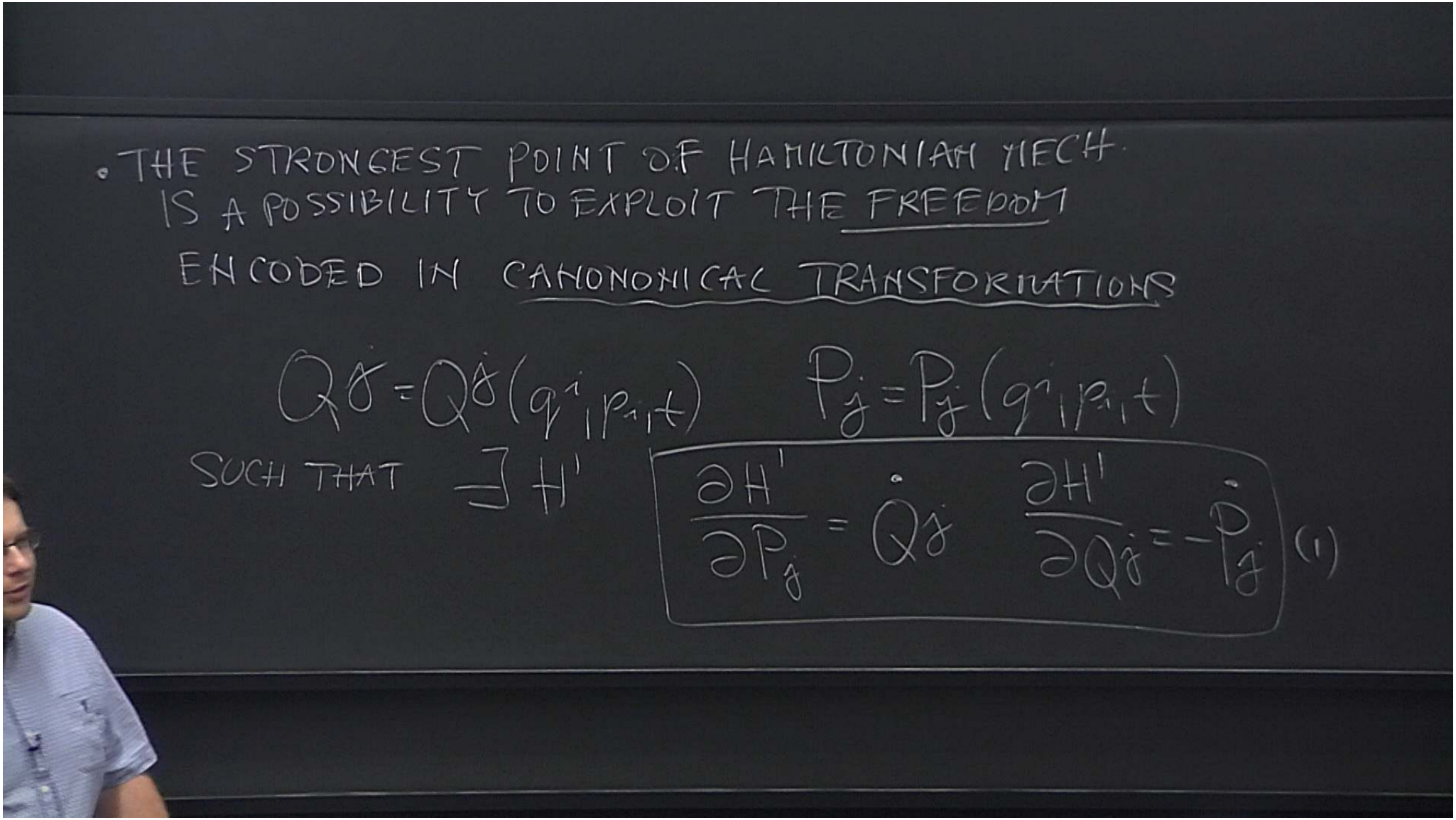
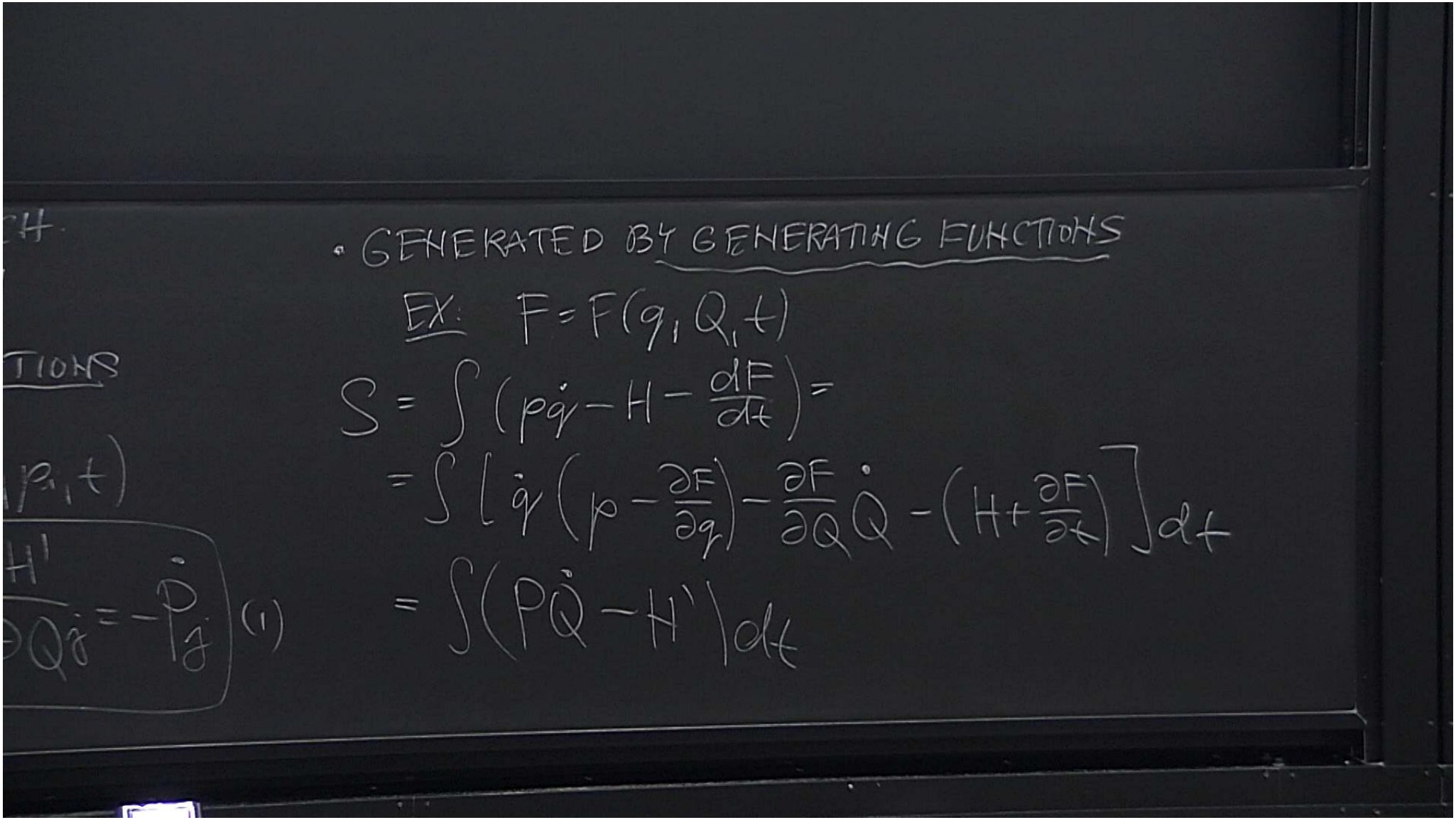
Title: PSI 17/18 Front End - Theoretical Mechanics (Kubiznak)

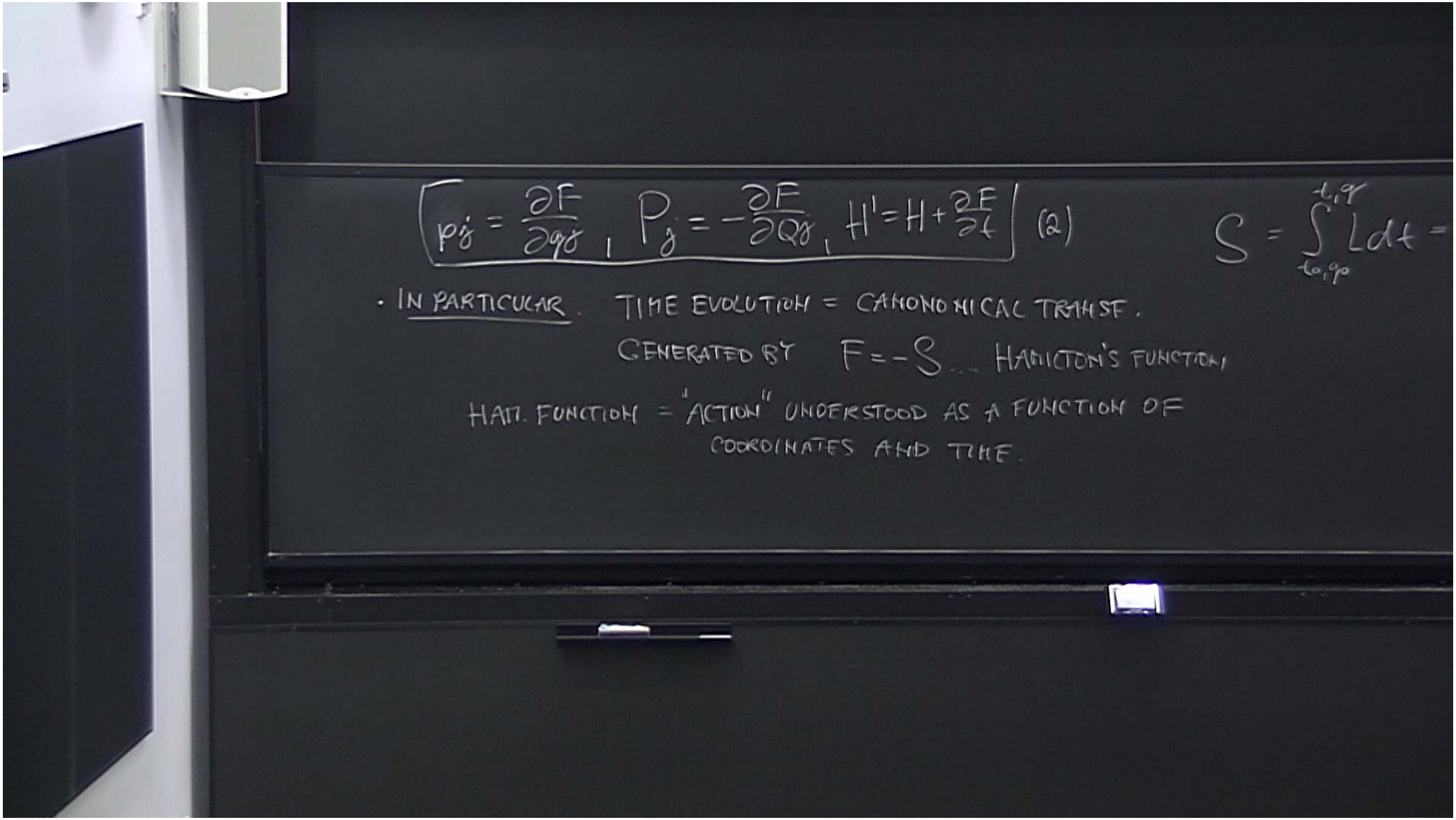
Date: Aug 16, 2017 09:00 AM

URL: http://pirsa.org/17080047

Abstract:


• THE STRONGEST POINT OF HAMILTONIAN MECH.
IS A POSSIBILITY TO EXPLOIT THE FREEDOM
ENCODED IN CANONONICAL TRANSFORMATIONS
$Q^j = Q^j(q^i, p_i, t)$ $P_j = P_j(q^i, p_i, t)$
SUCH THAT $\exists H'$
$\frac{\partial H'}{\partial P_j} = \dot{Q}^j$ $\frac{\partial H'}{\partial Q^j} = -\dot{P}_j$ (1)


• GENERATED BY GENERATING FUNCTIONS
EX: $F = F(q, Q, t)$
$S = \int \left(p\dot{q} - H - \frac{dF}{dt} \right) =$
$= \int \left[\dot{q}\left(p - \frac{\partial F}{\partial q} \right) - \frac{\partial F}{\partial Q}\dot{Q} - \left(H + \frac{\partial F}{\partial t} \right) \right] dt$
$= \int \left(P\dot{Q} - H' \right) dt$


$p_i = \frac{\partial F}{\partial q_i}$, $P_i = -\frac{\partial F}{\partial Q_i}$, $H' = H + \frac{\partial F}{\partial t}$ (2)
$S = \int_{t_0,q_0}^{t,q} L\,dt =$
• IN PARTICULAR. TIME EVOLUTION = CANONICAL TRANSF.
GENERATED BY $F = -S$ ... HAMILTON'S FUNCTION
HAM. FUNCTION = "ACTION" UNDERSTOOD AS A FUNCTION OF
COORDINATES AND TIME.

$$p_j = \frac{\partial F}{\partial q_j}, \quad P_j = -\frac{\partial F}{\partial Q_j}, \quad H' = H + \frac{\partial F}{\partial t} \qquad (2)$$

$$S = \int_{t_0, q_0}^{t, q} L\,dt =$$

- IN PARTICULAR. TIME EVOLUTION = CANONICAL TRANSF.
  GENERATED BY $F = -S$ ... HAMILTON'S FUNCTION

HAM. FUNCTION = "ACTION" UNDERSTOOD AS A FUNCTION OF COORDINATES AND TIME.

(2)

$$S = \int_{t_0,q_0}^{t,q} L\,dt = \left| \begin{array}{l} \text{ON SHELL - FIXED TRAJ} \\ q = q(t_0, q_0, t) \end{array} \right| = S(q_0, t_0, q, t)$$

$$\simeq \int_{t_0}^{t} (p\,dq - H\,dt)$$

TRANSF.

ICTON'S FUNCTION

NCTION OF

$$dS = (p_i\,dq^i - H\,dt)_{\text{FINAL}} - (p_i\,dq^i - H\,dt)_{\text{INITIAL}}.$$

(2)

$$S = \int_{t_0,q_0}^{t,q} L\,dt = \left| \begin{array}{l} \text{ON SHELL - FIXED TRAJ} \\ q = q(t_0,q_0,t) \end{array} \right| = S(q_0,t_0,q,t)$$

$$\simeq \int_{t_0}^{t} (p\,dq - H\,dt)$$

AL TRANSF.

HAMILTON'S FUNCTION

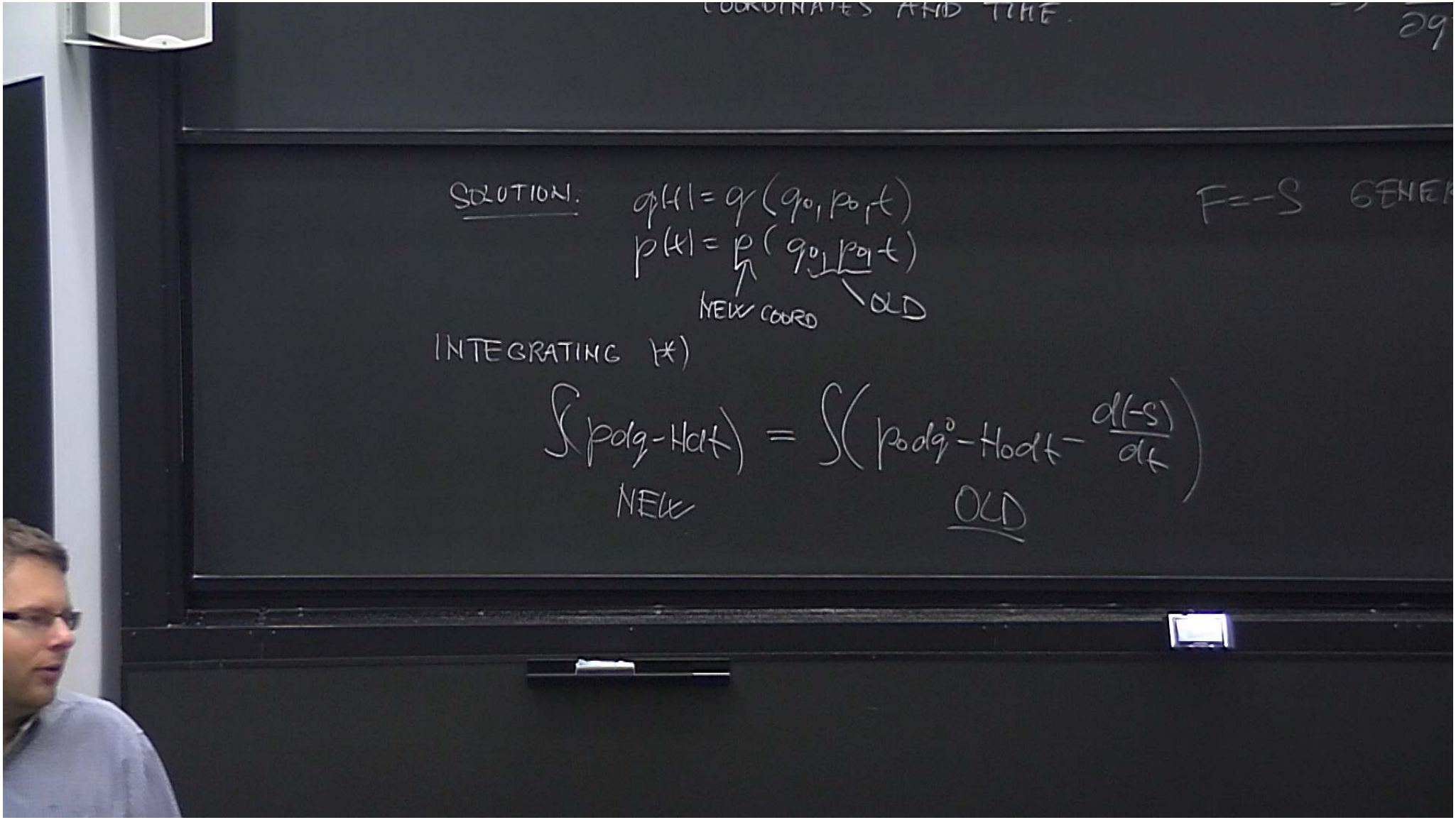
FUNCTION OF

F.

$$dS = (p_i\,dq^i - H\,dt)_{\text{FINAL}} - (p_i\,dq^i - H\,dt)_{\text{INITIAL}}$$

$$\Rightarrow \frac{\partial S}{\partial q^i} = p_i \qquad \underline{\frac{\partial S}{\partial t} = -H}$$

H-J EQUATION


COORDINATES AND TIME.
SOLUTION:
$q(t) = q(q_0, p_0, t)$
$p(t) = p(q_0, p_0, t)$
NEW COORD
OLD
F=-S GENE
INTEGRATING (*)
$\int (p\,dq - H\,dt) = \int (p_0\,dq^0 - H_0\,dt - \frac{d(-S)}{dt})$
NEW
OLD

$H' = H + \frac{\partial F}{\partial t}$ (2)

$S = \int_{t_0,q_0}^{t,q} L\,dt = \left| \text{ON SHELL - FIXED TIMES} \right| = S(q_0, t_0, q, t)$

$q = q(t_0, q_0, t)$

$= \int_{t_0}^{t} (p\,dq - H\,dt)$

...OH = CANONICAL TRANSF.

...T $F = -S$ ... HAMILTON'S FUNCTION

...UNDERSTOOD AS A FUNCTION OF

...INATES AND TIME.

$dS = (p_i\,dq^i - H\,dt)_{\text{final}} - (p_i\,dq^i - H\,dt)_{\text{initial}}$ (*)

$\Rightarrow \frac{\partial S}{\partial q^i} = p_i \qquad \underline{\frac{\partial S}{\partial t} = -H}$ H-J EQUATION

$F = -S$ GENERATES TIME EVOLUTION. $S = S(q_0, q, t)$

(q_0, p_0, t)

(q_0, p_0, t) — OLD

...URED

$= S\left(p_0\,dq^0 - H_0\,dt - \frac{d(-S)}{dt}\right)$

OLD

$H' = H + \frac{\partial F}{\partial t}$ (2)

$S = \int_{t_0, q_0}^{t,q} L\,dt = \left| \text{ON SHELL - FIXED TIMES } q = q(t_0, q_0, t) \right| = S(q_0, t_0, q, t)$

$= \int_{t_0}^{t} (p\dot{q} - H)\,dt$

OH = CANONICAL TRANSF.

F = -S ... HAMILTON'S FUNCTION

UNDERSTOOD AS A FUNCTION OF

INATES AND TIME.

$dS = (p_i\,dq^i - H\,dt)_{\text{final}} - (p_i\,dq^i - H\,dt)_{\text{initial}}$ (*)

$\Rightarrow \frac{\partial S}{\partial q^i} = p_i \qquad \underline{\frac{\partial S}{\partial t} = -H}$ H-J EQUATION

(q_0, p_0, t)

(q_0, p_q, t)

OLD

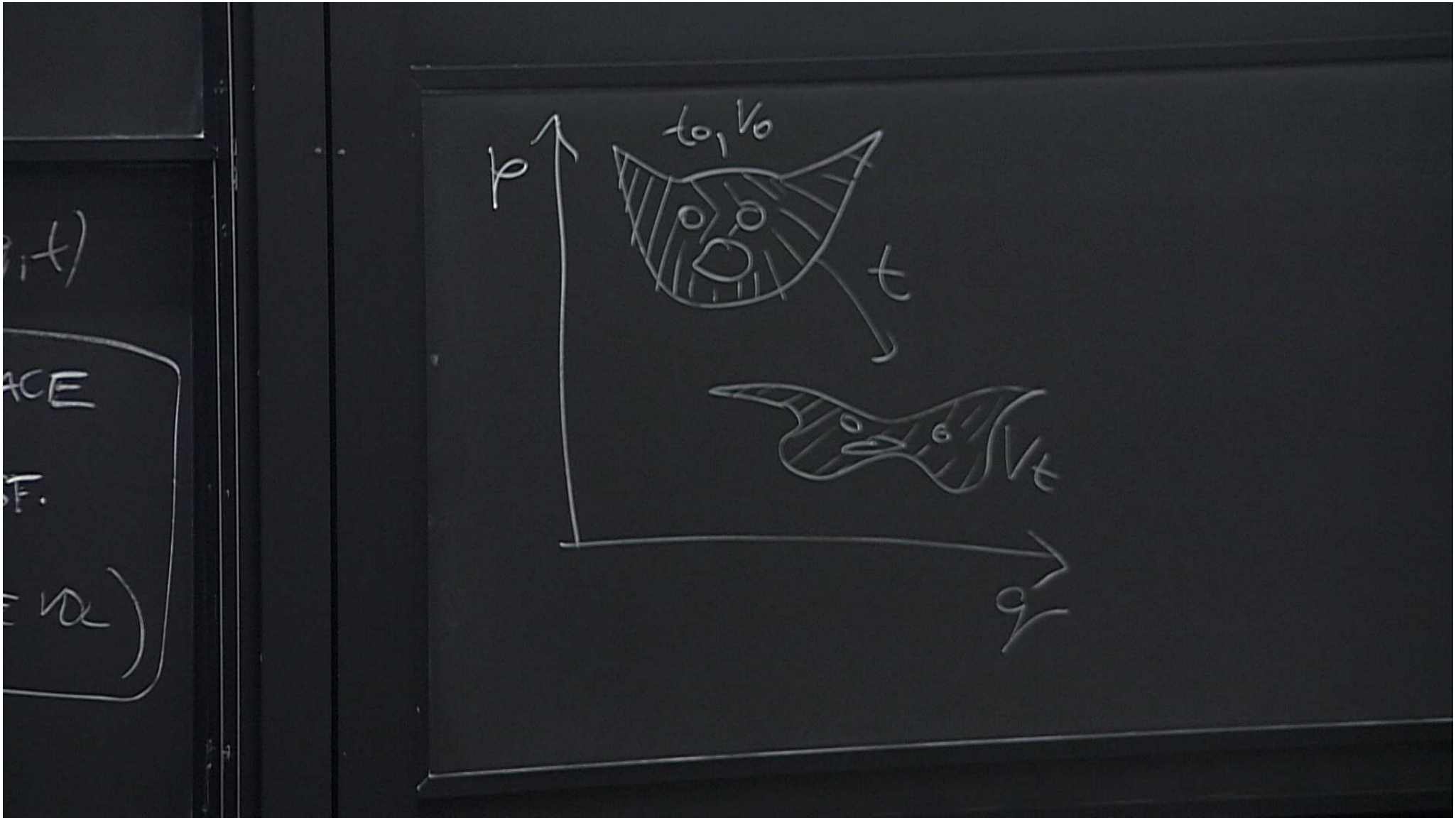
URED

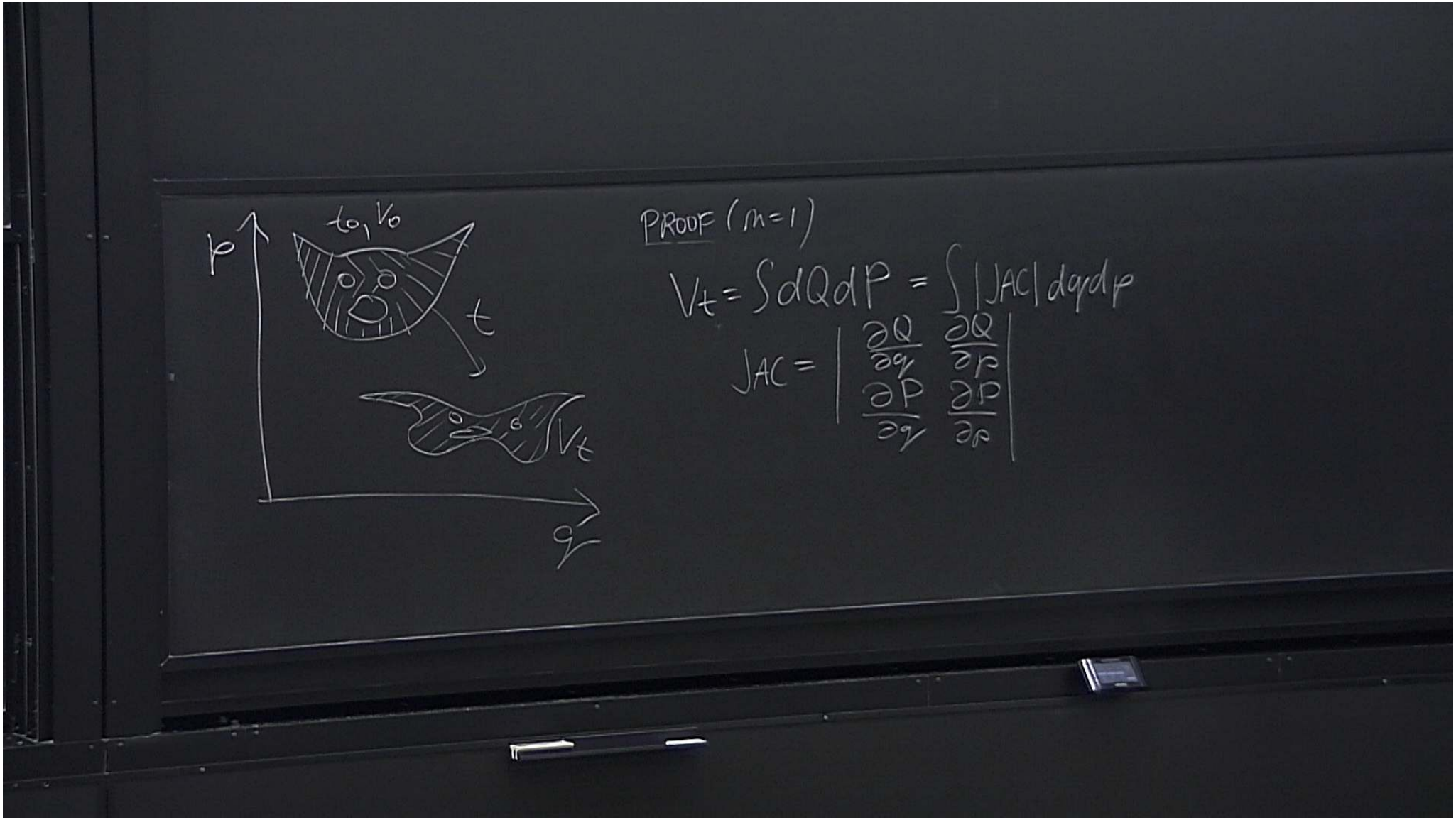
$= S\left(p_0 dq^0 - H_0 dt - \frac{d(-S)}{dt}\right)$

OLD

F = -S GENERATES TIME EVOLUTION. $S = S(q_0, q, t)$

LIOUVILLE'S THEOREM: VOLUME OF THE PHASE SPACE REMAINS INVARIANT UNDER CANONICAL TRANSF. (TIME EVOLUTION PRESERVES THE PHASE SPACE VOL)

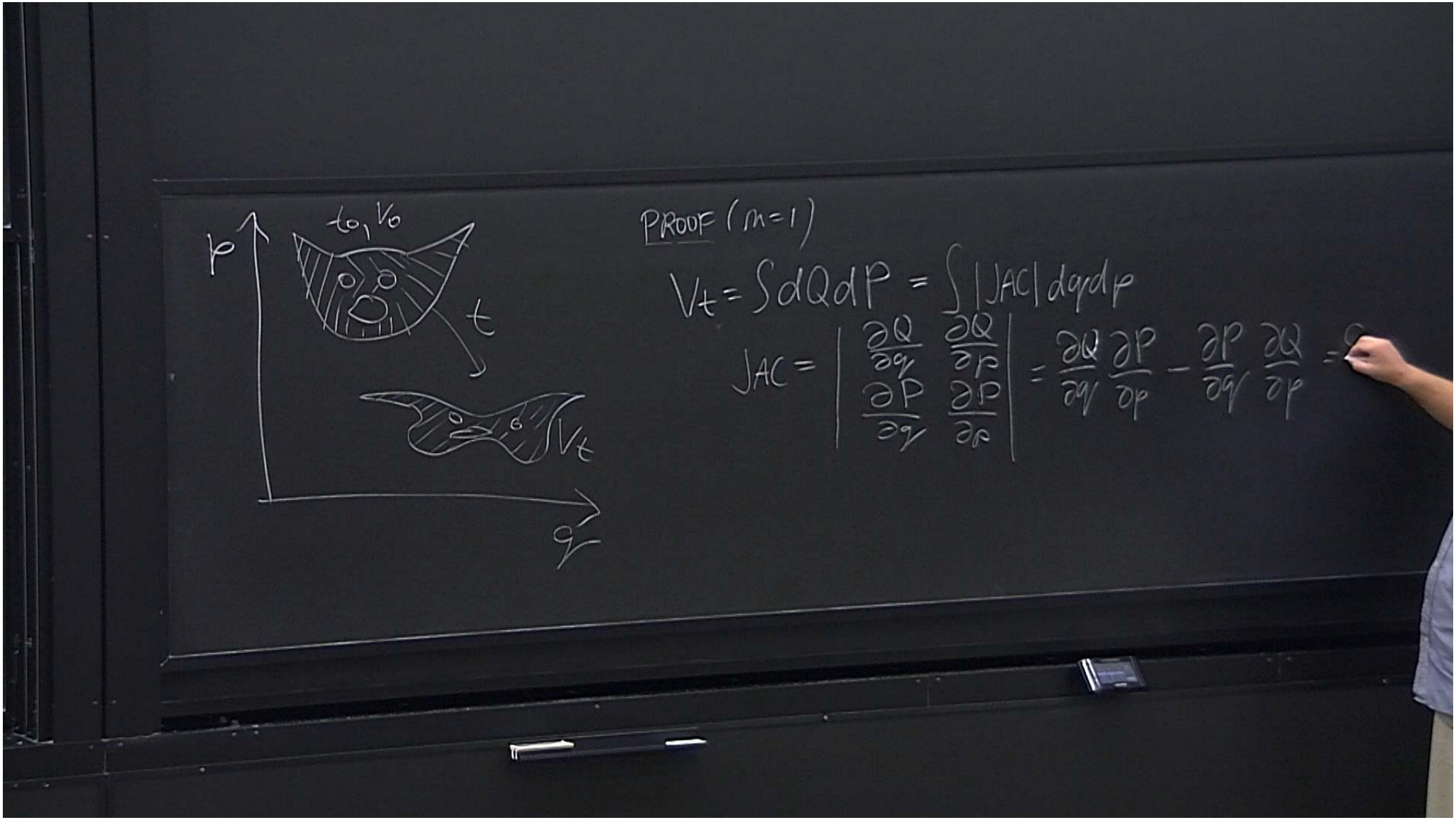

p
t_0, V_0
t
V_t
q



PROOF ($n=1$)

$$V_t = \int dQ\,dP = \int |JAC|\,dq\,dp$$

$$JAC = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix}$$



PROOF ($n=1$)

$$V_t = \int dQ\,dP = \int |JAC|\,dq\,dp$$

$$JAC = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 0$$

PROOF ($n=1$)

$$V_t = \int dQ\,dP = \int |JAC|\, dq\,dp = \int dq\,dp = V_0 .$$

$$JAC = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = \{Q, P\} = 1$$

d) HAMILTON-JACOBI THEORY

IDEA. FIND SPECIAL GENERATING FUNCTION (S)
SO THAT $H' = 0$.

(1) $\Rightarrow$ $Q^j = \alpha^j = \text{CONST}$, $P_j = \beta_j = \text{CONST}$

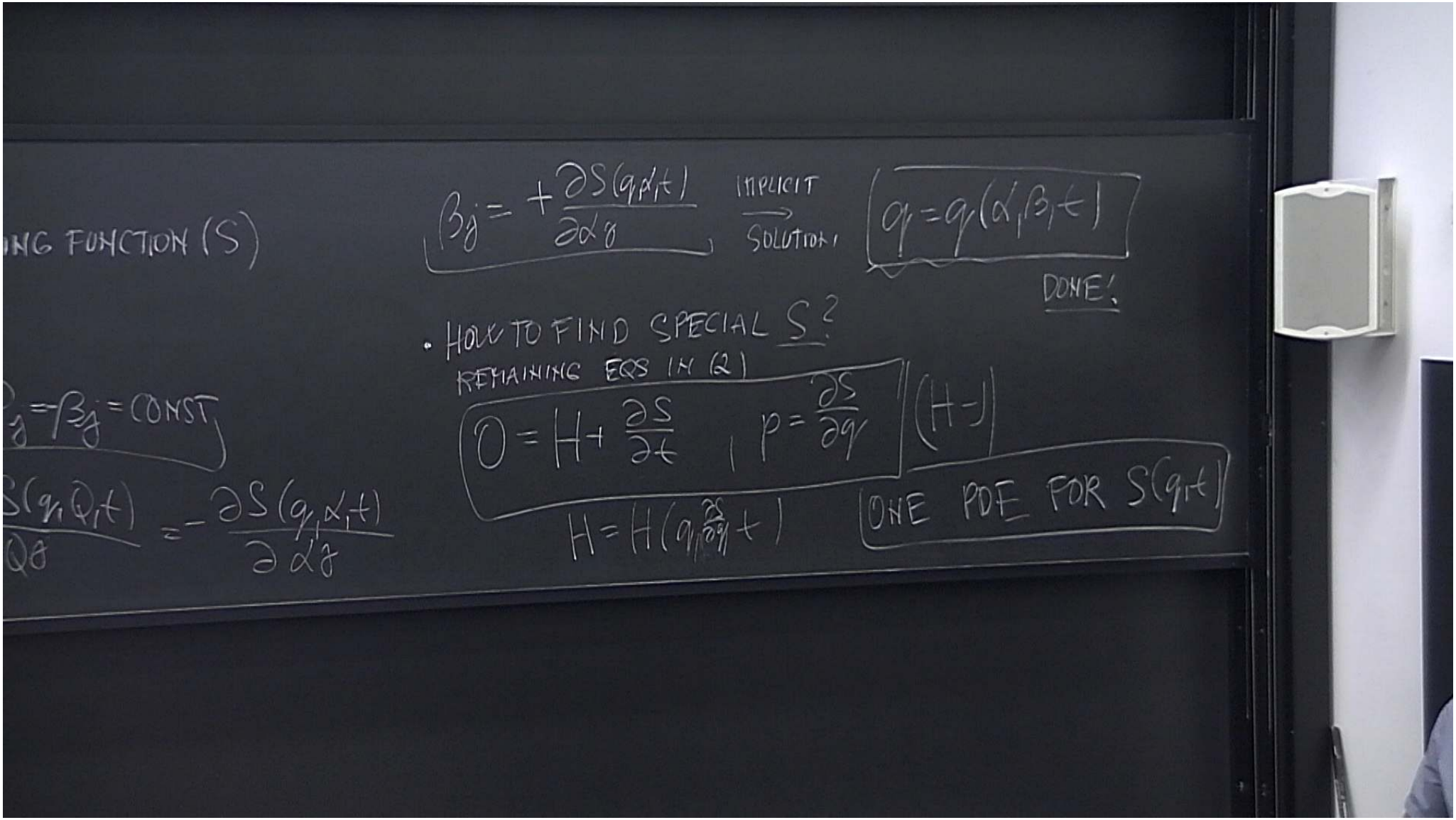
d) HAMILTON-JACOBI THEORY

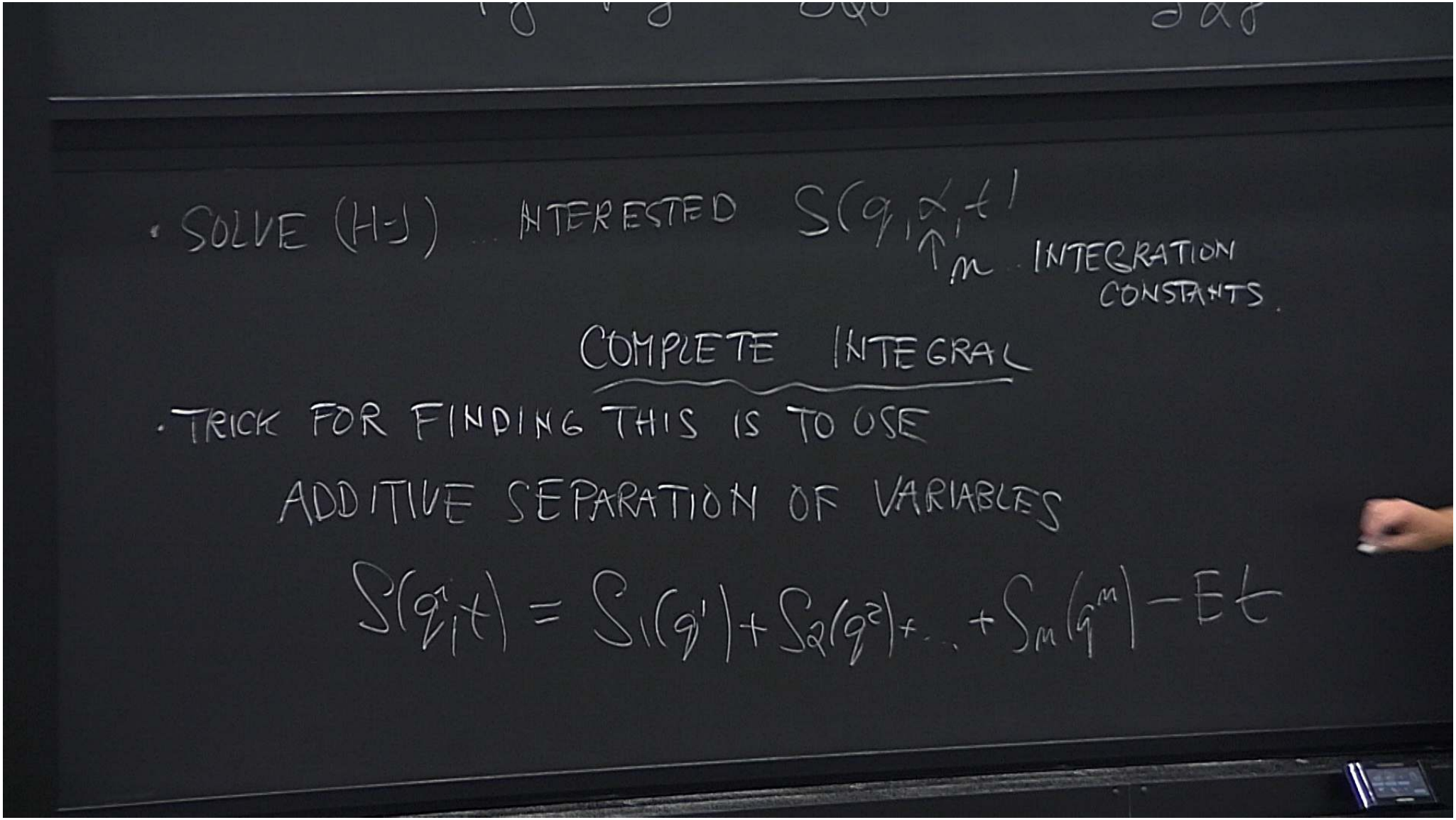
IDEA: FIND SPECIAL GENERATING FUNCTION (S)
SO THAT $H' = 0$.

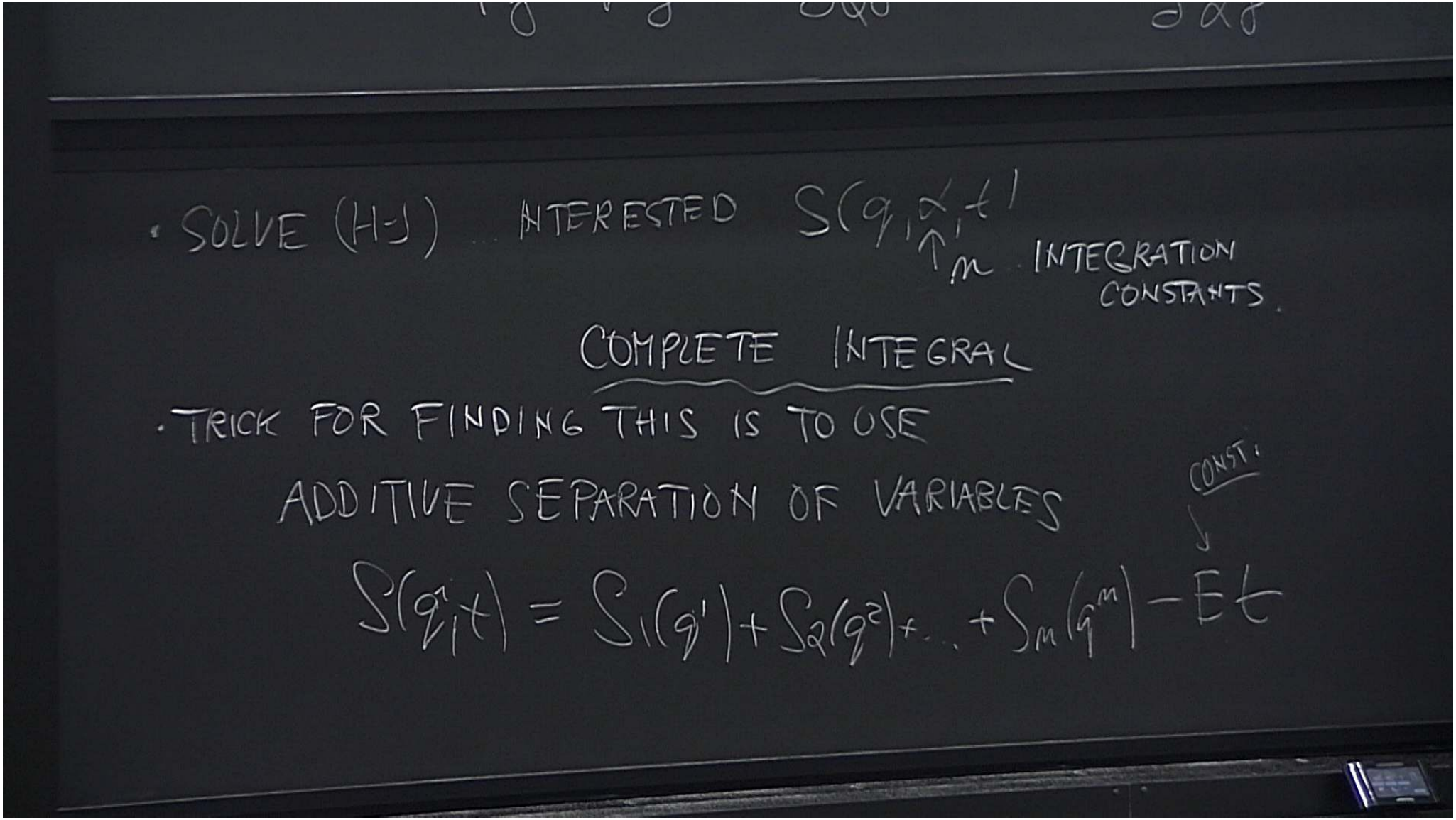
(1) $\Rightarrow$ $\dot{Q}_\delta = \dot{\alpha}_\delta = \text{CONST}$, $\dot{P}_j = \dot{\beta}_j = \text{CONST}$

THE MIDDLE (2)

$$P_j = -\beta_j = -\frac{\partial S(q_i, Q_i, t)}{\partial Q_\delta} = -\frac{\partial S(q_i, \alpha_i, t)}{\partial \alpha_\delta}$$


NG FUNCTION (S)
$\beta_j = +\frac{\partial S(q,\alpha,t)}{\partial \alpha_j}$
IMPLICIT
SOLUTION
$q_i = q_i(\alpha,\beta,t)$
DONE!
• HOW TO FIND SPECIAL S?
REMAINING EQS IN (2)
$0 = H + \frac{\partial S}{\partial t}$, $p = \frac{\partial S}{\partial q}$
(H-J)
$H = H(q, \frac{\partial S}{\partial q}, t)$
ONE PDE FOR S(q,t)
$\beta_j = \text{CONST}$
$\frac{\partial S(q,Q,t)}{\partial Q} = -\frac{\partial S(q,\alpha,t)}{\partial \alpha_j}$


• SOLVE (H-J) ... INTERESTED $S(q, \alpha, t)$
α: m INTEGRATION CONSTANTS.
COMPLETE INTEGRAL
• TRICK FOR FINDING THIS IS TO USE
ADDITIVE SEPARATION OF VARIABLES
$S(q,t) = S_1(q^1) + S_2(q^2) + \ldots + S_m(q^m) - Et$


• SOLVE (H-J) ... INTERESTED $S(q, \alpha, t)$
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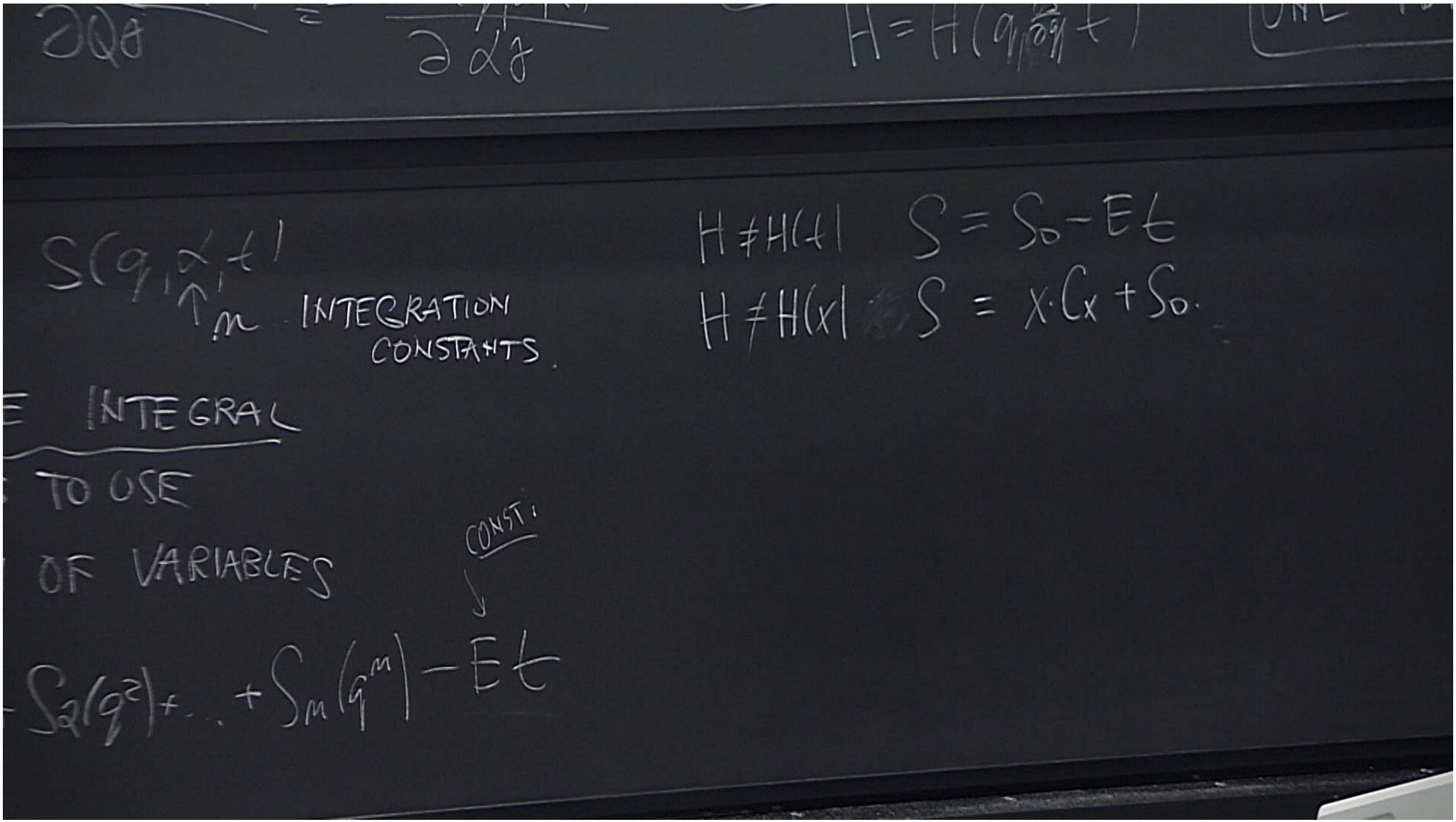
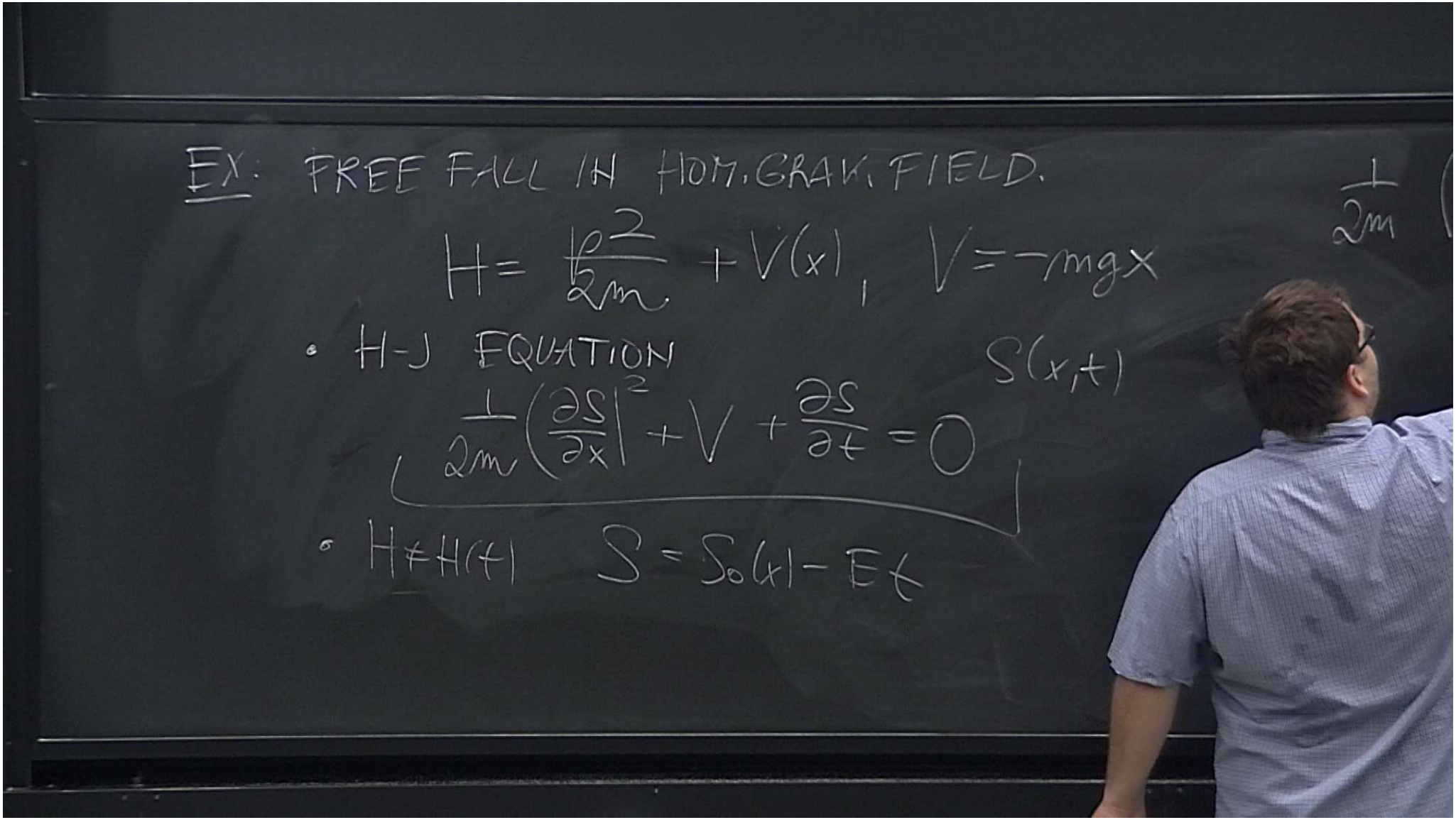
$S(q,t) = S_1(q^1) + S_2(q^2) + \ldots + S_m(q^m) - Et$


$\frac{\partial Q_\theta}{\partial \alpha_\gamma}$
$H = H(q, p, t)$
$S(q, \alpha, t)$
n INTEGRATION CONSTANTS
$H \neq H(t)$ $S = S_0 - Et$
$H \neq H(x)$ $S = x \cdot C_x + S_0$
INTEGRAL
TO USE
OF VARIABLES
CONST:
$S_2(q^2) + \ldots + S_n(q^n) - Et$


EX: FREE FALL IN HOM. GRAV. FIELD.
$H = \frac{p^2}{2m} + V(x), \quad V = -mgx$
• H-J EQUATION
$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V + \frac{\partial S}{\partial t} = 0$
$S(x,t)$
• $H \neq H(t) \quad S = S_0(x) - Et$
$\frac{1}{2m}$

$\frac{\partial S}{\partial t} = -H$ H-J EQUATION

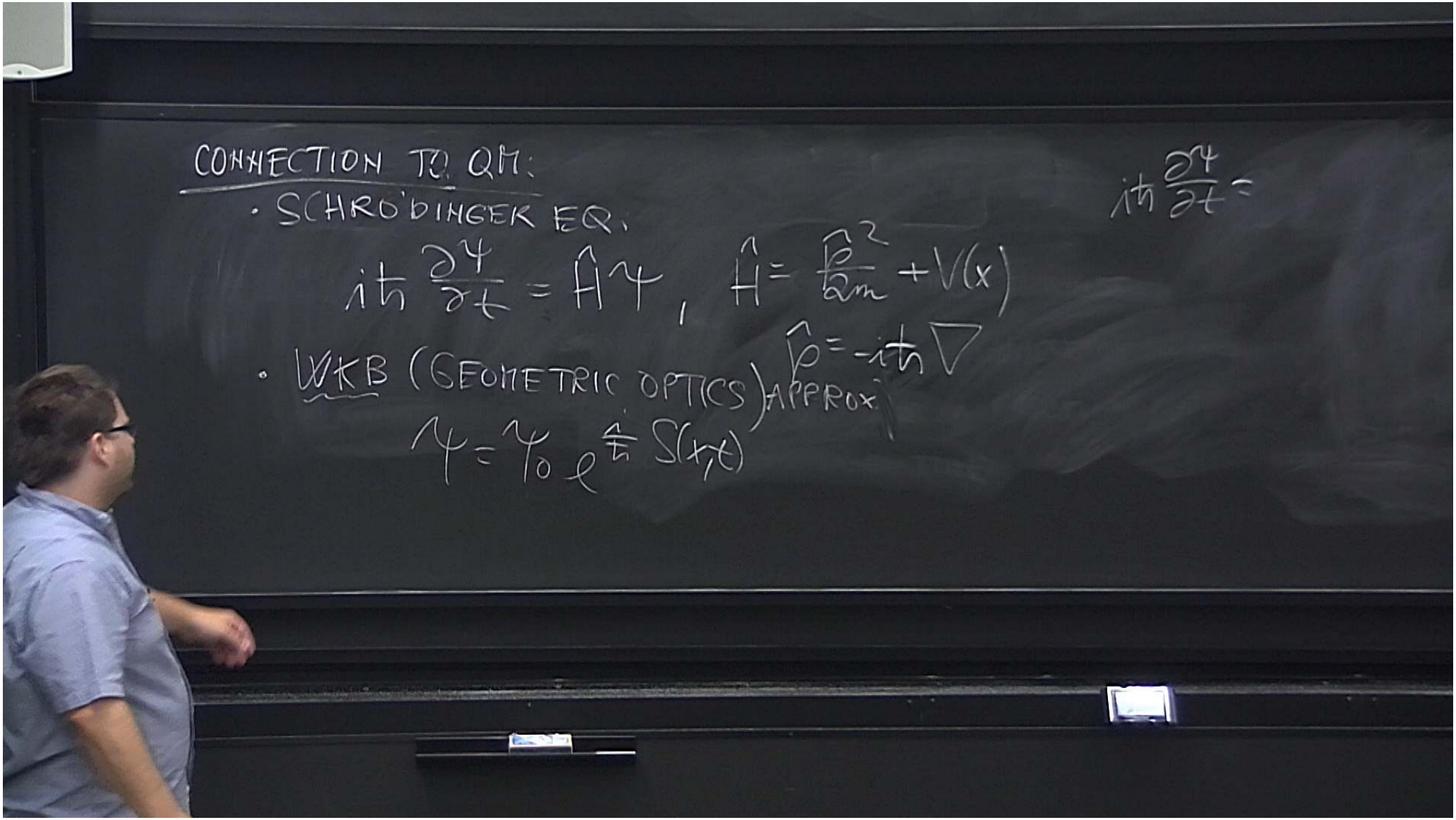
$$\frac{1}{2m}\left(\frac{dS_0}{dx}\right)^2 + V - E = 0 \Rightarrow S_0 = \int \sqrt{2m(E-V)}\,dx$$

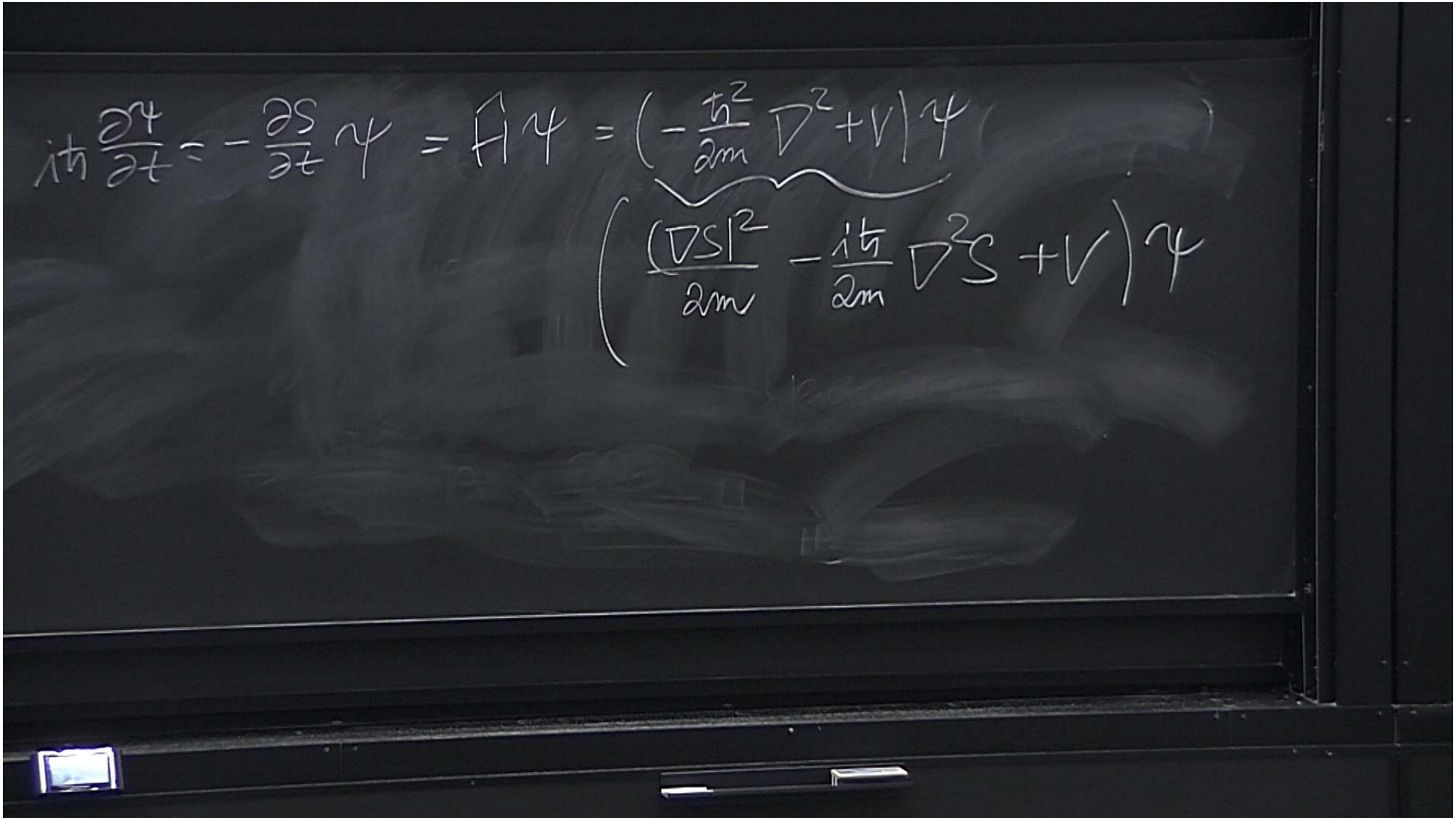
$$\boxed{S = \frac{1}{3gm^2}\left[2m(E+mgx)\right]^{3/2} - Et}$$

$$\frac{\partial S}{\partial E} = \beta = \frac{1}{mg}\sqrt{2m(E-V)} - t$$

BY INVERSION

$$\Rightarrow \boxed{x = \frac{g}{2}(t+\beta)^2 - \frac{E}{mg}}$$


CONNECTION TO QM:
• SCHRÖDINGER EQ.
$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$, $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$
$\hat{p} = -i\hbar\nabla$
$i\hbar \frac{\partial \psi}{\partial t} =$
• WKB (GEOMETRIC OPTICS) APPROX:
$\psi = \psi_0 e^{\frac{i}{\hbar} S(x,t)}$

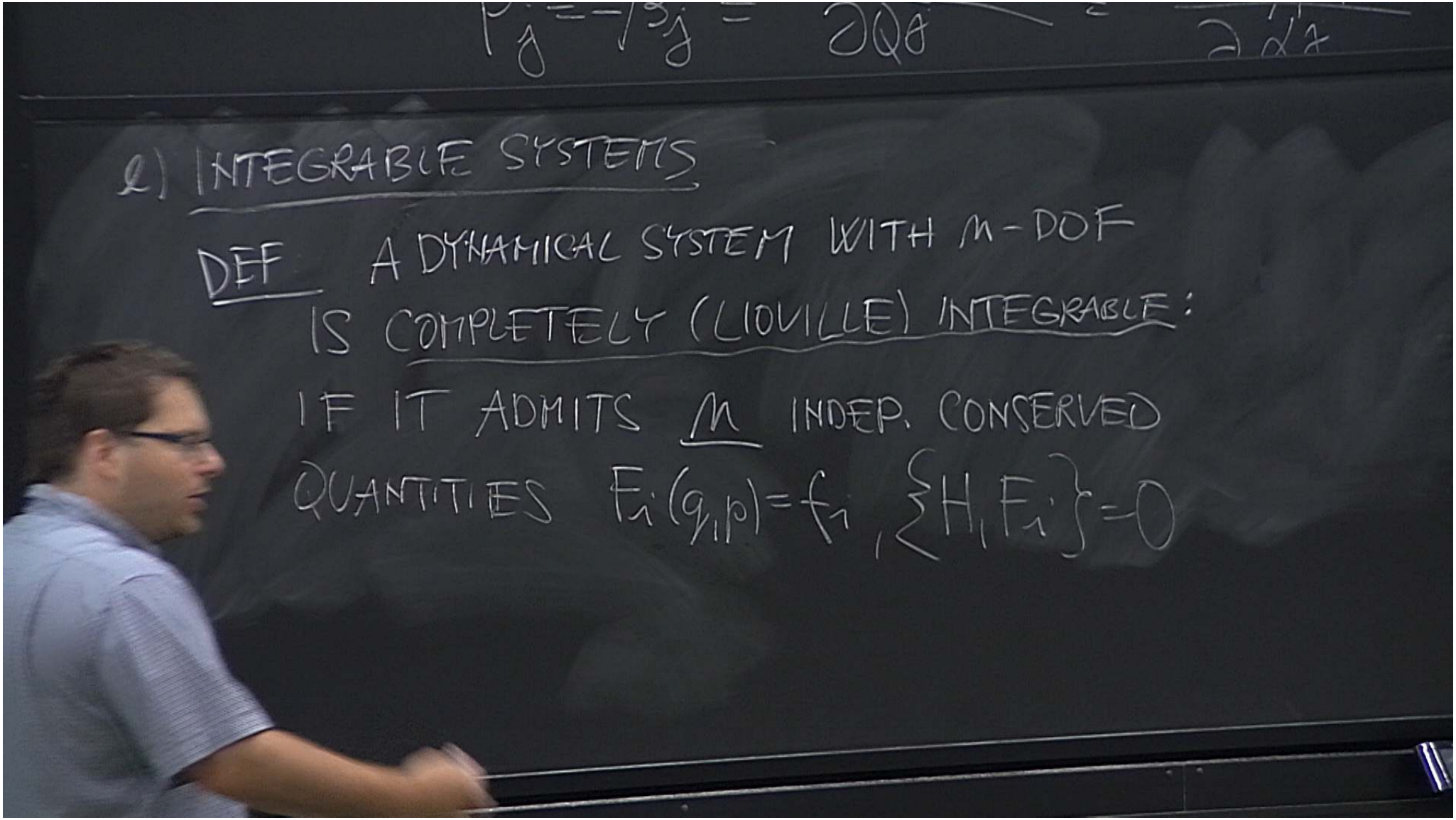

$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi = \hat{H}\psi = \left(-\frac{\hbar^2}{2m}D^2 + V\right)\psi$
$\left(\frac{|DS|^2}{2m} - \frac{i\hbar}{2m}D^2 S + V\right)\psi$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi = \hat{H}\psi = \left(-\frac{\hbar^2}{2m}D^2 + V\right)\psi$$

$$\left(\frac{|DS|^2}{2m} - \frac{i\hbar}{2m}D^2 S + V\right)\psi$$

IN THE LIMIT $\hbar \to 0$

$$\boxed{-\frac{\partial S}{\partial t} = \frac{|DS|^2}{2m} + V} \quad \ldots (H\text{-}J)$$


2) INTEGRABLE SYSTEMS
DEF A DYNAMICAL SYSTEM WITH n-DOF
IS COMPLETELY (LIOUVILLE) INTEGRABLE:
IF IT ADMITS n INDEP. CONSERVED
QUANTITIES $F_i(q,p)=f_i$, $\{H,F_i\}=0$

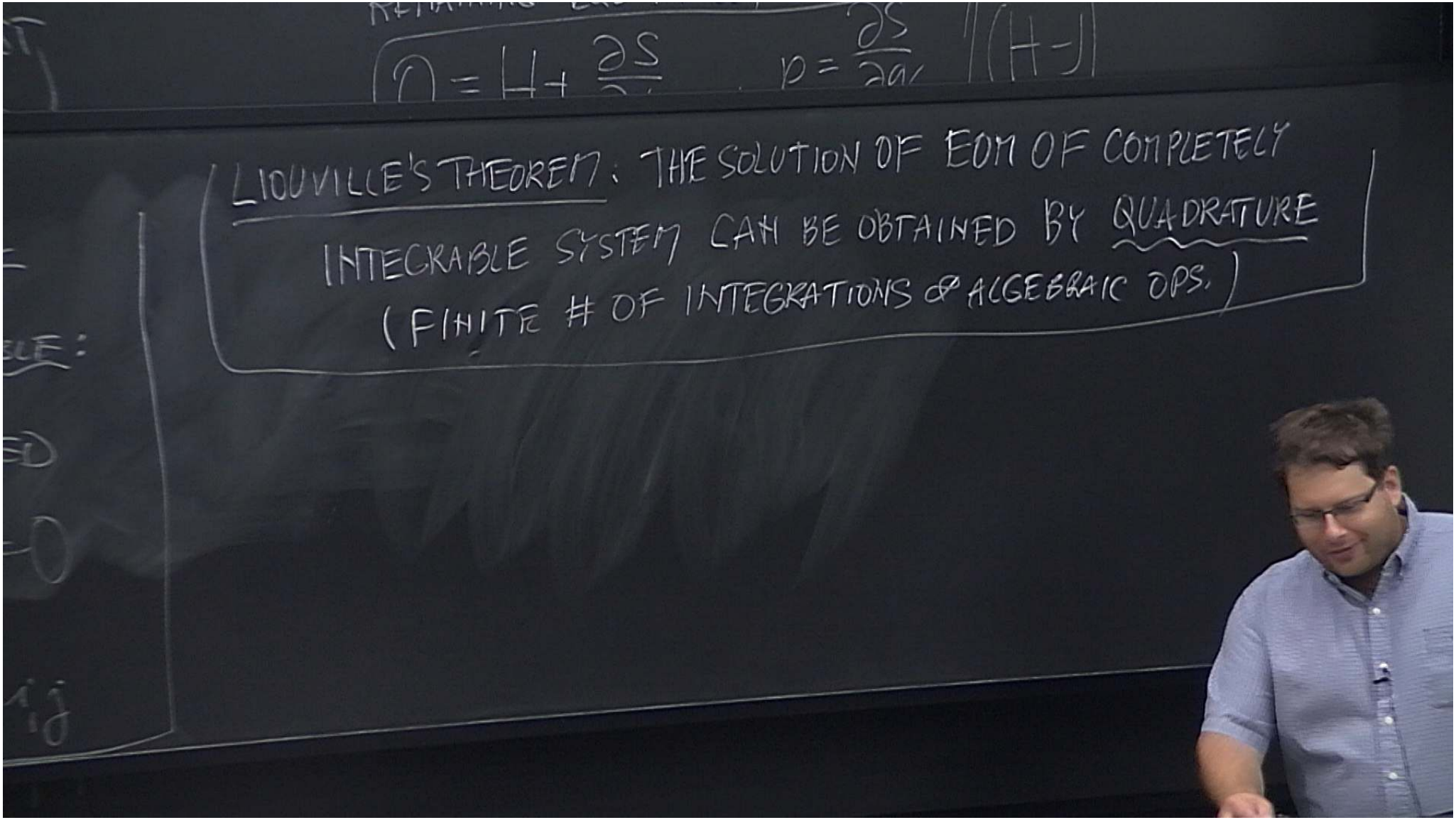
$Q_i = \alpha_i t = \text{CONST}$ $P_i = \beta_i = \text{CONST}$

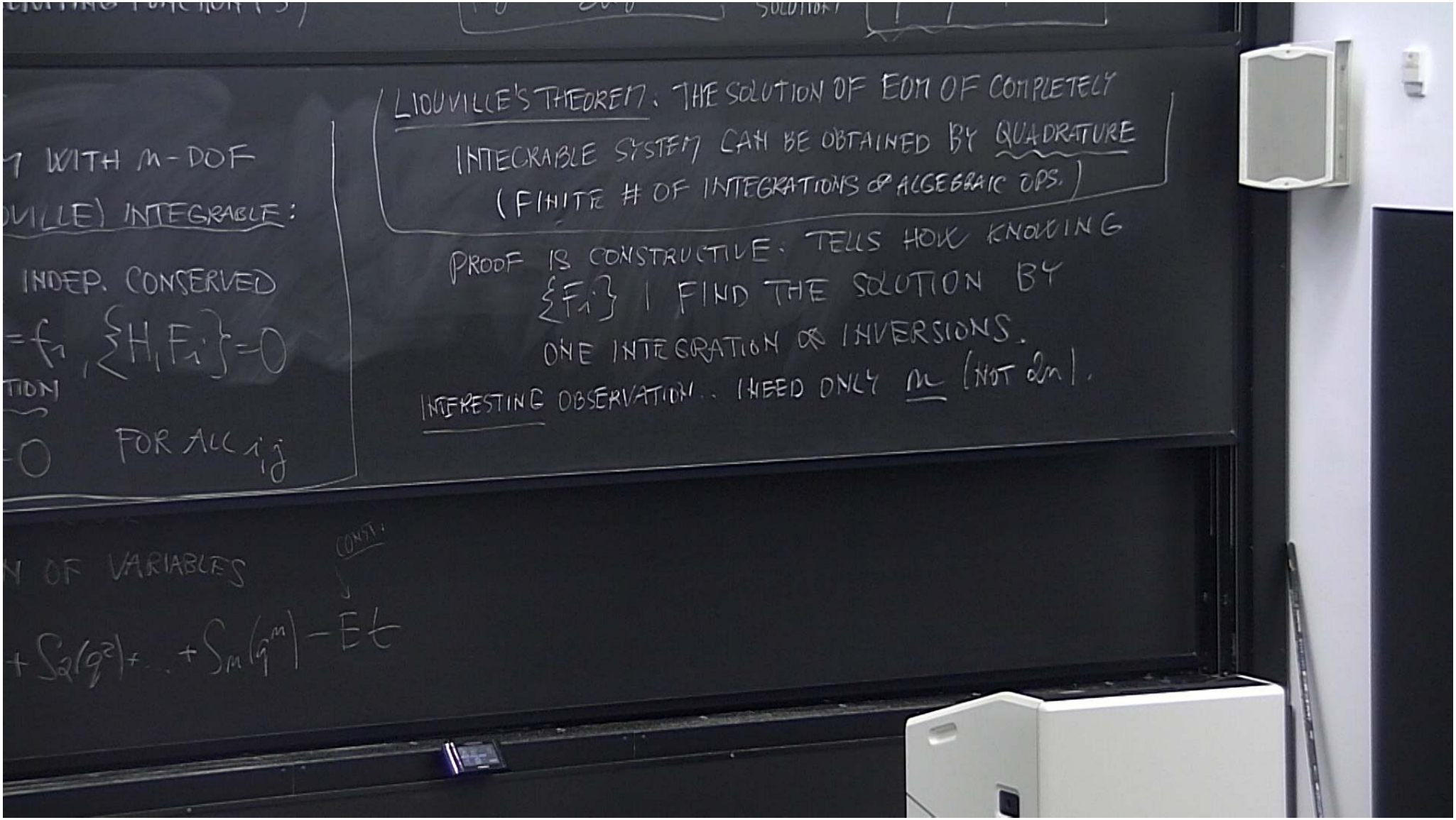
2) INTEGRABLE SYSTEMS

DEF A DYNAMICAL SYSTEM WITH n-DOF IS COMPLETELY (LIOUVILLE) INTEGRABLE: IF IT ADMITS n INDEP. CONSERVED QUANTITIES $F_i(q,p) = f_i$, $\{H, F_i\} = 0$ THAT ARE IN INVOLUTION

$$\{F_i, F_j\} = 0 \quad \text{FOR ALL } i,j$$

$W(q) = W_1(q^1) + W_2(q^2) + \dots + W_n(q^n)$ $\quad E = E_0$


$\Omega = H + \frac{\partial S}{\partial t}$
$p = \frac{\partial S}{\partial q^i}$
LIOUVILLE'S THEOREM: THE SOLUTION OF EOM OF COMPLETELY INTEGRABLE SYSTEM CAN BE OBTAINED BY QUADRATURE (FINITE # OF INTEGRATIONS & ALGEBRAIC OPS.)


WITH n-DOF
OVILLE) INTEGRABLE:
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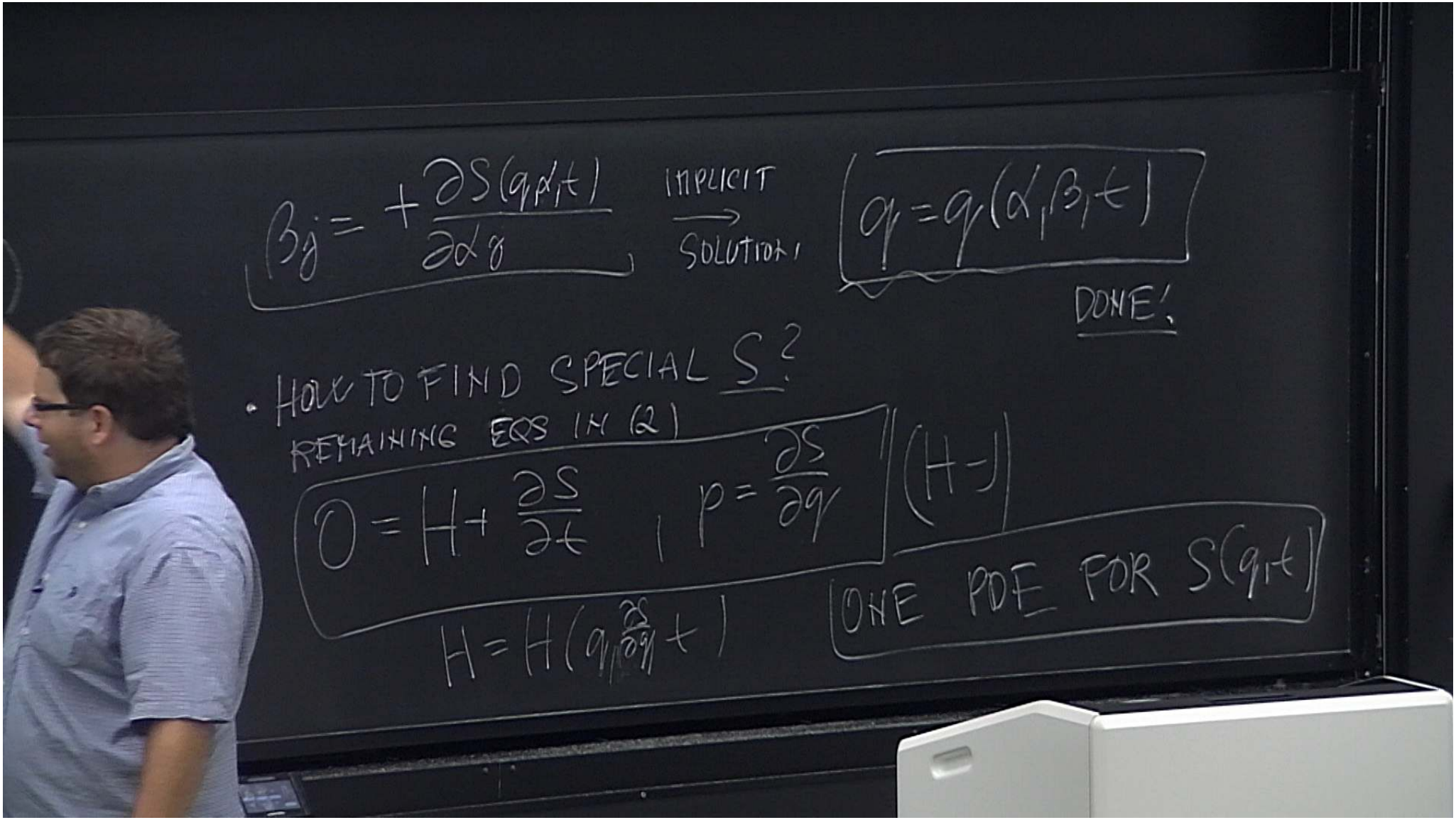
$=f_1, \{H, F_1\}=0$
TION
$=0$ FOR ALL i,j
LIOUVILLE'S THEOREM. THE SOLUTION OF EOM OF COMPLETELY
INTEGRABLE SYSTEM CAN BE OBTAINED BY QUADRATURE
(FINITE # OF INTEGRATIONS OR ALGEBRAIC OPS.)
PROOF IS CONSTRUCTIVE. TELLS HOW KNOWING
$\{F_i\}$ I FIND THE SOLUTION BY
ONE INTEGRATION OR INVERSIONS.
INTERESTING OBSERVATION.. I NEED ONLY n (NOT 2n).
N OF VARIABLES
CONST.
$+S_2(q^2)+\ldots+S_n(q^n)-Et$


$\beta_j = +\frac{\partial S(q,\alpha,t)}{\partial \alpha_j}$
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SOLUTION
$q_i = q_i(\alpha,\beta,t)$
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REMAINING EQS IN (2)
$0 = H + \frac{\partial S}{\partial t}, \quad p = \frac{\partial S}{\partial q}$
(H-J)
$H = H(q, \frac{\partial S}{\partial q}, t)$
ONE PDE FOR S(q,t)

ADMITS n INDEP. CONSERVED QUANTITIES $F_i(q,p)=f_i$, $\{H,F_i\}=0$ THAT ARE IN INVOLUTION $\{F_i,F_j\}=0$ FOR ALL i,j

PROOF IS CONSTRUCTIVE. TELLS HOW KNOWING $\{F_i\}$ I FIND THE SOLUTION BY ONE INTEGRATION OR INVERSIONS.

INTERESTING OBSERVATION. I NEED ONLY n (NOT $2n$). EACH INTEGRAL OF MOTION CAN BE USED TWICE.

SIMILAR STATEMENT IN FT: IN ANY GAUGE THEORY

TRUE DOF = #(APPARENT DOF) − 2×#(DOF OF A GAUGE FUNCTION)

EX EM: DESCRIBED IN TERM $(\varphi,\vec{A})$

$\varphi \to \varphi + \partial_t \Lambda$ (Λ FUNCTION!)

$\vec{A} \to \vec{A} + \nabla\Lambda$ $\Rightarrow$ E,B ARE INVARIANT.

Λ GAUGE FUNCTION (1 NUMBER)

4 APPARENT $(\varphi,\vec{A})$

1 DOF OF GAUGE F Λ

TRUE DOF = $4-2\times1=2$ POLARIZATIONS OF PHOTON.

• SOLVE (H-J) INTERESTED $S(q,\vec{\alpha},t)$ (n INTEGRATION CONSTANTS)

COMPLETE INTEGRAL

• TRICK FOR FINDING THIS IS TO USE

CONST.

$H \neq H(t)$ $S = S_0 - Et$

$H \neq H(x)$ $S = x\cdot C_x + S_0$

$\vec{A} \to \vec{A} + \nabla\Lambda$ INVARIANT.

GRAVITY: $g_{\mu\nu}$ 4×4 SYMMETRIC MATRIX

10 APP. DOF.

$g_{\mu\nu} \to g_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

↑ GAUGE F. WITH 4 DOF.

TRUE DOF = $10 - 2\times 4 = \underline{\underline{2}}$ POL. OF GRAVITON