

Title: PSI 17/18 Front End - Lie Groups and Lie Algebras (Dupuis)

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Abstract:

- I. Intro
- II. Physical motivations
- III. A first catalogue of Lie groups
- IV. Lie algebras and Lie groups
- V. $SU(2)$



II. Physical motivations

- QM

- $|\psi\rangle$

- physical observable $A = A^\dagger$

- $P(|\psi_1\rangle \rightarrow |\psi_2\rangle) = |\langle \psi_1 | \psi_2 \rangle|^2$

$$\begin{array}{l} |\Psi_1\rangle \text{ and } |\Psi_2\rangle \\ |\Psi'_1\rangle \text{ and } |\Psi'_2\rangle \end{array} \quad \rightarrow \quad P(|\Psi_1\rangle \rightarrow |\Psi_2\rangle) = P(|\Psi'_1\rangle \rightarrow |\Psi'_2\rangle)$$

Wigner 30's: $U / |\Psi'\rangle = U|\Psi\rangle$.

$\rightarrow$ unitary and linear $\xleftarrow{\text{id.}}$
 $\rightarrow$ anti-unitary and anti-linear

Unitary and linear

$$\langle U\Phi | U\Psi \rangle = \langle \Phi | \Psi \rangle$$

$$U(a|\Psi\rangle + b|\Phi\rangle) = aU|\Psi\rangle + bU|\Phi\rangle$$

$$\rightarrow \|T_2\| = |\langle \psi_1 | T_2 \rangle|$$

Unitary and linear

$$\cdot \langle U\phi | U\psi \rangle = \langle \phi | \psi \rangle$$

$$\cdot U(a|\psi\rangle + b|\phi\rangle) = aU|\psi\rangle + bU|\phi\rangle$$

$$|\langle \psi_1 | \psi_2 \rangle|^2 = |\langle U\psi_1 | U\psi_2 \rangle|^2$$

$$|\langle \psi_1 | U^\dagger U | \psi_2 \rangle|^2$$

$$\Rightarrow U^\dagger U = \mathbb{1} \Leftrightarrow U^\dagger = U^{-1}$$

$$\begin{aligned} & \langle U\phi | U\psi \rangle = \langle \phi | \psi \rangle \\ & U(a|\psi\rangle + b|\phi\rangle) = aU|\psi\rangle + bU|\phi\rangle \end{aligned}$$

$$|\langle \psi_1 | \psi_2 \rangle|^2 = \frac{|\langle U\psi_1 | U\psi_2 \rangle|^2}{|\langle \psi_1 | U^\dagger U | \psi_2 \rangle|^2} \Rightarrow U^\dagger U = \mathbb{1} \Leftrightarrow U^{-1} = U^\dagger$$

• Composition of operators
3 observers

$$\begin{aligned} |\psi'\rangle &= U_1 |\psi\rangle \\ |\psi''\rangle &= U_2 |\psi'\rangle \end{aligned} \quad \left. \vphantom{\begin{aligned} |\psi'\rangle &= U_1 |\psi\rangle \\ |\psi''\rangle &= U_2 |\psi'\rangle \end{aligned}} \right) |\psi''\rangle = U_2 U_1 |\psi\rangle \quad \text{so we have } U_3 = U_2 U_1$$

• Ex

Galilean transformations

- Galilean boost $\vec{x}'(t) \rightarrow \vec{x} + \vec{v}t \rightarrow$ group? yes
- Translation in space
- Rotation

• Ex

Galilean transformations

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$$\begin{pmatrix} 0 & 0 \\ \vec{v} & 1 \end{pmatrix} \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$$

• Special Relativity - D 3+1 D Minkowski space

$$V = (V^0, \vec{V}), \quad W = (W^0, \vec{W})$$

$$(V, W) = V^0 W^0 - \vec{V} \cdot \vec{W} = \eta_{\mu\nu} V^\mu W^\nu$$

$\hookrightarrow \eta_{\mu\nu} =$

Galilean transformations

- Galilean boost $\vec{x}'(t) \rightarrow \vec{x} + \vec{v}t \rightarrow$
- Translation in space
- Rotation

- Special Relativity - D 3+1 D Minkowski space

$$V = (v^0, \vec{V}), \quad W = (w^0, \vec{W})$$

group? yes $\begin{pmatrix} \vec{v} & 1 \end{pmatrix} \begin{pmatrix} \vec{w} \end{pmatrix}$

$$(v, w) = v^0 w^0 - \vec{v} \cdot \vec{w} = \eta_{\mu\nu} v^\mu w^\nu$$

$$\rightarrow \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1).$$

$\rightarrow$ invariant Lorentz group
 $SO(1,3)$.

Galilean transformations

- Galilean boost $\vec{x}'(t) \rightarrow \vec{x} + \vec{v}t \rightarrow$ group? yes
- Translation in space
- Rotation

$$\begin{pmatrix} 0 & 0 \\ \vec{v} & 1 \end{pmatrix} \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$$

relativity

- D 3+1 D Minkowski space

$$V = (V^0, \vec{V}), \quad W = (W^0, \vec{W})$$

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$\downarrow$
 η

Lorentz + translations = Poincaré group (see tutorial)

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• Gauge symmetries

- Electromagnetism
 - Weak
 - Strong
 - Gravity : General Relativity.
- } Yang-Mills theory.

translations = Poincaré group (see tutorial).

Lie groups : Lie groups are groups which are described by a set of continuous parameters.

Lie Group G is a Lie Group if it is a manifold and the inverse and product operators

- $\cdot^{-1} : G \rightarrow G$
- $\cdot : G \times G \rightarrow G$

are smooth maps.

ex. $SU(2)$, $U(1)$
 S^3 , S^1

Galileo = Poincaré group (see tutorial).

Lie groups : Lie groups are groups which are described by a set
of continuous parameters.
 $g(\varphi) \rightarrow g(\varphi=0) = 1$

Lie Group G is a Lie Group if it is a manifold
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IV Catalogue

$M_n(K)$ the set of $n \times n$ matrices with real & complex entries

$\hookrightarrow K = \mathbb{C}, \text{ or } K = \mathbb{R}$

$$GL(n, K) = \{ M \in M_n \mid \det(M) \neq 0 \}$$

General linear group

SL

complex entries

$$SL(n, K) = \{ M \in M_n / \det M = 1 \}$$

Special linear

$K = \mathbb{R}$
gp of all volume-preserving
linear transformations
of $\mathbb{R}^n$

complex entries

$$SL(n, K) = \{ M \in M_n / \det M = 1 \}$$

Special linear

$K = \mathbb{R}$
gp of all volume-preserving
linear transformation
of $\mathbb{R}^n$

$O(p, q)$ orthogonal group

$$O(n) = \{ T \in M_n(\mathbb{R}) / T^t T = T T^t = I \}$$

L_0 (Euclidean metric)

$SO(p, q)$ special orthogonal groups

$SO(n)$

→ $SO(3)$ rotations in 3d Euclidean space.

→ $SO(3, 1)$: Lorentz group

L_0 (Euclidean metric)

$SO(p, q)$ special orthogonal groups
 $SO(n)$

→ $SO(3)$ rotations in 3d Euclidean space

→ $SO(3, 1)$: Lorentz group 4x4 matrices preserving the standard Minkowski metric
spatial rotations + boosts

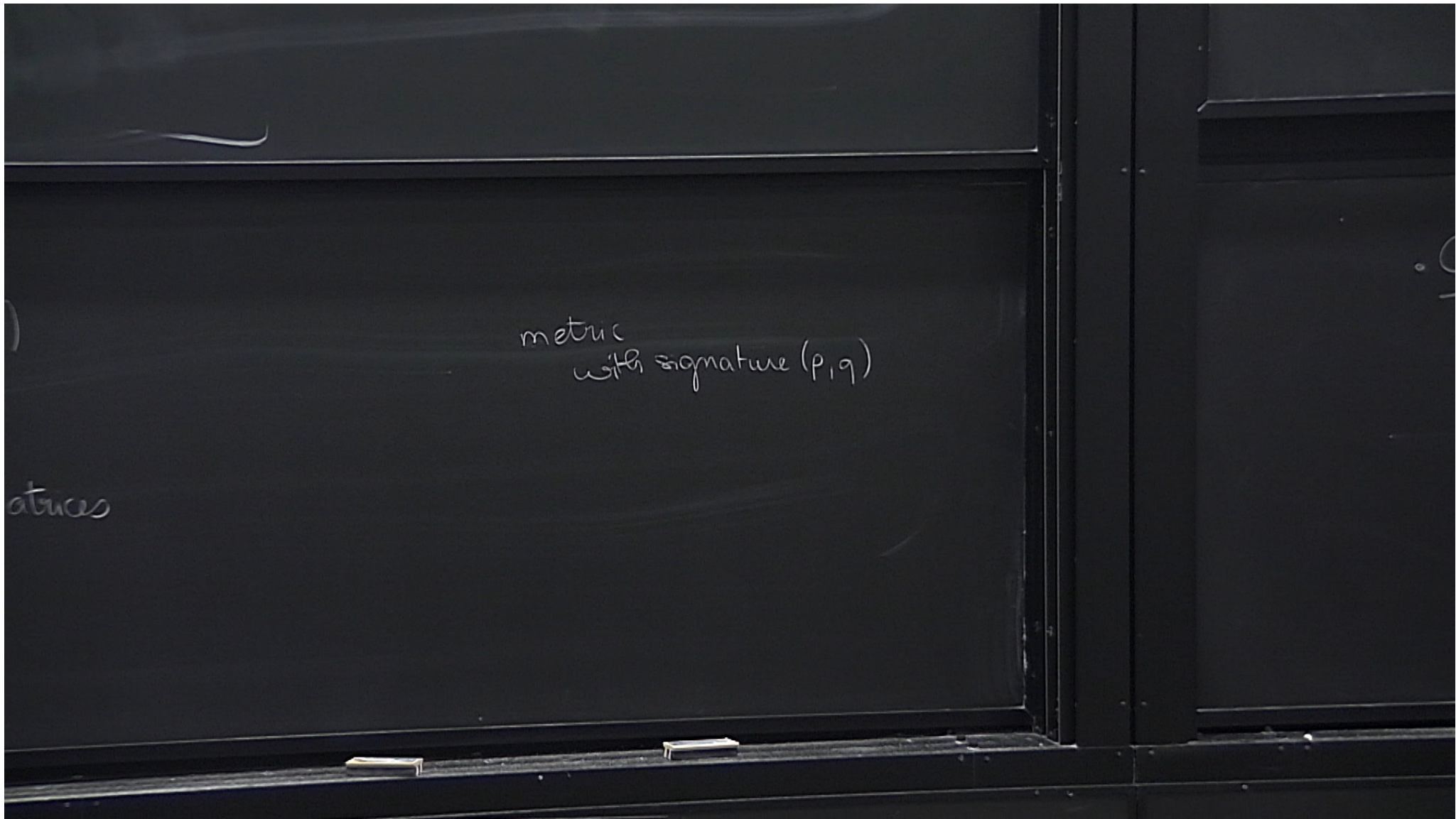
• Poincaré group

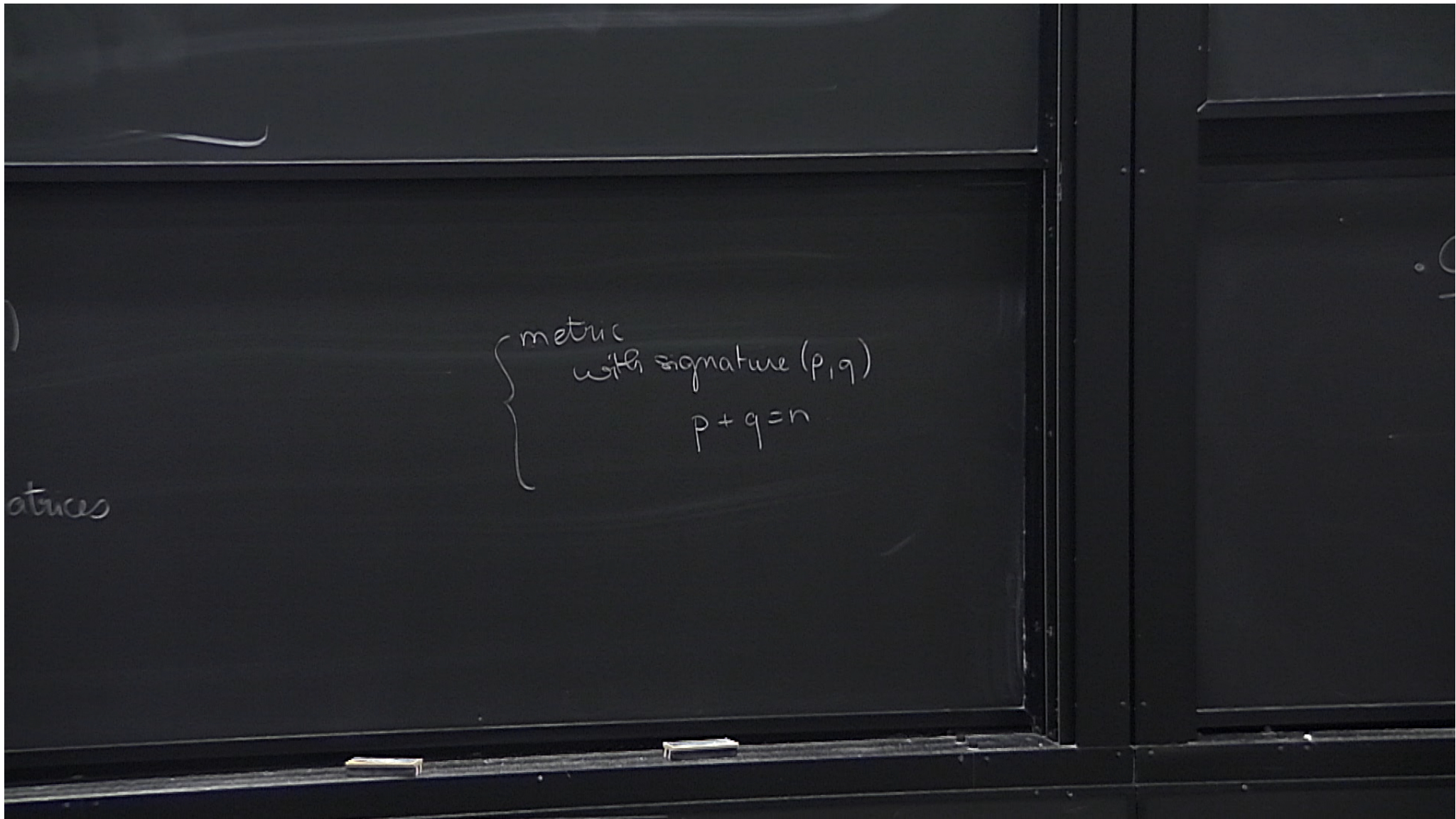
↳ preserve spacetime intervals. (Minkowski space)

↳ translation, Lorentz transformation

• $U(n)$: unitary group = gp of unitary $n \times n$ complex matrices
↳ preserves inner product on $\mathbb{C}^n$

$SU(n)$ special unitary group
+ $\det = 1$





V. Lie algebras and Lie groups.

• Exercise Matrices for 3d rotations by an angle θ (counterclockwise) about
x-axis
y-axis
z-axis

• Lie groups Lie groups are groups

→ What are the associated infinitesimal rotations ($\Theta \rightarrow 0$) J_x, J_y, J_z at first order in Θ

→ $g_z(\Theta)$, rotation by an angle Θ about the z-axis can be obtained by exponentiating J_z . $g_z(\Theta) \equiv \exp(\Theta J_z)$

Exponential of a $n \times n$ complex matrices M

$$\exp(M) = 1 + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots$$

Lie algebras and Lie groups.

• Exercise Matrices for 3d rotations by an angle θ (counterclockwise) about x , y , z .

$$g_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad g_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad g_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of θ about the z -axis can be obtained

$$J_z(\theta) = \begin{pmatrix} 0 & \cos\theta - \sin\theta & 0 \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \\ 0 & 0 & 1 \end{pmatrix}, \quad J_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Differentiate
at $\theta=0$

↳ What are the associated infinitesimal rotations ($\theta \rightarrow 0$) J_x, J_y, J_z are

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3)$$

↳ $g_z(\theta)$, rotation by an angle θ about the z -axis can be obtained by ex

Exponential of a $n \times n$ complex matrices M

$$\exp(M) = 1 + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots$$

ans $(\Theta \rightarrow 0)$ J_x, J_y, J_z at first order in Θ

$$J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \text{Lie algebra of } SO(3) = \mathfrak{so}(3)$$

g -axis can be obtained by exponentiating J_z . $g_z(\Theta) \equiv$

Exponential of a $n \times n$ complex matrices M

$$\exp(M) = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots$$

$$J_3, J_3^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, J_3^3 = -J_3, J_3^4 = -J_3^2 \text{ and so on } \dots$$

$$\exp(\theta J_3) = I + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \dots\right)}_{\sin \theta} J_3 + \underbrace{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \dots\right)}_{1 - \cos \theta} J_3^2 = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

{ metric
with signature (p, q)
 $p + q = n$

$SU(2)$
 $SO(3)$

Lie algebra

The Lie algebra of G, g , is the tangent space of the identity component of G

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Def Algebra

An algebra A is a vector space endowed with an additional binary op. $*$: A

ent of G

binary op. $*$: $A \times A \rightarrow A$ which is bilinear.

1a

$\mathfrak{g}$ is the tangent space of the identity component of G

A is a vector space endowed with an additional binary op. $*$: $A \times A \rightarrow A$ which is

$$(x+y) * z = x * y + y * z \quad \forall x, y, z \in A$$

$$x * (y+z) = x * y + x * z$$

$$(\lambda x) * (\eta y) = (\lambda \eta) x * y \quad \forall x, y \in A, \lambda, \eta \in K \quad (\mathbb{R}, \mathbb{C})$$

metric
with sig

$$\begin{aligned}
 (x+y)*z &= x*y + y*z \quad \forall x, y, z \in A \\
 x*(y+z) &= x*y + x*z \\
 (\lambda x)*(\eta y) &= (\lambda\eta) x*y \quad \forall x, y \in A, \lambda, \eta \in K \quad (\mathbb{R}, \mathbb{C})
 \end{aligned}$$

Def: (Lie algebra)

↗ Lie bracket

A Lie algebra $\mathfrak{g}$ is an algebra for which the bilinear op, denoted $[\cdot, \cdot]$, is neither commutative or associative, but rather possesses two special properties

$$\begin{aligned}
 \bullet [x, y] &= -[y, x] \quad \forall x, y \in \mathfrak{g} \\
 \bullet [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \quad (\text{Jacobi identity})
 \end{aligned}$$