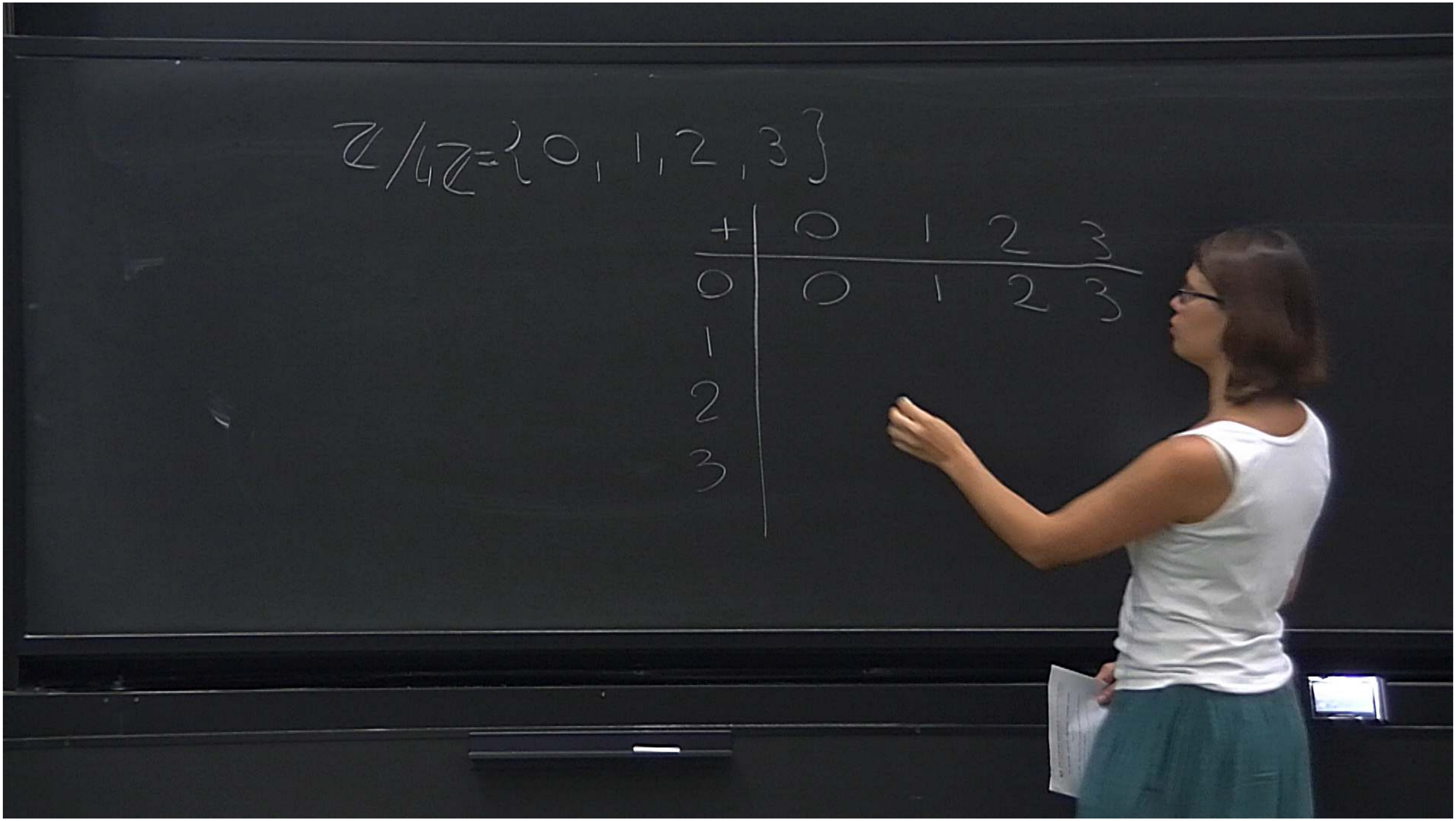
Title: PSI 17/18 Front End - Lie Groups and Lie Algebras (Dupuis)

Date: Aug 09, 2017 10:30 AM

URL: http://pirsa.org/17080050

Abstract:


Z/4Z = {0, 1, 2, 3}
+ | 0 1 2 3
0 | 0 1 2 3
1
2
3

$\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\circ$	e	a	b	c
e	e	a	b	c

$+ \to \circ$

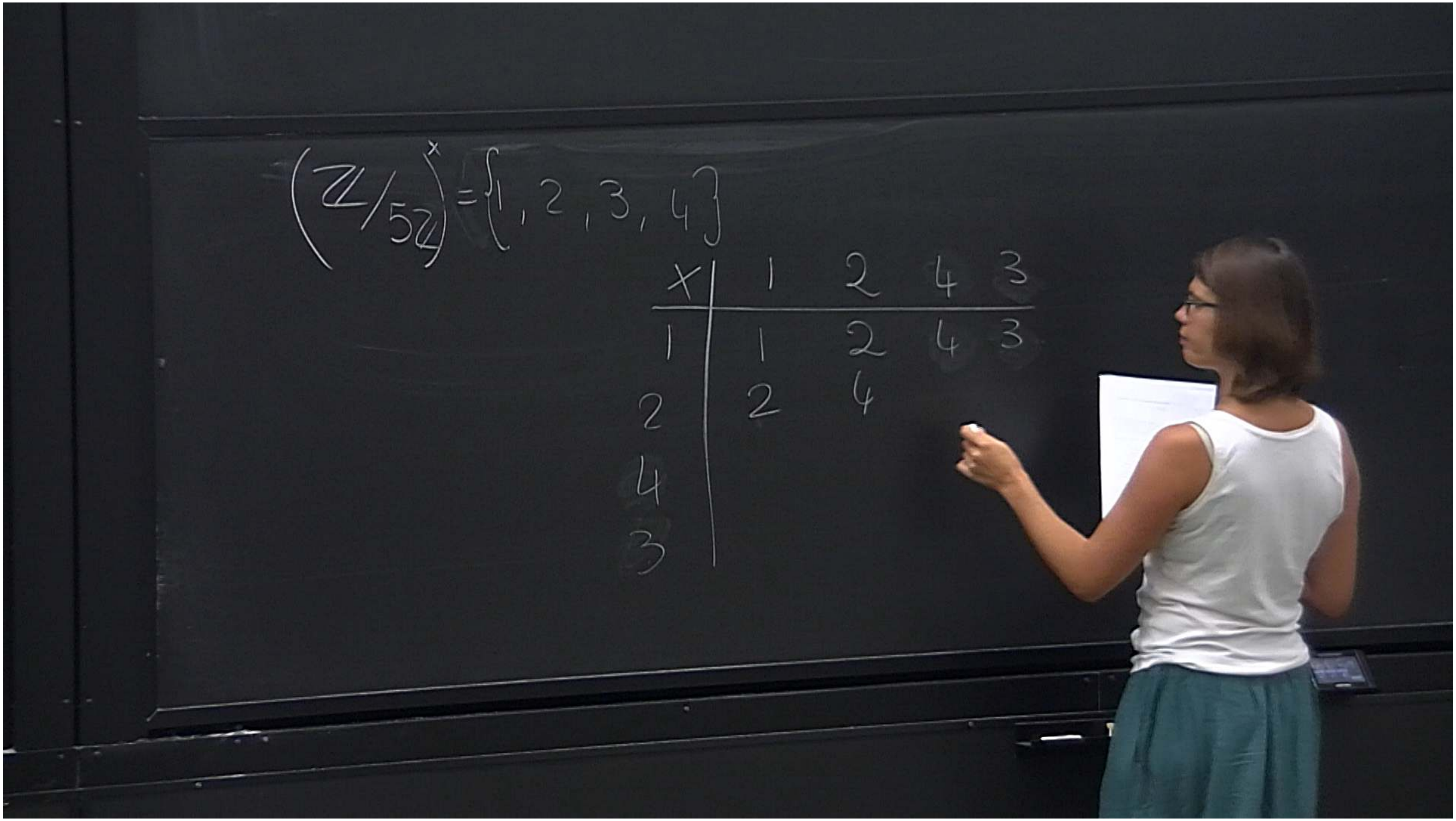
$0 \to e$

$1 \to a$

$2 \to b$

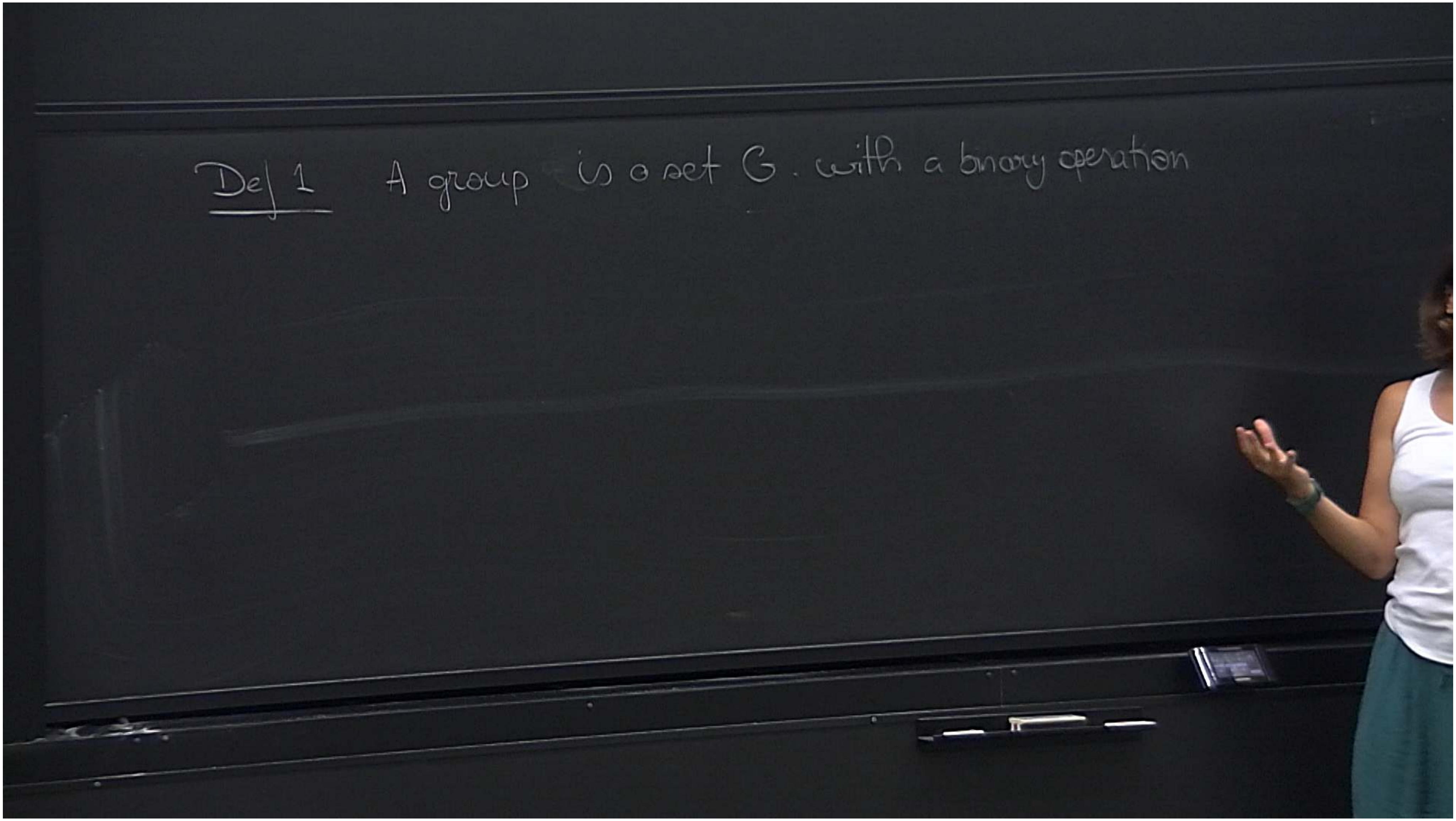
$3 \to c$

$\circ$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b


(Z/5Z)^× = {1, 2, 3, 4}
× | 1 2 4 3
1 | 1 2 4 3
2 | 2 4
4
3

$$\left(\mathbb{Z}/5\mathbb{Z}\right)^{\times}=\{1,2,3,4\}$$

$\times$	1	2	4	3
1	1	2	4	3
2	2	4	3	1
4	4	3	1	2
3	3	1	2	4


Def 1 A group is a set G. with a binary operation

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- closed under the binary op. $\forall g_1, g_2 \in G, \; g_1 \circ g_2 \in G$.
- $\circ$ associative $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.
- identity, $\exists e \,/\, \forall g \in G \quad e \circ g = g \circ e = g$

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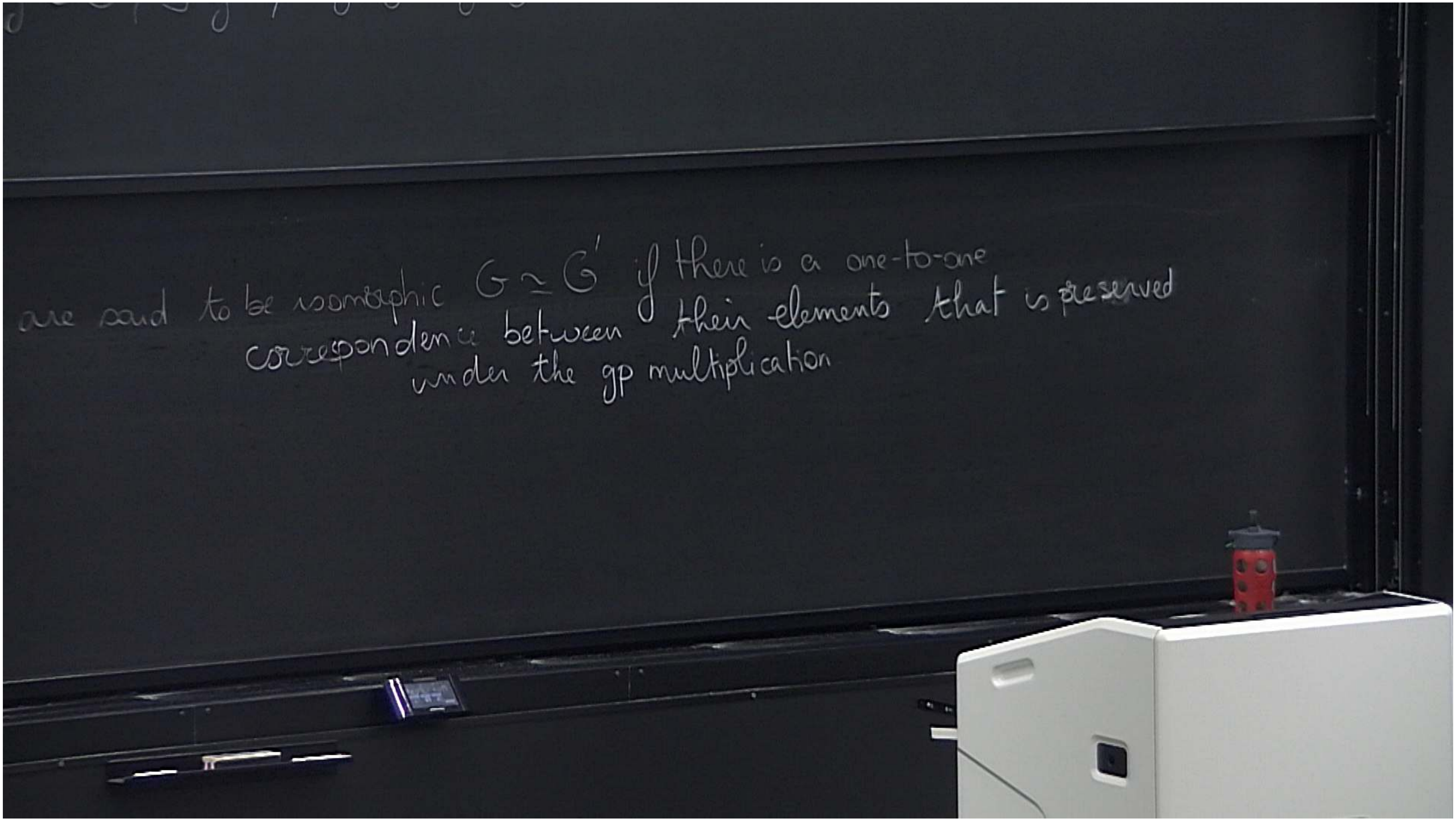
- associative $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$
- identity: $\exists e \,/\, \forall g \in G \quad e \circ g = g \circ e = g$
- inverse: $\forall g \in G, \exists g^{-1} \,/\, g \circ g^{-1} = g^{-1} \circ g = e$

• inverse : $\forall g \in G, \exists g^{-1}$

Def 2 (Isomorphic)

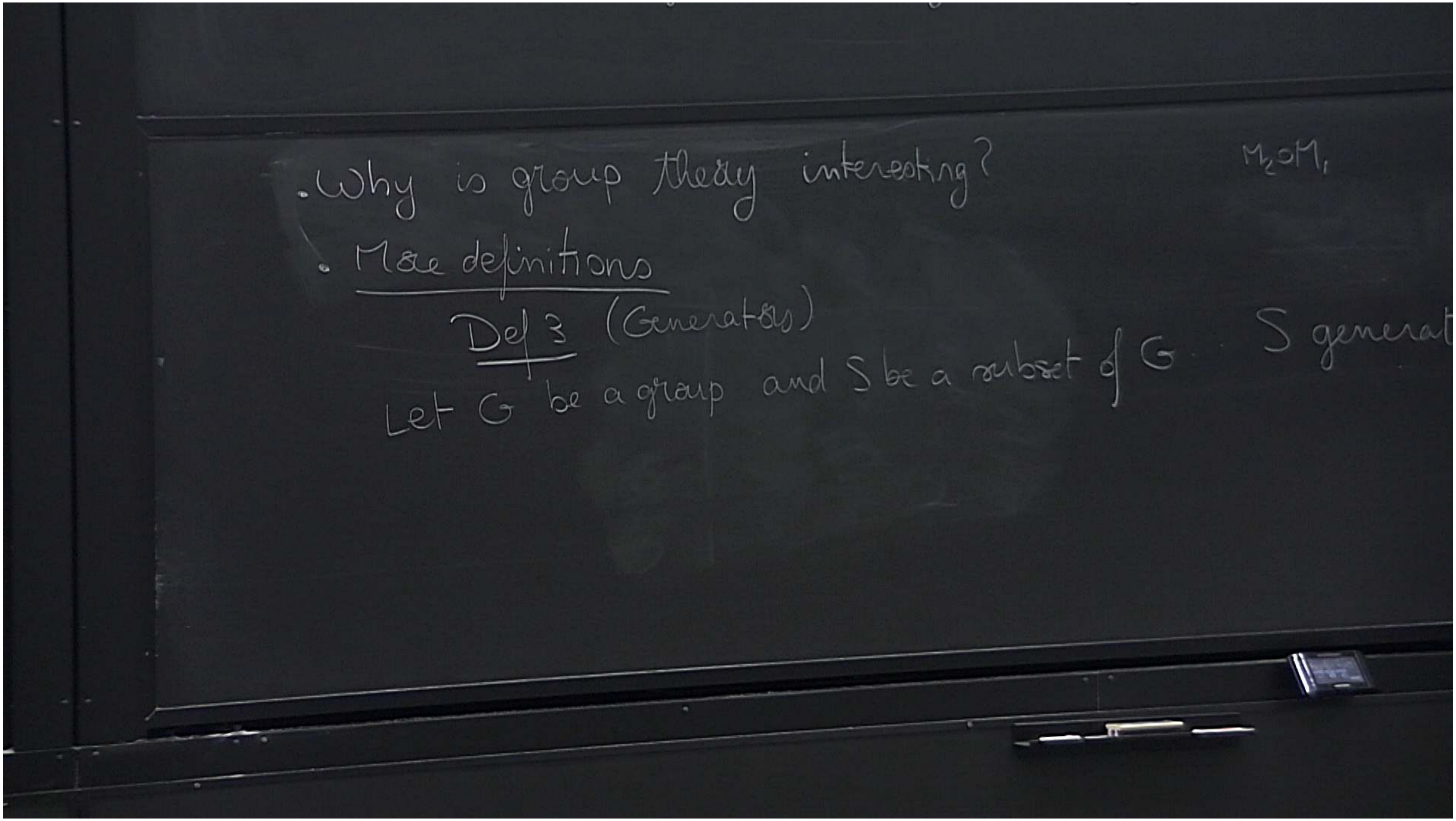
Two groups G and G' are said to be isomorphic $G \simeq G'$ if there is a

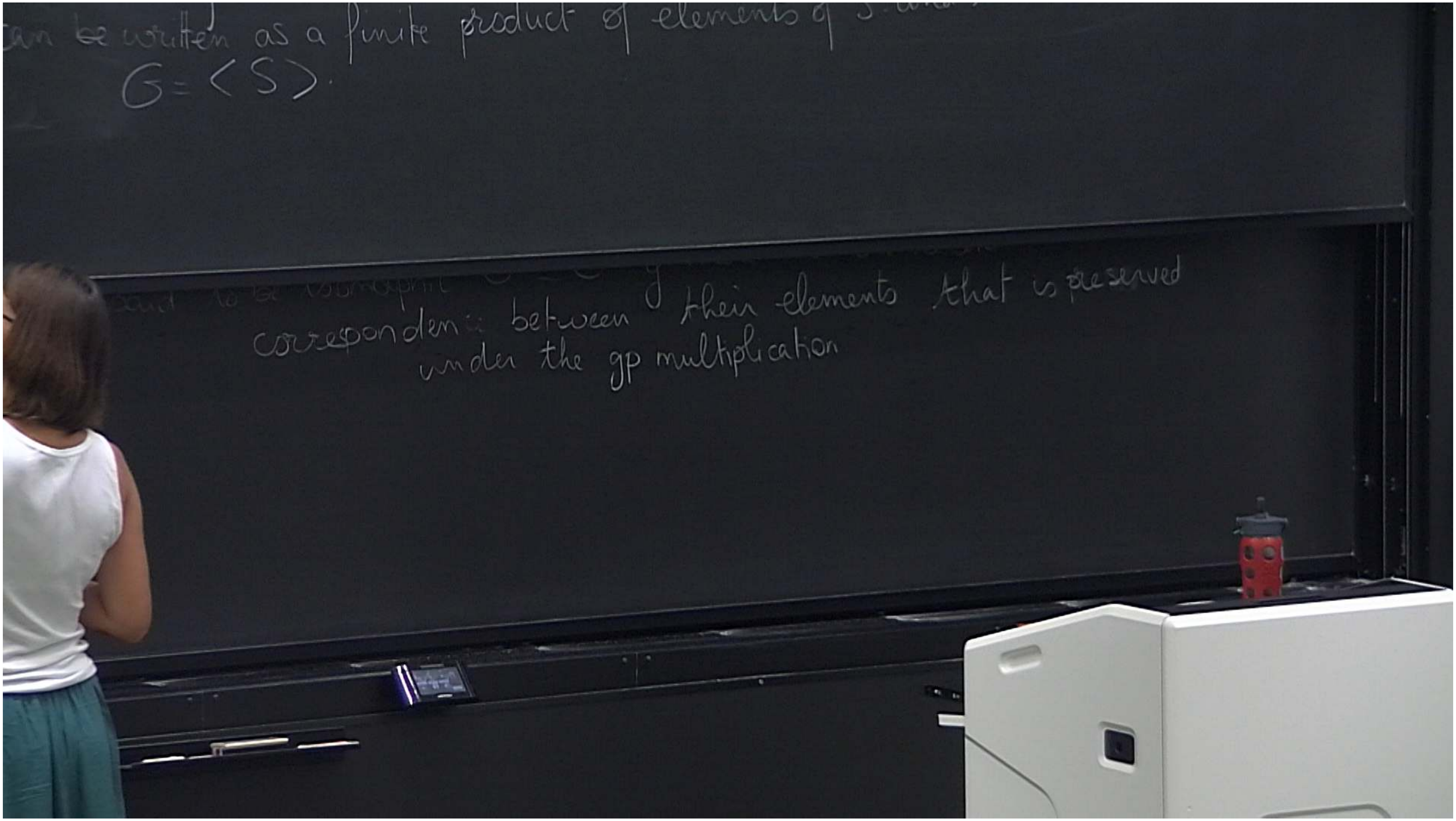

are said to be isomorphic $G \simeq G'$ if there is a one-to-one
correspondence between their elements that is preserved
under the gp multiplication

- $(\mathbb{Z}, +)$
- $(\mathbb{Z}, -)$
- $(\mathbb{R}, \cdot)$
- $(\mathbb{R} \setminus \{0\}, \cdot)$
- the set of negative numbers with multiplication

Why is group theory

- $(\mathbb{Z}, +)$ ✓
- $(\mathbb{Z}, -)$ No $a-(b-c) \neq (a-b)-c$ not associative
- $(\mathbb{R}, \cdot)$ No 0 has no inverse.
- $(\mathbb{R}\setminus\{0\}, \cdot)$ ✓
- the set of negative numbers with multiplication NO


. Why is group theory interesting?
$M_2 \circ M_1$
. More definitions
Def 3 (Generators)
S generat
Let G be a group and S be a subset of G.


an be written as a finite product of elements of
G = < S >.
correspondence between their elements that is preserved
under the gp multiplication

can be written as a finite product of elements of S

$G = \langle S \rangle$.

Def 4 (Cyclic group)

A group G is cyclic if t

- $\mathbb{Z} = \langle 1 \rangle$, $\mathbb{Z}/4\mathbb{Z} = \langle 1 \rangle$.

correspondence between their elements that

under the gp multiplication

Def 5 (Abelian group)

if $g_1 \circ g_2 = g_2 \circ g_1 \quad \forall g_1, g_2 \in G$ then G is abelian inverse of 1 = 3

$\to \circ$

$0 \to e$

$1 \to a$

$2 \to b$

$3 \to c$

$\circ$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$$(\mathbb{Z}/5\mathbb{Z})^\times$$

$\times \to \circ$

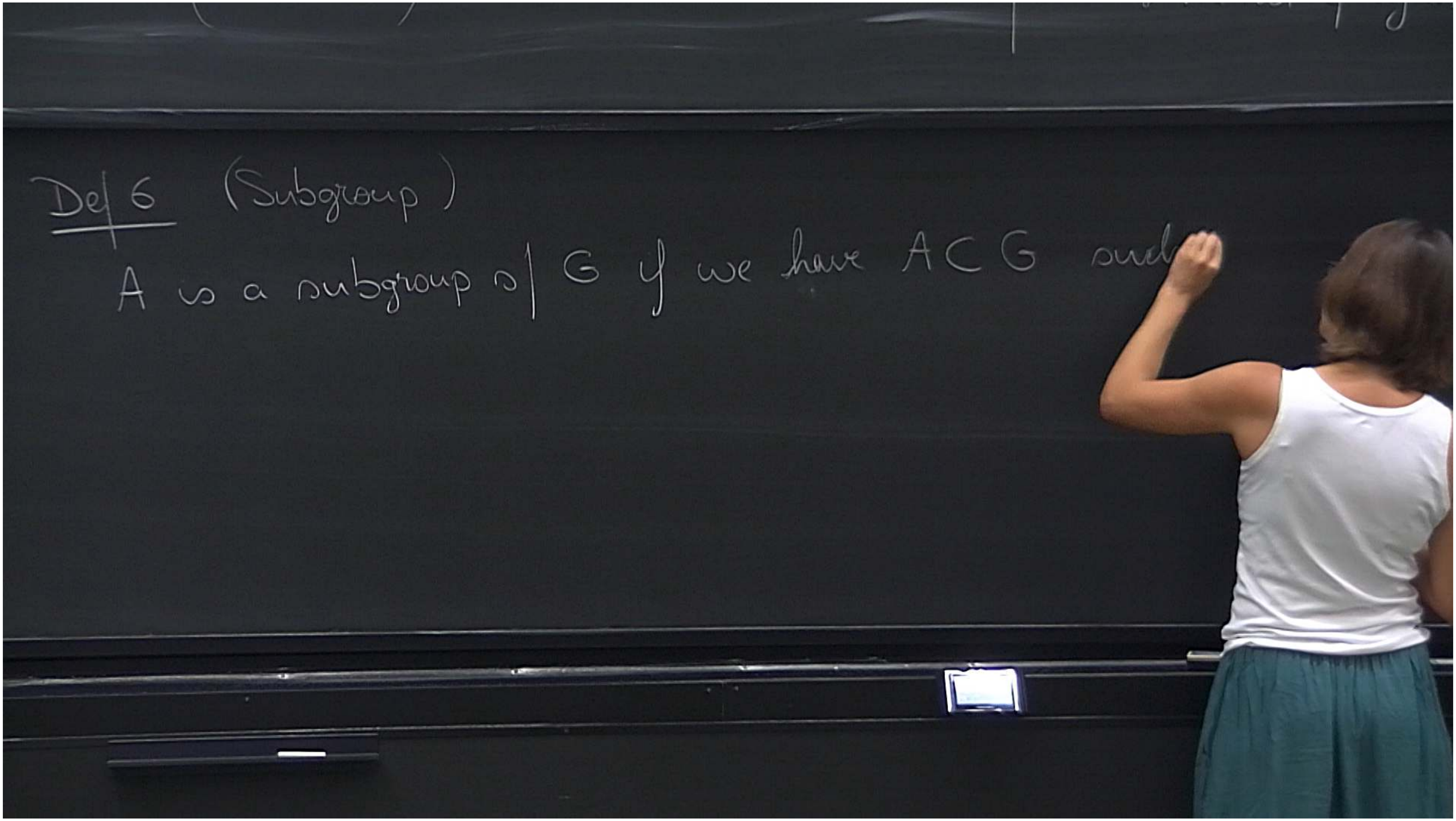
$1 \to e$

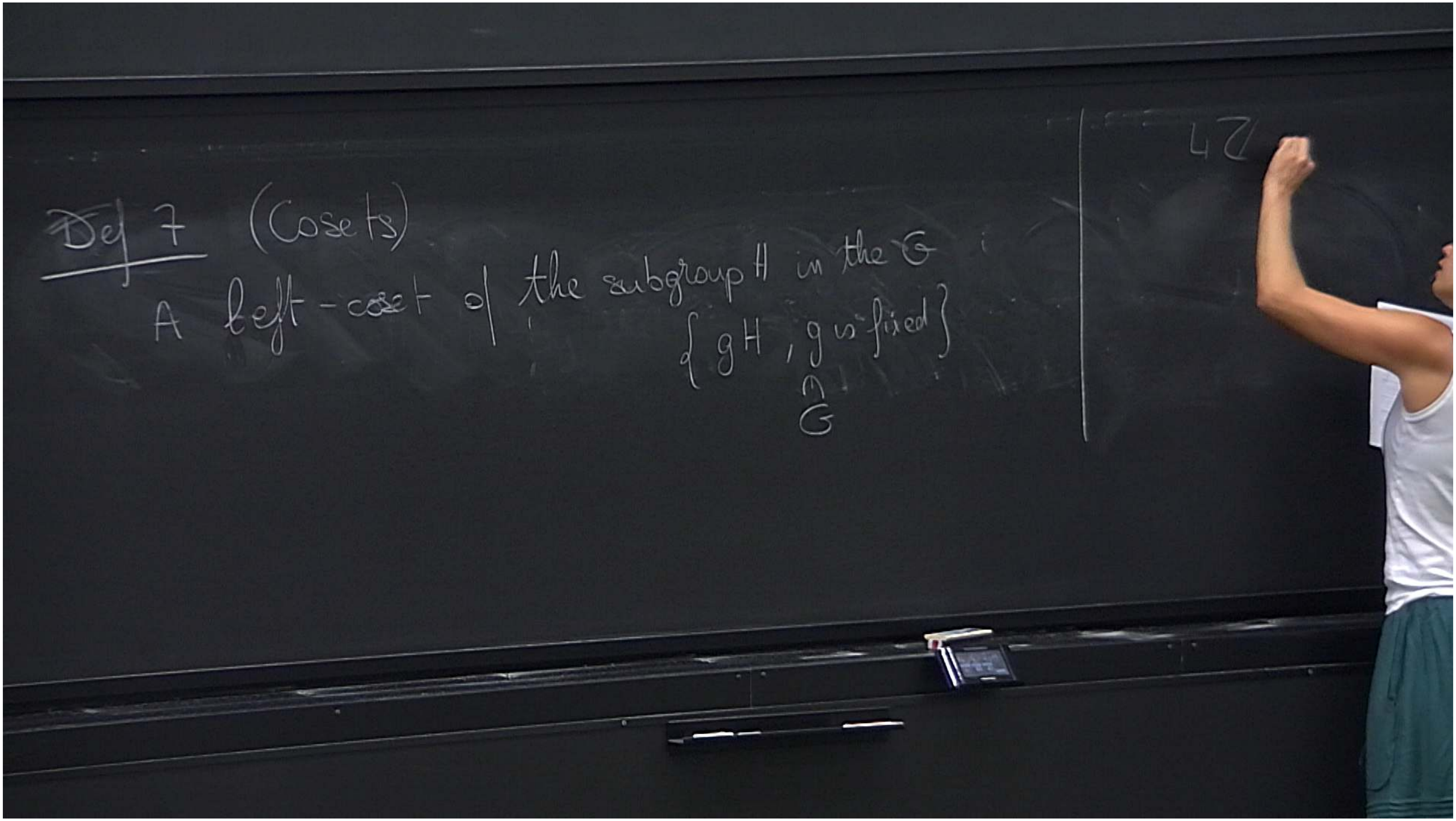
$a \to 2$

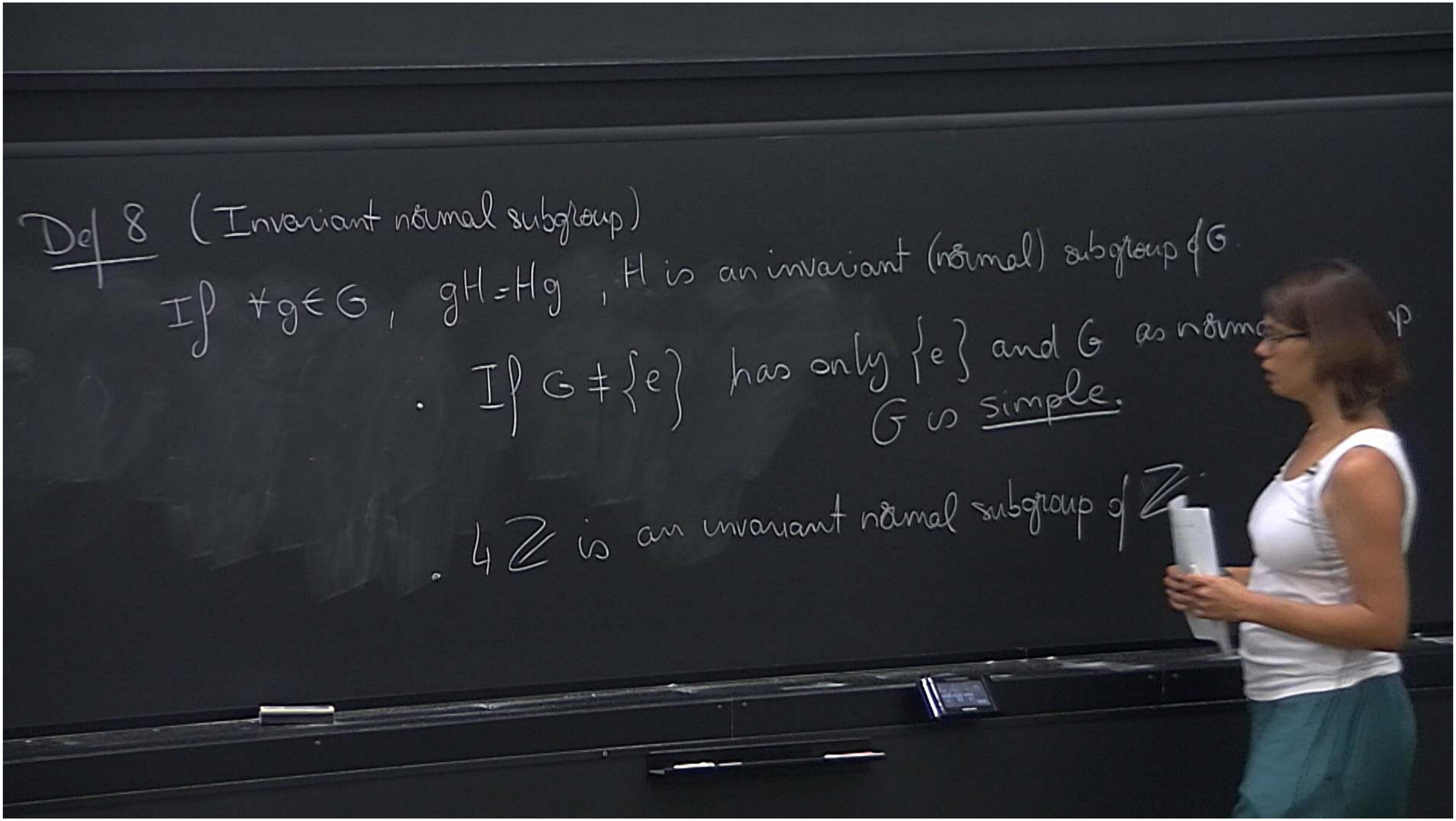
$b \to 4$

$c \to 3$

the


Def 6 (Subgroup)
A is a subgroup of G if we have A ⊂ G such


Def 7 (Cosets)
A left-coset of the subgroup H in the G
{gH, g is fixed}
G
42


Def 8 (Invariant normal subgroup)
If ∀g∈G, gH=Hg, H is an invariant (normal) subgroup of G.
. If G ≠ {e} has only {e} and G as
G is simple.
. 4Z is an invariant normal subgroup of Z.

Def 8 (Invariant normal subgroup)

If $\forall g \in G$, $gH = Hg$, H is an invariant (normal) subgroup of G.

. If $G \neq \{e\}$ has only $\{e\}$ and G as normal subgroup G is simple.

. $4\mathbb{Z}$ is an invariant normal subgroup of $\mathbb{Z}$.

• The set of negative numbers with multiplication

Def 9 (Factor group)

If H is an invariant subgroup, then $G/H = \{gH, g \in G\}$ is a group and it is called the quotient group or factor group

inverse of $1 = 3$ $3+1 = 1+3 = 0$

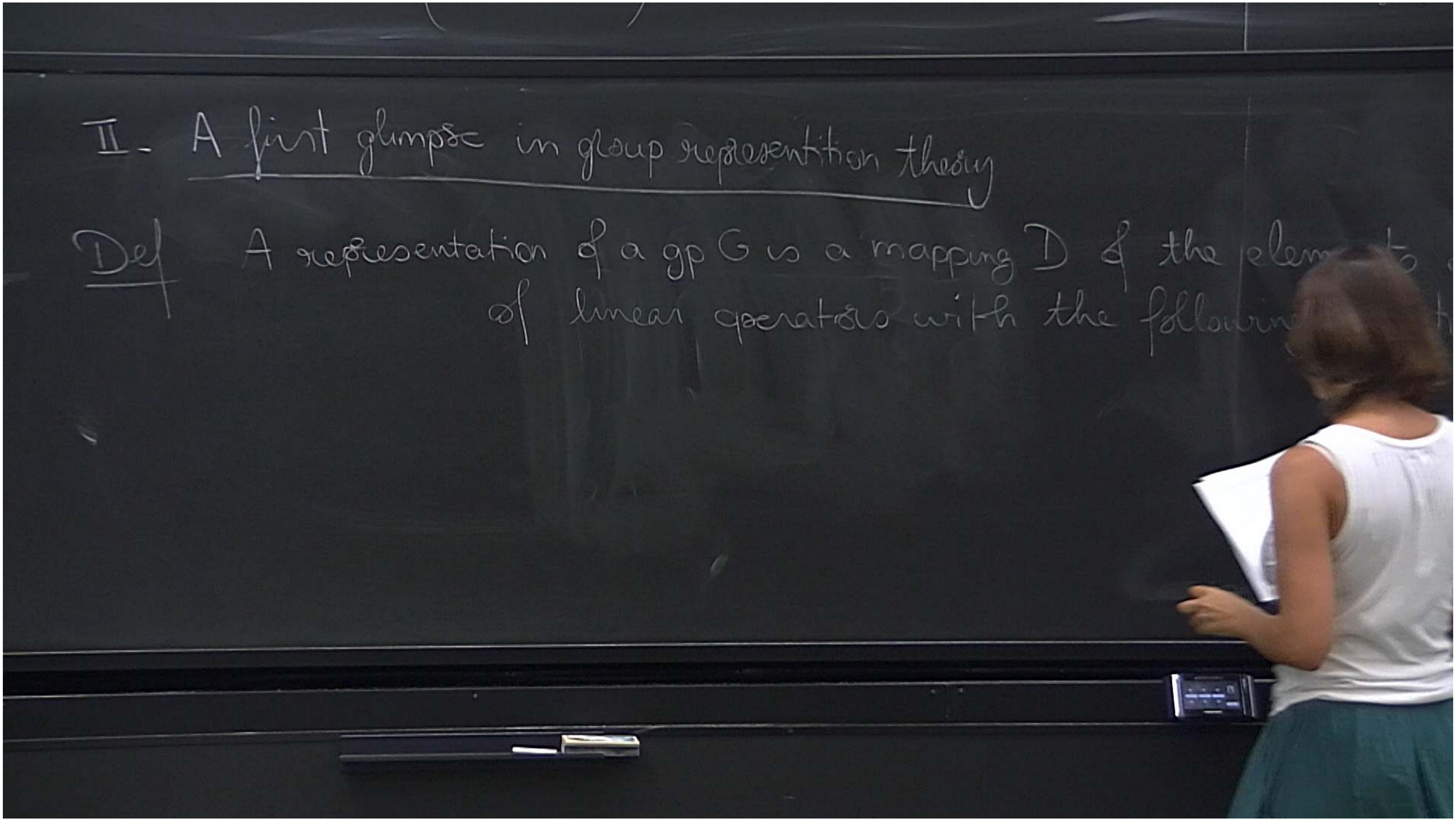
• The set of negative numbers with multiplication

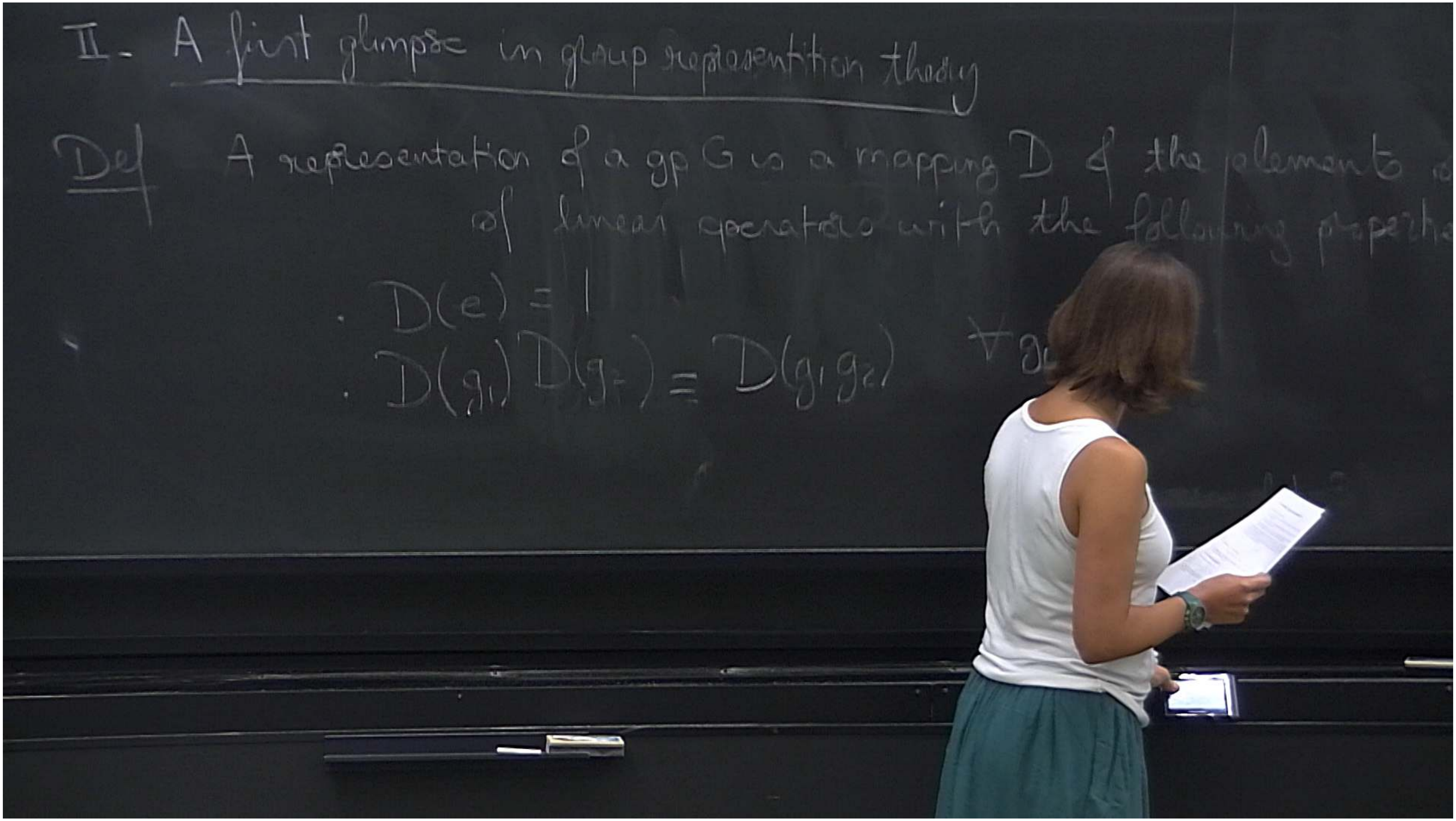
Def 9 (Factor group)

If H is an invariant subgroup, then $G/H = \{gH, g \in G\}$ is a group and it is called the quotient group or factor group

$$\Rightarrow \mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}.$$

inverse of $1 = 3$ $\quad 3+1 = 1+3 = 0$


II. A first glimpse in group representation theory
Def
A representation of a gp G is a mapping D of the
of linear operators with the


II. A first glimpse in group representation theory
Def A representation of a gp G is a mapping D of the elements
of linear operators with the following
. D(e) = 1
. D(g1) D(g2) = D(g1 g2) ∀

. C_n : cyclic gp.

. C_4

Representation of C_4

$D(e) = 1$, $D(a) = e^{i\pi/2} = i$, $D(b) = e^{i\pi} = -1$, $D(c)$

. C_n : cyclic gp

. C_4

Representation of C_4

$D(e) = 1$, $D(a) = e^{i\pi/2} = i$, $D(b) = e^{i\pi} = -1$, $D(c) = e^{i\frac{3\pi}{2}} = -i$

1 dimensional representation.

A left-coset of the subgroup H in the ...

$\{aH \quad a$ is fixed?

Regular representation for C_4

$$D(e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \quad D(a) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \quad D(b)$$

Regular representation for $\mathbb{Z}_4$

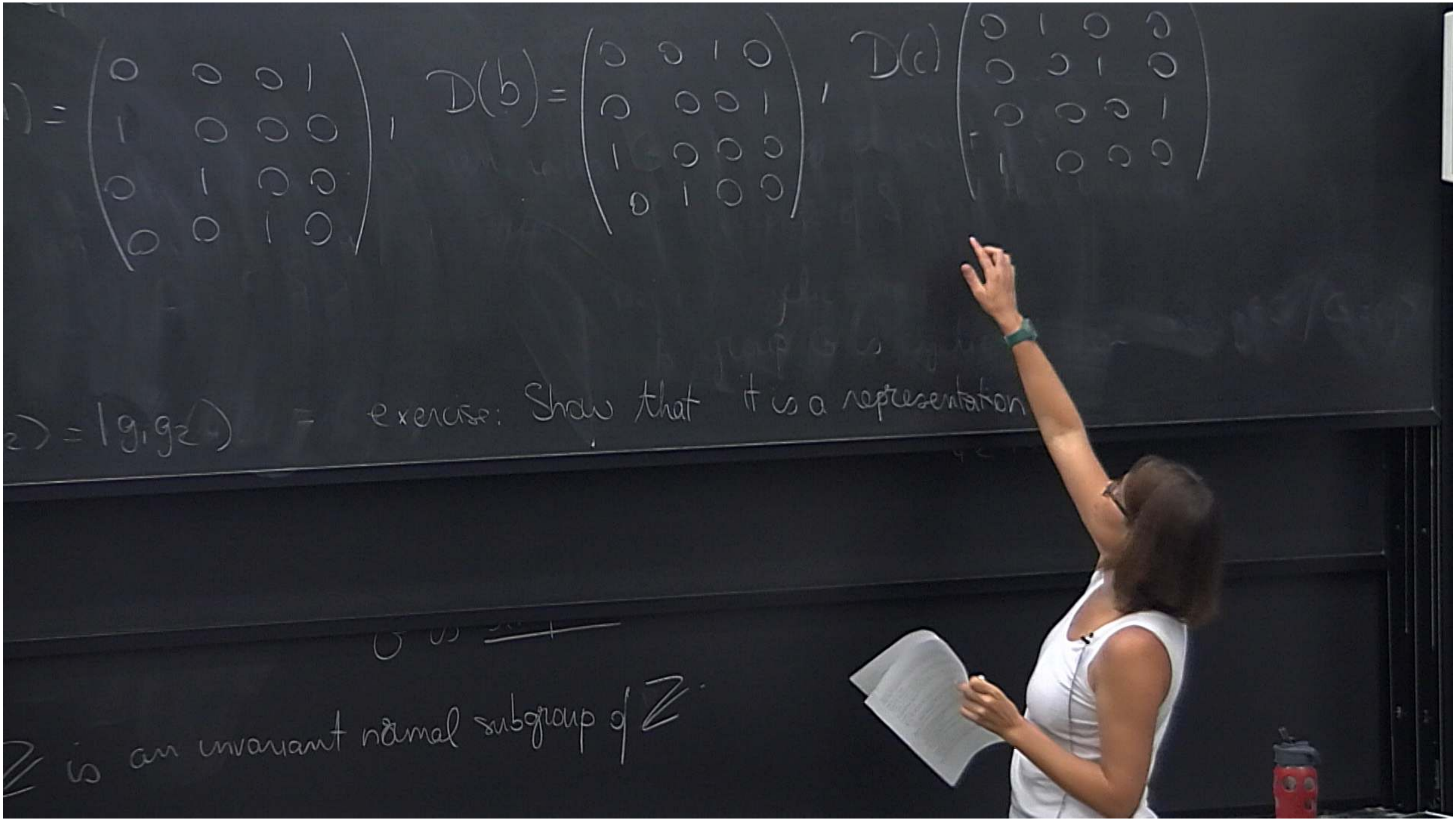
$$D(e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \quad D(a) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad D(b) =$$

$|e\rangle, |a\rangle, |b\rangle, |c\rangle$

- $D(g_1)|g_2\rangle = |g_1 g_2\rangle$

exercise:

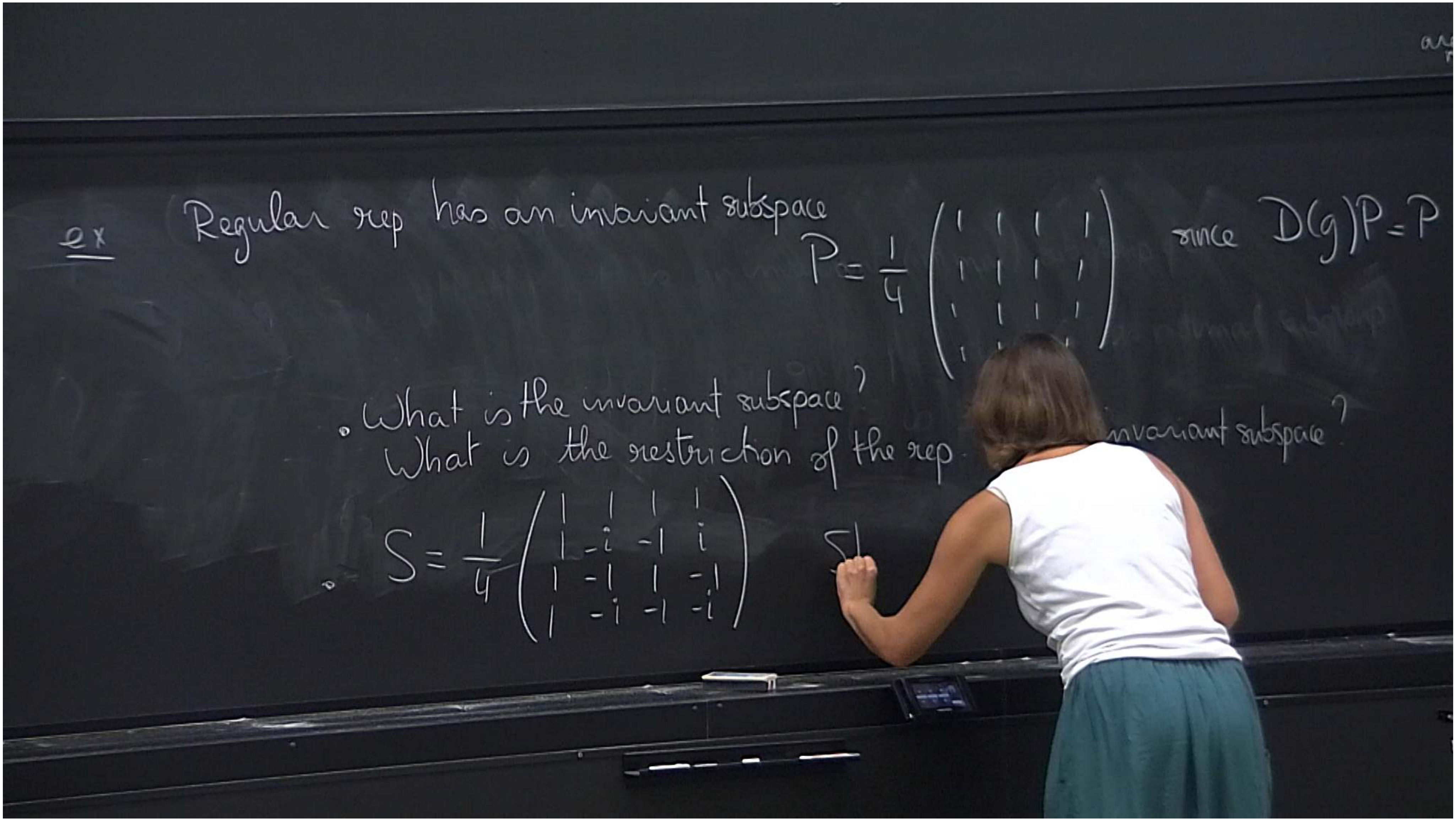
- $4\mathbb{Z}$ is an invariant normal subgroup


D(b) =
D(c)
exercise: Show that it is a representation
is an invariant normal subgroup of

- $D(g) \longrightarrow D'(g) = S^{-1} D(g) S$ $\quad$ $D'(g), D(g)$ are equivalent
- Unitary operators $(O \,/\, O^{\dagger} = O^{-1})$: $D(g)$s are unitary
- A representation is reducible if it has an <u>invariant subspace</u>

projector onto the subspace, P

$$P D(g) P = D(g) P \quad \forall g \in G$$


ex Regular rep has an invariant subspace
P = 1/4
since D(g)P = P
What is the invariant subspace?
What is the restriction of the rep
invariant subspace?
S = 1/4