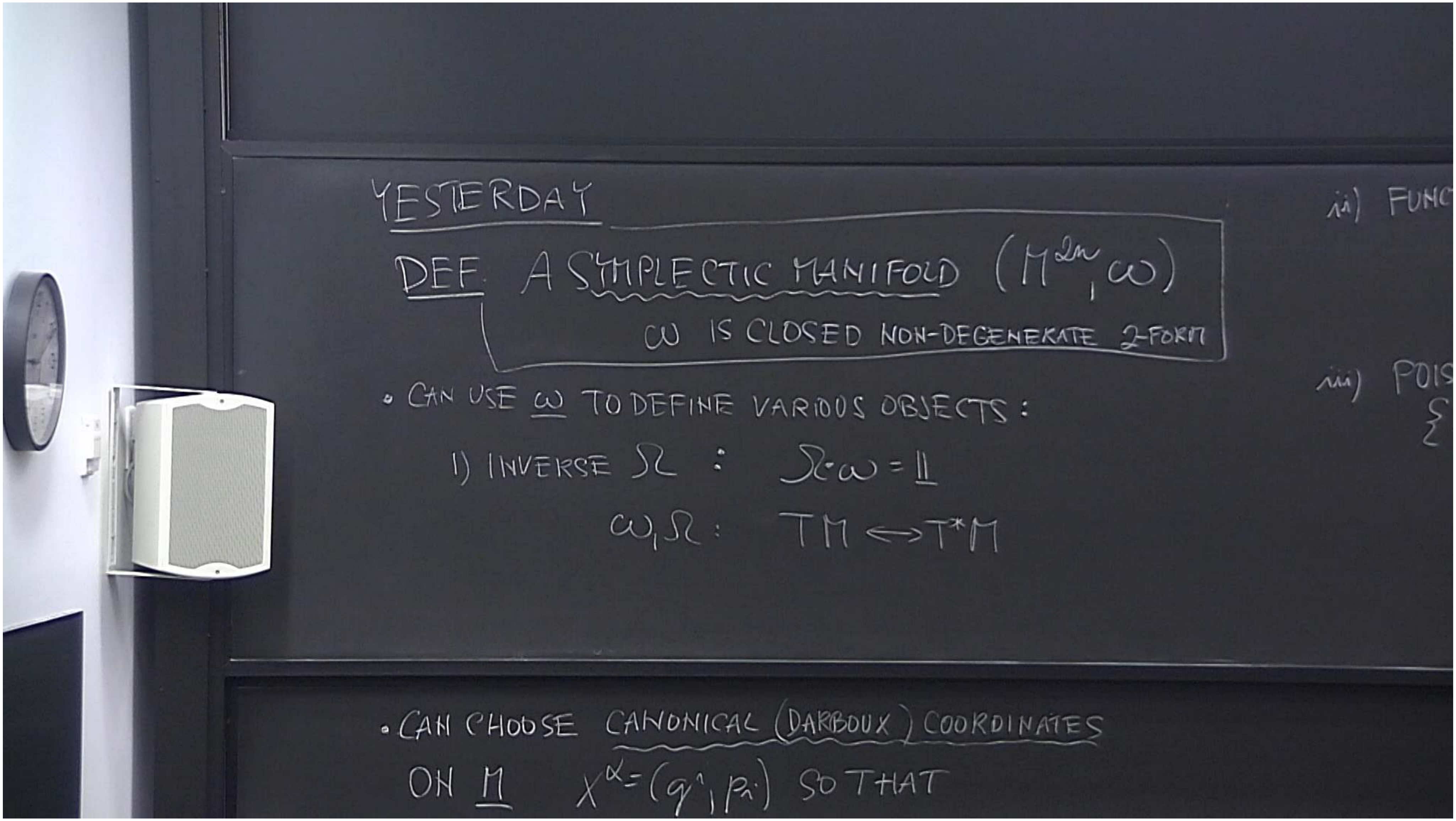
Title: PSI 17/18 Front End - Theoretical Mechanics (Kubiznak)

Date: Aug 18, 2017 09:00 AM

URL: http://pirsa.org/17080049

Abstract:


YESTERDAY
DEF. A SYMPLECTIC MANIFOLD (M^{2n}, ω)
ω IS CLOSED NON-DEGENERATE 2-FORM
• CAN USE ω TO DEFINE VARIOUS OBJECTS:
1) INVERSE Ω : $\Omega \cdot \omega = \mathbb{1}$
ω, Ω : $TM \Longleftrightarrow T^*M$
ii) FUNC
iii) POIS
• CAN CHOOSE CANONICAL (DARBOUX) COORDINATES
ON M $x^\alpha = (q^i, p_i)$ SO THAT

ii) FUNCTION f $\longrightarrow$ HAMILTONIAN VECTOR FIELD X_f

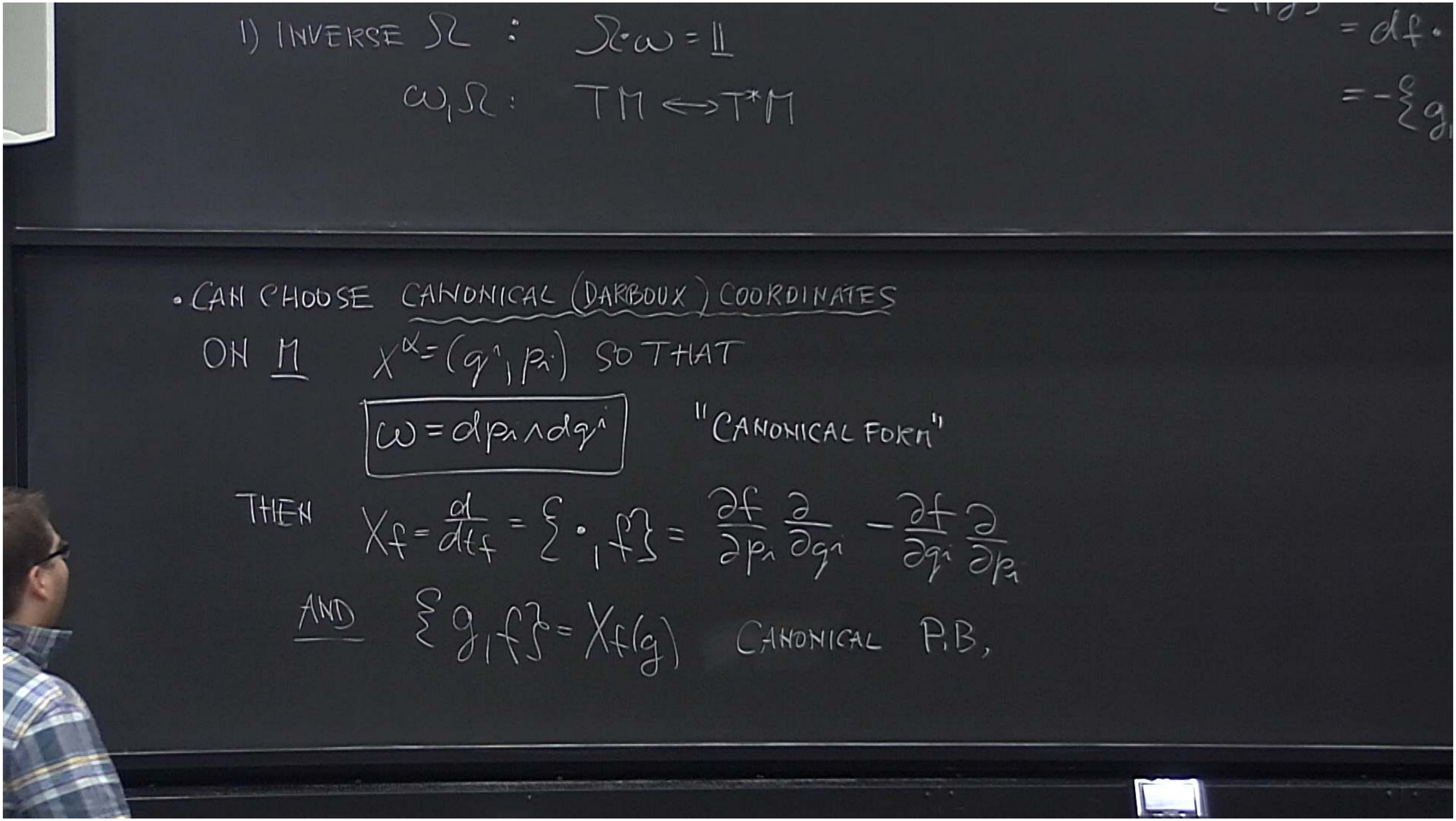
$$X_f \equiv \frac{d}{dt_f} = \Omega \cdot df \iff X_f \lrcorner\, \omega = -df$$

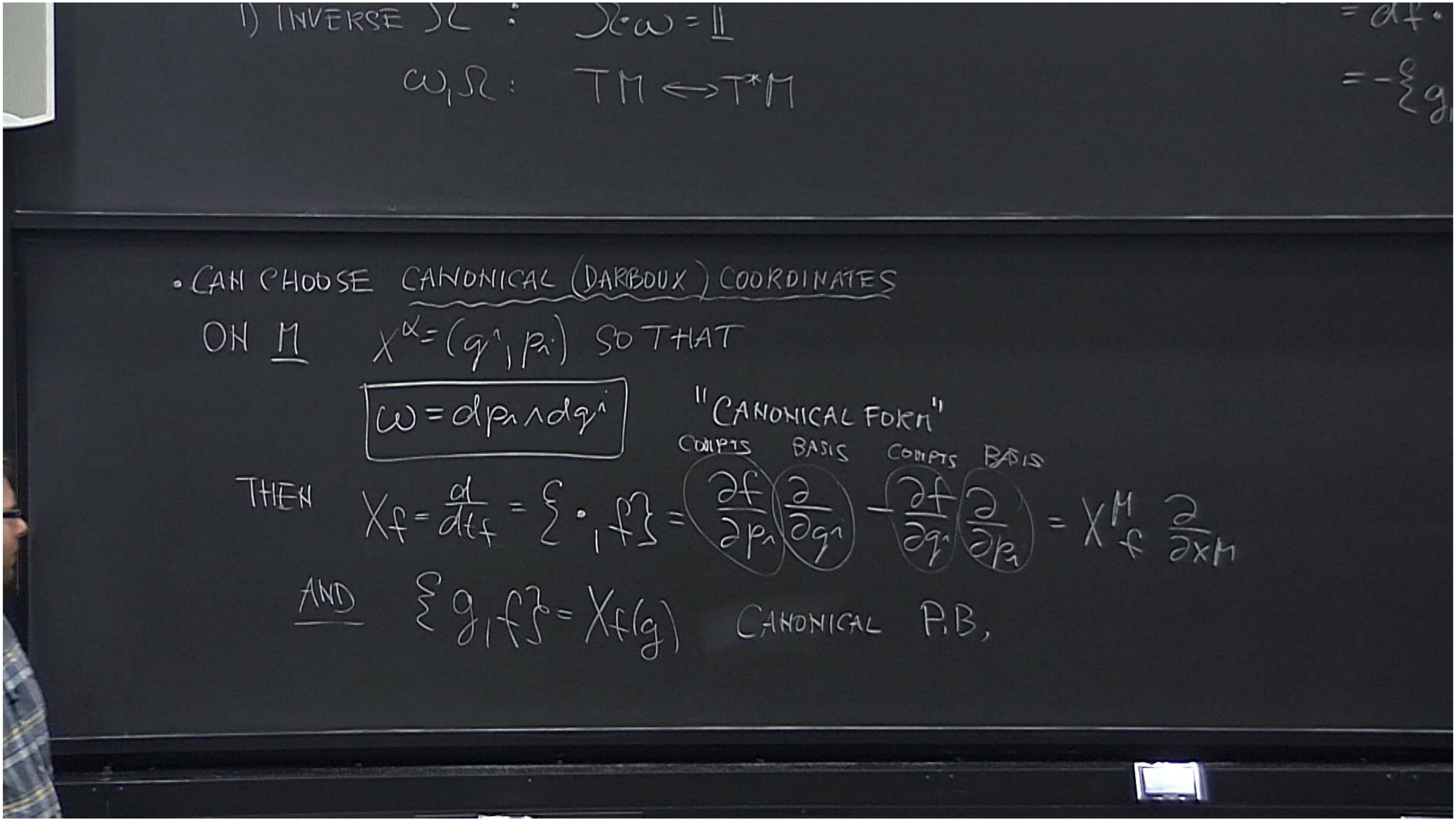
X_f PRESERVES ω. $\quad \mathcal{L}_{X_f}\omega = 0$ (LIOUVILLE)

iii) POISSON BRACKET

$$\{f,g\} = df \cdot \Omega \cdot dg = -\omega(X_f, X_g)$$
$$= df \cdot X_g = X_g(f) = \frac{df}{dt_g}$$
$$= -\{g,f\} = -X_f(g) = -\frac{dg}{dt_f}$$

$$\boxed{X_f = \frac{d}{dt_f} = \{\,\cdot\,, f\}}$$


1) INVERSE Ω : $\Omega \cdot \omega = \mathbb{1}$
ω, Ω : $TM \leftrightarrow T^*M$
$= df \cdot$
$= -\{g,$
• CAN CHOOSE CANONICAL (DARBOUX) COORDINATES
ON M $x^\alpha = (q^i, p_i)$ SO THAT
$\omega = dp_i \wedge dq^i$ "CANONICAL FORM"
THEN $X_f = \frac{d}{dt_f} = \{\cdot, f\} = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}$
AND $\{g, f\} = X_f(g)$ CANONICAL P.B.


1) INVERSE Ω : $\Omega \cdot \omega = \mathbb{1}$
$\omega, \Omega : TM \longleftrightarrow T^*M$
$= df \cdot$
$= -\{g,$
• CAN CHOOSE CANONICAL (DARBOUX) COORDINATES
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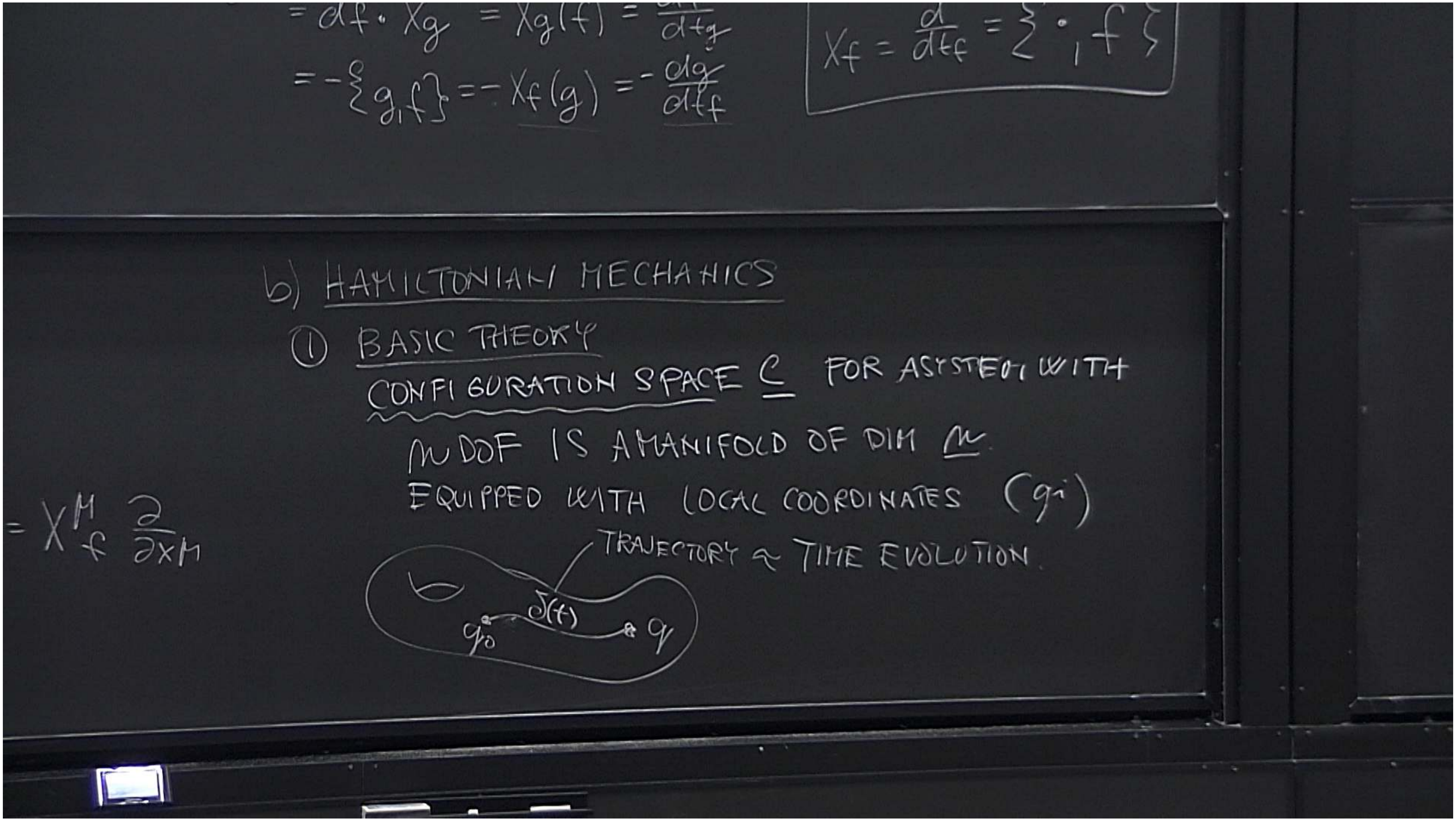
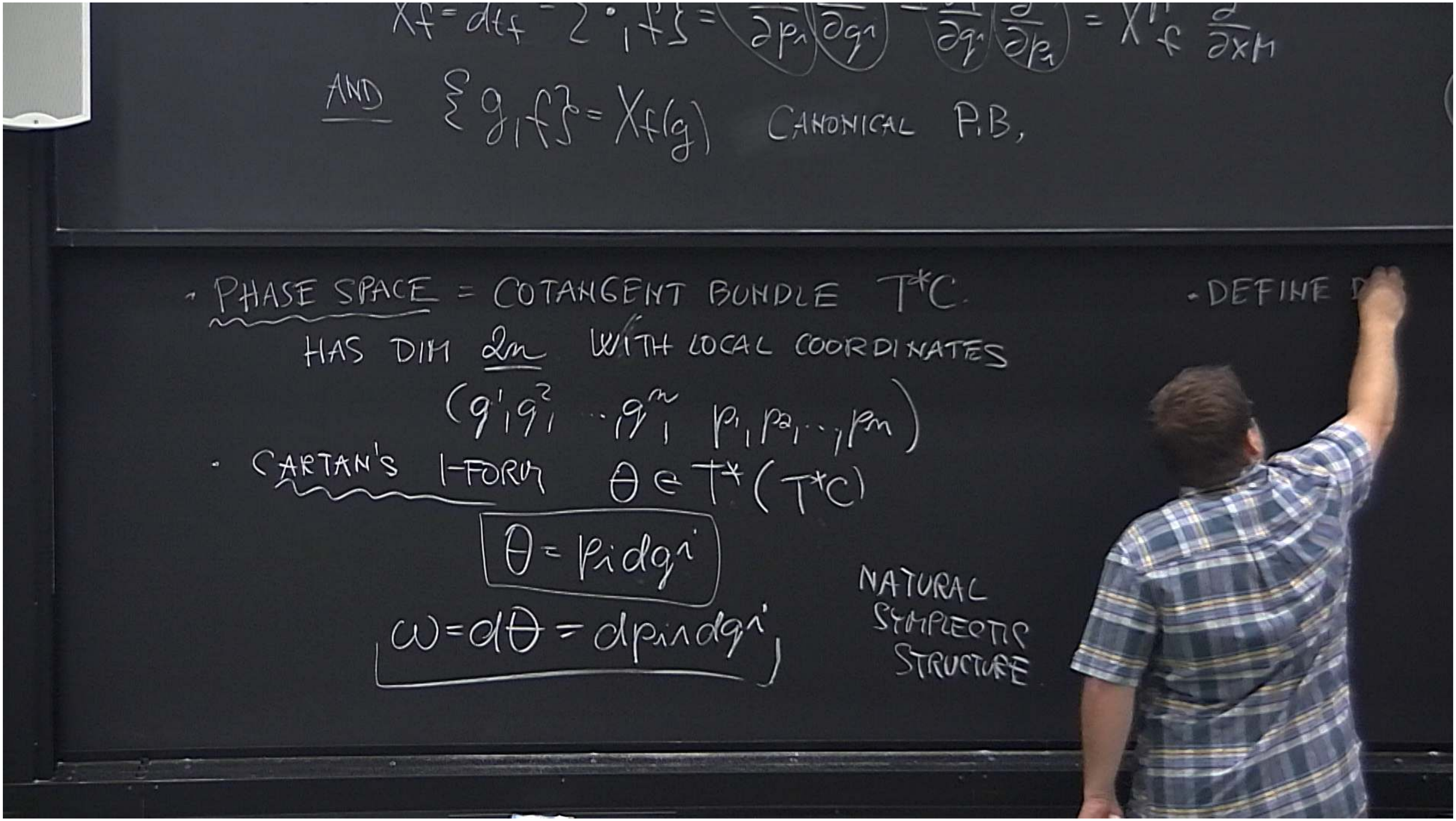
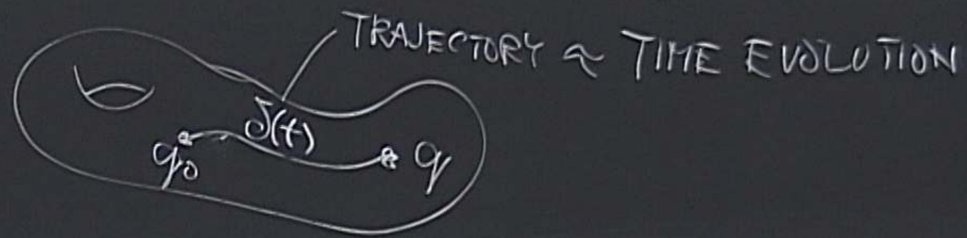
"CANONICAL FORM"
COMPTS BASIS COMPTS BASIS
THEN $X_f = \frac{d}{dt_f} = \{\cdot, f\} = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} = X^\mu_f \frac{\partial}{\partial x^\mu}$
AND $\{g, f\} = X_f(g)$ CANONICAL P.B.


$= df \cdot X_g = X_g(f) = \frac{d f}{d t_g}$
$= -\{g, f\} = -X_f(g) = -\frac{dg}{dt_f}$
$X_f = \frac{d}{dt_f} = \{\cdot, f\}$
$= X^{\mu}_f \frac{\partial}{\partial x^{\mu}}$
b) HAMILTONIAN MECHANICS
① BASIC THEORY
CONFIGURATION SPACE C FOR A SYSTEM WITH
n DOF IS A MANIFOLD OF DIM n.
EQUIPPED WITH LOCAL COORDINATES (q^i)
TRAJECTORY ~ TIME EVOLUTION.
$\gamma(t)$
q_0
q


AND {g, f} = X_f(g) CANONICAL P.B.
· PHASE SPACE = COTANGENT BUNDLE T*C.
HAS DIM 2n WITH LOCAL COORDINATES
(q^1, q^2, ..., q^n, p_1, p_2, ..., p_n)
· CARTAN'S 1-FORM θ ∈ T*(T*C)
θ = p_i dq^i
ω = dθ = dp_i ∧ dq^i
NATURAL SYMPLECTIC STRUCTURE
· DEFINE

$= X^{\mu}_{f} \frac{\partial}{\partial x^{\mu}}$

EQUIPPED WITH LOCAL COORDINATES (q^i)



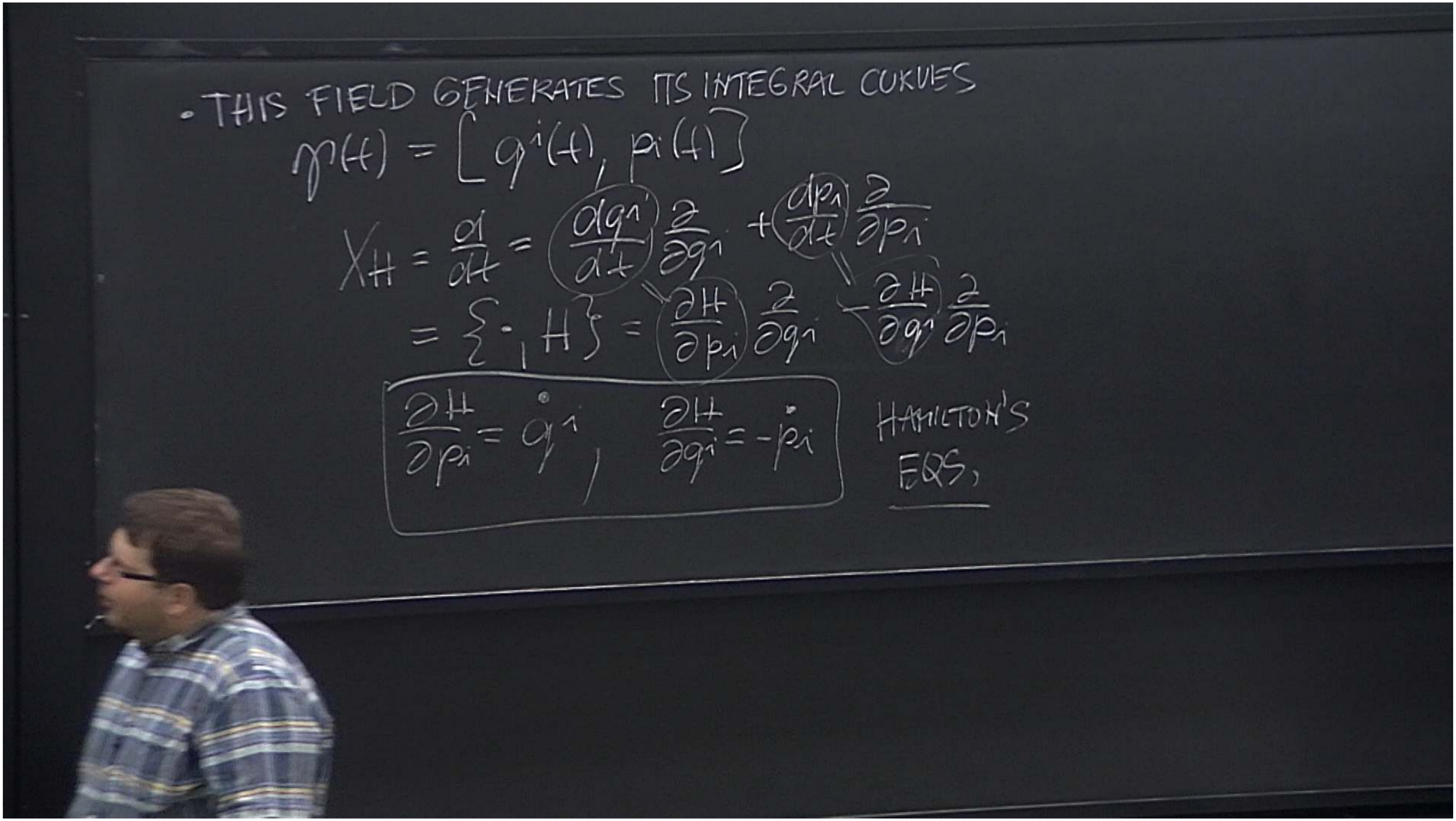
• DEFINE DYNAMICS BY SPECIFYING HAMILTONIAN

$$H: T^*C \to \mathbb{R} \qquad H(q^i, p_i)$$

NO EXPLICIT t-DEP.
AUTONOMOUS SYSTEMS

THIS DEFINES
DYNAMICAL HAMILTONIAN VECTOR FIELD

$$X_H = \frac{d}{dt} = \Omega \cdot dH = \{\cdot, H\}$$


• THIS FIELD GENERATES ITS INTEGRAL CURVES
$\gamma(t) = [q^i(t), p_i(t)]$
$X_H = \frac{d}{dt} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i}$
$= \{\cdot, H\} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$
$\frac{\partial H}{\partial p_i} = \dot{q}^i, \quad \frac{\partial H}{\partial q^i} = -\dot{p}_i$
HAMILTON'S EQS.

- THIS FIELD GENERATES ITS INTEGRAL CURVES

$$\gamma(t) = [q^i(t), p_i(t)]$$

$$X_H = \frac{d}{dt} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i}$$

$$= \{\cdot, H\} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

$$\boxed{\frac{\partial H}{\partial p_i} = \dot{q}^i, \quad \frac{\partial H}{\partial q^i} = -\dot{p}_i}$$

HAMILTON'S EQS. → DETERMINE $\gamma(t)$

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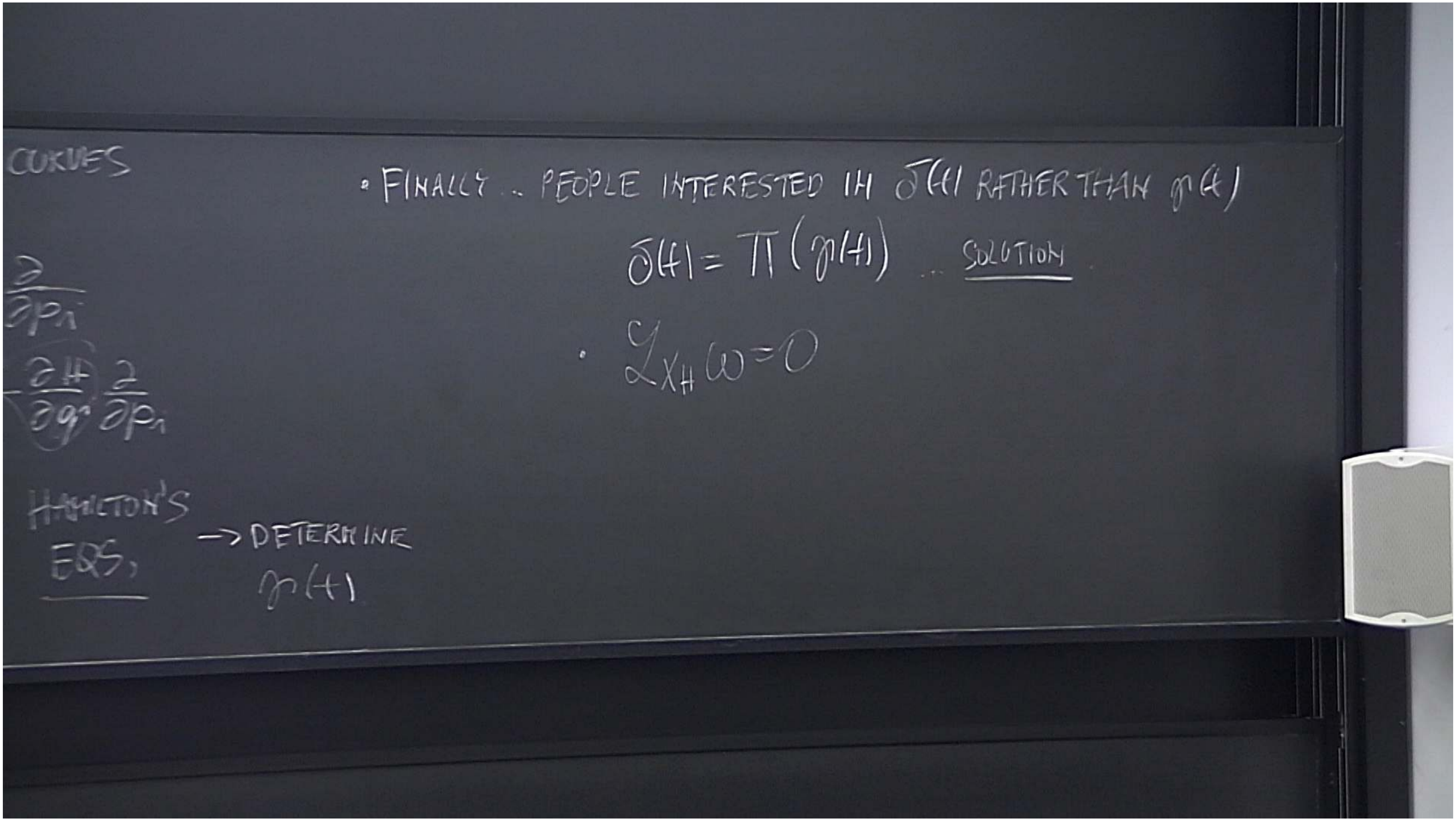
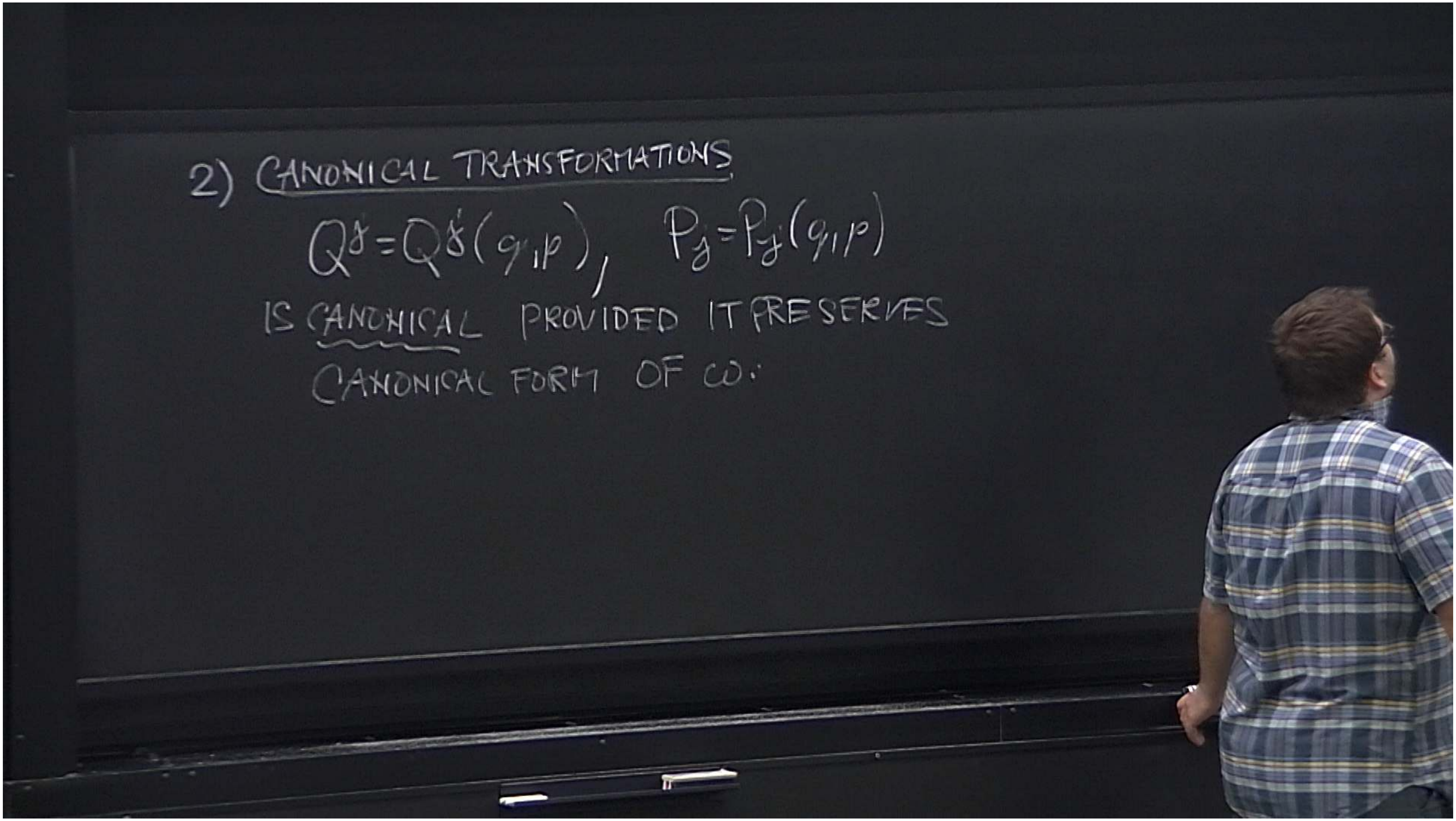
HAMILTON'S EQS, → DETERMINE $\gamma(t)$


CURVES
• FINALLY .. PEOPLE INTERESTED IN σ(t) RATHER THAN γ(t)
σ(t) = π(γ(t)) ... SOLUTION
• $\mathcal{L}_{X_H}\omega = 0$
HAMILTON'S EQS,
→ DETERMINE γ(t)


2) CANONICAL TRANSFORMATIONS
$Q^j = Q^j(q,p)$, $P_j = P_j(q,p)$
IS CANONICAL PROVIDED IT PRESERVES
CANONICAL FORM OF ω:

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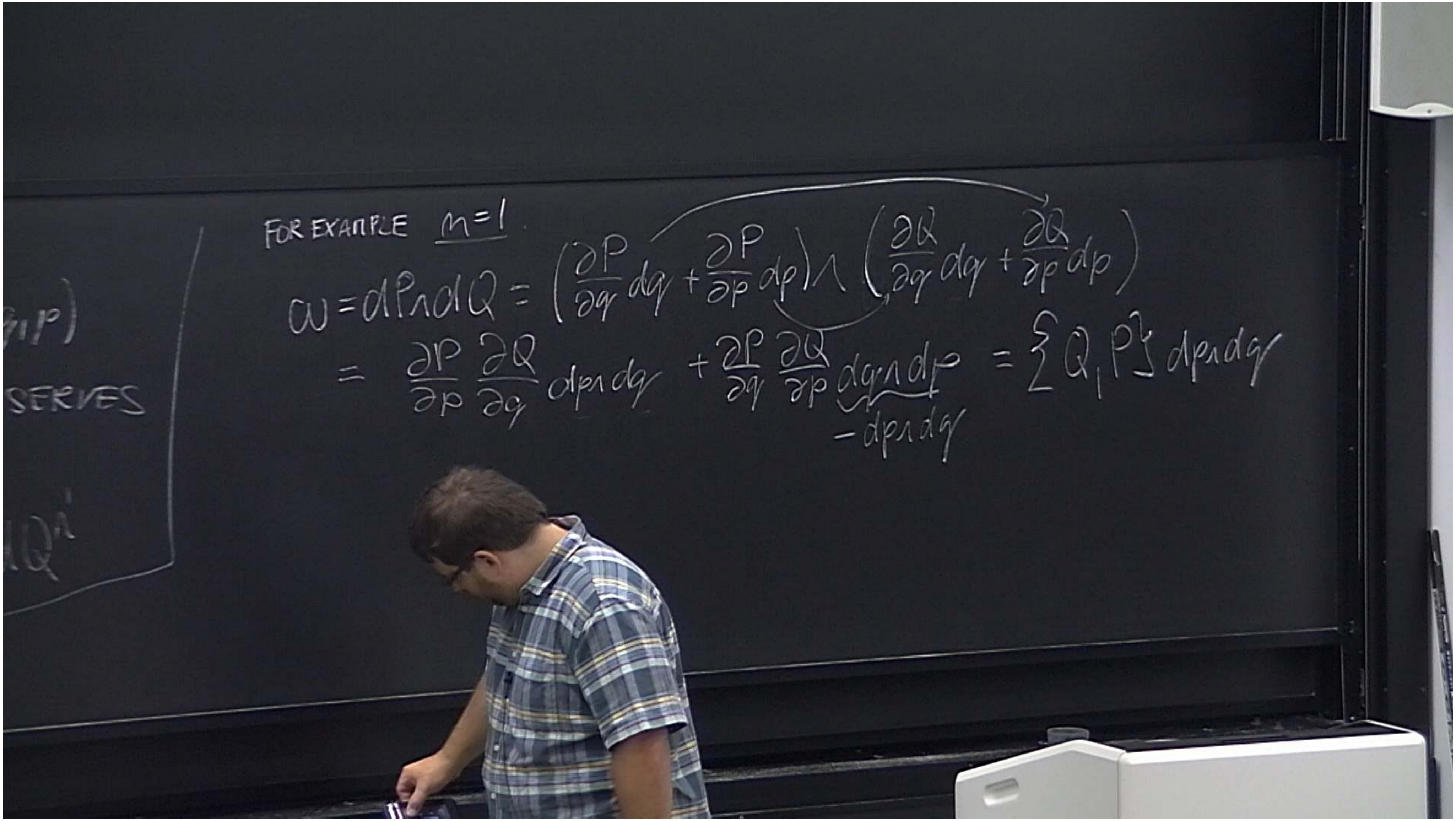
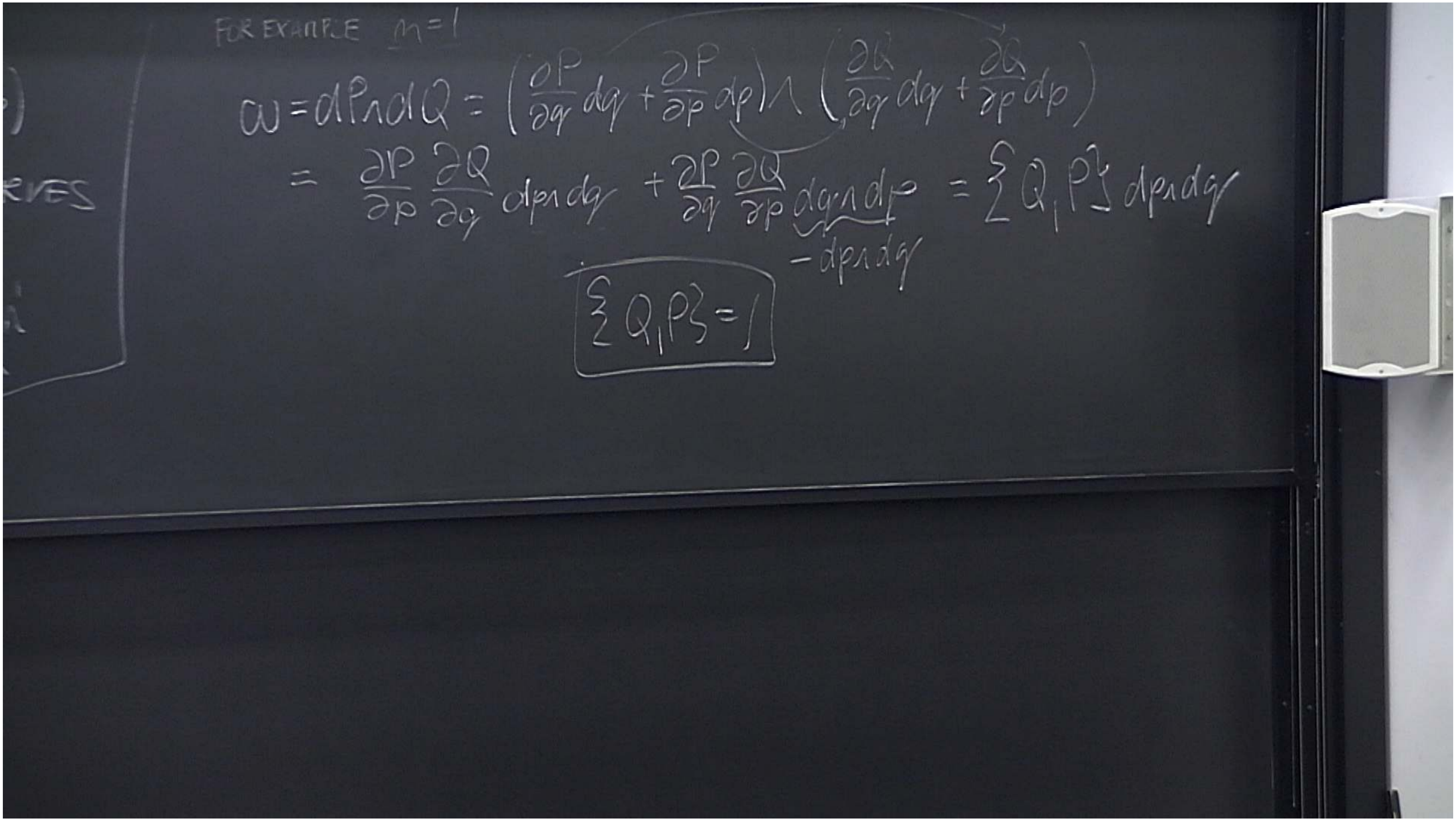
$$Q^j = Q^j(q,p), \quad P_j = P_j(q,p)$$

IS CANONICAL PROVIDED IT PRESERVES CANONICAL FORM OF ω:

$$\omega = dp_i \wedge dq^i = dP_i \wedge dQ^i$$


FOR EXAMPLE n=1
$\omega = dP \wedge dQ = \left(\frac{\partial P}{\partial q} dq + \frac{\partial P}{\partial p} dp\right) \wedge \left(\frac{\partial Q}{\partial q} dq + \frac{\partial Q}{\partial p} dp\right)$
$= \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} dp \wedge dq + \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} dq \wedge dp = \{Q, P\} dp \wedge dq$
$-dp \wedge dq$
SERVES


FOR EXAMPLE n=1
ω = dP∧dQ = (∂P/∂q dq + ∂P/∂p dp) ∧ (∂Q/∂q dq + ∂Q/∂p dp)
= ∂P/∂p ∂Q/∂q dp∧dq + ∂P/∂q ∂Q/∂p dq∧dp = {Q, P} dp∧dq
−dp∧dq
{Q, P} = 1

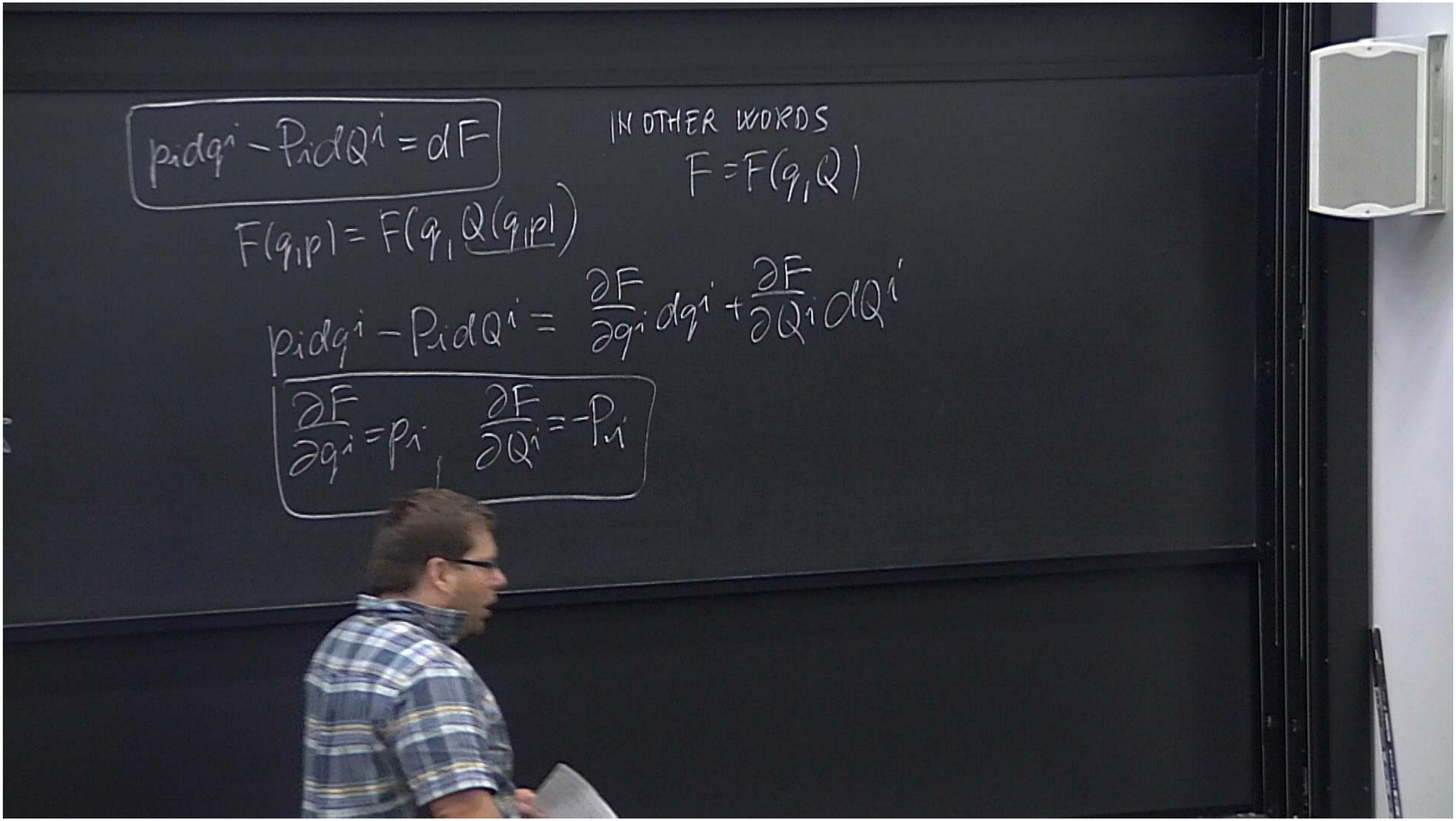
$$\omega = dp_i \wedge dq^i = dP_i \wedge dQ^i$$

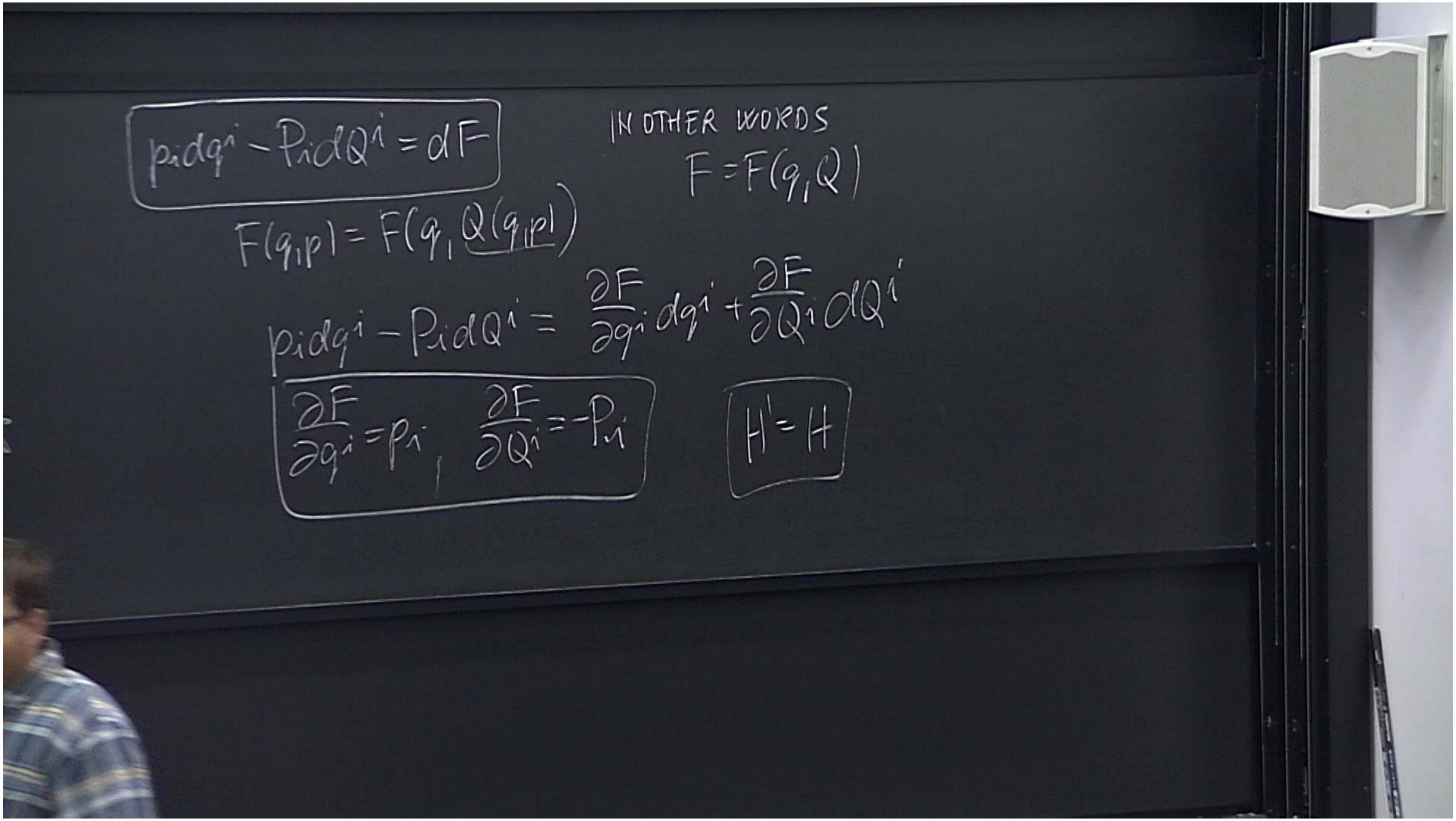
$\theta = p_i dq^i$ $\qquad$ $\tilde{\theta} = P_i dQ^i$

$d\theta = \omega$ REQUIRE $d\tilde{\theta} = \omega$

$$0 = d(\theta - \tilde{\theta})$$

dF LOCALLY THERE EXISTS A GENERATING FUNCTION $F(q,p)$


$p_i dq^i - P_i dQ^i = dF$
IN OTHER WORDS
$F = F(q, Q)$
$F(q,p) = F(q, Q(q,p))$
$p_i dq^i - P_i dQ^i = \frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial Q^i} dQ^i$
$\frac{\partial F}{\partial q^i} = p_i, \quad \frac{\partial F}{\partial Q^i} = -P_i$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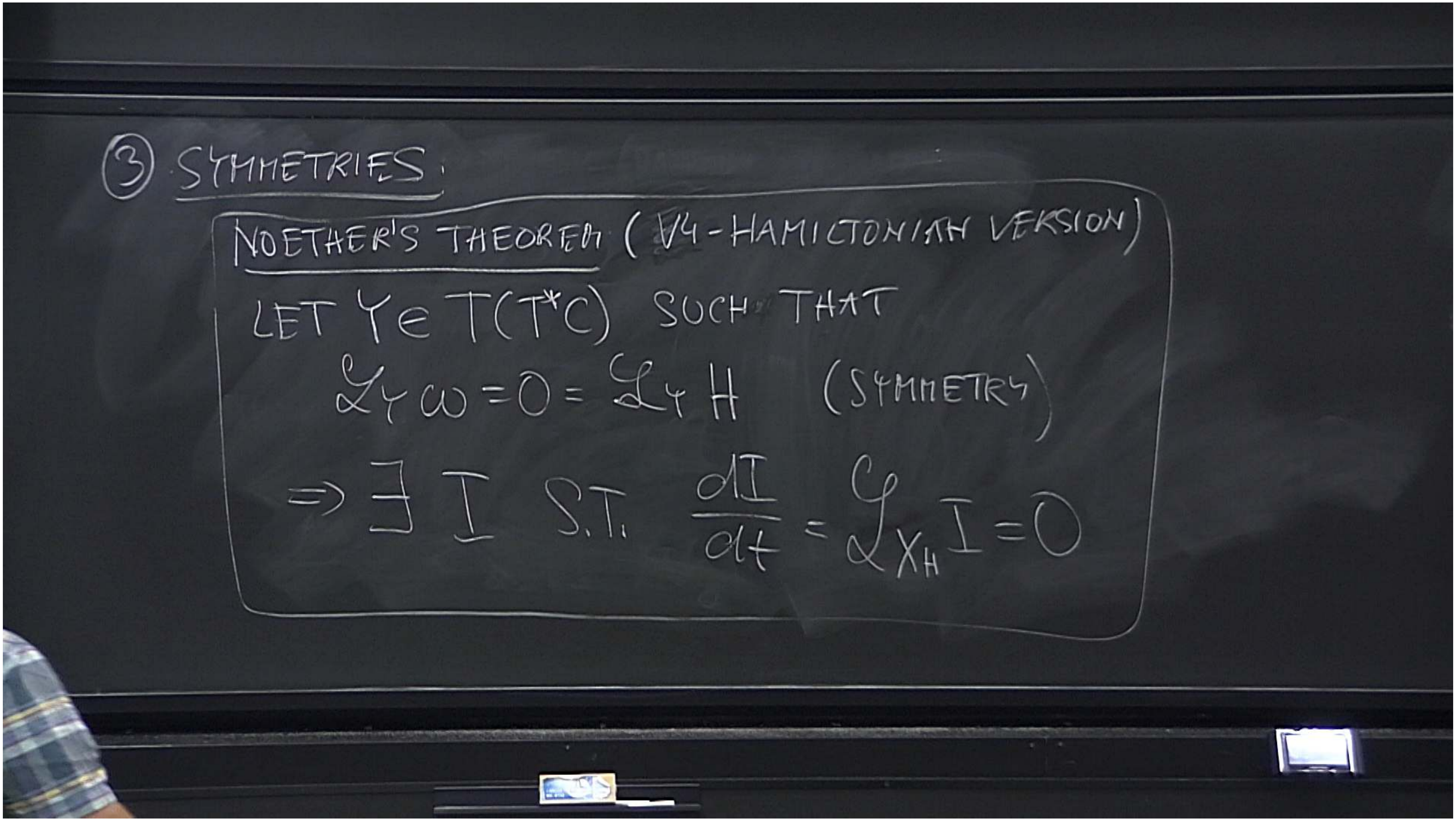
$H' = H$


③ SYMMETRIES:
NOETHER'S THEOREM (V4-HAMILTONIAN VERSION)
LET $Y \in T(T^*C)$ SUCH THAT
$\mathcal{L}_Y \omega = 0 = \mathcal{L}_Y H$ (SYMMETRY)
$\Rightarrow \exists\, I$ S.T. $\frac{dI}{dt} = \mathcal{L}_{X_H} I = 0$

PROOF: • $\mathcal{L}_Y \omega = 0 = Y \lrcorner \underbrace{d\omega}_{0} + d(\underbrace{Y \lrcorner \omega}_{-dI \text{ LOCALLY}})$

THERE EXISTS I S.T. $\boxed{Y \lrcorner \omega = -dI}$ $\quad \underline{Y = Y_I}$

• $0 = \mathcal{L}_Y H = \mathcal{L}_{Y_I} H = \{H, I\}$

$= -\{I, H\} = -\frac{dI}{dt} = 0$

$X_H = \frac{d}{dt} = \Omega \cdot dH = \{\cdot, H\}$

PROOF: • $\mathcal{L}_Y \omega = 0 = Y \lrcorner \underbrace{d\omega}_{0} + d(\underbrace{Y \lrcorner \omega}_{-dI \text{ LOCALLY}})$

THERE EXISTS I S.T. $\boxed{Y \lrcorner \omega = -dI}$ $\quad$ $\underline{Y = Y_I}$.

• $0 = \mathcal{L}_Y H = \mathcal{L}_{Y_I} H = \{H, I\}$

$= -\{I, H\} = -\frac{dI}{dt} = 0$ ∎

$X_H = \frac{d}{dt} = \Omega \cdot dH = \{\cdot, H\}$

PROOF: • $\mathcal{L}_Y \omega = 0 = Y \lrcorner \underbrace{d\omega}_{0} + d(\underbrace{Y \lrcorner \omega}_{-dI \text{ LOCALLY}})$

THERE EXISTS I S.T. $\boxed{Y \lrcorner \omega = -dI}$ $\quad Y = Y_I$

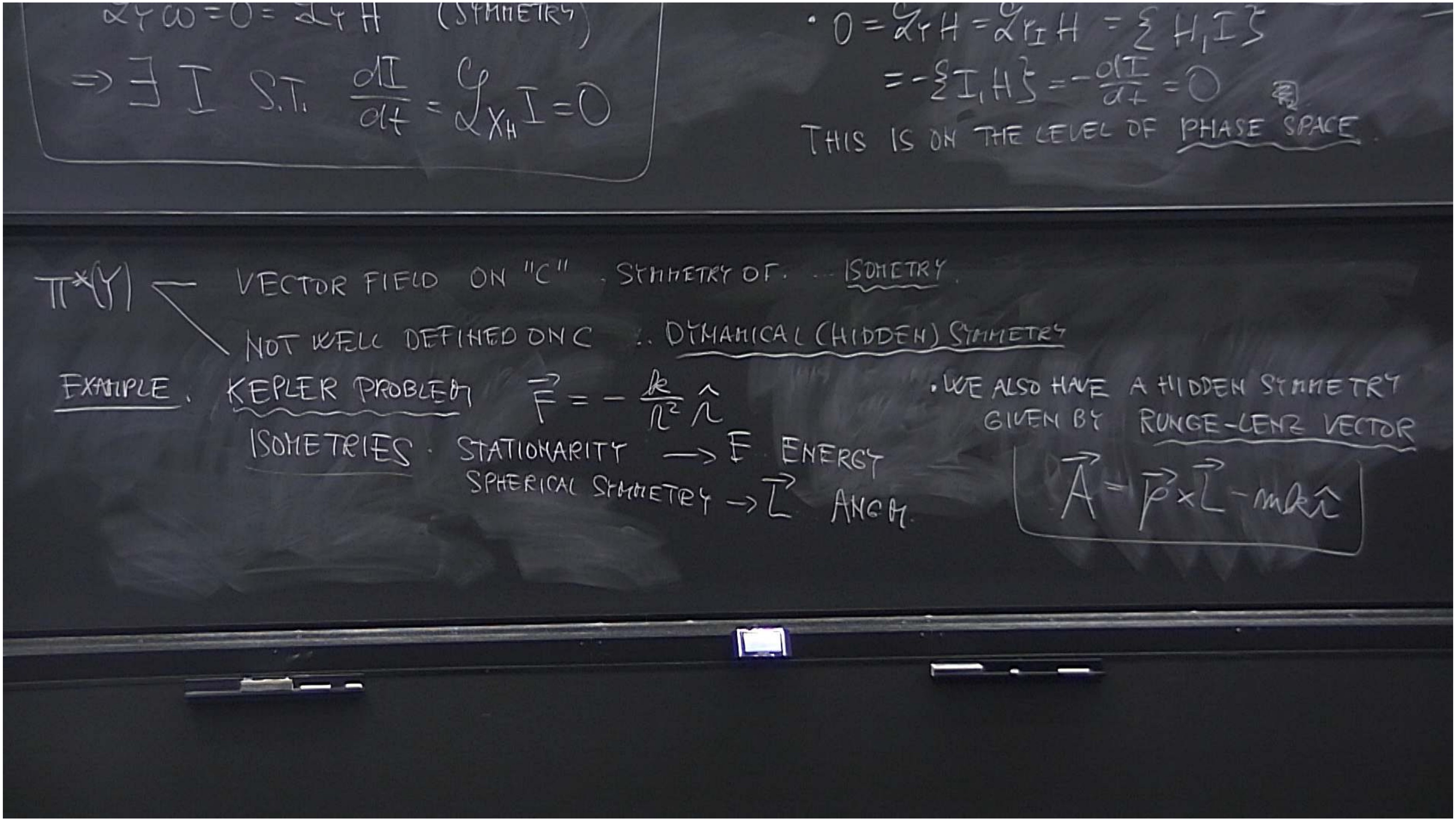
• $0 = \mathcal{L}_Y H = \mathcal{L}_{Y_I} H = \{H, I\}$

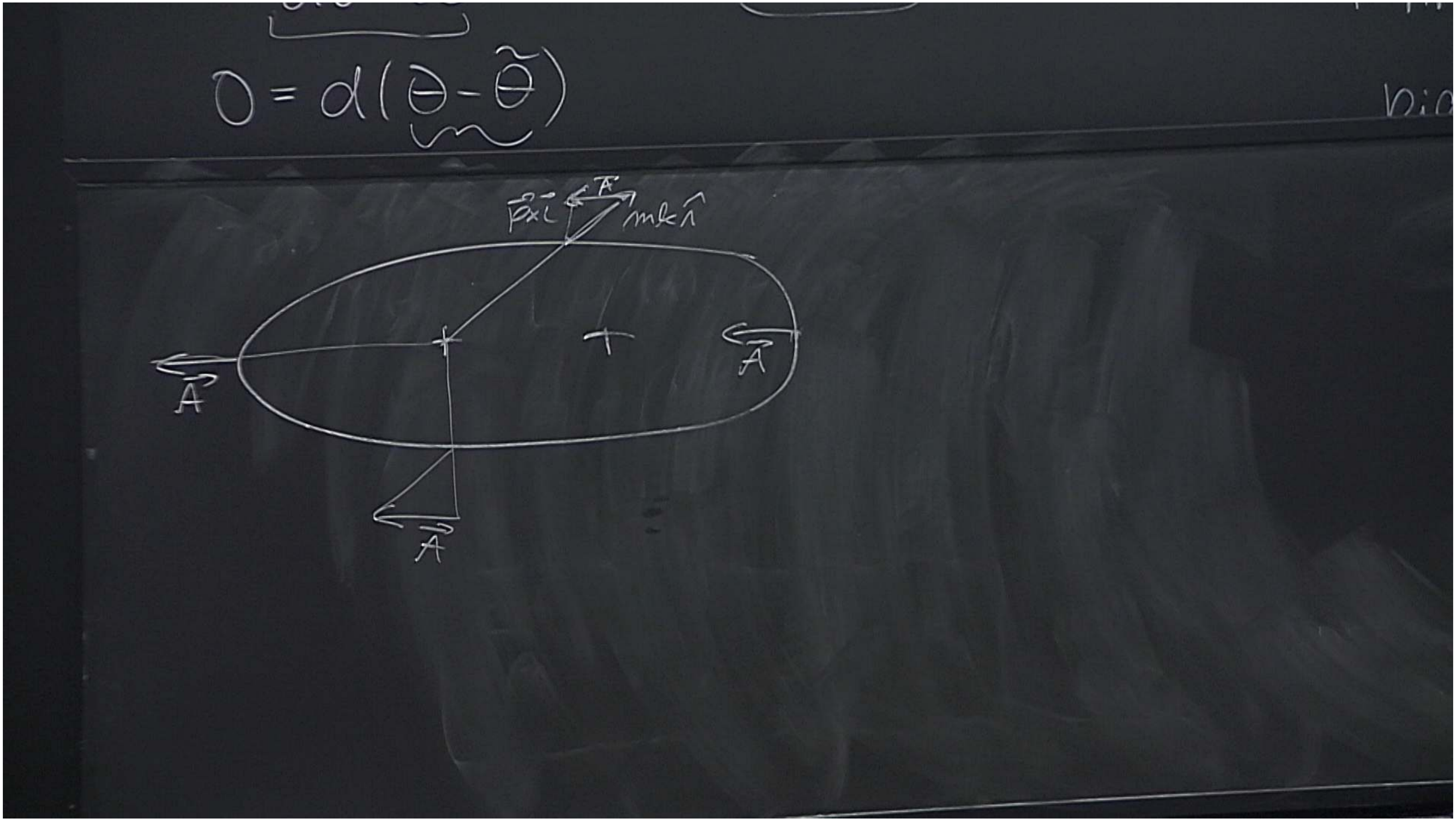
$= -\{I, H\} = -\frac{dI}{dt} = 0$ ∎

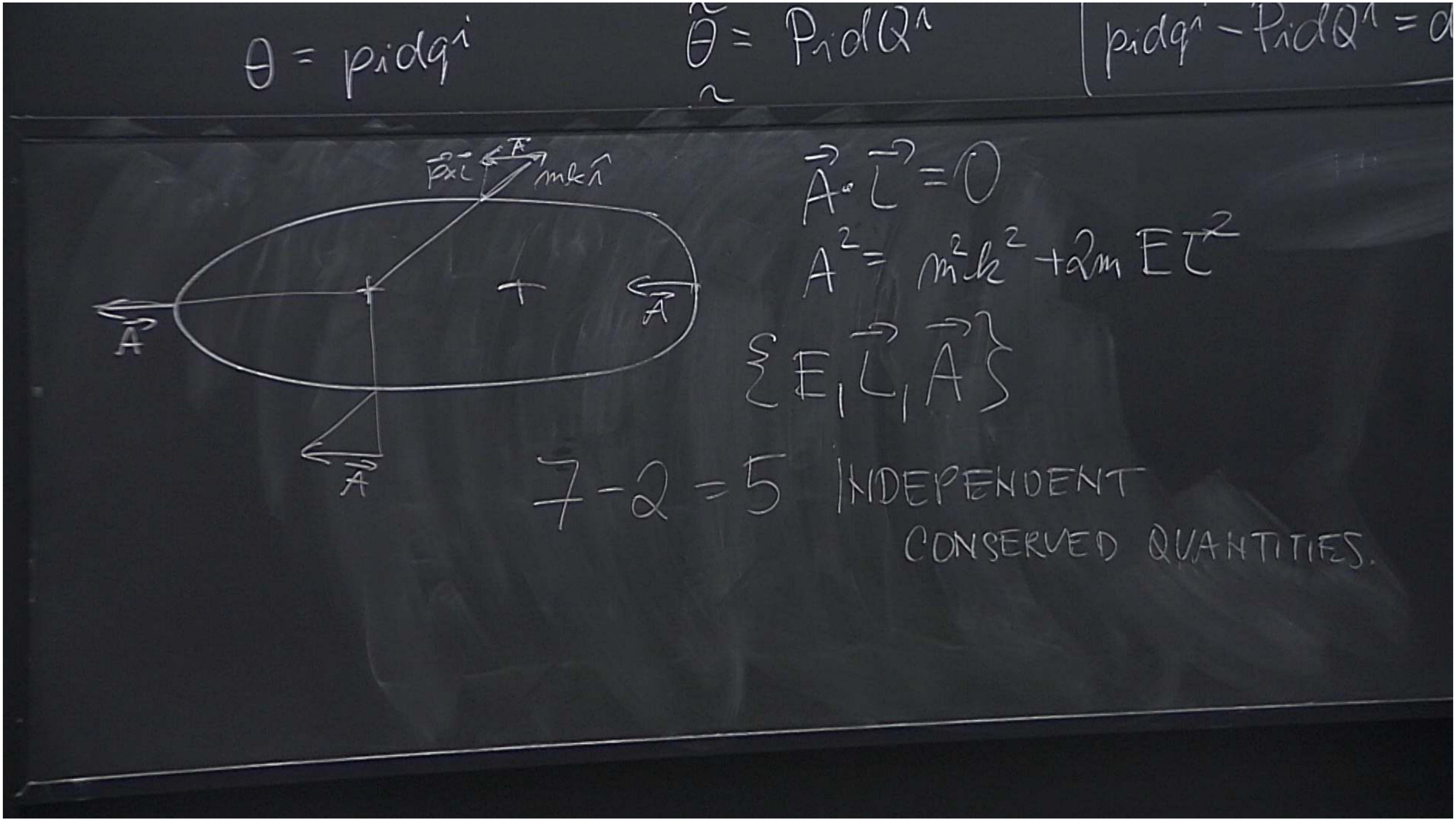
THIS IS ON THE LEVEL OF PHASE SPACE.

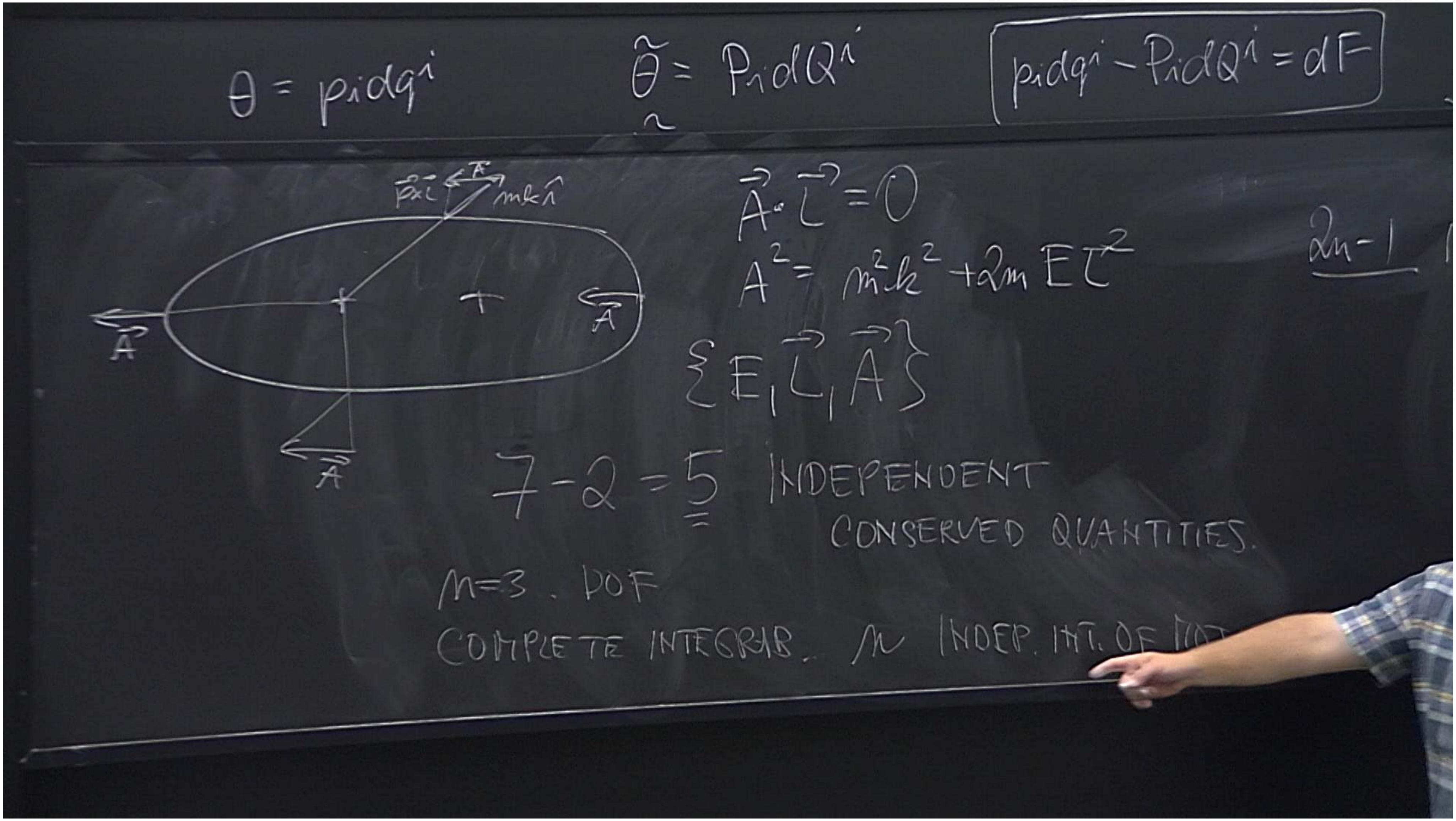
• DEFINE DYNAMICS BY SPECIFYING HAMILTONIAN

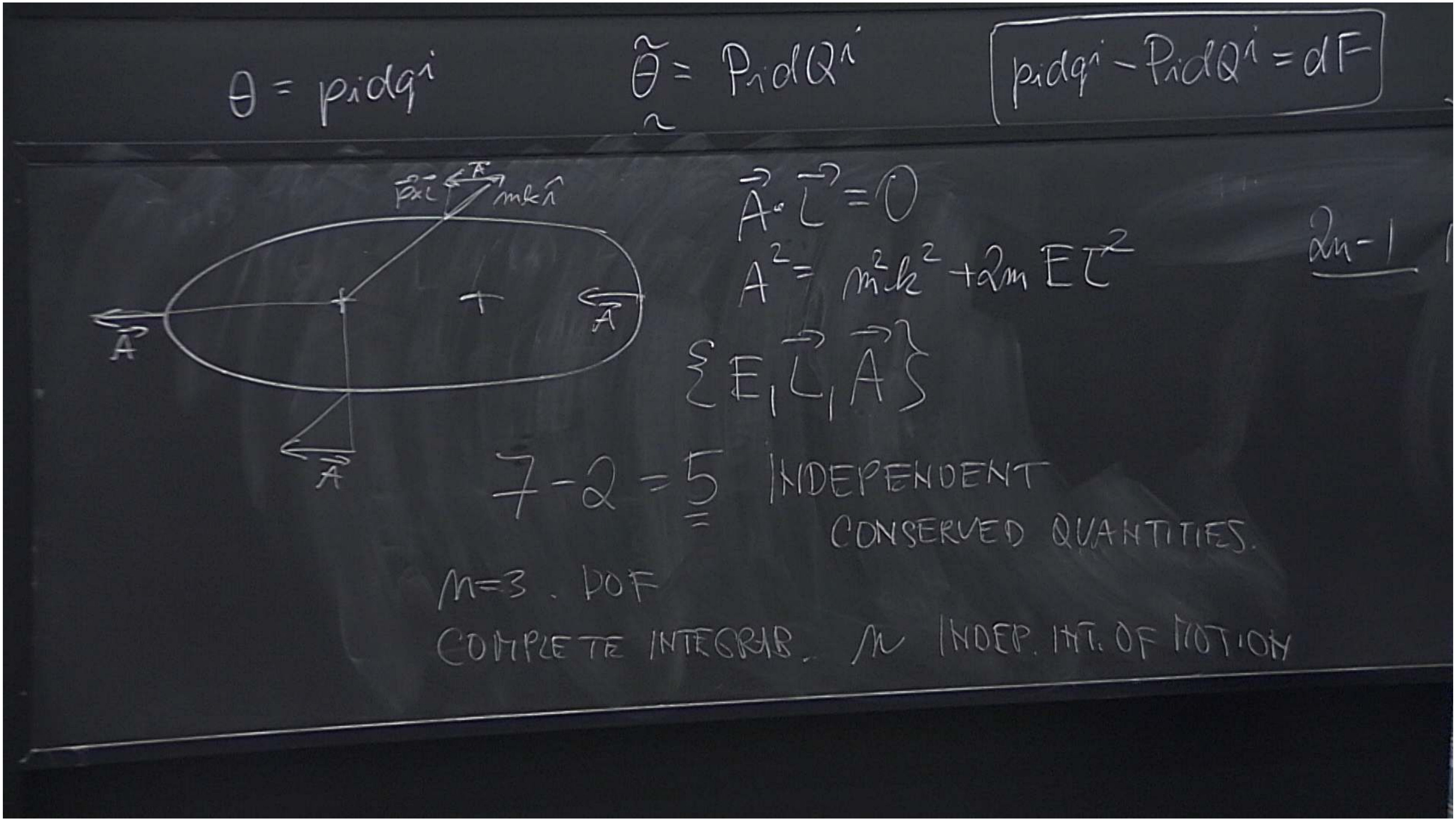
$H: T^*C \to \mathbb{R} \qquad H(q^i, p_i)$

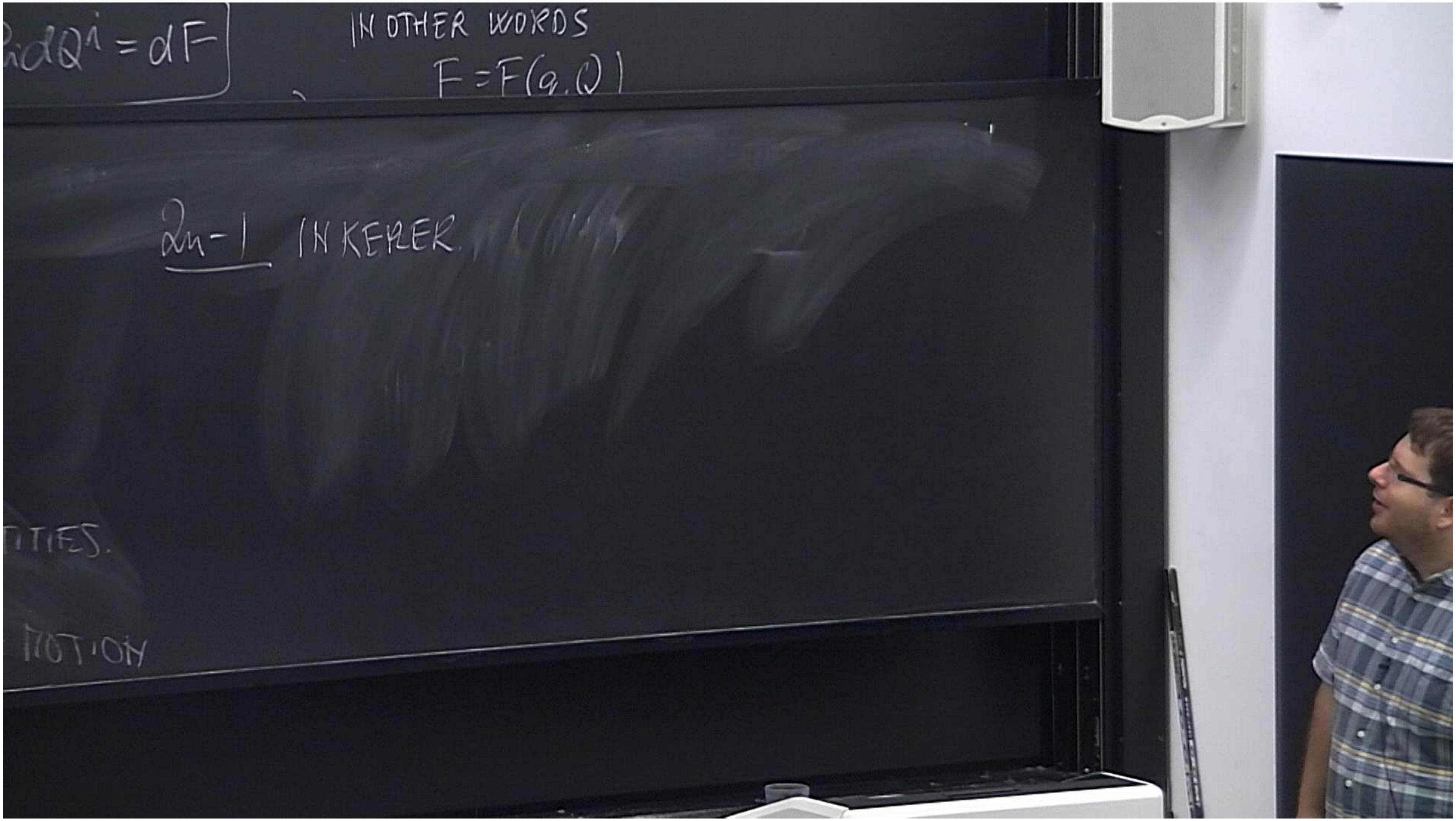

$\mathcal{L}_Y \omega = 0 = \mathcal{L}_Y H$ (SYMMETRY)
$\Rightarrow \exists I$ S.T. $\frac{dI}{dt} = \mathcal{L}_{X_H} I = 0$
$\cdot\ 0 = \mathcal{L}_Y H = \mathcal{L}_{Y_I} H = \{H, I\}$
$= -\{I, H\} = -\frac{dI}{dt} = 0$
THIS IS ON THE LEVEL OF PHASE SPACE
$\pi^*(Y)$
VECTOR FIELD ON "C" . SYMMETRY OF . ... ISOMETRY
NOT WELL DEFINED ON C ∴ DYNAMICAL (HIDDEN) SYMMETRY
EXAMPLE. KEPLER PROBLEM $\vec{F} = -\frac{k}{r^2}\hat{r}$
ISOMETRIES · STATIONARITY $\rightarrow E$. ENERGY
SPHERICAL SYMMETRY $\rightarrow \vec{L}$. ANG M.
• WE ALSO HAVE A HIDDEN SYMMETRY
GIVEN BY RUNGE-LENZ VECTOR
$\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}$


0 = d(θ - θ̃)
A
A
A
A

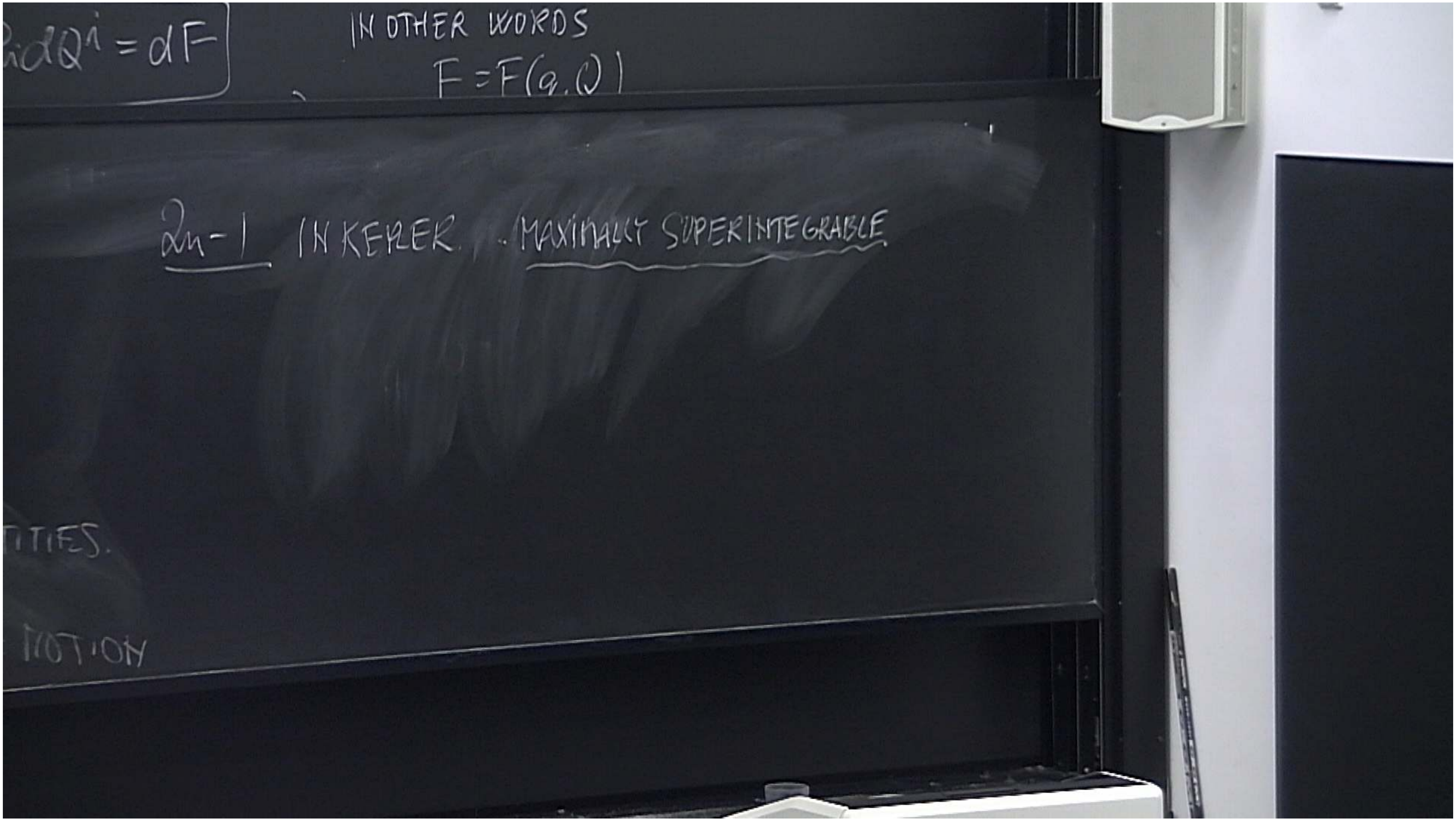

θ = p_i dq^i
θ̃ = P_i dQ^i
p_i dq^i − P_i dQ^i = d
p × L
A
mk r̂
A
A
A
A
A · L = 0
A² = m²k² + 2mEL²
{E, L, A}
7 − 2 = 5 INDEPENDENT
CONSERVED QUANTITIES.

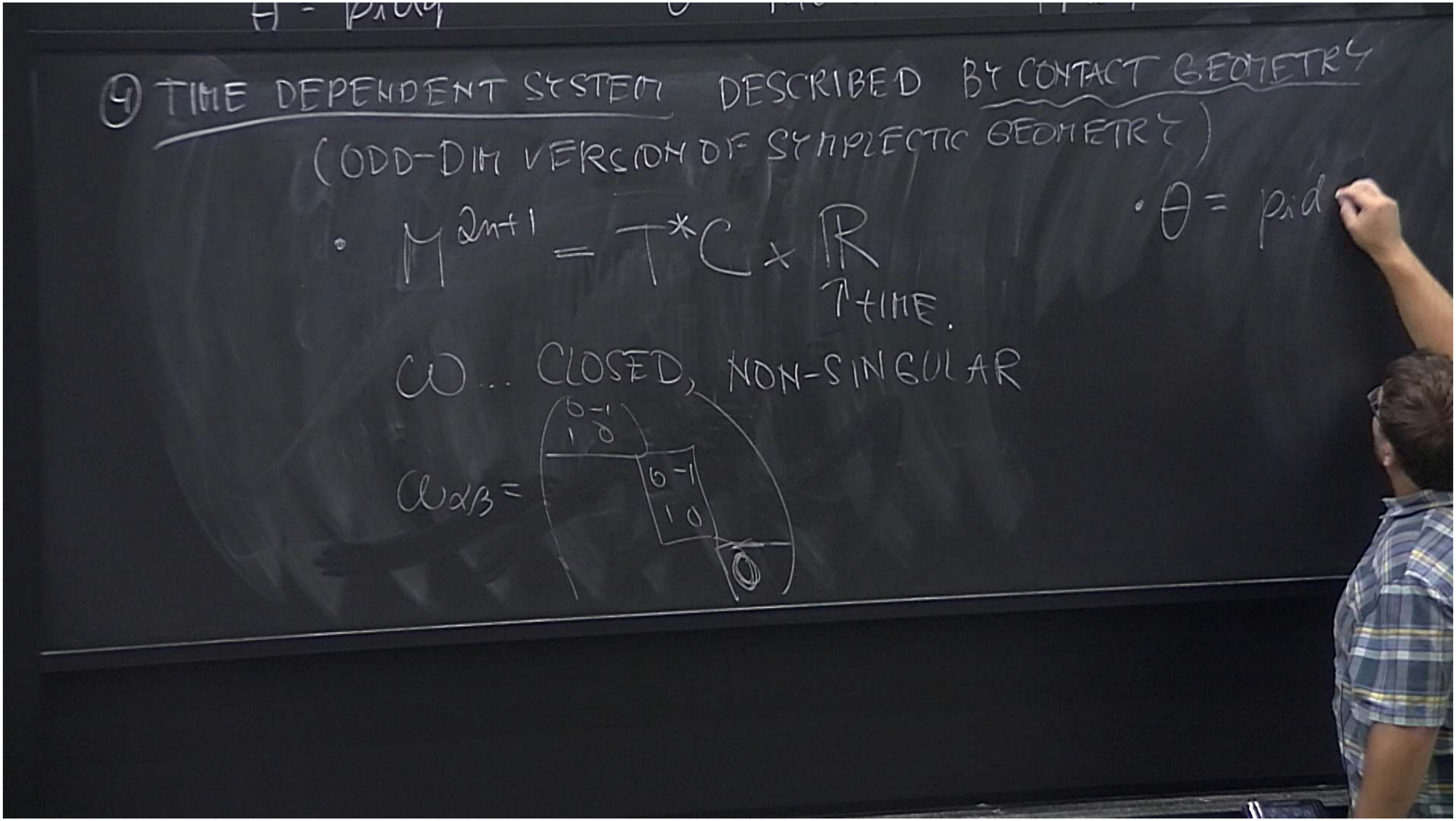

θ = p_i dq^i
θ̃ = P_i dQ^i
p_i dq^i − P_i dQ^i = dF
p × L
A
mkr̂
A · L = 0
A² = m²k² + 2mEL²
{E, L, A}
2n-1
7 − 2 = 5 INDEPENDENT
CONSERVED QUANTITIES.
n=3. DOF
COMPLETE INTEGRAB. n INDEP. INT. OF

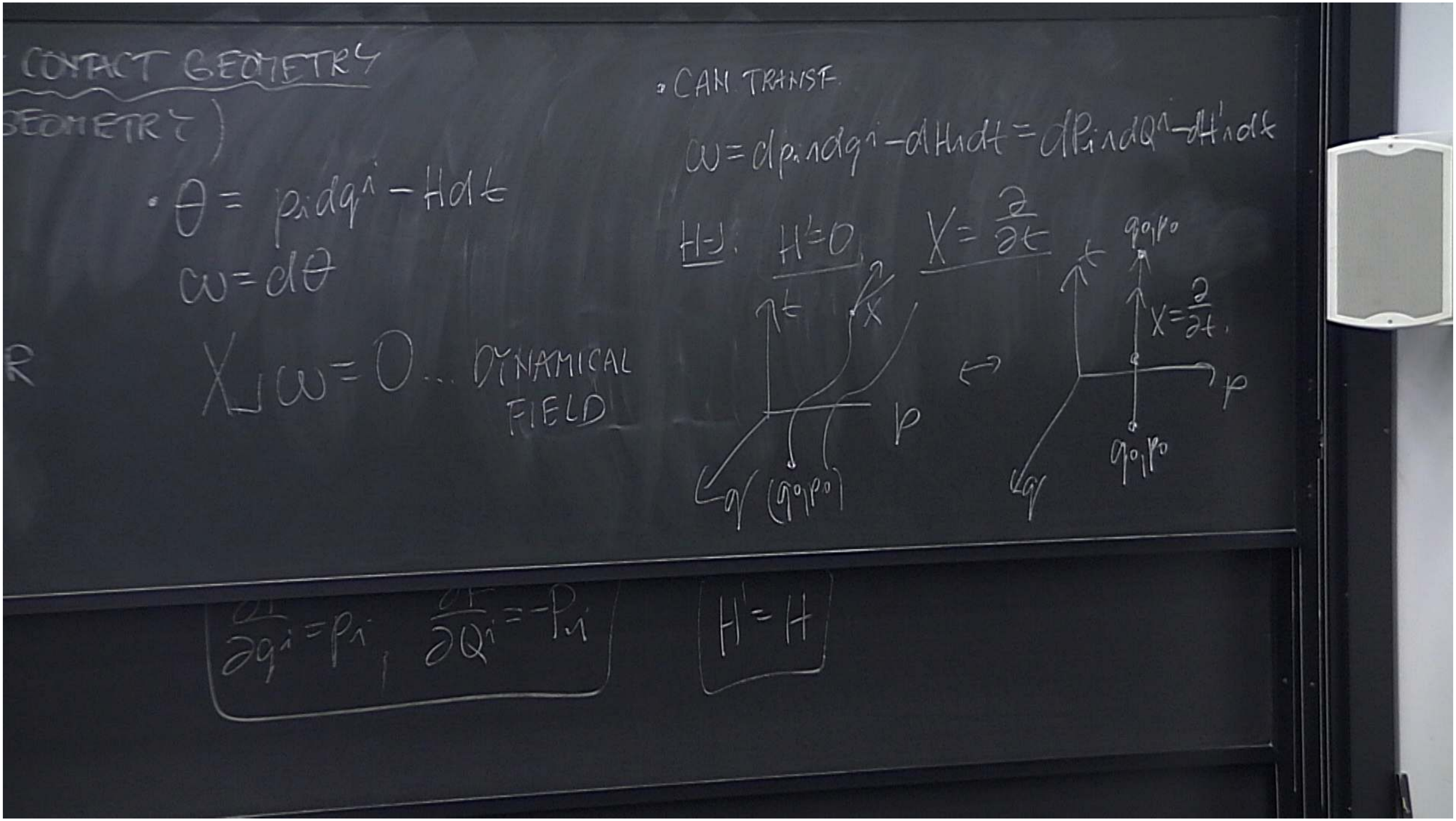

$\theta = p_i dq^i$
$\tilde{\theta} = P_i dQ^i$
$p_i dq^i - P_i dQ^i = dF$
$\vec{p}\times\vec{L}$
$\vec{A}$
$mk\hat{r}$
$\vec{A}$
$\vec{A}$
$\vec{A}$
$\vec{A}\cdot\vec{L} = 0$
$A^2 = m^2k^2 + 2mEL^2$
$\{E, \vec{L}, \vec{A}\}$
$2n-1$
$7-2=5$ INDEPENDENT CONSERVED QUANTITIES.
$n=3$. DOF
COMPLETE INTEGRAB. n INDEP. INT. OF MOTION

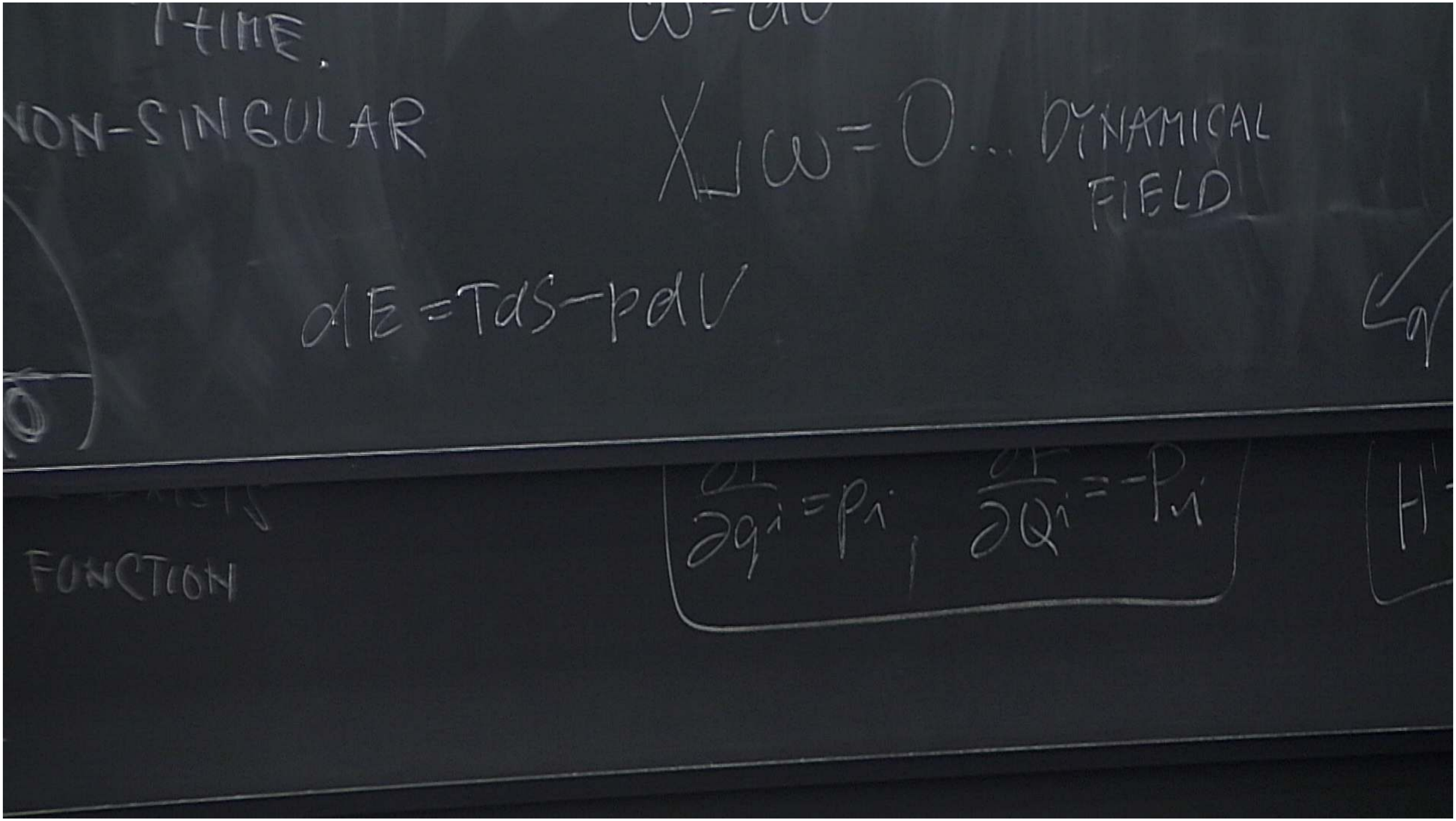


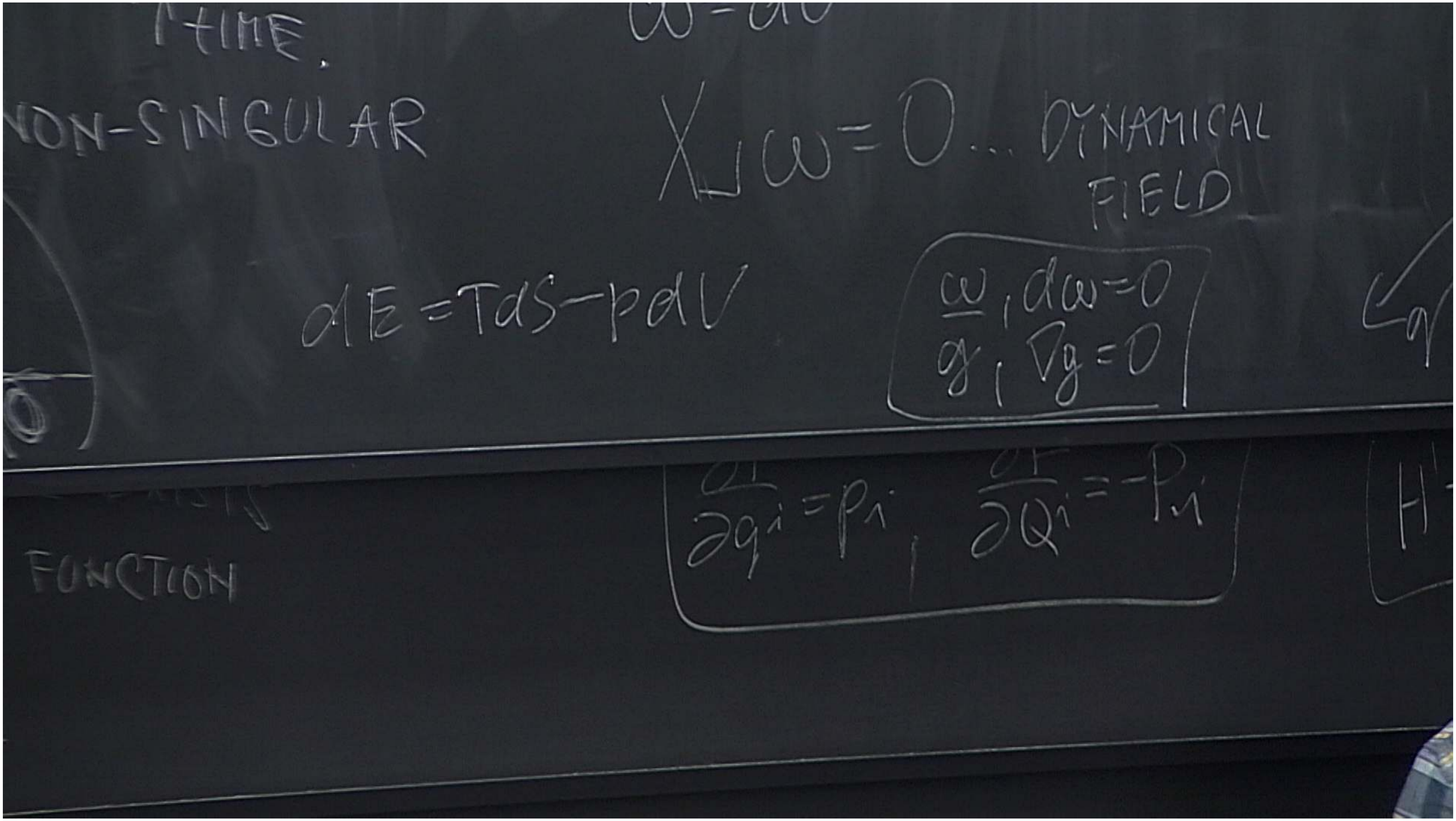
IN OTHER WORDS
F = F(q,Q)
2n-1 IN KERER
TITIES.
MOTION


dQ^i = dF
IN OTHER WORDS
F = F(q,Q)
2n-1 IN KEPLER .. MAXIMALLY SUPERINTEGRABLE
TITIES.
MOTION


④ TIME DEPENDENT SYSTEM DESCRIBED BY CONTACT GEOMETRY
(ODD-DIM VERSION OF SYMPLECTIC GEOMETRY)
• M^{2n+1} = T*C × R
↑ time.
• θ = p_i d
ω… CLOSED, NON-SINGULAR
ω_{αβ} =


CONTACT GEOMETRY
GEOMETRY
• θ = p_i dq^i − H dt
ω = dθ
X ⌟ ω = 0 ... DYNAMICAL FIELD
• CAN. TRANSF.
ω = dp_i ∧ dq^i − dH ∧ dt = dP_i ∧ dQ^i − dH' ∧ dt
H-J: H' = 0, X = ∂/∂t
t
X
p
q
(q_0, p_0)
↔
t
q_0, p_0
X = ∂/∂t
p
q
q_0, p_0
∂q^i = p_i , ∂Q^i = −P_i
H' = H


time.
NON-SINGULAR
$X \lrcorner \omega = 0$ ... DYNAMICAL FIELD
$dE = TdS - pdV$
FUNCTION
$\frac{\partial F}{\partial q^i} = p_i , \frac{\partial F}{\partial Q^i} = -P_i$



TIME.
ON-SINGULAR
X⌟ω = 0 ... DYNAMICAL FIELD
dE = TdS − pdV
ω, dω = 0
g, ∇g = 0
∂q^i = p_i, ∂Q^i = −P_i
FUNCTION