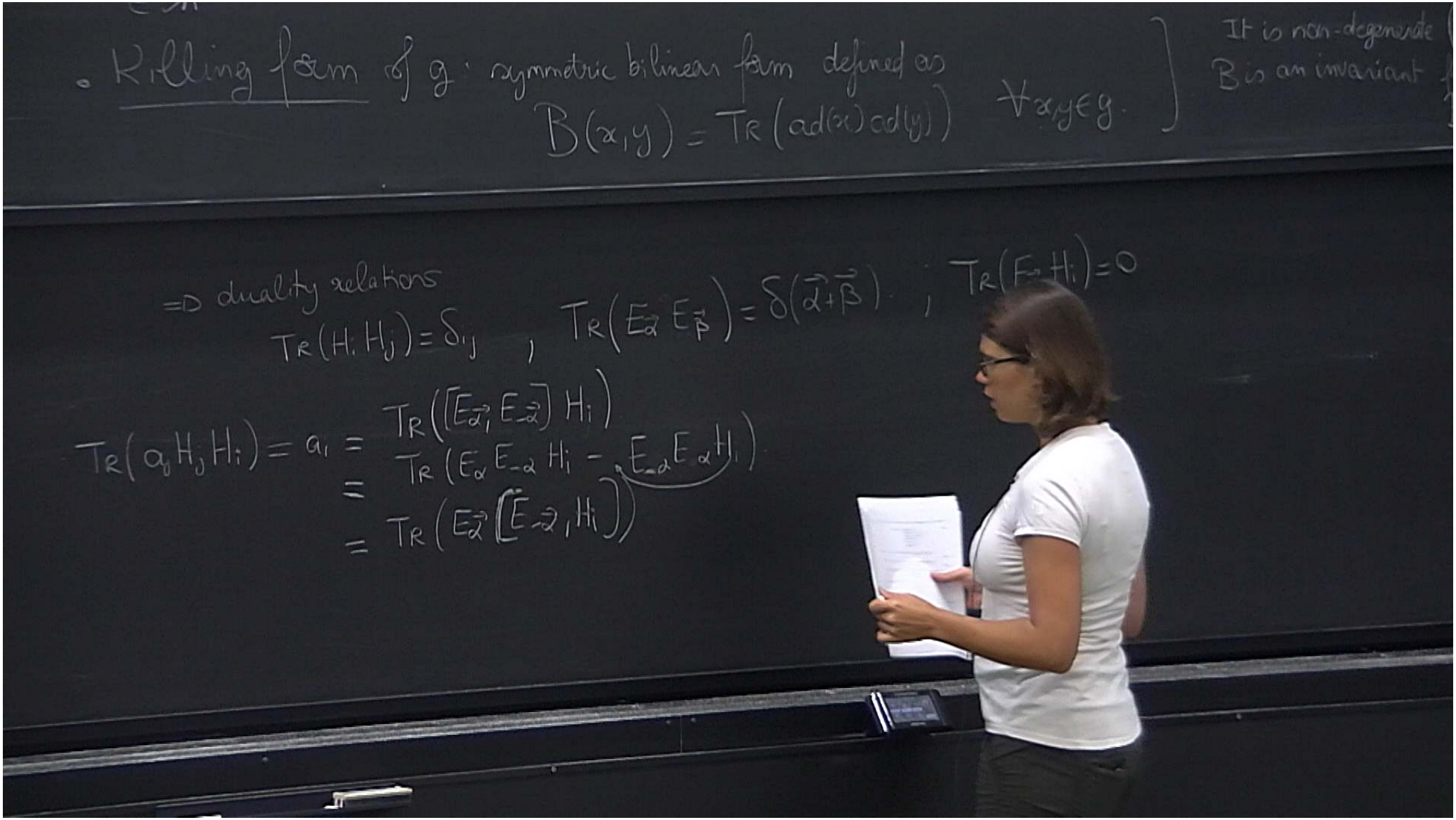
⇒ duality relations
$\mathrm{Tr}(H_i H_j) = \delta_{ij}$, $\mathrm{Tr}(E_{\vec{\alpha}} E_{\vec{\beta}}) = \delta(\vec{\alpha}+\vec{\beta})$, $\mathrm{Tr}(E_{\vec{\alpha}} H_i) = 0$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$\mathrm{Tr}(a_j H_j H_i) = a_i = \mathrm{Tr}([E_{\vec{\alpha}}, E_{-\vec{\alpha}}] H_i)$
$= \mathrm{Tr}(E_\alpha E_{-\alpha} H_i - E_{-\alpha} E_\alpha H_i)$
$= \mathrm{Tr}(E_{\vec{\alpha}} [E_{-\vec{\alpha}}, H_i])$

- Killing form of g: symmetric bilinear form defined as

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It is non-degenerate
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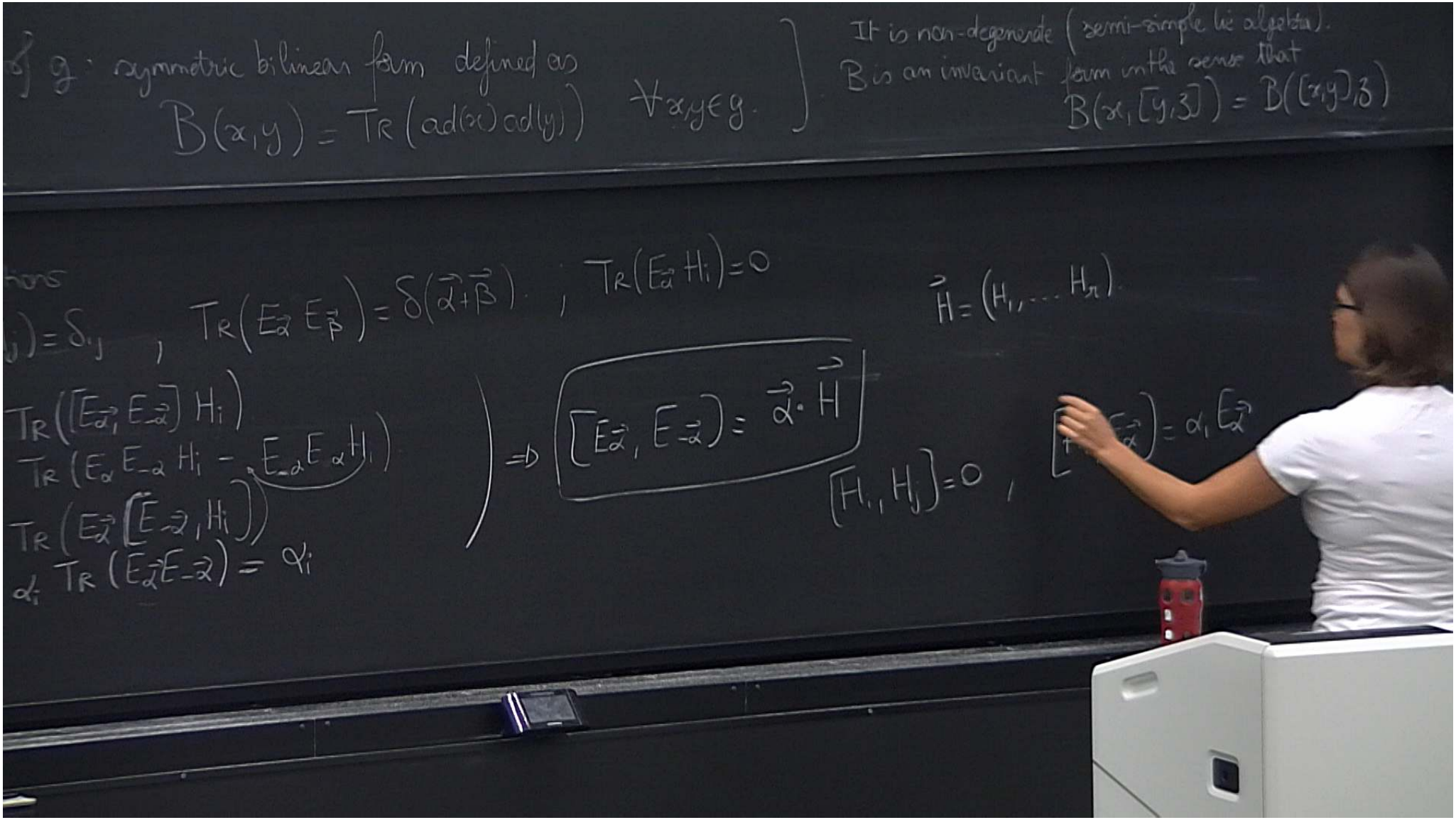
$$\mathrm{Tr}(H_i H_j) = \delta_{ij} \;, \quad \mathrm{Tr}(E_{\vec{\alpha}} E_{\vec{\beta}}) = \delta(\vec{\alpha}+\vec{\beta}) \;, \quad \mathrm{Tr}(E_{\vec{\alpha}} H_i) = 0$$

$$\mathrm{Tr}(\alpha_j H_j H_i) = \alpha_i = \mathrm{Tr}\left([E_{\vec{\alpha}}, E_{-\vec{\alpha}}] H_i\right) = \mathrm{Tr}(E_\alpha E_{-\alpha} H_i - E_{-\alpha} E_\alpha H_i) = \mathrm{Tr}\left(E_{\vec{\alpha}} [E_{-\vec{\alpha}}, H_i]\right) = \alpha_i \,\mathrm{Tr}(E_{\vec{\alpha}} E_{-\vec{\alpha}}) = \alpha_i$$

$$\Rightarrow [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \vec{\alpha}\cdot\vec{H}$$

$\vec{H}$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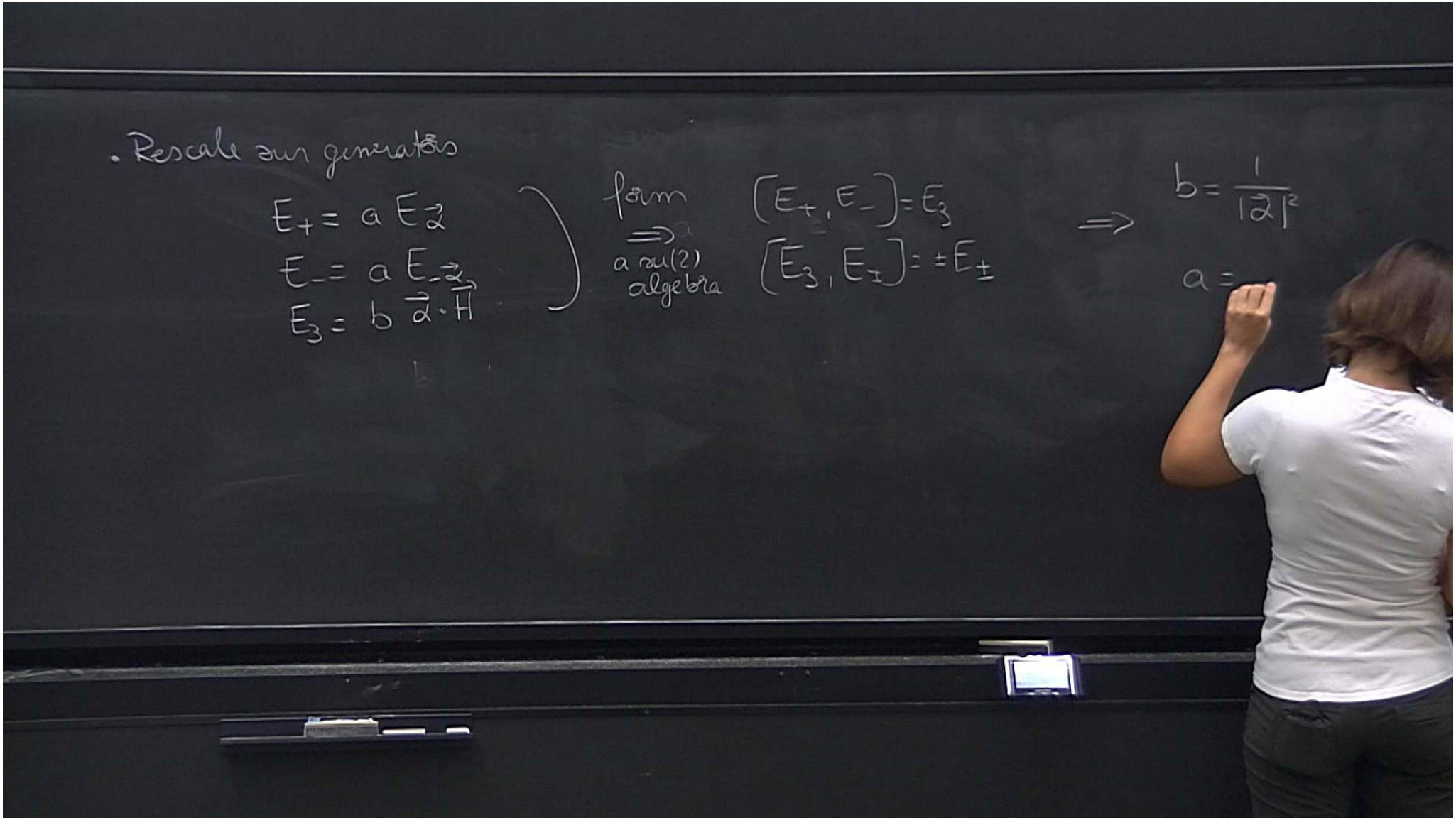
$B(x,[y,z]) = B([x,y],z)$
$Tr(E_{\vec{\alpha}} E_{\vec{\beta}}) = \delta(\vec{\alpha}+\vec{\beta})$
$Tr(E_{\vec{\alpha}} H_i) = 0$
$\vec{H} = (H_1, \dots H_r)$
$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \vec{\alpha}\cdot\vec{H}$
$[H_i, H_j] = 0$
$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$
$Tr([E_{\vec{\alpha}}, E_{-\vec{\alpha}}] H_i)$
$Tr(E_{\vec{\alpha}} E_{-\vec{\alpha}} H_i - E_{-\vec{\alpha}} E_{\vec{\alpha}} H_i)$
$Tr(E_{\vec{\alpha}} [E_{-\vec{\alpha}}, H_i])$
$\alpha_i\, Tr(E_{\vec{\alpha}} E_{-\vec{\alpha}})$

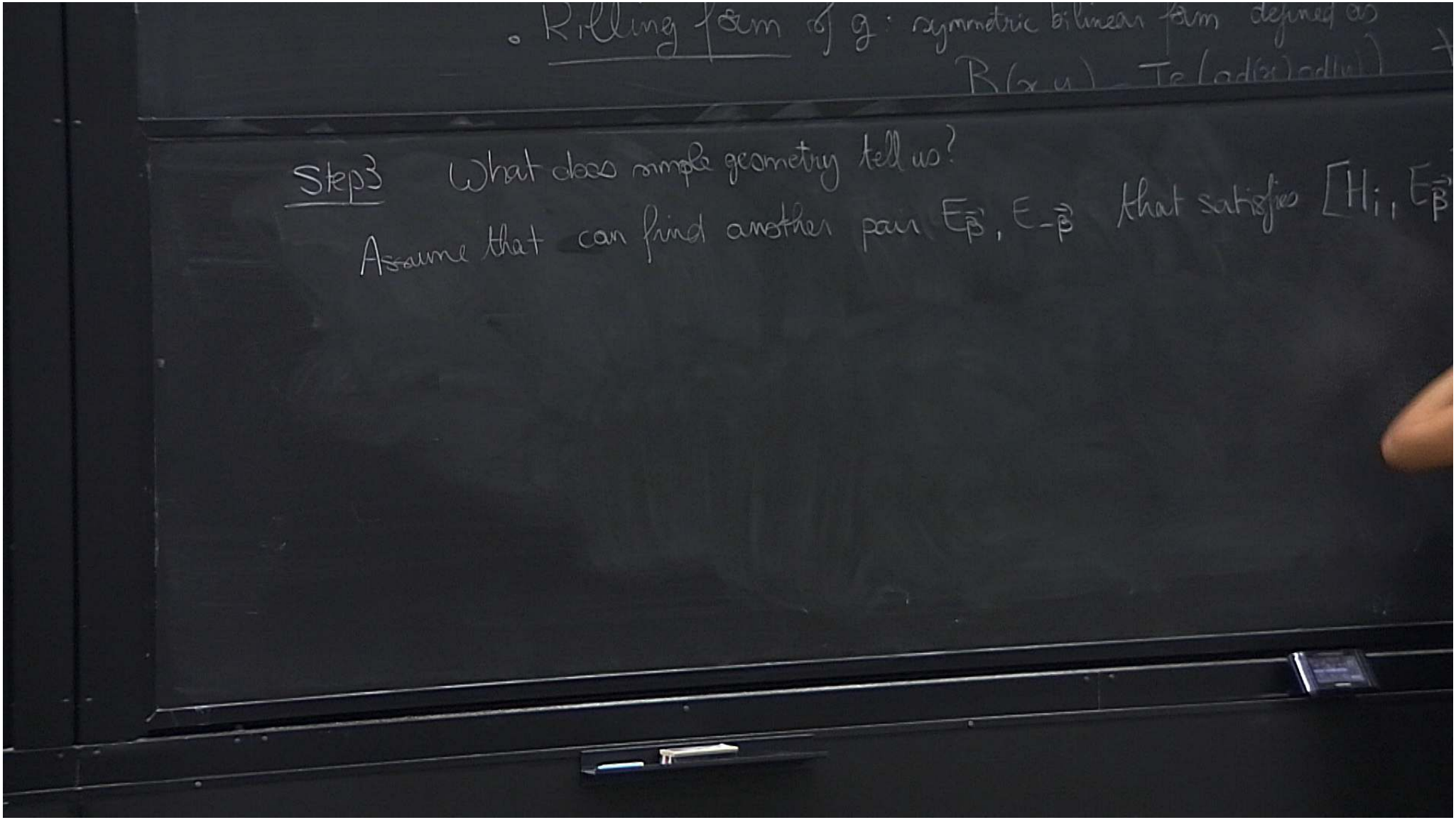

. Rescale sur generateurs
$E_+ = a E_{\vec{\alpha}}$
$E_- = a E_{-\vec{\alpha}}$
$E_3 = b\, \vec{\alpha}\cdot\vec{H}$
form
$\Rightarrow$
a su(2) algebra
$[E_+, E_-] = E_3$
$[E_3, E_\pm] = \pm E_\pm$
$\Rightarrow$
$b = \frac{1}{|\vec{\alpha}|^2}$
$a =$

$= E_3$

$= \pm E_\pm$

$\Rightarrow$ $b = \frac{1}{|\vec{\alpha}|^2}$ $\Rightarrow$ we have $E_3 = \frac{\vec{\alpha}\,\vec{H}}{|\vec{\alpha}|^2}$

$a = \frac{1}{|\vec{\alpha}|}$ $E_\pm = \frac{1}{|\vec{\alpha}|} E_{\pm\vec{\alpha}}$


• Killing form of g: symmetric bilinear form defined as
B(x,y) = Tr(ad(x)ad(y))
Step3 What does simple geometry tell us?
Assume that can find another pair $E_{\vec{\beta}}, E_{-\vec{\beta}}$ that satisfies $[H_i, E_{\vec{\beta}}$

Killing form of g: symmetric bilinear form defined as

$B(x,y) = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y)) \quad \forall x,y \in g$.

B is an invariant form on the

does simple geometry tell us?

can find another pair $E_{\vec{\beta}}, E_{-\vec{\beta}}$ that satisfies $[H_i, E_{\vec{\beta}}) = \beta_i E_{\vec{\beta}}$

hat is $[E_3, E_{\vec{\beta}}]$?

• Killing form of g: symmetric bilinear form defined as

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Step 3 What does simple geometry tell us?

Assume that can find another pair $E_{\vec{\beta}}, E_{-\vec{\beta}}$ that satisfies $[H_i, E_{\vec{\beta}}] = \beta_i E_{\vec{\beta}}$

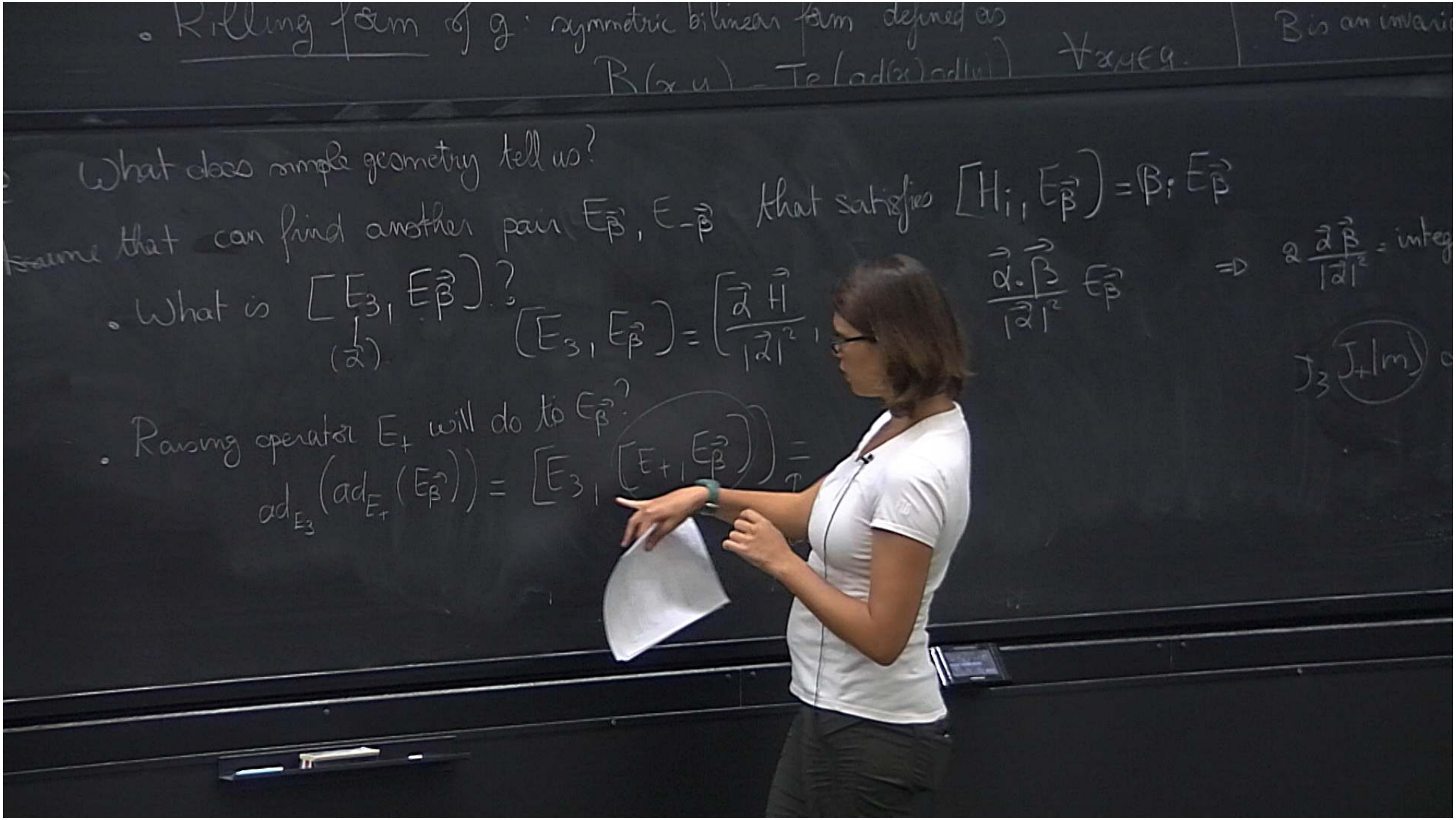
• What is $[E_3, E_{\vec{\beta}}]$? ($E_3 \to \vec{\alpha}$)

$$(E_3, E_{\vec{\beta}}) = \left(\frac{\vec{\alpha}\cdot\vec{H}}{|\vec{\alpha}|^2}, E_{\vec{\beta}}\right) = \frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} E_{\vec{\beta}} \Rightarrow 2$$

• Raising operator E_+ will do to $E_{\vec{\beta}}$?

$$\mathrm{ad}_{E_3}\left(\mathrm{ad}_{E_+}(E_{\vec{\beta}})\right) = [E_3, [E_+, E_{\vec{\beta}}]) = \text{?}$$

Jacobi identity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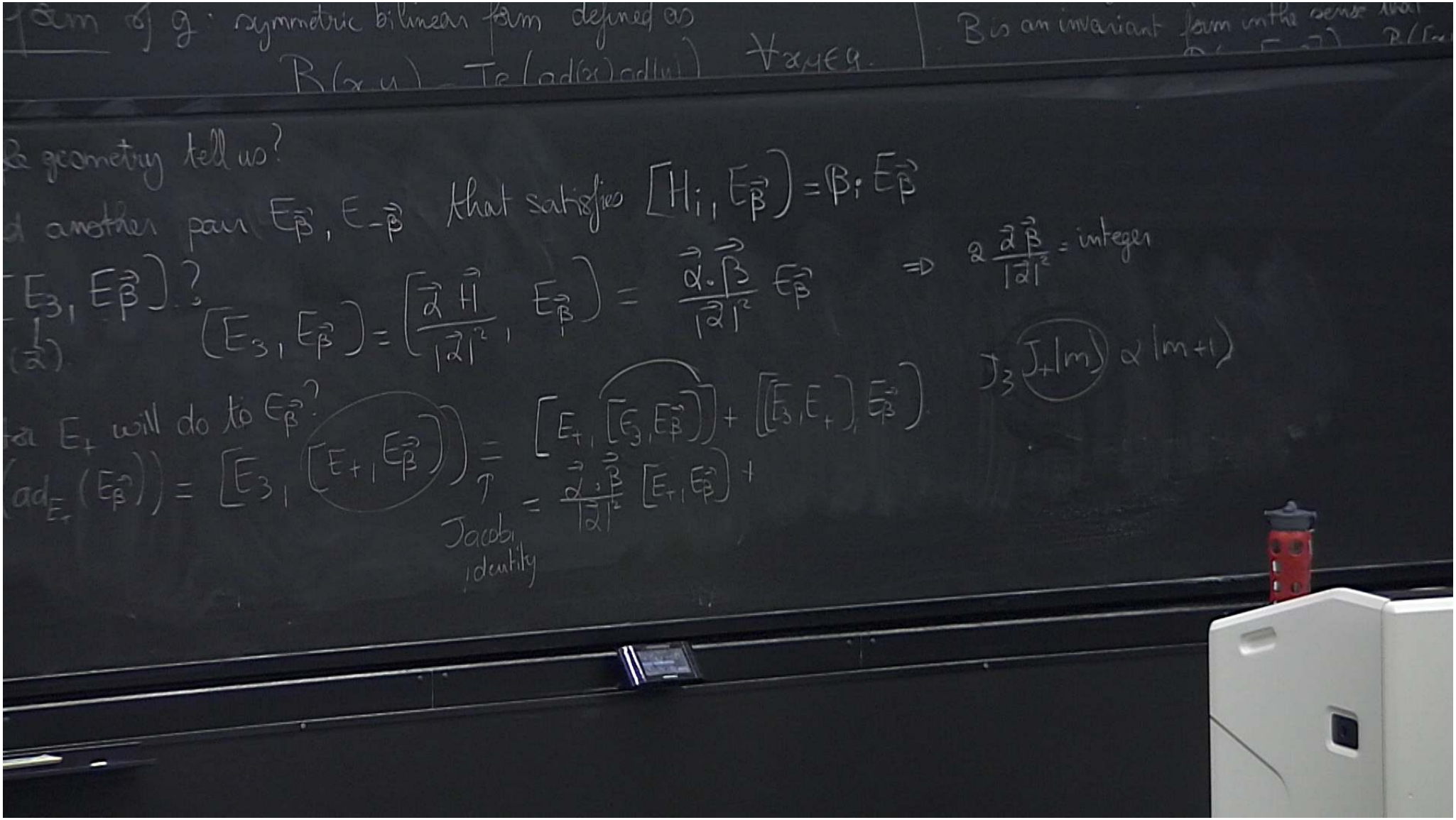
What is $[E_3, E_{\vec{\beta}}]$?
$(E_3, E_{\vec{\beta}}) = \left(\frac{\vec{\alpha}\cdot\vec{H}}{|\vec{\alpha}|^2}, \frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} E_{\vec{\beta}}\right)$ ⇒ $2\frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2}$ = integ
$J_3 J_+|m\rangle$
Raising operator E_+ will do to $E_{\vec{\beta}}$?
$ad_{E_3}(ad_{E_+}(E_{\vec{\beta}})) = [E_3, [E_+, E_{\vec{\beta}}]] =$


sum of g: symmetric bilinear form defined as
$B(x,y) = \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) \quad \forall x,y \in g$
B is an invariant form in the sense that
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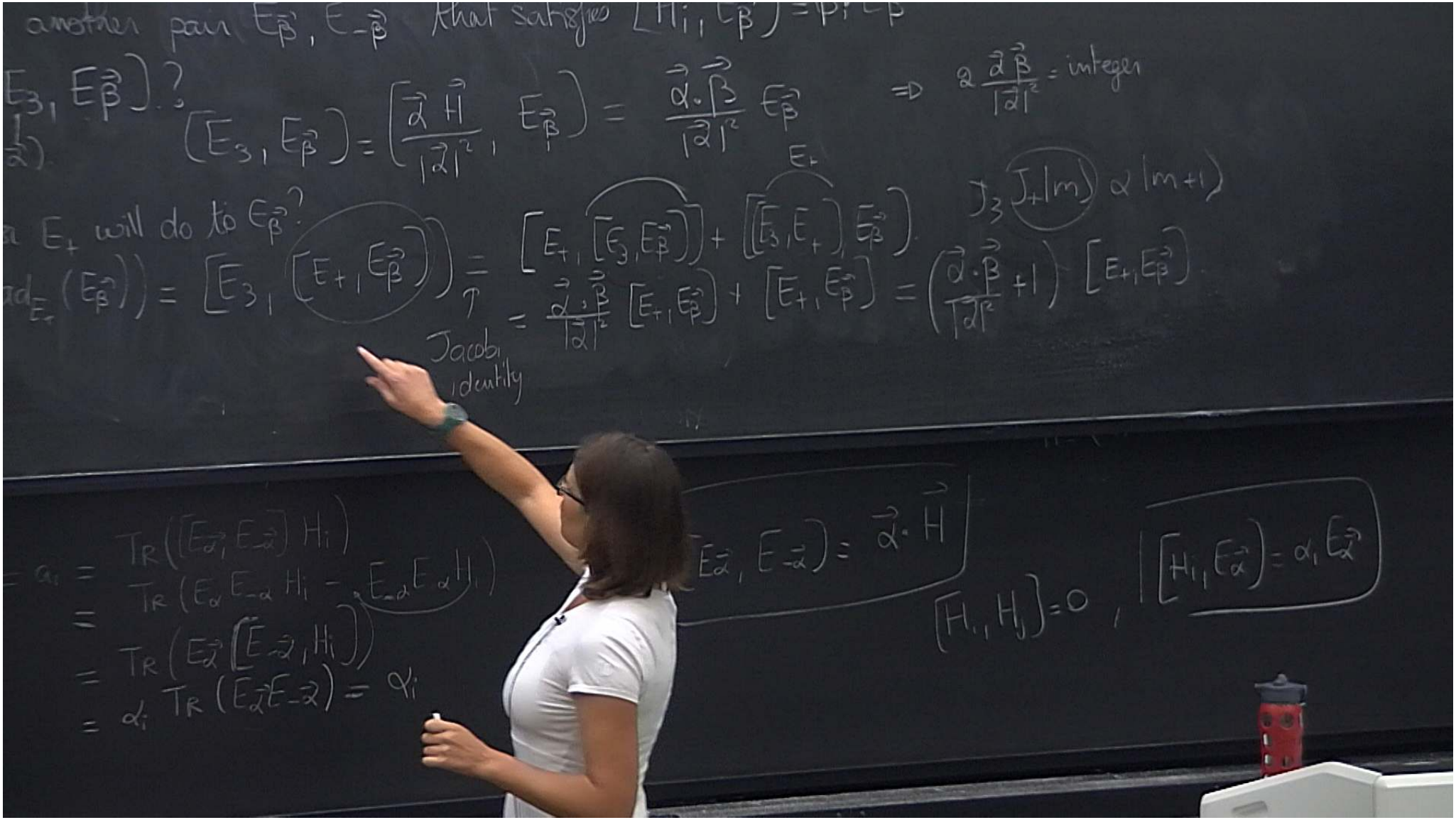
another pair $E_{\vec{\beta}}, E_{-\vec{\beta}}$ that satisfies $[H_i, E_{\vec{\beta}}] = \beta_i E_{\vec{\beta}}$
$[E_3, E_{\vec{\beta}}]$?
$(E_3, E_{\vec{\beta}}) = \left(\frac{\vec{\alpha}\cdot\vec{H}}{|\vec{\alpha}|^2}, E_{\vec{\beta}}\right) = \frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} E_{\vec{\beta}}$
$\Rightarrow 2\frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2}$ = integer
$J_3 J_+|m\rangle \propto |m+1\rangle$
E_+ will do to $E_{\vec{\beta}}$?
$\mathrm{ad}_{E_+}(E_{\vec{\beta}}) = [E_3, [E_+, E_{\vec{\beta}}]] = [E_+, [E_3, E_{\vec{\beta}}]] + [[E_3, E_+], E_{\vec{\beta}}]$
$= \frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2}[E_+, E_{\vec{\beta}}] +$
Jacobi identity


another pair $E_{\vec{\beta}}, E_{-\vec{\beta}}$ that satisfies $[H_i, E_{\vec{\beta}}] = \beta_i E_{\vec{\beta}}$
$[E_3, E_{\vec{\beta}}]$?
$(E_3, E_{\vec{\beta}}) = \left(\frac{\vec{\alpha}\cdot\vec{H}}{|\vec{\alpha}|^2}, E_{\vec{\beta}}\right) = \frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} E_{\vec{\beta}}$ $\Rightarrow 2\frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2}$ = integer
E_+ will do to $E_{\vec{\beta}}$?
$J_3 J_+|m\rangle \propto |m+1\rangle$
$ad_{E_+}(E_{\vec{\beta}}) = [E_3, [E_+, E_{\vec{\beta}}]] = [E_+, [E_3, E_{\vec{\beta}}]] + [[E_3, E_+], E_{\vec{\beta}}]$
$= \frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2}[E_+, E_{\vec{\beta}}] + [E_+, E_{\vec{\beta}}] = \left(\frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} + 1\right)[E_+, E_{\vec{\beta}}]$
Jacobi identity
$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \vec{\alpha}\cdot\vec{H}$
$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$
$[H_i, H_j] = 0$,
$\alpha_i = Tr([E_{\vec{\alpha}}, E_{-\vec{\alpha}}] H_i)$
$= Tr(E_{\vec{\alpha}} E_{-\vec{\alpha}} H_i - E_{-\vec{\alpha}} E_{\vec{\alpha}} H_i)$
$= Tr(E_{\vec{\alpha}} [E_{-\vec{\alpha}}, H_i])$
$= \alpha_i Tr(E_{\vec{\alpha}} E_{-\vec{\alpha}}) = \alpha_i$

another pair $E_{\vec{\beta}}, E_{-\vec{\beta}}$ that satisfies $[H_i, E_{\vec{\beta}}] = \beta_i E_{\vec{\beta}}$

$[E_3, E_{\vec{\beta}}]$?

$$(E_3, E_{\vec{\beta}}) = \left(\frac{\vec{\alpha}\cdot\vec{H}}{|\vec{\alpha}|^2}, E_{\vec{\beta}}\right) = \frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} E_{\vec{\beta}} \Rightarrow 2\frac{\vec{\alpha}\vec{\beta}}{|\vec{\alpha}|^2} = \text{integer}$$

E_+ will do to $E_{\vec{\beta}}$?

$$ad_{E_+}(E_{\vec{\beta}}) = [E_3, [E_+, E_{\vec{\beta}}]] = -[E_+, [E_3, E_{\vec{\beta}}]] + [[E_3, E_+], E_{\vec{\beta}}]$$

Jacobi identity

$$= \frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2}[E_+, E_{\vec{\beta}}] + [E_+, E_{\vec{\beta}}] = \left(\frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} + 1\right)[E_+, E_{\vec{\beta}}]$$

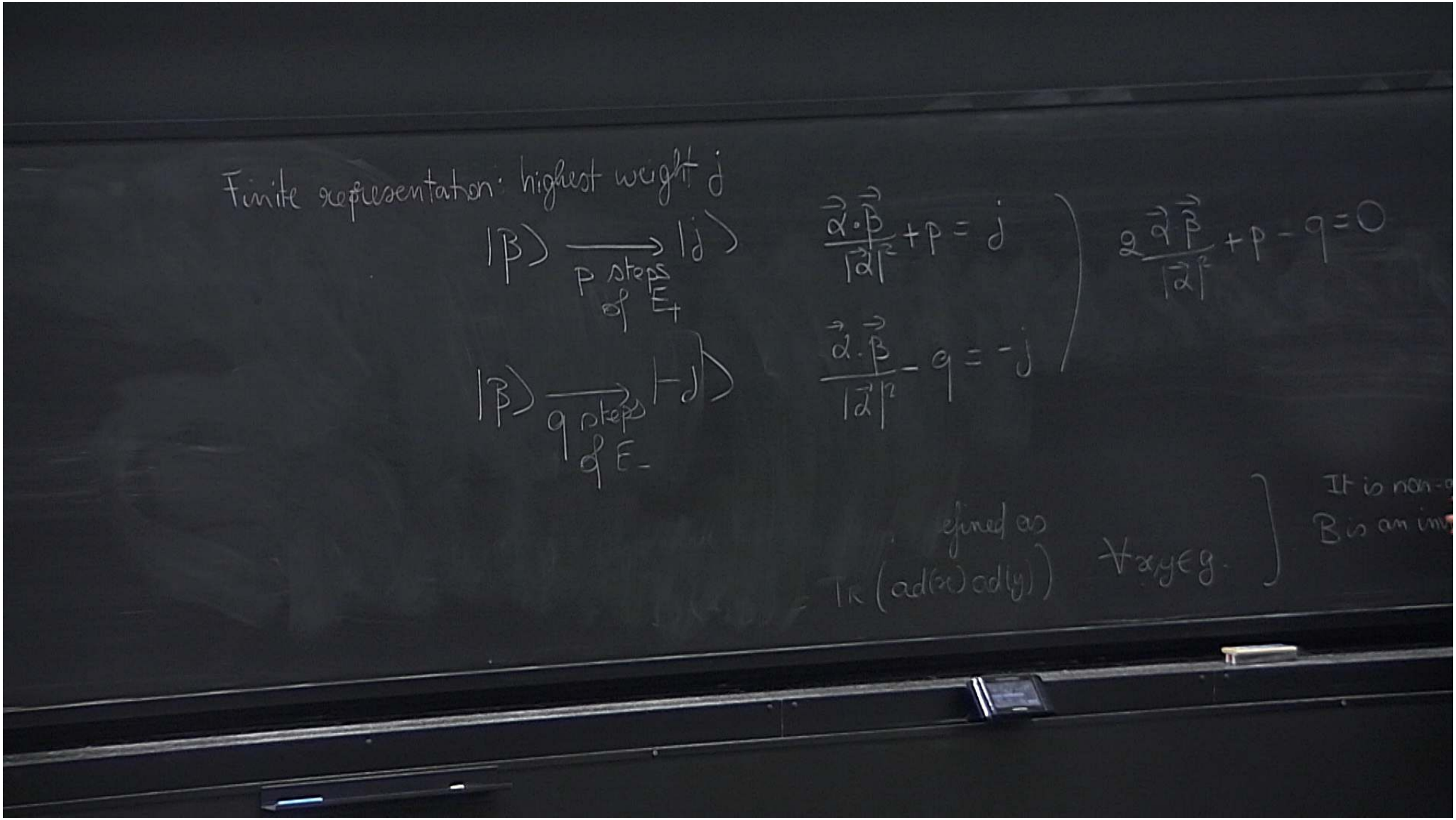
$$\frac{\vec{\alpha}\cdot(\vec{\beta}+\vec{\alpha})}{|\vec{\alpha}|^2}$$

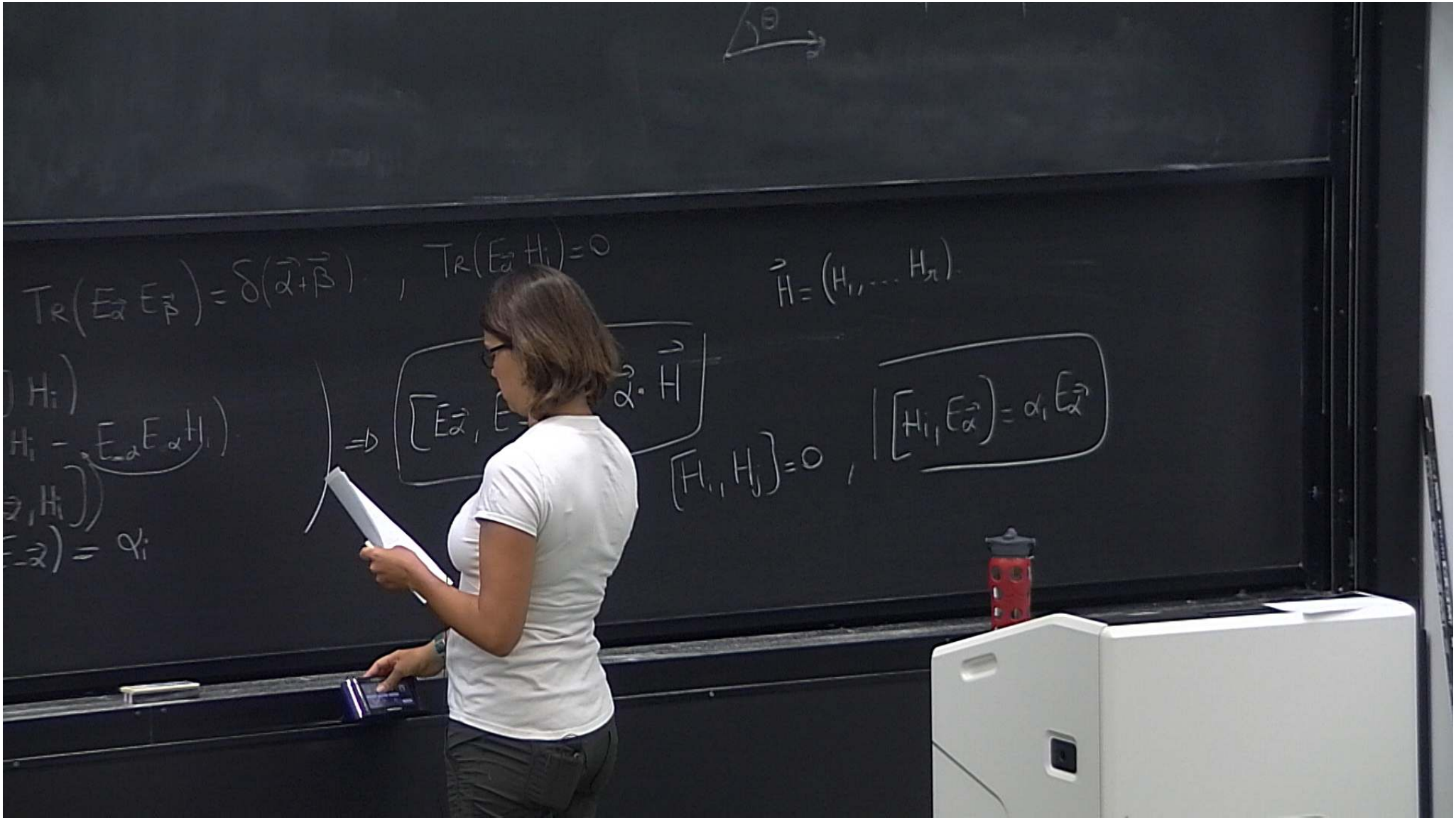
$J_3 J_+|m\rangle \propto |m+1\rangle$

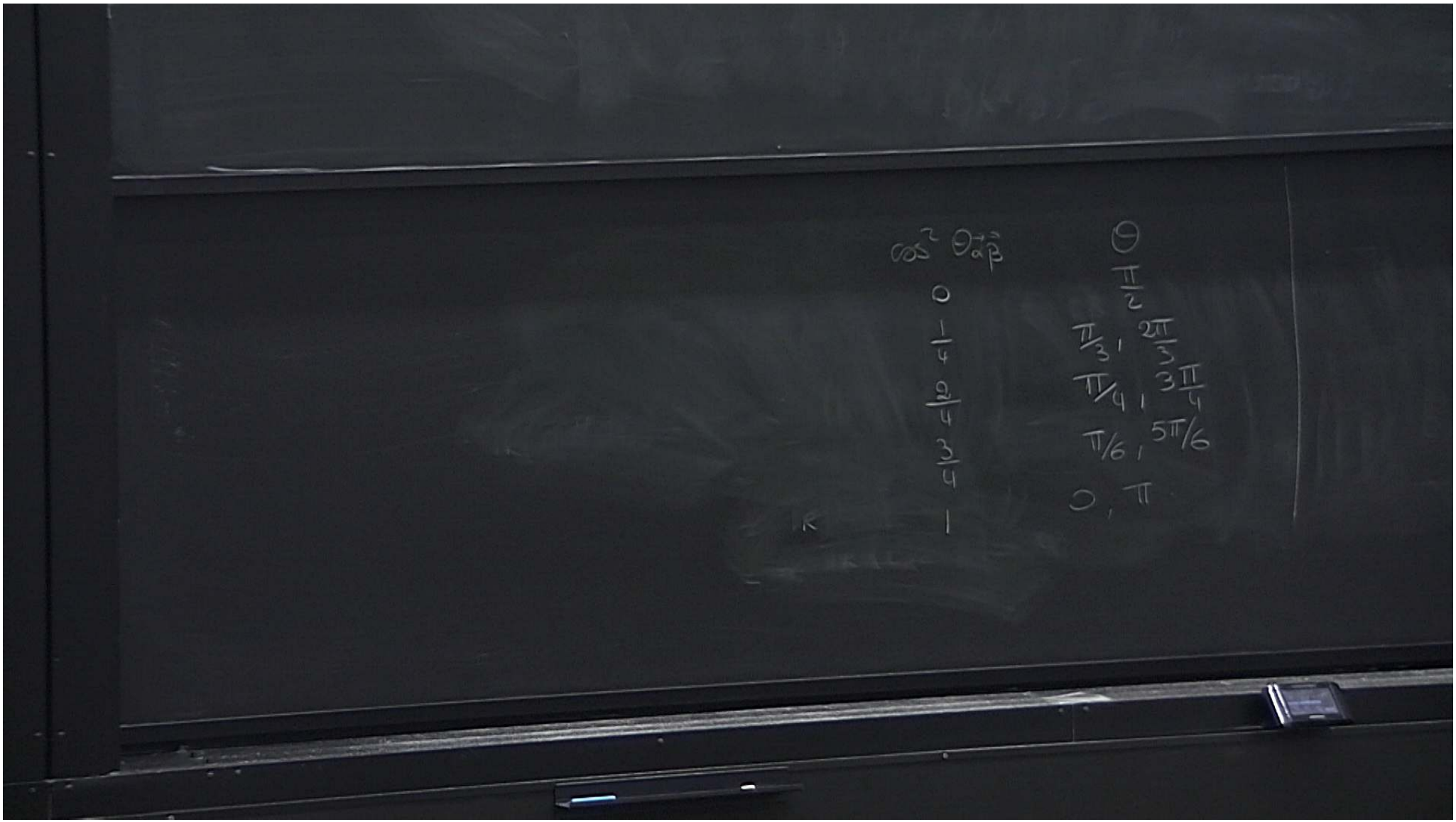
$$\alpha_i = \mathrm{Tr}([E_{\vec{\alpha}}, E_{-\vec{\alpha}}] H_i)$$
$$= \mathrm{Tr}(E_{\vec{\alpha}} E_{-\vec{\alpha}} H_i - E_{-\vec{\alpha}} E_{\vec{\alpha}} H_i)$$
$$= \mathrm{Tr}(E_{\vec{\alpha}}[E_{-\vec{\alpha}}, H_i])$$
$$= \alpha_i \mathrm{Tr}(E_{\vec{\alpha}} E_{-\vec{\alpha}}) = \alpha_i$$

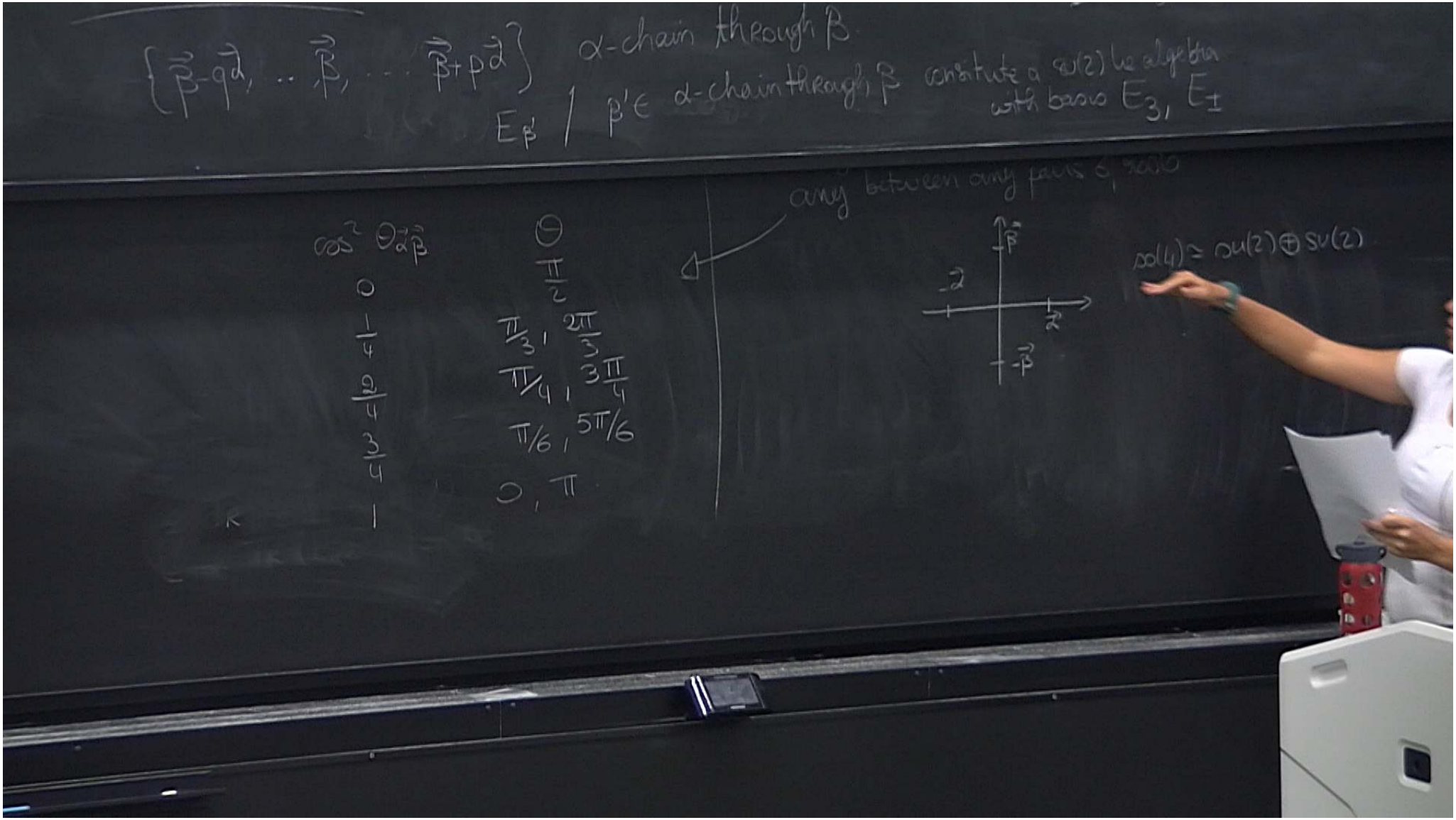
$$\Rightarrow [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \vec{\alpha}\cdot\vec{H}$$

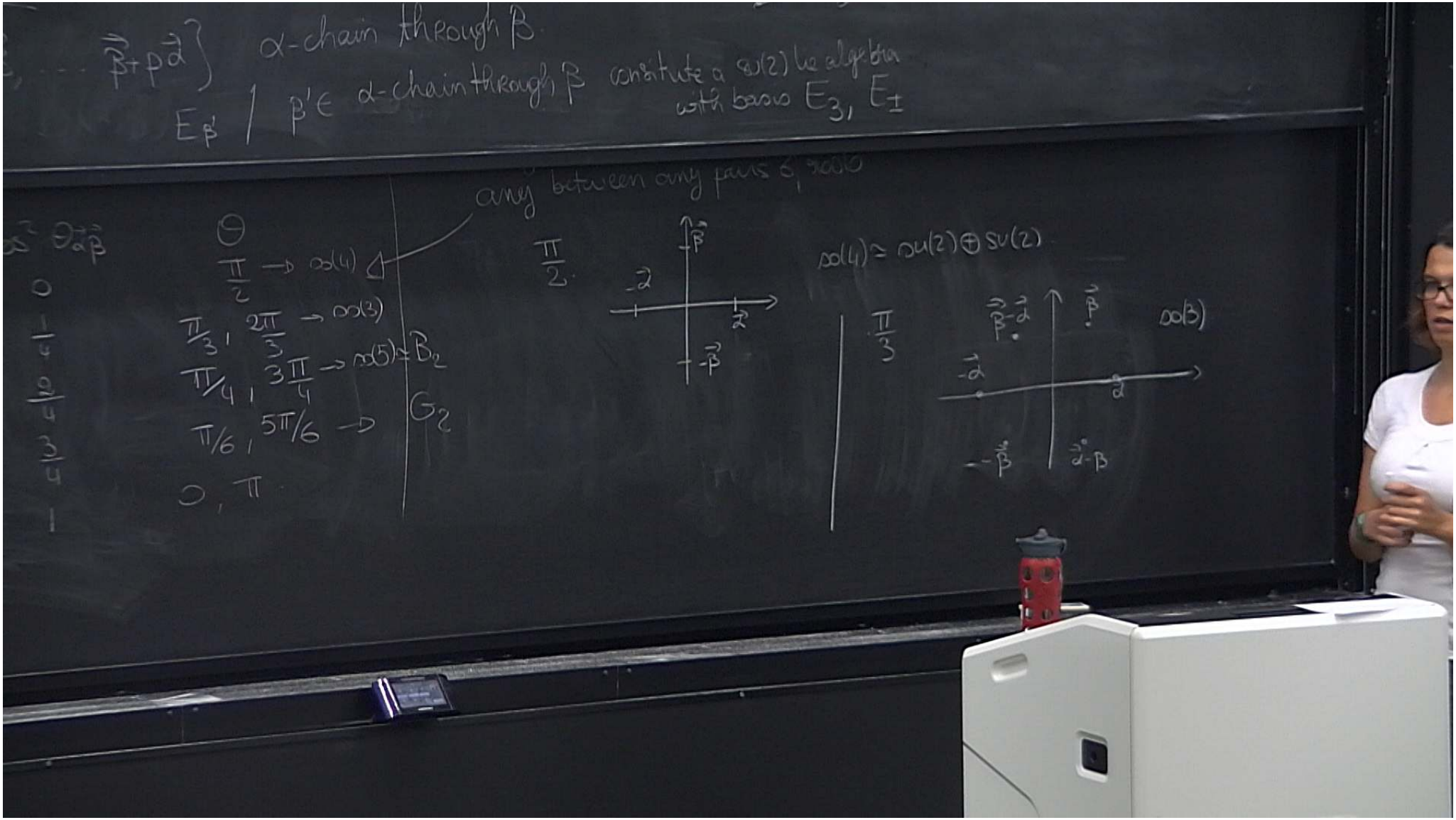
$[H_i, H_j] = 0$, $[H_i, E_{\vec{\alpha}}] = \alpha_i$


Finite representation: highest weight j
$|\beta\rangle \xrightarrow[p \text{ steps of } E_+]{} |j\rangle$
$|\beta\rangle \xrightarrow[q \text{ steps of } E_-]{} |-j\rangle$
$\frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} + p = j$
$\frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} - q = -j$
$2\frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} + p - q = 0$
$\forall x, y \in g.$
It is non-
B is an in


$\vec{H} = (H_1, \ldots H_r)$
$[H_i, H_j] = 0$
$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$


$\cos^2\theta_{\vec{\alpha}\vec{\beta}}$
θ
0
$\frac{\pi}{2}$
$\frac{1}{4}$
$\pi/3,\ \frac{2\pi}{3}$
$\frac{2}{4}$
$\pi/4,\ \frac{3\pi}{4}$
$\frac{3}{4}$
$\pi/6,\ 5\pi/6$
1
$0,\ \pi$


$\{\vec{\beta}-q\vec{\alpha}, \ldots \vec{\beta}, \ldots \vec{\beta}+p\vec{\alpha}\}$ α-chain through β
$E_{\beta'}$ / $\beta' \in$ α-chain through β constitute a $su(2)$ lie algebra with basis E_3, $E_\pm$
$\cos^2\theta = \frac{\vec{\alpha}\cdot\vec{\beta}}{\alpha^2}$
0
$\frac{1}{4}$
$\frac{2}{4}$
$\frac{3}{4}$
1
θ
$\frac{\pi}{2}$
$\pi/3$, $2\pi/3$
$\pi/4$, $3\pi/4$
$\pi/6$, $5\pi/6$
0, π
angle between any pairs of roots
$\vec{\beta}$
-2
2
$-\vec{\beta}$
$so(4) \cong su(2) \oplus su(2)$


$\vec{\beta}+p\vec{\alpha}$ } α-chain through β
$E_{\beta'}$ / $\beta' \in$ α-chain through β constitute a su(2) lie algebra with basis $E_3, E_\pm$
angle between any pairs of roots
$\theta_{\vec{\alpha}\vec{\beta}}$
0
1/4
2/4
3/4
1
θ
π/2 → so(4)
π/3, 2π/3 → so(3)
π/4, 3π/4 → so(5) ≅ B_2
π/6, 5π/6 → G_2
0, π
π/2
$\vec{\beta}$
$-\vec{\alpha}$
$\vec{\alpha}$
$-\vec{\beta}$
so(4) ≅ su(2) ⊕ su(2)
π/3
so(3)
$\vec{\beta}-\vec{\alpha}$
$\vec{\beta}$
$-\vec{\alpha}$
$\vec{\alpha}$
$-\vec{\beta}$
$\vec{\alpha}-\vec{\beta}$