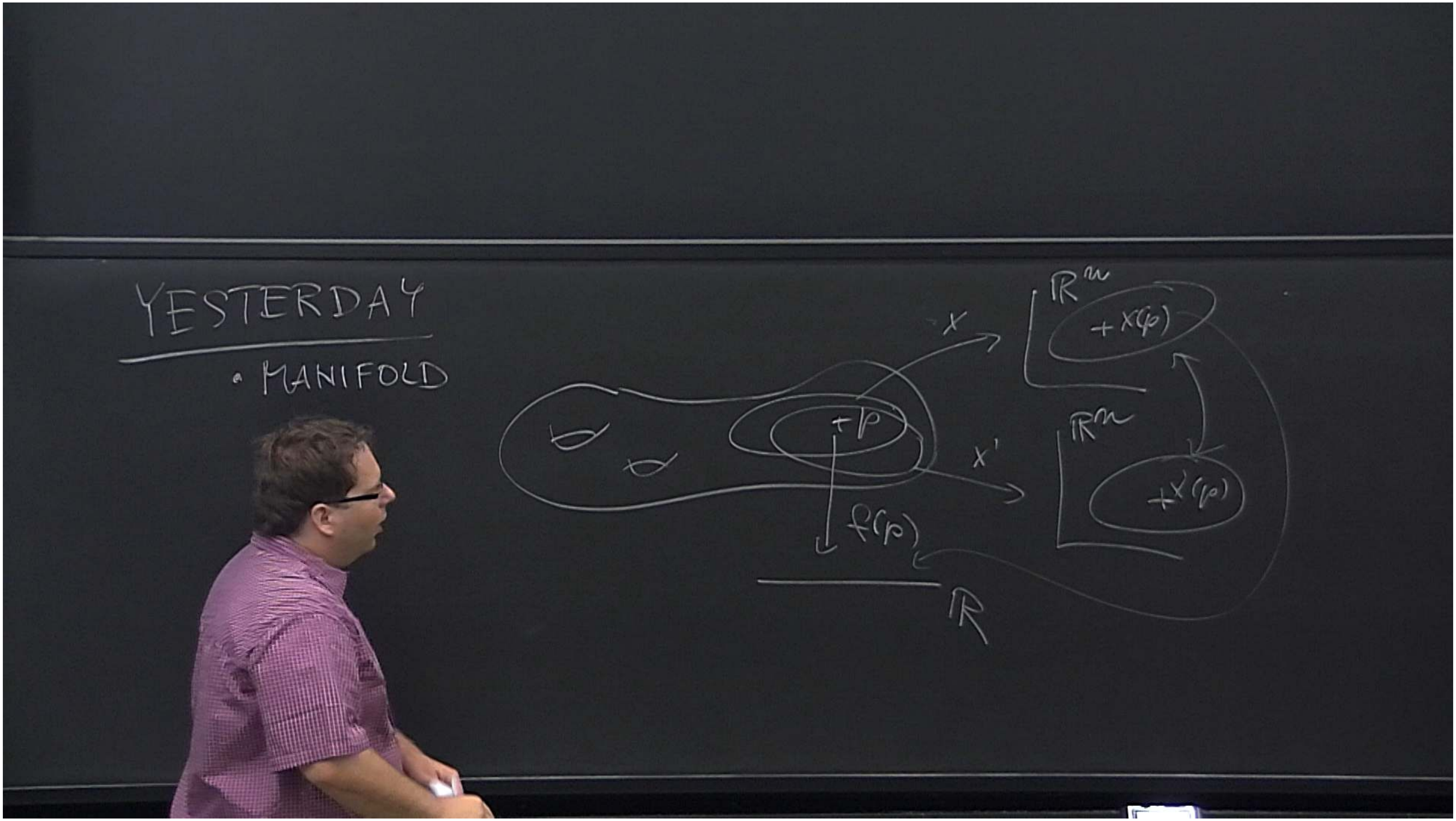
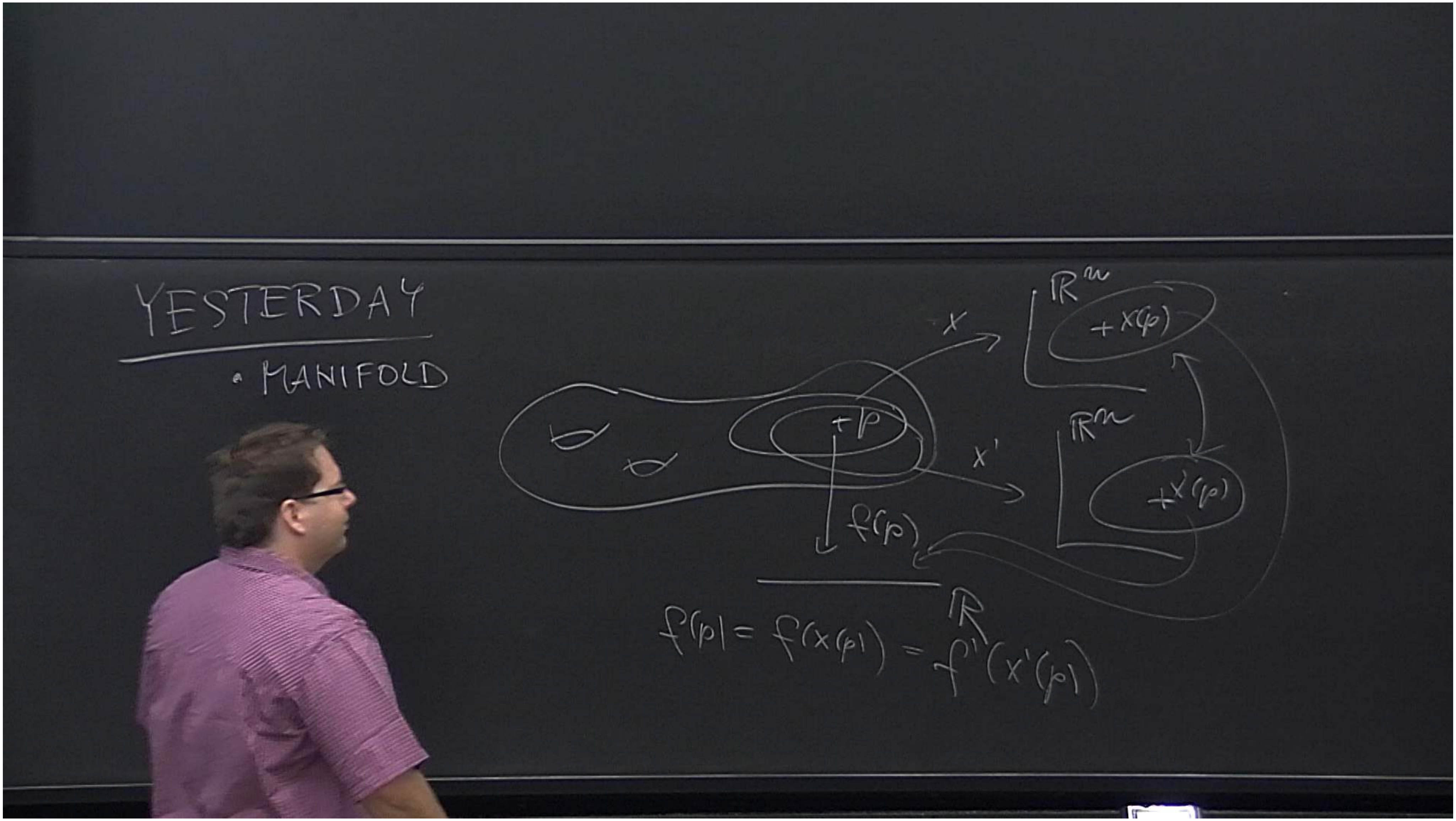
Title: PSI 17/18 Front End - Theoretical Mechanics (Kubiznak)

Date: Aug 11, 2017 09:00 AM

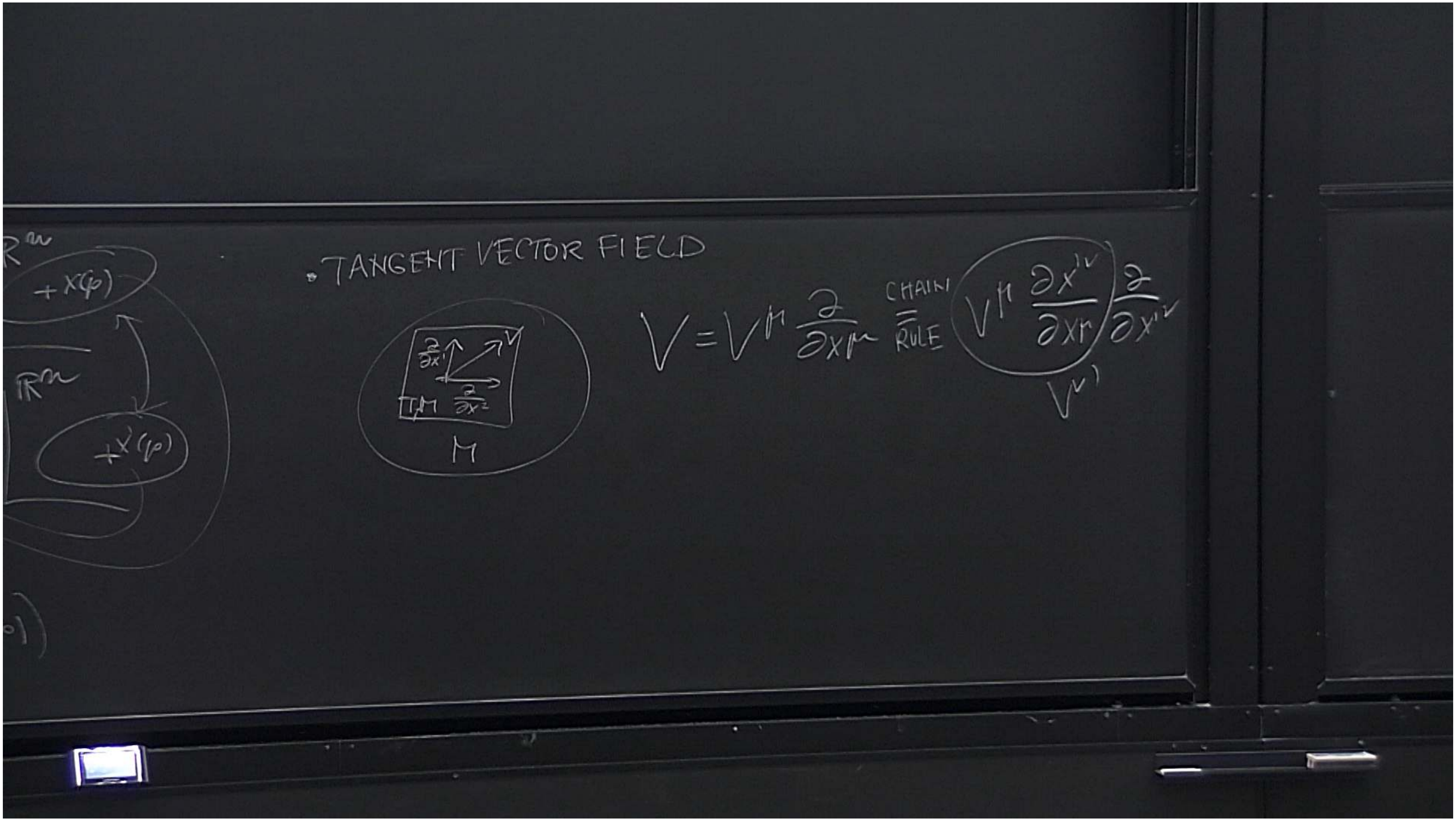
URL: http://pirsa.org/17080044

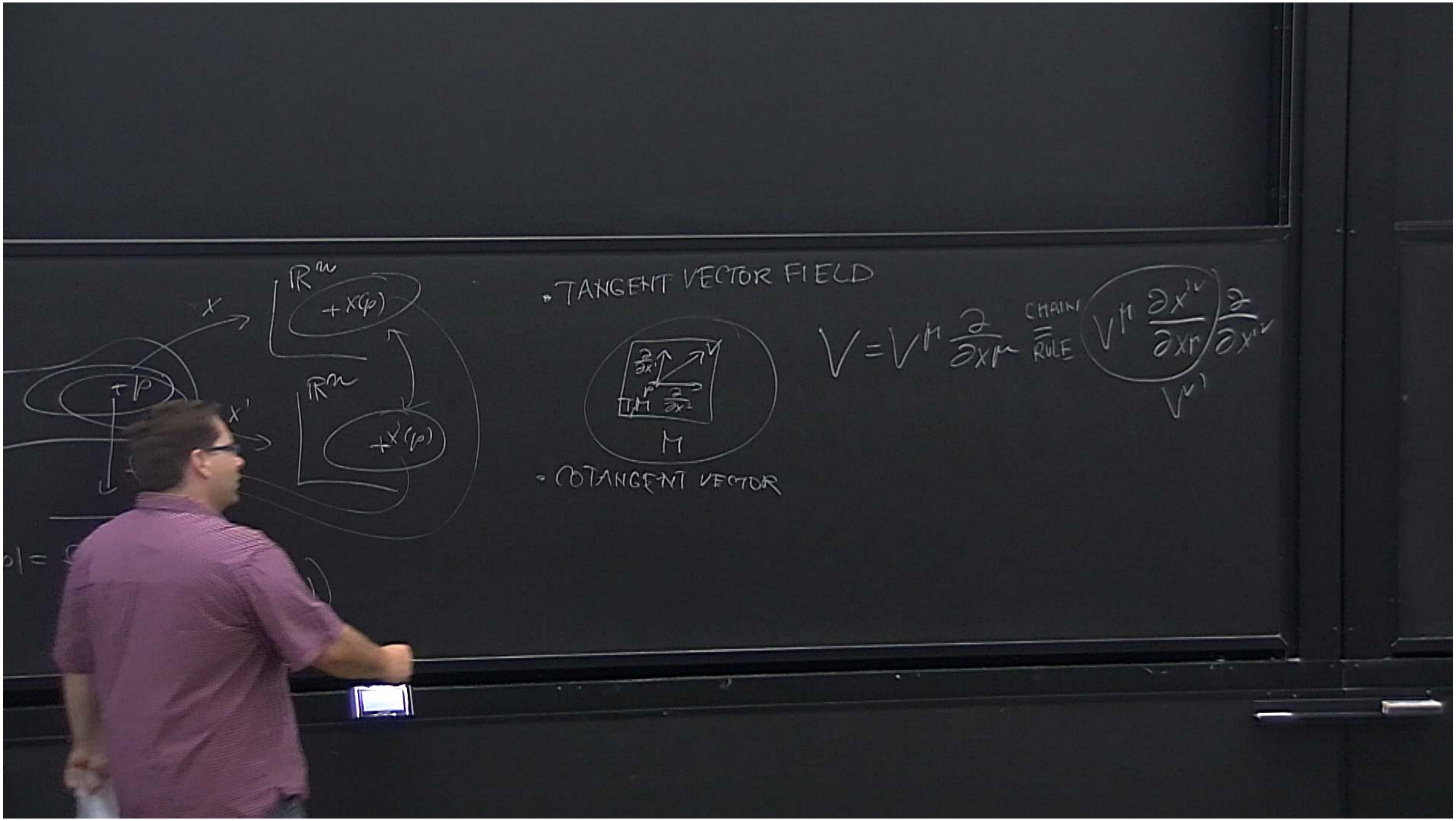
Abstract:


YESTERDAY
• MANIFOLD
+p
x
x'
f(p)
ℝ
ℝ^n
+x(p)
ℝ^n
+x'(p)



YESTERDAY
• MANIFOLD
ℝⁿ
+x(p)
x
+p
x'
ℝⁿ
+x'(p)
f(p)
ℝ
f(p) = f(x(p)) = f'(x'(p))


TANGENT VECTOR FIELD
$V = V^\mu \frac{\partial}{\partial x^\mu} \overset{\text{CHAIN RULE}}{=} V^\mu \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu}$
V'^ν
$\frac{\partial}{\partial x^1}$
$\frac{\partial}{\partial x^2}$
V
T_pM
M
$\mathbb{R}^n$
$x(p)$
$\mathbb{R}^n$
$x'(p)$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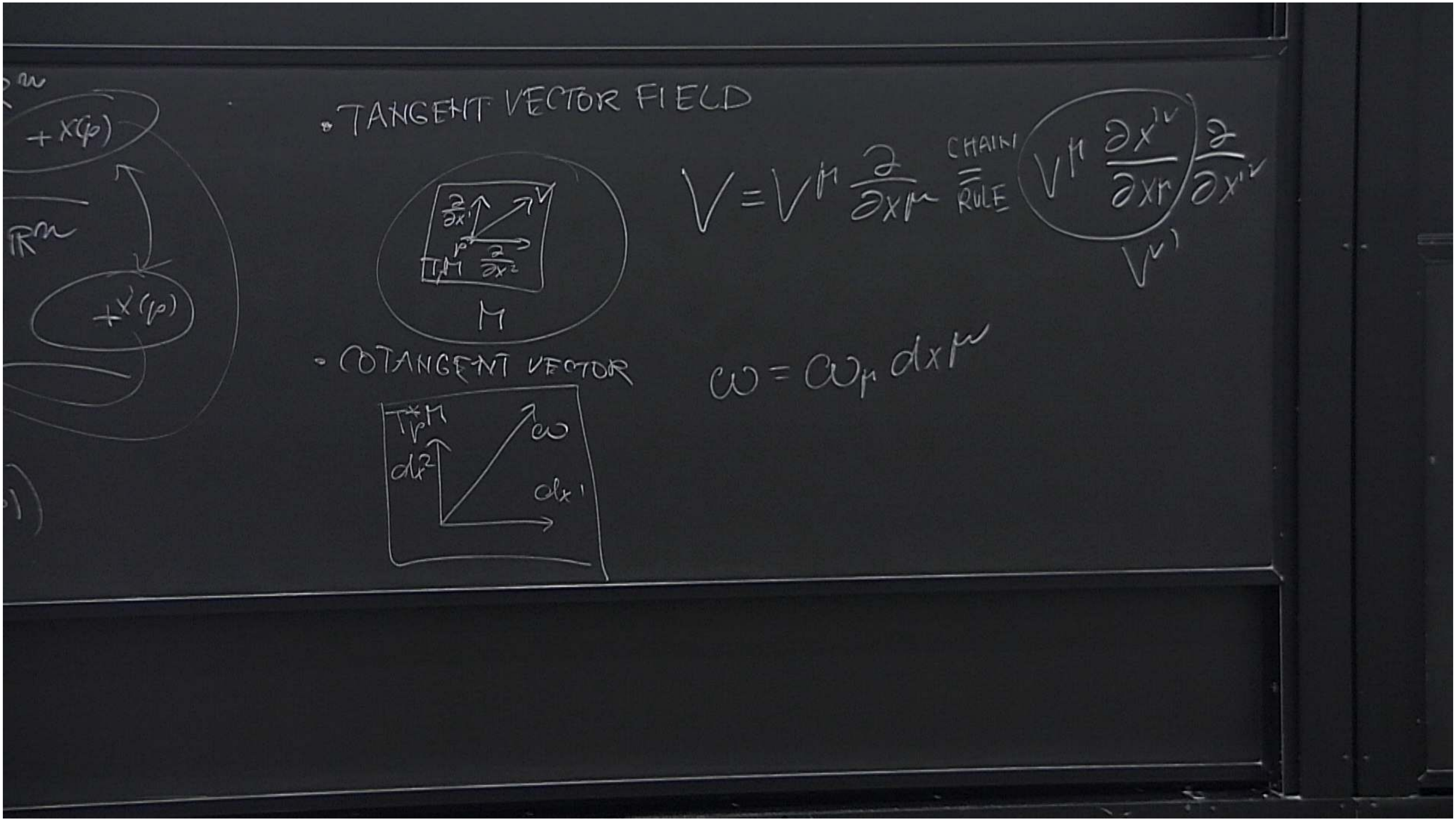
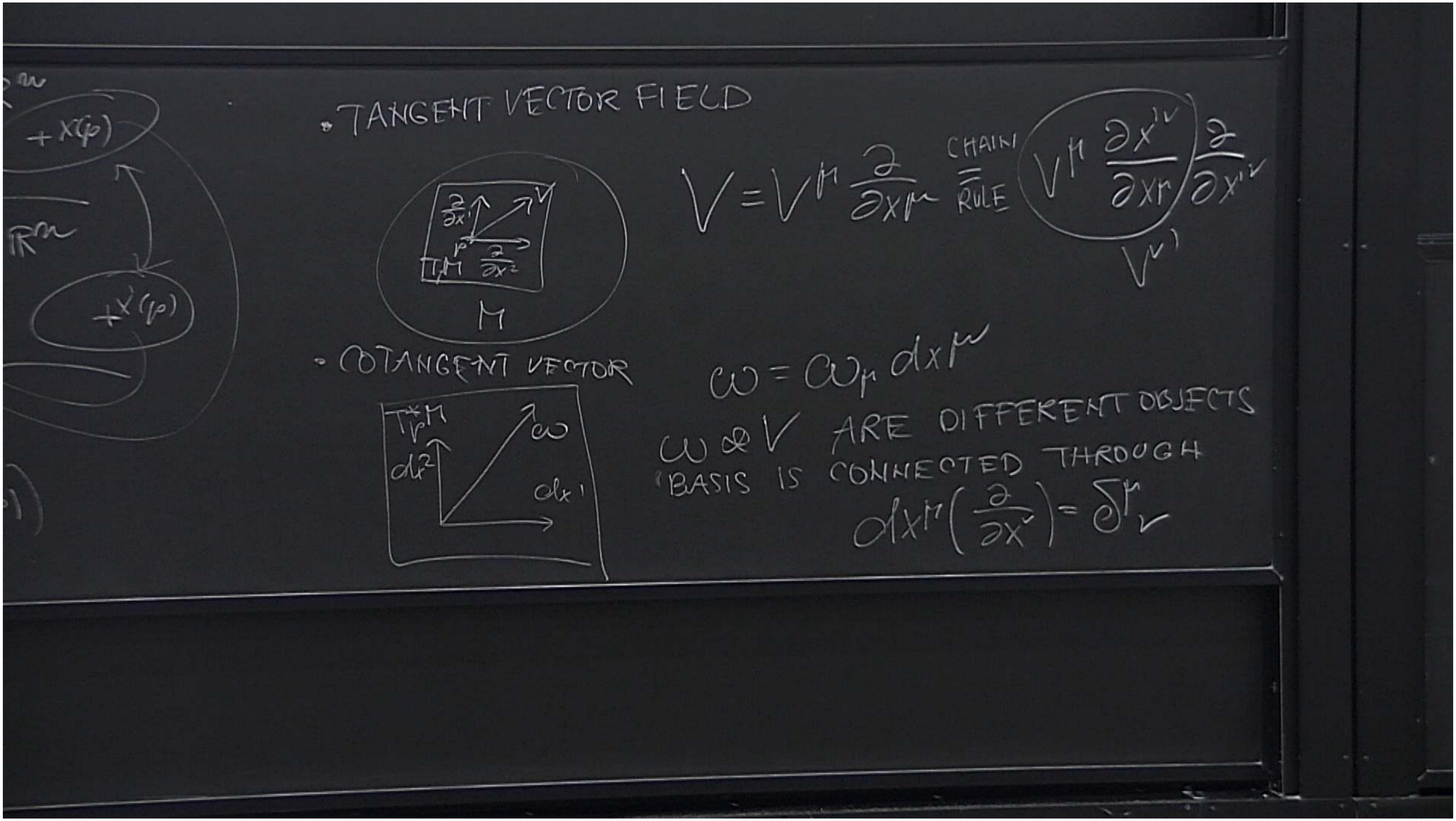
V'^ν
T_pM
M
COTANGENT VECTOR
$\mathbb{R}^n$
$x(p)$
$x'(p)$


$+x(p)$
$\mathbb{R}^n$
$+x'(p)$
• TANGENT VECTOR FIELD
$\frac{\partial}{\partial x^1}$
V
p
T_pM
$\frac{\partial}{\partial x^2}$
M
$V = V^\mu \frac{\partial}{\partial x^\mu} \overset{\text{CHAIN}}{\underset{\text{RULE}}{=}} V^\mu \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu}$
V'^ν
• COTANGENT VECTOR
$\omega = \omega_\mu dx^\mu$
T_p^*M
dx^2
ω
dx^1


• TANGENT VECTOR FIELD
$V = V^\mu \frac{\partial}{\partial x^\mu} \overset{\text{CHAIN}}{\underset{\text{RULE}}{=}} V^\mu \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu}$
V'^ν
T_pM
$\frac{\partial}{\partial x^1}$
$\frac{\partial}{\partial x^2}$
M
• COTANGENT VECTOR
$\omega = \omega_\mu dx^\mu$
T_p^*M
dx^2
dx^1
ω & V ARE DIFFERENT OBJECTS
BASIS IS CONNECTED THROUGH
$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu$

- TENSOR FIELD OF TYPE (k,l) IS A MAP

$$T: \underbrace{T^* \times \dots \times T^*}_{k} \times \underbrace{T \times \dots \times T}_{l} \to \mathcal{F}$$

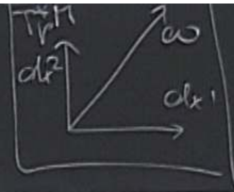
EX: TYPE $(2,1)$ RANK 3

$$T = T^{\alpha\beta}{}_{\gamma} \; \underbrace{\partial_{x_\alpha} \otimes \partial_{x_\beta} \otimes dx^\gamma}_{\text{BASIS}}$$

COMPTS

- MEA

$f(p) = f(x(p)) = f'(x'(p))$ $\mathbb{R}$



ω & V ARE DIFFERENT OBJECTS
BASIS IS CONNECTED THROUGH

$$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu$$

TYPE (k,l) IS A MAP

$$\times T^* \times \underbrace{T \times \ldots \times T}_{l} \to \mathbb{R}$$

) RANK 3

$$\alpha\beta{}_\gamma \; \underbrace{\partial x_\alpha \otimes \partial x_\beta \otimes dx^\gamma}_{\text{BASIS}}$$

COMPTS

• MEANING OF COMPTS

$$T^{\kappa\delta}{}_2 = T\left(dx^\kappa, dx^\delta, \frac{\partial}{\partial x^2}\right)$$

$$= T^{\alpha\beta}{}_\gamma \underbrace{dx^\kappa\left(\frac{\partial}{\partial x^\alpha}\right)}_{\delta^\kappa{}_\alpha} \underbrace{dx^\delta\left(\frac{\partial}{\partial x^\beta}\right)}_{\delta^\delta{}_\beta} \underbrace{dx^\gamma\left(\frac{\partial}{\partial x^2}\right)}_{\delta^\gamma{}_2}$$

• MEANING OF COMPTS.

$$T^{\kappa\delta}{}_{\gamma} = T\left(dx^{\kappa}, dx^{\delta}, \frac{\partial}{\partial x^{\gamma}}\right)$$

$$= T^{\alpha\beta}{}_{\eta}\, \underbrace{dx^{\kappa}\left(\frac{\partial}{\partial x^{\alpha}}\right)}_{\delta^{\kappa}_{\alpha}} \underbrace{dx^{\delta}\left(\frac{\partial}{\partial x^{\beta}}\right)}_{\delta^{\delta}_{\beta}} \underbrace{dx^{\eta}\left(\frac{\partial}{\partial x^{\gamma}}\right)}_{\delta^{\eta}_{\gamma}}$$

$$T(\omega, v, W) = T\left(\omega_{\mu}dx^{\mu}, v_{\nu}dx^{\nu}, W^{\kappa}\frac{\partial}{\partial x^{\kappa}}\right)$$

$$= \omega_{\mu} v_{\nu} W^{\kappa}\, T\left(dx^{\mu}, dx^{\nu}, \frac{\partial}{\partial x^{\kappa}}\right) = \underline{T^{\mu\nu}{}_{\kappa}\, \omega_{\mu} v_{\nu} W^{\kappa}}$$

• MEANING OF COMPTS.

$$T^{\varkappa\delta}{}_{2} = T\left(dx^{\varkappa}, dx^{\delta}, \frac{\partial}{\partial x^{2}}\right)$$

$$= T^{\alpha\beta}{}_{\gamma}\, \underbrace{dx^{\varkappa}\left(\frac{\partial}{\partial y^{\alpha}}\right)}_{\delta^{\varkappa}_{\alpha}} \underbrace{dx^{\delta}\left(\frac{\partial}{\partial x^{\beta}}\right)}_{\delta^{\delta}_{\beta}} \underbrace{dx^{\gamma}\left(\frac{\partial}{\partial x^{2}}\right)}_{\delta^{\gamma}_{2}}$$

dx^{γ}

$$T(\omega, V, W) = T\left(\omega_{\mu}\, dx^{\mu}, V_{\nu}\, dx^{\nu}, W^{\varkappa} \frac{\partial}{\partial x^{\varkappa}}\right)$$

$$= \omega_{\mu} V_{\nu} W^{\varkappa}\, T\left(dx^{\mu}, dx^{\nu}, \frac{\partial}{\partial x^{\varkappa}}\right) = \underline{T^{\mu\nu}{}_{\varkappa}\, \omega_{\mu} V_{\nu} W^{\varkappa}}$$

• MEANING OF COMPTS.

$$T^{\varkappa\delta}{}_{\gamma} = T\left(dx^{\varkappa}, dx^{\delta}, \frac{\partial}{\partial x^{\gamma}}\right)$$

$$= T^{\alpha\beta}{}_{\gamma}\, \underbrace{dx^{\varkappa}\left(\frac{\partial}{\partial y^{\alpha}}\right)}_{\delta^{\varkappa}_{\alpha}} \underbrace{dx^{\delta}\left(\frac{\partial}{\partial x^{\beta}}\right)}_{\delta^{\delta}_{\beta}} \underbrace{dx^{\gamma}\left(\frac{\partial}{\partial x^{\gamma}}\right)}_{\delta^{\gamma}_{\gamma}}$$

dx^{γ}

$$T(\omega, v, W) = T\left(\omega_{\mu} dx^{\mu}, v_{\nu} dx^{\nu}, W^{\varkappa} \frac{\partial}{\partial x^{\varkappa}}\right)$$

$$= \omega_{\mu} v_{\nu} W^{\varkappa}\, T\left(dx^{\mu}, dx^{\nu}, \frac{\partial}{\partial x^{\varkappa}}\right) = \underline{T^{\mu\nu}{}_{\varkappa}\, \omega_{\mu} v_{\nu} W^{\varkappa}}$$

• MEANING OF COMPTS.

$$T^{\varkappa\delta}{}_{2} = T\left(dx^{\varkappa}, dx^{\delta}, \frac{\partial}{\partial x^{2}}\right)$$

$$= T^{\alpha\beta}{}_{\gamma}\, \underbrace{dx^{\varkappa}\left(\frac{\partial}{\partial x^{\alpha}}\right)}_{\delta^{\varkappa}{}_{\alpha}} \underbrace{dx^{\delta}\left(\frac{\partial}{\partial x^{\beta}}\right)}_{\delta^{\delta}{}_{\beta}} \underbrace{dx^{\gamma}\left(\frac{\partial}{\partial x^{2}}\right)}_{\delta^{\gamma}{}_{2}}$$

dx^{γ}

$$T(\omega, v, W) = T\left(\omega_{\mu} dx^{\mu}, v_{\nu} dx^{\nu}, W^{\varkappa}\frac{\partial}{\partial x^{\varkappa}}\right)$$

$$= \omega_{\mu} v_{\nu} W^{\varkappa}\, T\left(dx^{\mu}, dx^{\nu}, \frac{\partial}{\partial x^{\varkappa}}\right) = \underline{T^{\mu\nu}{}_{\varkappa}\, \omega_{\mu} v_{\nu} W^{\varkappa}}$$

TENSOR ALGEBRA.

i) SUM $T+S$ (SAME TYPE)

ii) TENSOR PRODUCT $\otimes$. CREATES BIGGER TENSORS

$$S = S_\delta \, dx^\delta$$

$$T\otimes S = \underbrace{T^{\alpha\beta}{}_{\gamma} S_\delta}_{(T\otimes S)^{\alpha\beta}{}_{\gamma\delta}} \, \partial_{x_\alpha}\otimes\partial_{x_\beta}\otimes dx^\gamma\otimes dx^\delta$$

$\gamma \quad \underbrace{\partial x_\alpha \otimes \partial x_\beta \otimes dx^\gamma}_{\text{BASIS}}$

$T(\omega, v, W) = T(\omega_\mu dx^\mu, v_\nu dx^\nu, W^\varkappa \frac{\partial}{\partial x^\varkappa})$

$= \omega_\mu v_\nu W^\varkappa \, T(dx^\mu, dx^\nu, \frac{\partial}{\partial x^\varkappa}) = \underline{T^{\mu\nu}{}_\varkappa \, \omega_\mu v_\nu W^\varkappa}$

SAME TYPE)

CT $\otimes$. CREATES BIGGER TENSORS

$\delta \, dx^\delta$

$T^{\alpha\beta}{}_\gamma S_\delta \; \partial x_\alpha \otimes \partial x_\beta \otimes dx^\gamma \otimes dx^\delta$

$(T \otimes S)^{\alpha\beta}{}_{\gamma\delta}$

• CONTRACTION → "SMALLER TENSORS"

EX. $T_{CONTR} = T^{\alpha\beta}{}_\gamma \; \partial x_\alpha \otimes \partial x_\beta \otimes dx^\gamma$ $\qquad dx^\gamma(\partial x_\alpha) = \delta^\gamma{}_\alpha$

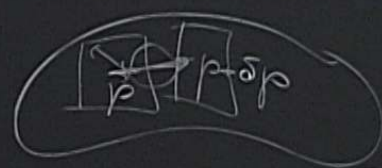
$= T^{\alpha\beta}{}_\alpha \frac{\partial}{\partial x^\beta}$

$(T_{Const})^\beta \frac{\partial}{\partial x^\beta}$

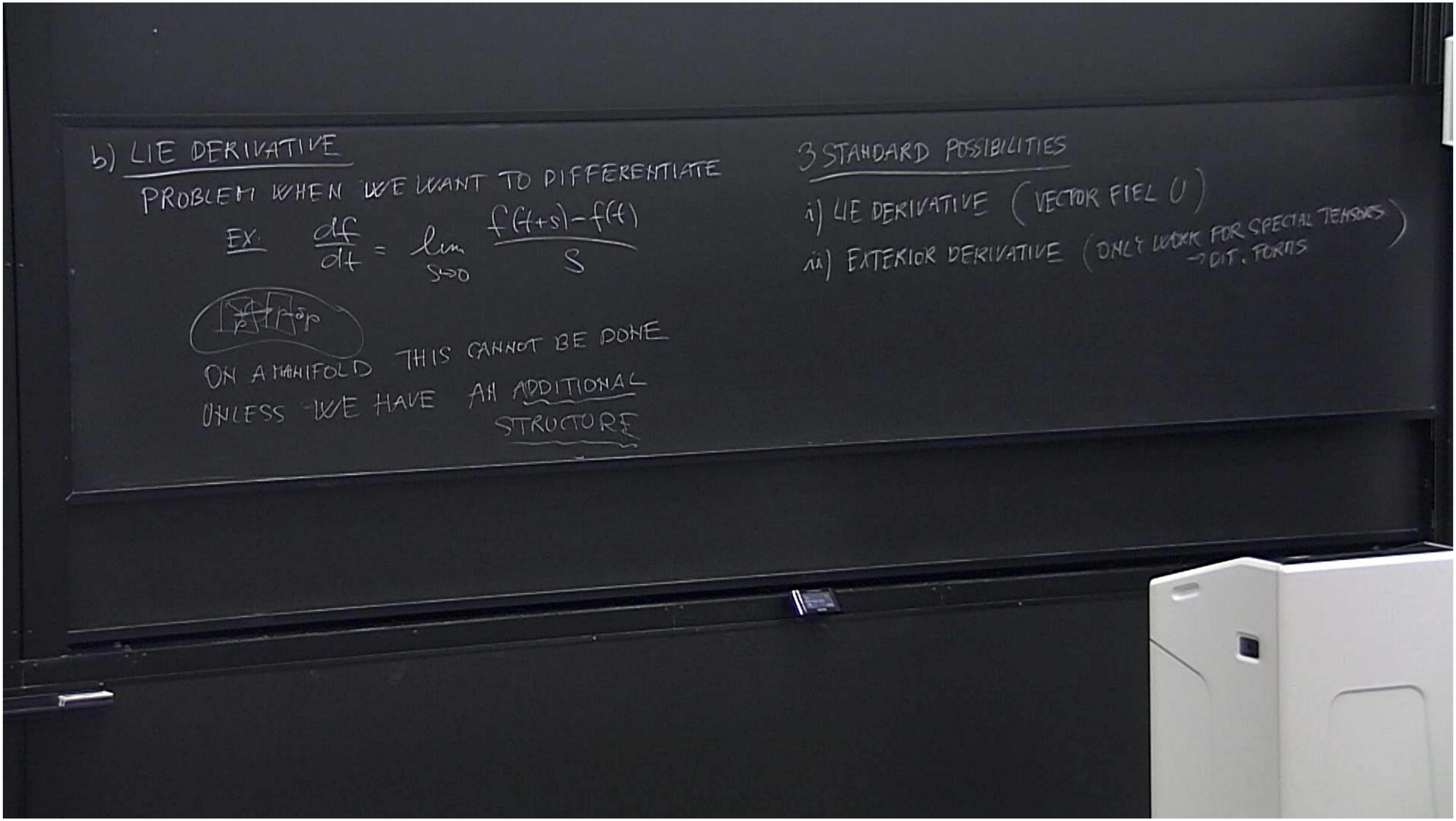
b) LIE DERIVATIVE

PROBLEM WHEN WE WANT TO DIFFERENTIATE

EX. $\dfrac{df}{dt} = \lim_{s \to 0} \dfrac{f(t+s) - f(t)}{s}$



ON A MANIFOLD THIS CANNOT BE DONE UNLESS WE HAVE AN ADDITIONAL STRUCTURE


b) LIE DERIVATIVE
PROBLEM WHEN WE WANT TO DIFFERENTIATE
EX. $\frac{df}{dt} = \lim_{s \to 0} \frac{f(t+s)-f(t)}{s}$
ON A MANIFOLD THIS CANNOT BE DONE
UNLESS WE HAVE AN ADDITIONAL STRUCTURE
3 STANDARD POSSIBILITIES
i) LIE DERIVATIVE (VECTOR FIELD)
ii) EXTERIOR DERIVATIVE (ONLY WORK FOR SPECIAL TENSORS
→ DIF. FORMS)

E

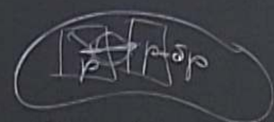
3 STANDARD POSSIBILITIES

i) LIE DERIVATIVE (VECTOR FIEL U)

ii) EXTERIOR DERIVATIVE (ONLY WORK FOR SPECIAL TENSORS $\to$ DIF. FORMS)

iii) COVARIANT DERIVATIVE (CONNECTION $\Gamma^{\alpha}{}_{\beta\gamma}$)

Ex. $\frac{df}{dt} = \lim_{s \to 0} \frac{f(t+s)-f(t)}{s}$



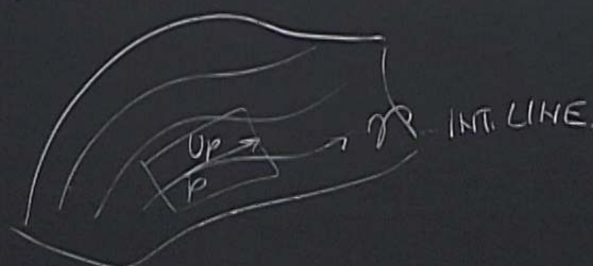
ON A MANIFOLD THIS CANNOT BE DONE UNLESS WE HAVE AN ADDITIONAL STRUCTURE

i) LIE DERIVATIVE (VECTOR FIEL U

ii) EXTERIOR DERIVATIVE (ONLY WORK

iii) COVARIANT DERIVATIVE (CONNECTI

• A VECTOR FIELD U DEFINES INTEGRAL CURVES ON M (TANGENT VECTOR COINCIDES WITH U_p



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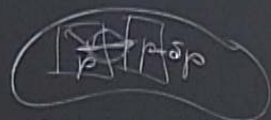


ON A MANIFOLD THIS CANNOT BE DONE UNLESS WE HAVE AN ADDITIONAL STRUCTURE

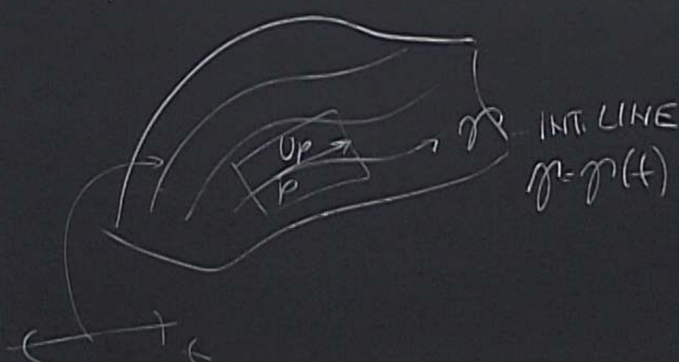
i) LIE DERIVATIVE (VECTOR FIEL[D] U

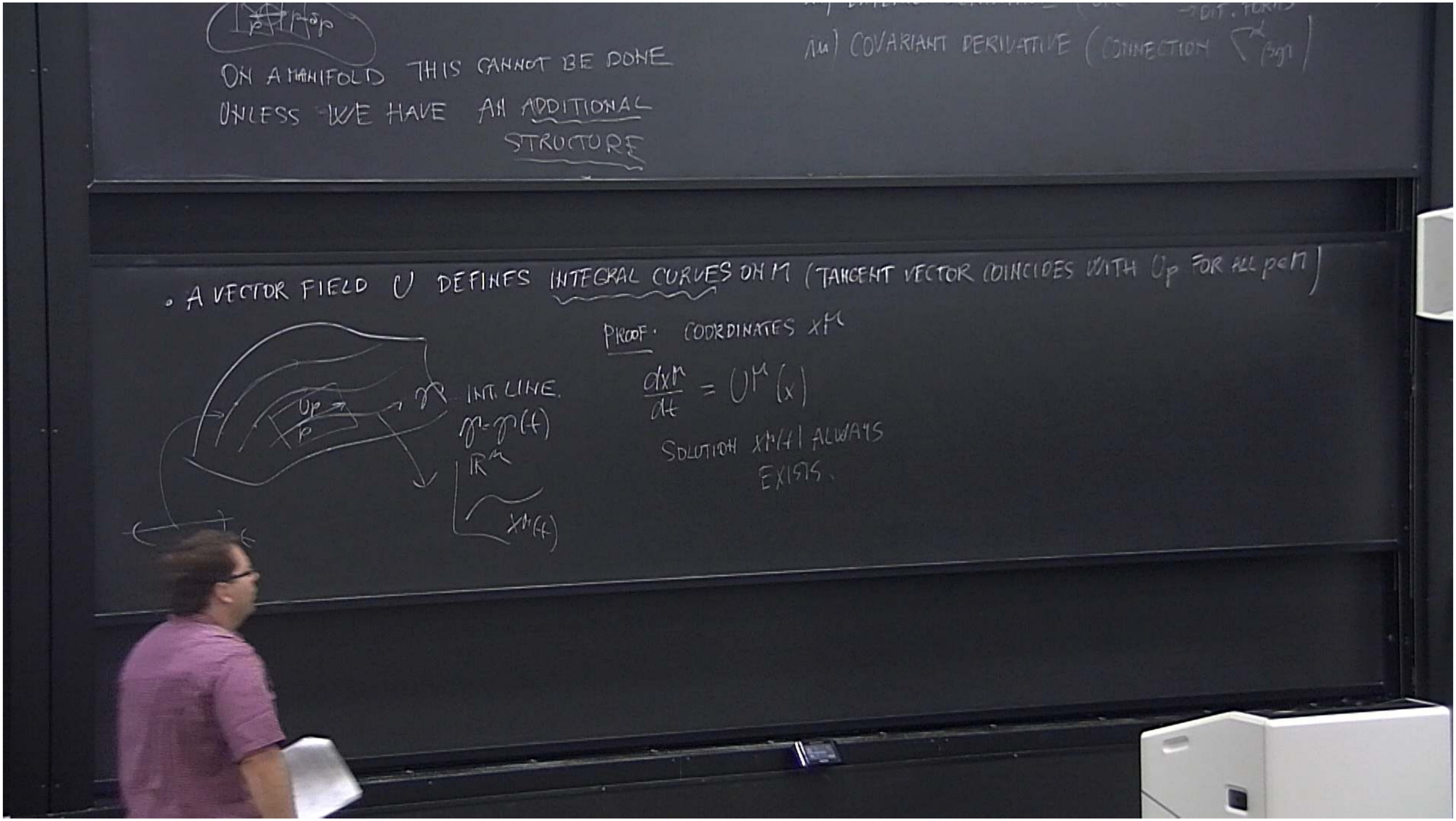
ii) EXTERIOR DERIVATIVE (ONLY WORK...

iii) COVARIANT DERIVATIVE (CONNECTI...

• A VECTOR FIELD U DEFINES INTEGRAL CURVES ON M (TANGENT VECTOR COINCIDES WITH U_p

PROOF: COORDINATES x^μ

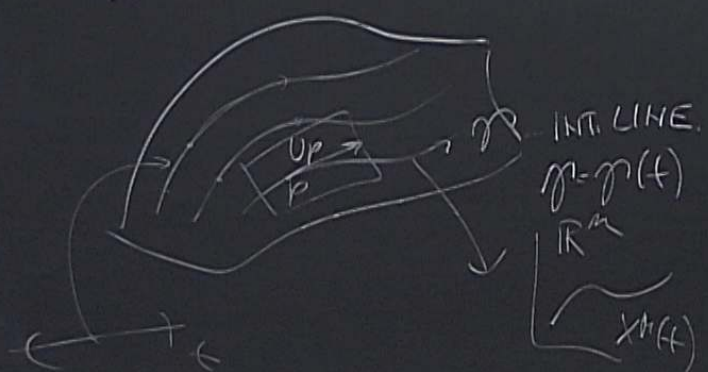



ON A MANIFOLD THIS CANNOT BE DONE
UNLESS WE HAVE AN ADDITIONAL
STRUCTURE
iii) COVARIANT DERIVATIVE (CONNECTION Γ
• A VECTOR FIELD U DEFINES INTEGRAL CURVES ON M (TANGENT VECTOR COINCIDES WITH Up FOR ALL p∈M)
PROOF. COORDINATES x^μ
dx^μ/dt = U^μ(x)
SOLUTION x^μ(t) ALWAYS
EXISTS.
Up
p
INT. LINE
γ = γ(t)
R^n
x^μ(t)

ON A MANIFOLD THIS CANNOT BE DONE
UNLESS WE HAVE AN ADDITIONAL
STRUCTURE

iii) COVARIANT DERIVATIVE (CONNECTION $\Gamma^{\mu}_{\nu\rho}$)

• A VECTOR FIELD U DEFINES INTEGRAL CURVES ON M (TANGENT VECTOR COINCIDES WITH U_p FOR ALL $p \in M$)

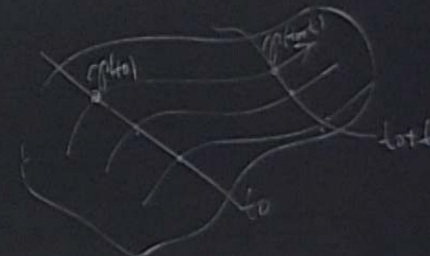


PROOF: COORDINATES x^μ

$$\frac{dx^\mu}{dt} = U^\mu(x)$$

SOLUTION $x^\mu(t)$ ALWAYS EXISTS.

THIS DEFINES A MAP $\phi_t : M \to M$
BY $\phi_t : \gamma(t_0) \to \gamma(t_0 + t)$



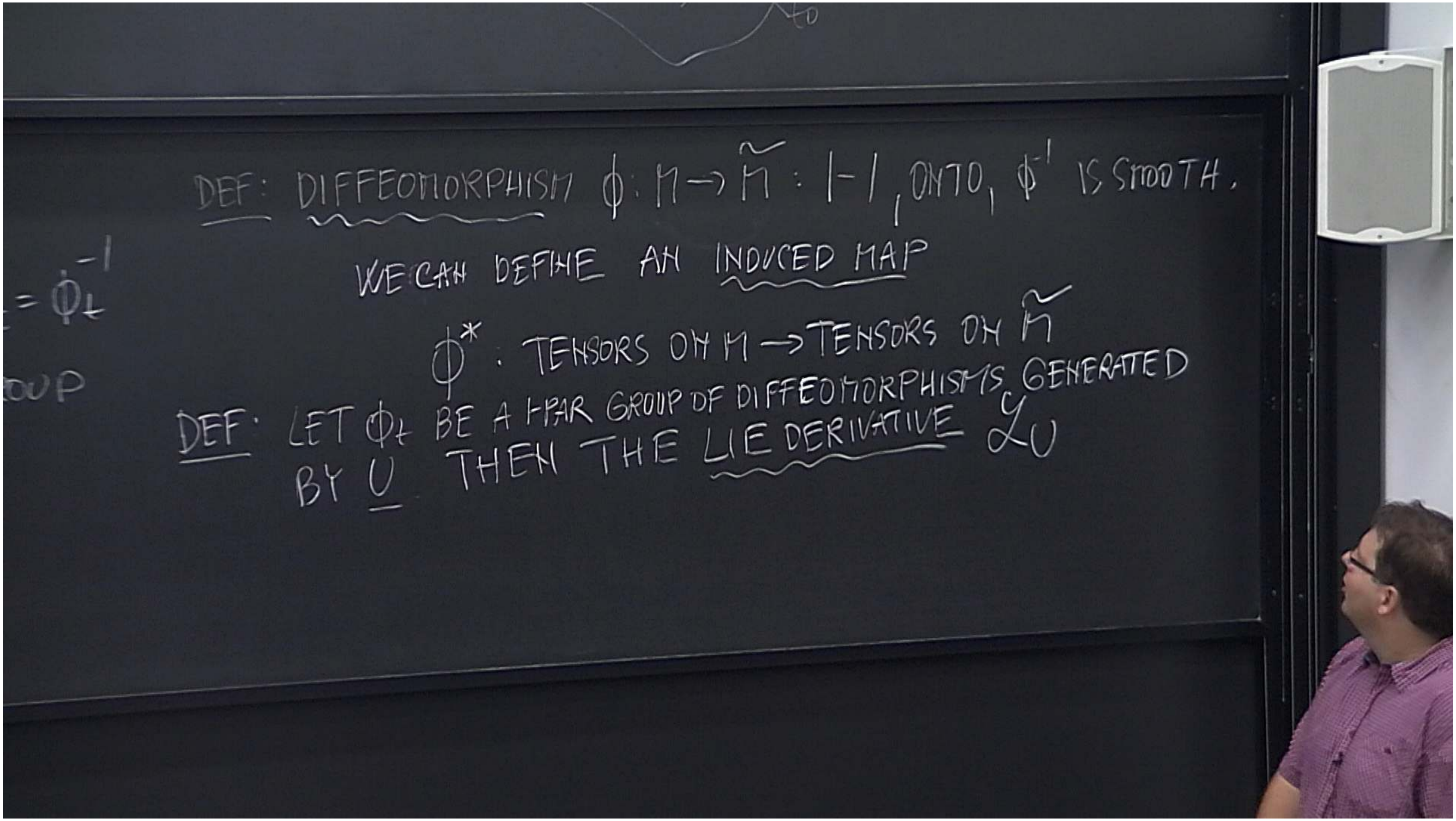
ϕ_t IS REALLY NICE:

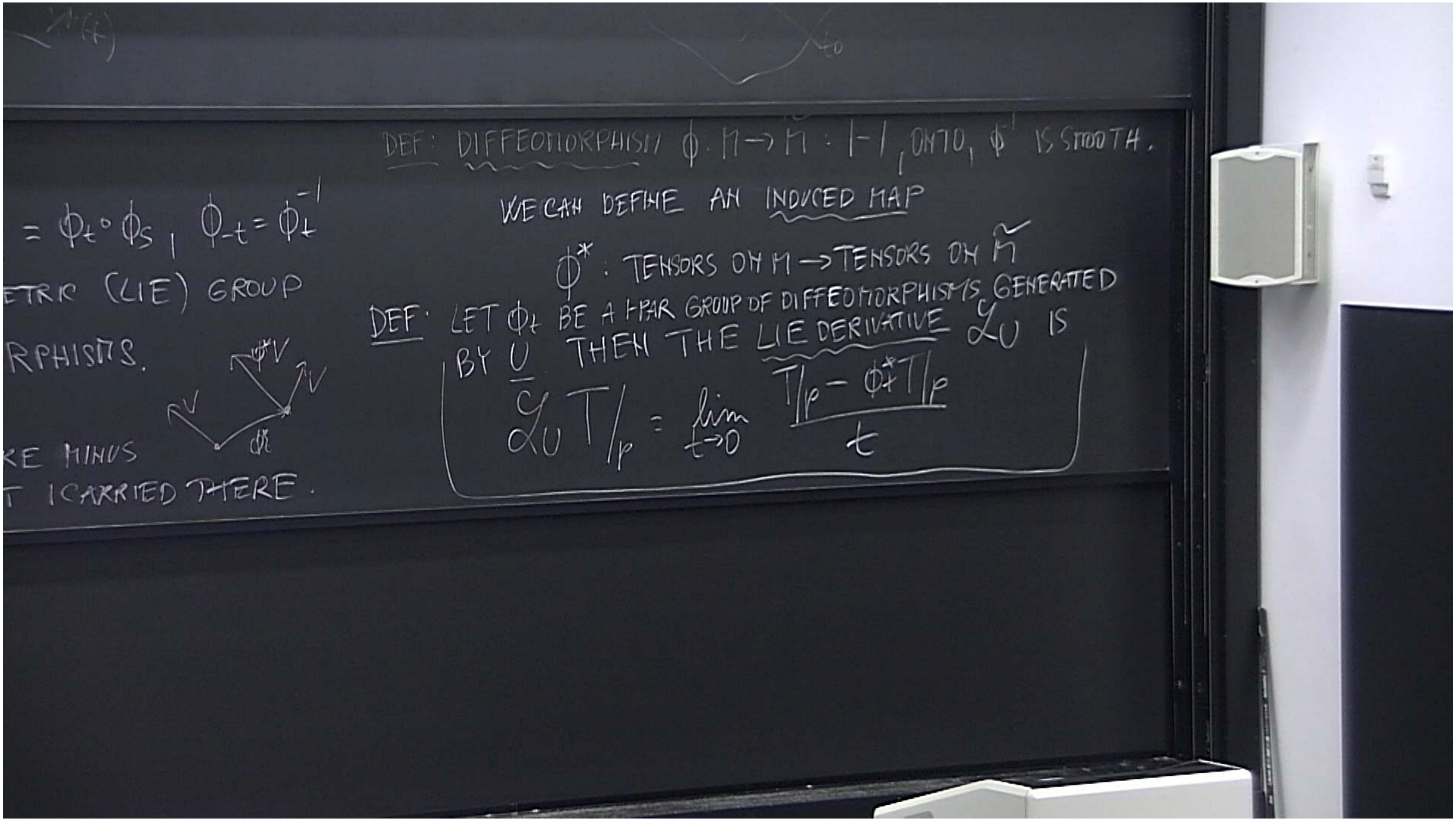
- CONTINUOUS IN t.

$$\phi_0 = \mathrm{Id}, \quad \phi_{t+s} = \phi_t \circ \phi_s, \quad \phi_{-t} = \phi_t^{-1}$$

DEFINES A 1-PARAMETRIC (LIE) GROUP OF DIFFEOMORPHISMS.

DEF: DIFFEOM


DEF: DIFFEOMORPHISM $\phi : M \to \tilde{M}$: 1-1, ONTO, ϕ^{-1} IS SMOOTH.
WE CAN DEFINE AN INDUCED MAP
ϕ^{*} : TENSORS ON $M \to$ TENSORS ON $\tilde{M}$
DEF: LET ϕ_t BE A 1-PAR GROUP OF DIFFEOMORPHISMS GENERATED BY U. THEN THE LIE DERIVATIVE $\mathcal{L}_U$


= $\phi_t \circ \phi_s$, $\phi_{-t} = \phi_t^{-1}$
ETRIC (LIE) GROUP
RPHISMS.
KE MINUS
T ICARRIED THERE.
DEF: DIFFEOMORPHISM $\phi : M \to \tilde{M}$: 1-1, ONTO, ϕ^{-1} IS SMOOTH.
WE CAN DEFINE AN INDUCED MAP
ϕ^* : TENSORS ON $M \to$ TENSORS ON $\tilde{M}$
DEF: LET ϕ_t BE A 1-PAR GROUP OF DIFFEOMORPHISMS GENERATED BY U. THEN THE LIE DERIVATIVE $\mathcal{L}_U$ IS
$\mathcal{L}_U T|_p = \lim_{t\to 0} \frac{T|_p - \phi_t^* T|_p}{t}$

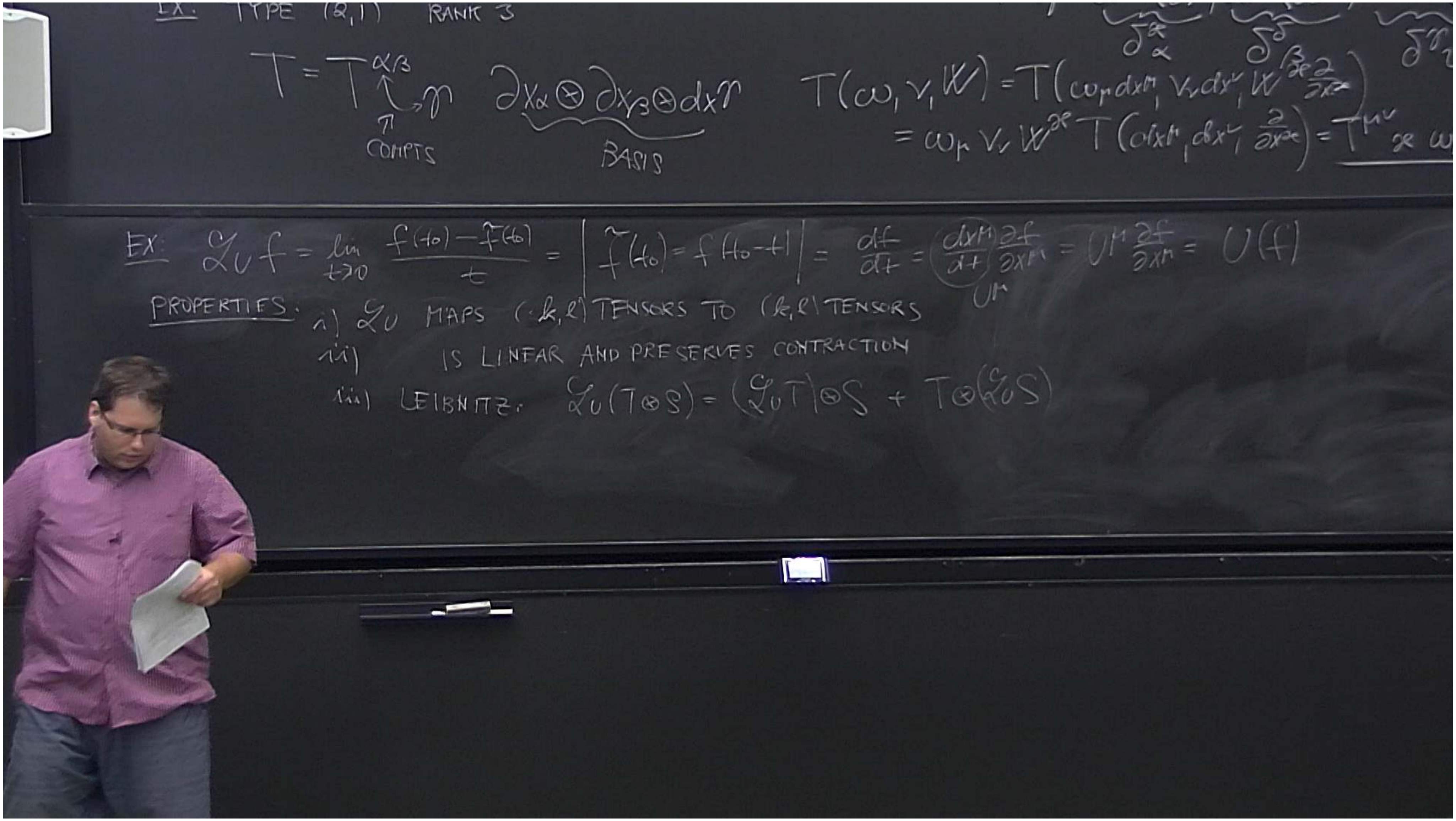
DEF: DIFFEOMORPHISM $\phi : M \to \tilde{M}$: 1-1, ONTO, ϕ^{-1} IS SMOOTH.

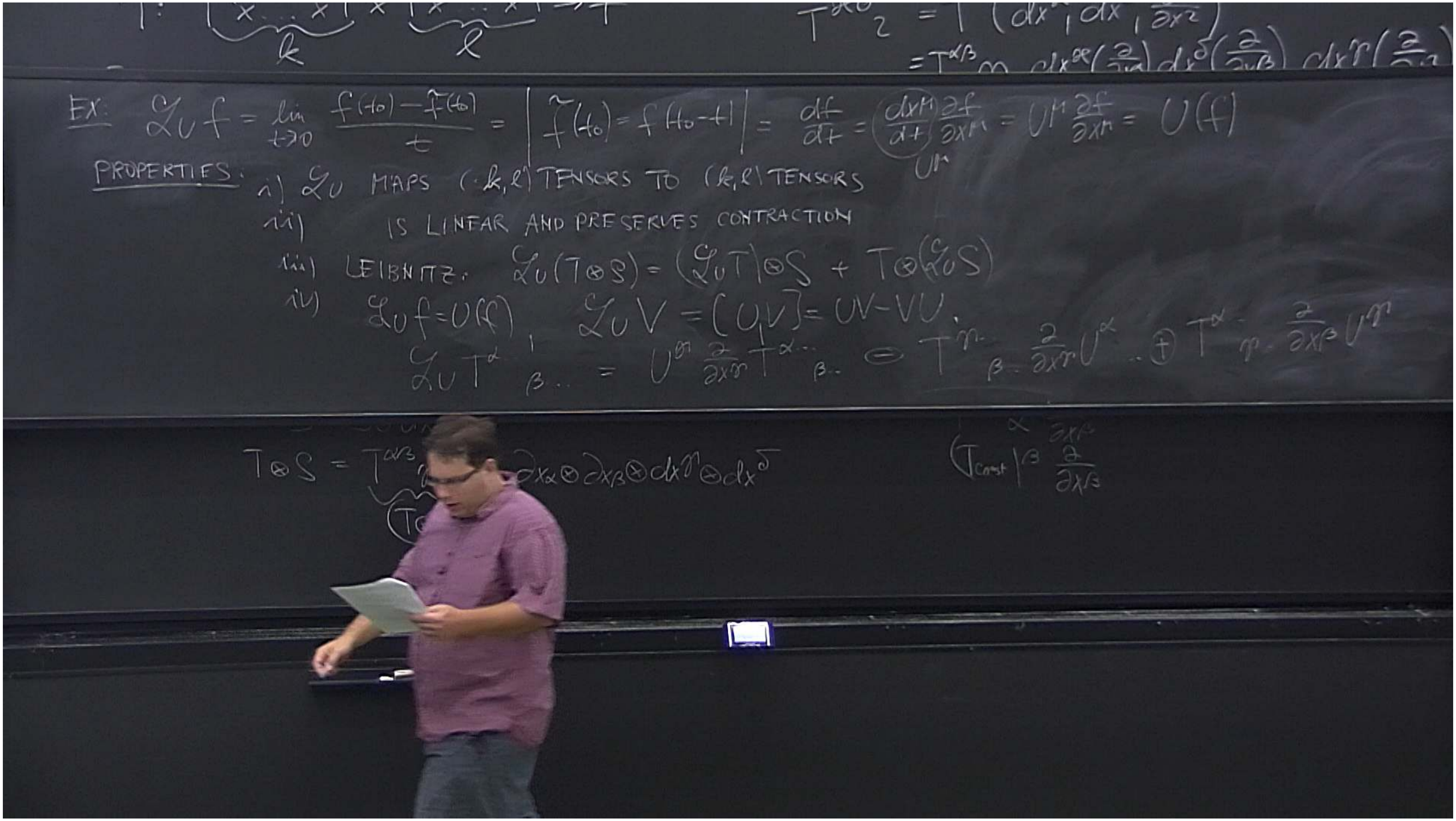
WE CAN DEFINE AN INDUCED MAP

$$\phi^* : \text{TENSORS ON } M \to \text{TENSORS ON } \tilde{M}$$

DEF: LET ϕ_t BE A 1-PAR GROUP OF DIFFEOMORPHISMS GENERATED BY $\underline{v}$. THEN THE LIE DERIVATIVE $\mathcal{L}_v$ IS

$$\boxed{\mathcal{L}_v T\big|_p = \lim_{t \to 0} \frac{T\big|_p - \phi_t^* T\big|_p}{t}}$$


EX: TYPE (2,1) RANK 3
$T = T^{\alpha\beta}{}_{\gamma}\, \partial x_\alpha \otimes \partial x_\beta \otimes dx^\gamma$
COMPTS
BASIS
$T(\omega, v, W) = T(\omega_\mu dx^\mu, v_\nu dx^\nu, W^{\varkappa} \frac{\partial}{\partial x^\varkappa})$
$= \omega_\mu v_\nu W^\varkappa T(dx^\mu, dx^\nu, \frac{\partial}{\partial x^\varkappa}) = T^{\mu\nu}{}_{\varkappa}\, \omega$
EX: $\mathcal{L}_U f = \lim_{t \to 0} \frac{f(t_0) - \tilde{f}(t_0)}{t} = \left| \tilde{f}(t_0) = f(t_0 - t) \right| = \frac{df}{dt} = \frac{dx^\mu}{dt} \frac{\partial f}{\partial x^\mu} = U^\mu \frac{\partial f}{\partial x^\mu} = U(f)$
PROPERTIES: i) $\mathcal{L}_U$ MAPS (k, ℓ) TENSORS TO (k, ℓ) TENSORS
ii) IS LINEAR AND PRESERVES CONTRACTION
iii) LEIBNITZ: $\mathcal{L}_U(T \otimes S) = (\mathcal{L}_U T) \otimes S + T \otimes (\mathcal{L}_U S)$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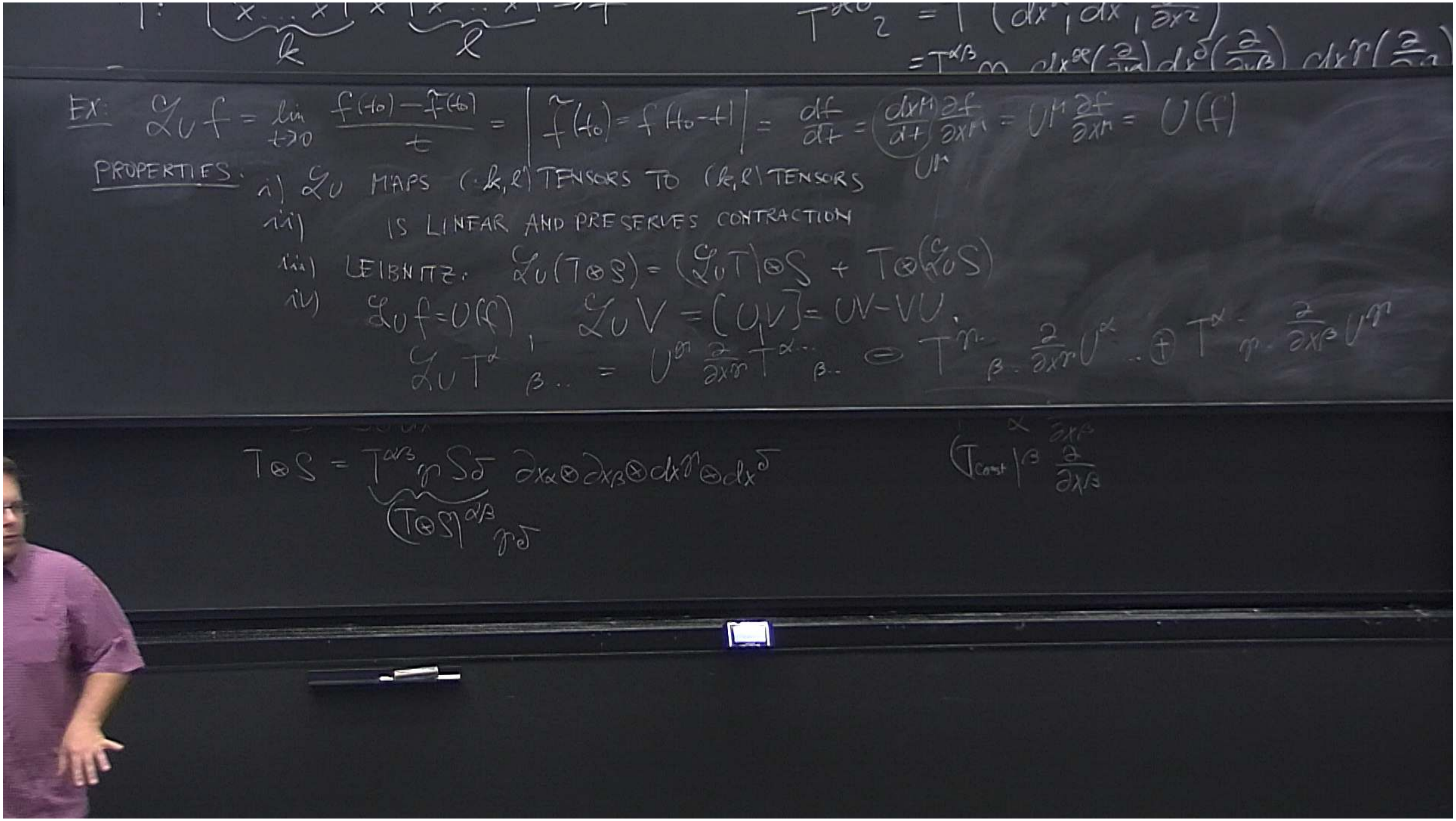
PROPERTIES:
i) $\mathcal{L}_U$ MAPS (k,l) TENSORS TO (k,l) TENSORS
ii) IS LINEAR AND PRESERVES CONTRACTION
iii) LEIBNITZ: $\mathcal{L}_U(T \otimes S) = (\mathcal{L}_U T) \otimes S + T \otimes (\mathcal{L}_U S)$
iv) $\mathcal{L}_U f = U(f)$, $\mathcal{L}_U V = [U, V] = UV - VU$
$\mathcal{L}_U T^{\alpha}{}_{\beta\ldots} = U^\gamma \frac{\partial}{\partial x^\gamma} T^{\alpha\ldots}{}_{\beta\ldots} - T^{\gamma}{}_{\beta\ldots} \frac{\partial}{\partial x^\gamma} U^\alpha \ldots + T^{\alpha}{}_{\gamma\ldots} \frac{\partial}{\partial x^\beta} U^\gamma$
$T \otimes S = T^{\alpha\beta} \ldots \partial x_\alpha \otimes \partial x_\beta \otimes dx^\gamma \otimes dx^\delta$
$(T_{\text{const}})^\beta \frac{\partial}{\partial x^\beta}$



Ex: $\mathcal{L}_U f = \lim_{t\to 0} \frac{f(t_0) - \tilde{f}(t_0)}{t} = \left| \tilde{f}(t_0) = f(t_0 - t) \right| = \frac{df}{dt} = \frac{dx^\mu}{dt}\frac{\partial f}{\partial x^\mu} = U^\mu \frac{\partial f}{\partial x^\mu} = U(f)$
PROPERTIES: i) $\mathcal{L}_U$ MAPS (k,l) TENSORS TO (k,l) TENSORS
ii) IS LINEAR AND PRESERVES CONTRACTION
iii) LEIBNITZ: $\mathcal{L}_U(T\otimes S) = (\mathcal{L}_U T)\otimes S + T\otimes(\mathcal{L}_U S)$
iv) $\mathcal{L}_U f = U(f)$, $\mathcal{L}_U V = [U,V] = UV - VU$
$\mathcal{L}_U T^{\alpha}{}_{\beta..} = U^\gamma \frac{\partial}{\partial x^\gamma} T^{\alpha..}{}_{\beta..} - T^{\gamma..}{}_{\beta..} \frac{\partial}{\partial x^\gamma} U^\alpha .. + T^{\alpha..}{}_{\gamma..} \frac{\partial}{\partial x^\beta} U^\gamma$
$T\otimes S = \underbrace{T^{\alpha\beta}{}_\gamma S_\delta}_{(T\otimes S)^{\alpha\beta}{}_{\gamma\delta}} \partial x_\alpha \otimes \partial x_\beta \otimes dx^\gamma \otimes dx^\delta$
$(T_{Const})^\beta \frac{\partial}{\partial x^\beta}$

C) DIFFERENTIAL FORMS

DEF: A DIFFERENTIAL p-FORM ω IS A TOTALLY ANTISYMMETRIC TENSOR OF TYPE $(0,p)$

$$\omega_{\alpha_1 \dots \alpha_p} = \omega_{[\alpha_1 \dots \alpha_p]} = \frac{1}{p!} \sum_{\text{PERM } \pi} \text{SIGN}(\pi)\, \omega_{\alpha_{\pi(1)} \dots \alpha_{\pi(p)}}$$

$$\omega_{[\alpha\beta]} = \frac{1}{2!}\left(\omega_{\alpha\beta} - \omega_{\beta\alpha}\right)$$

Λ^p_x .. VECTOR SPACE OF p-FORMS AT $x \in M$.

(CHOOSING p DIFFERENT INDICES OUT OF m)

$$\binom{m}{p} = \dim \Lambda^p_x$$

$\pi) \, \omega_{\alpha_{\pi(1)} \dots \alpha_{\pi(p)}}$

DEF: A WEDGE PRODUCT $\wedge : \Lambda^p_x \times \Lambda^q_x \to \Lambda^{p+q}_x$

$$(\omega \wedge \nu)_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q} = \frac{(p+q)!}{p!\,q!}\, \omega_{[\alpha_1 \ldots \alpha_p} \nu_{\beta_1 \ldots \beta_q]}$$

$$\omega \wedge \nu = (-1)^{pq}\, \nu \wedge \omega$$

IN COORDINATE BASIS

$$\omega = \frac{1}{p!}\, \omega_{\alpha_1 \ldots \alpha_p}\, dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_p}$$

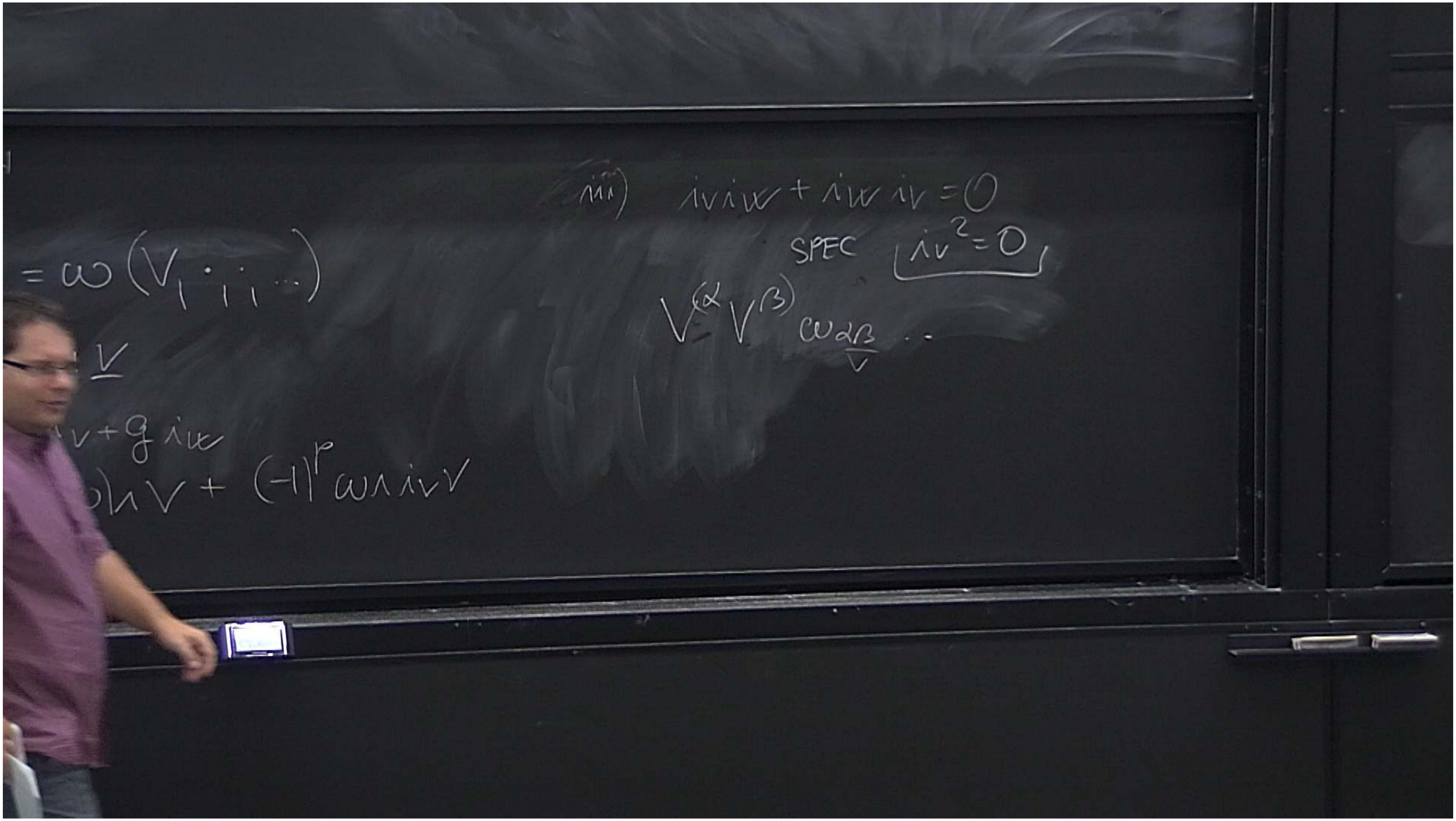
• FOR ANY VECTOR FIELD $\underline{V}$ WE DEFINE AN <u>INNER DERIVATIVE</u> $i_V : \Lambda^p \to \Lambda^{p-1}$

$$i_V \omega = V \lrcorner \omega = V \cdot \omega = \omega(V, \cdot, \cdot, \ldots)$$

<u>PROP</u>: i) i_V ... LINEAR & LINEAR IN $\underline{V}$

$$i_{fV+gW} = f i_V + g i_W$$

ii) <u>LEIBNITZ</u> (GRADED): $i_V(\omega \wedge \nu) = (i_V \omega) \wedge \nu + (-1)^p \omega \wedge i_V \nu$

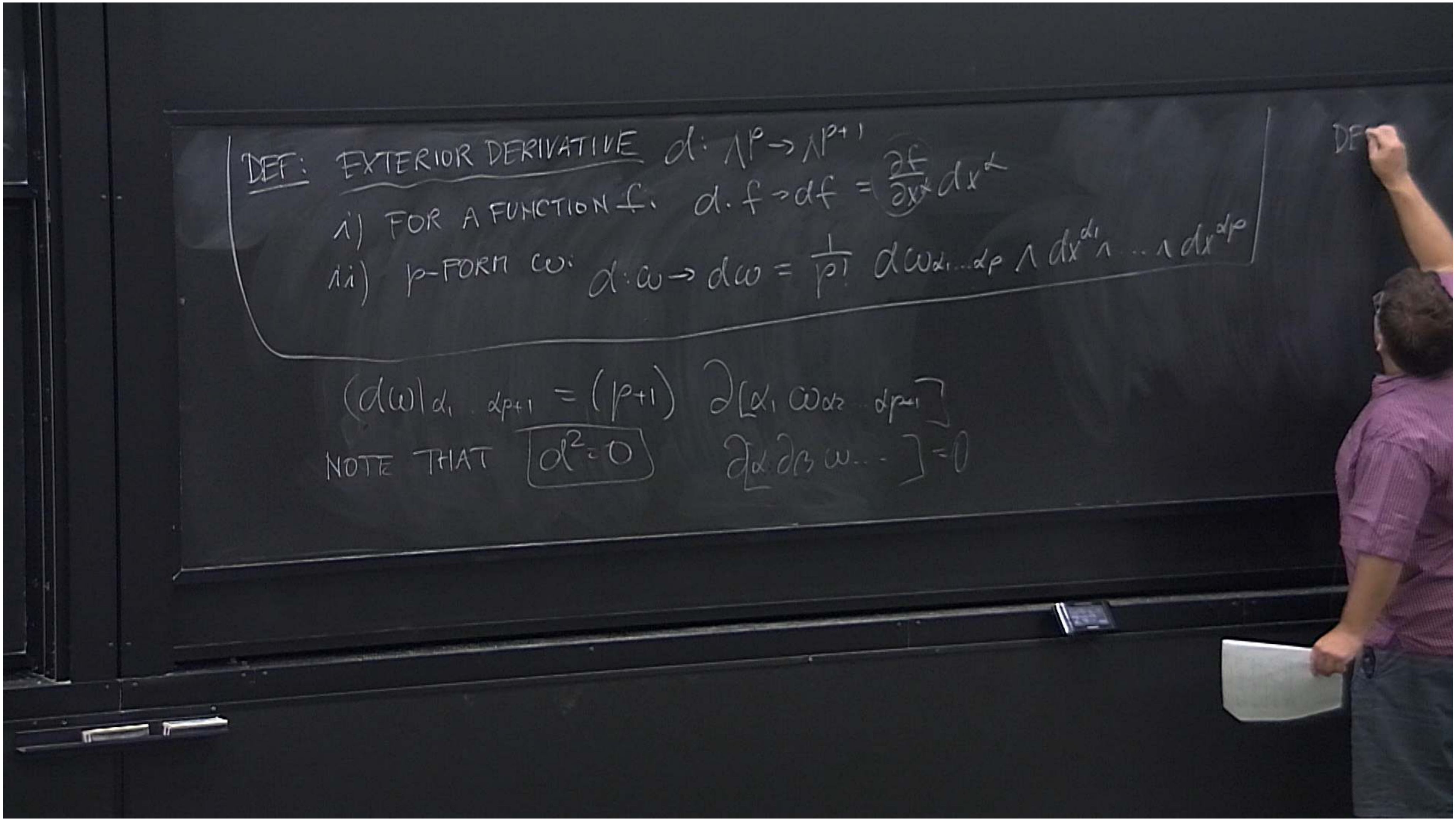

iii) $i_v i_w + i_w i_v = 0$
SPEC $i_v^2 = 0$

DEF: EXTERIOR DERIVATIVE $d : \Lambda^p \to \Lambda^{p+1}$

i) FOR A FUNCTION f: $d \cdot f \to df = \frac{\partial f}{\partial x^\alpha} dx^\alpha$

ii) p-FORM ω: $d : \omega \to d\omega = \frac{1}{p!} d\omega_{\alpha_1 \ldots \alpha_p} \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_p}$

$$(d\omega)_{\alpha_1 \ldots \alpha_{p+1}} = (p+1)\, \partial_{[\alpha_1} \omega_{\alpha_2 \ldots \alpha_{p+1}]}$$

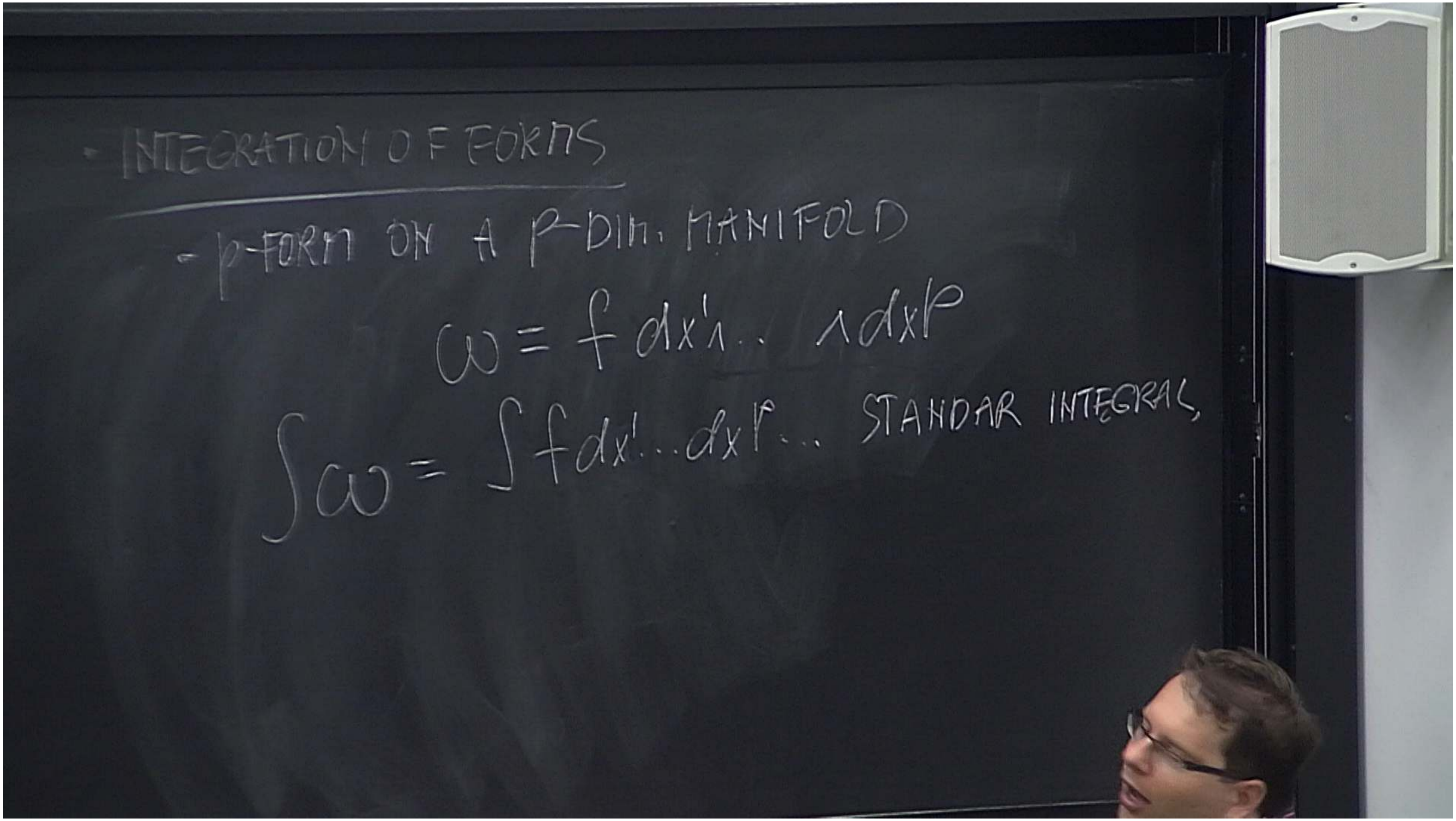

DEF: EXTERIOR DERIVATIVE $d: \Lambda^p \to \Lambda^{p+1}$
i) FOR A FUNCTION f: $d: f \to df = \frac{\partial f}{\partial x^\alpha} dx^\alpha$
ii) p-FORM ω: $d: \omega \to d\omega = \frac{1}{p!} d\omega_{\alpha_1 \dots \alpha_p} \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p}$
$(d\omega)_{\alpha_1 \dots \alpha_{p+1}} = (p+1)\, \partial_{[\alpha_1} \omega_{\alpha_2 \dots \alpha_{p+1}]}$
NOTE THAT $\boxed{d^2 = 0}$ $\partial_{[\alpha} \partial_\beta \omega_{\dots]} = 0$

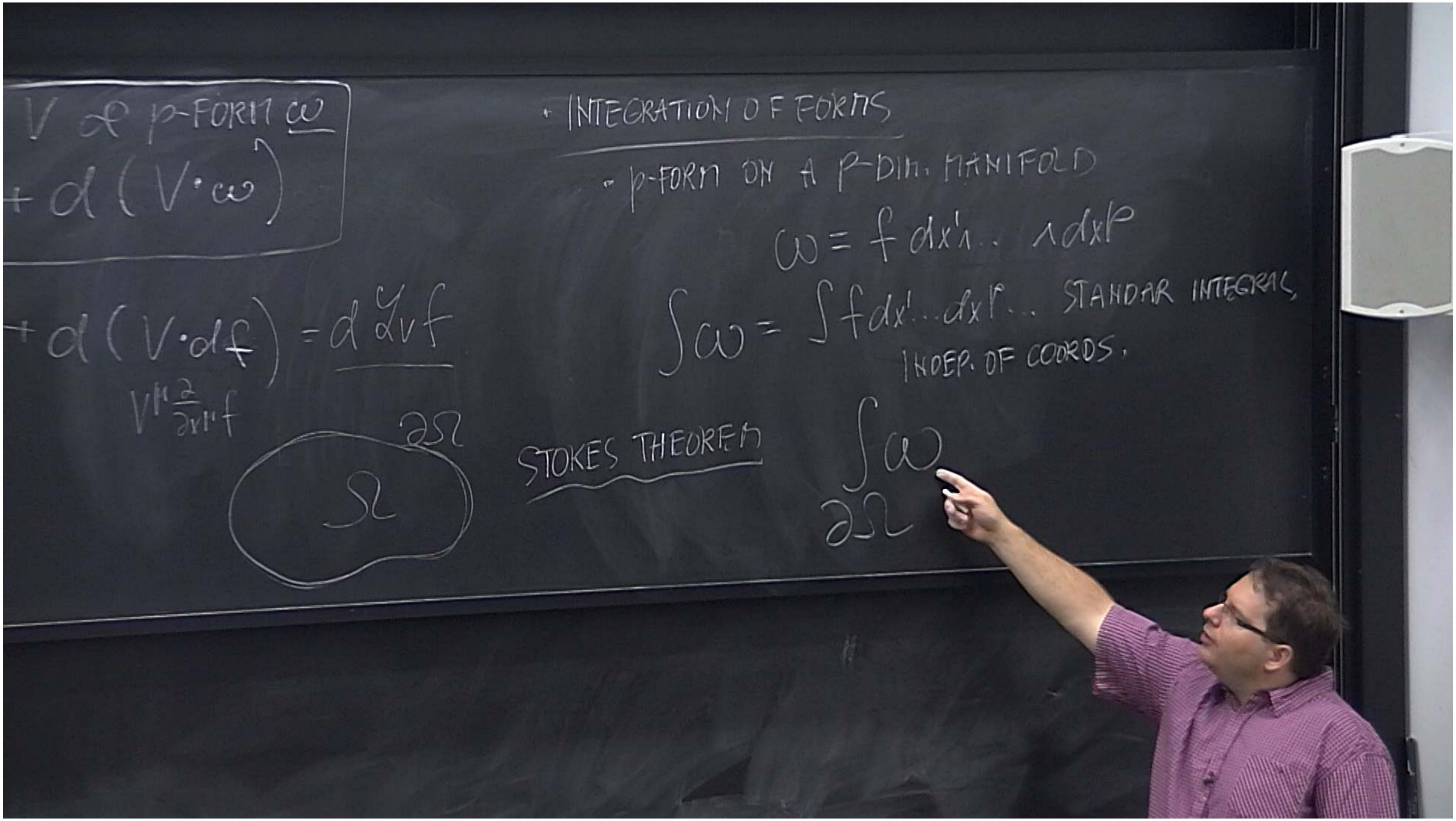
NOTE THAT $\boxed{d^2 = 0}$ $\quad d[d\alpha\, d\beta\, \omega \ldots] = 0$

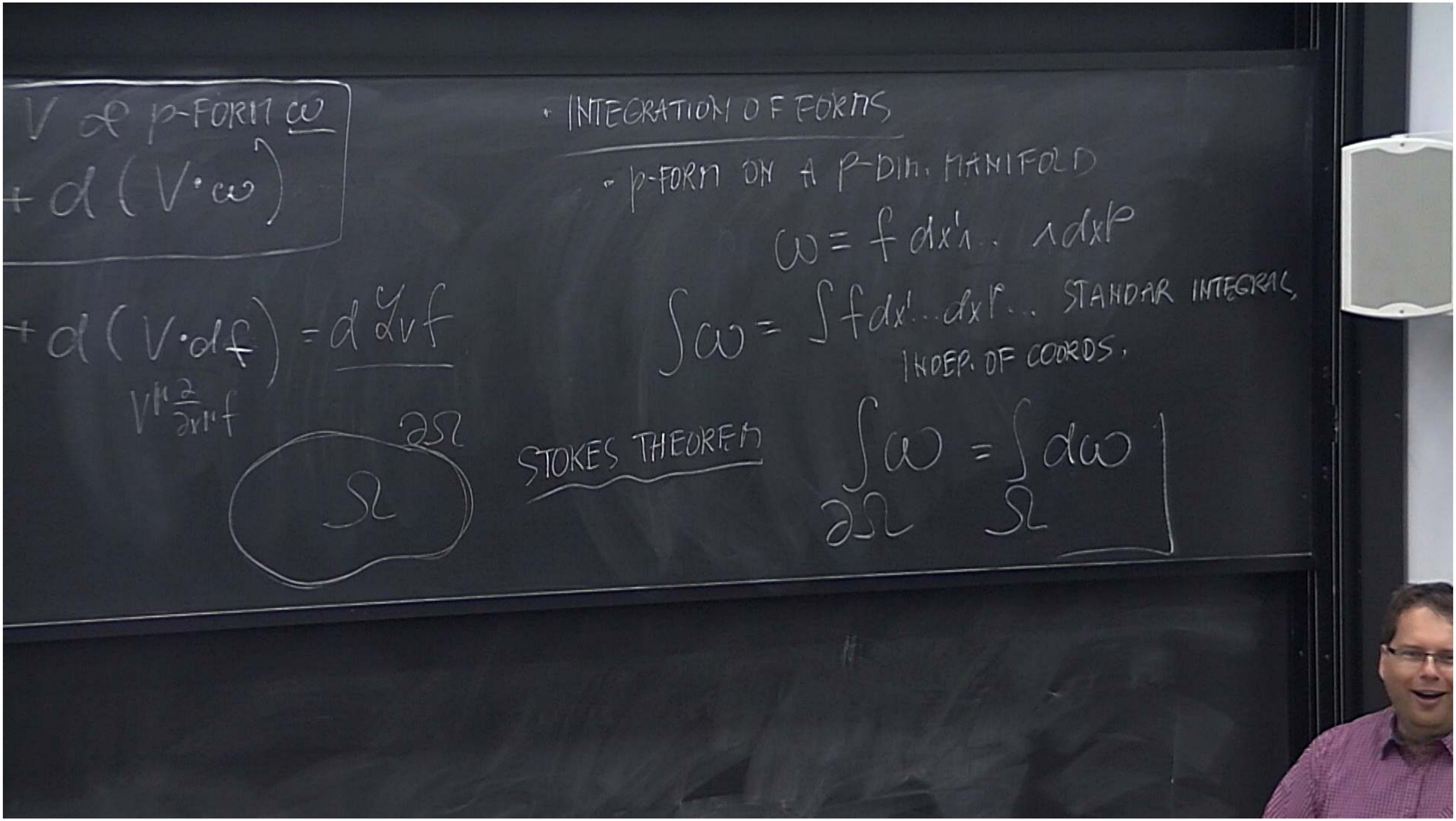
CARTAN'S LEMMA: FOR A VECTOR V & p-FORM ω

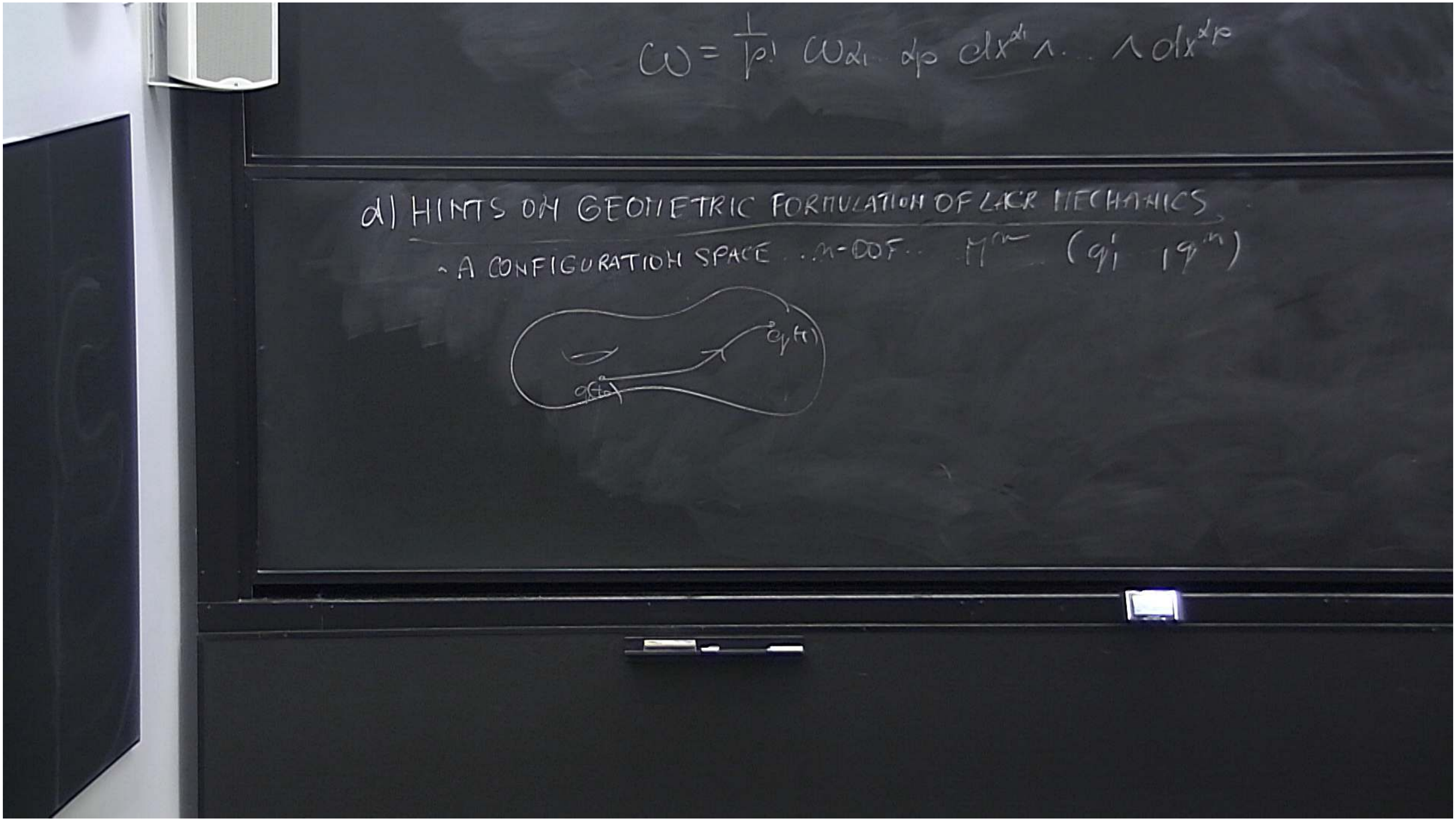
$$\boxed{\mathcal{L}_V \omega = V\cdot d\omega + d(V\cdot\omega)}$$

SPEC: $\mathcal{L}_V df = \underbrace{V\cdot ddf}_{0} + d(V\cdot df)$


- INTEGRATION OF FORMS
- p-FORM ON A p-DIM. MANIFOLD
$\omega = f\, dx^1 \wedge \dots \wedge dx^p$
$\int \omega = \int f\, dx^1 \dots dx^p$ ... STANDAR INTEGRAL,


V ∝ p-FORM ω
+ d(V·ω)
+ d(V·df) = d £V f
V^μ ∂/∂x^μ f
∂Ω
Ω
· INTEGRATION OF FORMS
· p-FORM ON A p-DIM. MANIFOLD
ω = f dx¹∧ .. ∧dx^p
∫ω = ∫ f dx¹...dx^p .. STANDAR INTEGRAL,
INDEP. OF COORDS.
STOKES THEOREM
∫∂Ω ω


V & p-FORM ω
+ d (V·ω)
+ d (V·df) = d 𝓛_V f
V^μ ∂/∂x^μ f
∂Ω
Ω
• INTEGRATION OF FORMS
• p-FORM ON A p-DIM. MANIFOLD
ω = f dx^1 ∧ … ∧ dx^p
∫ω = ∫ f dx^1 … dx^p … STANDAR INTEGRAL,
INDEP. OF COORDS.
STOKES THEOREM
∫_{∂Ω} ω = ∫_Ω dω

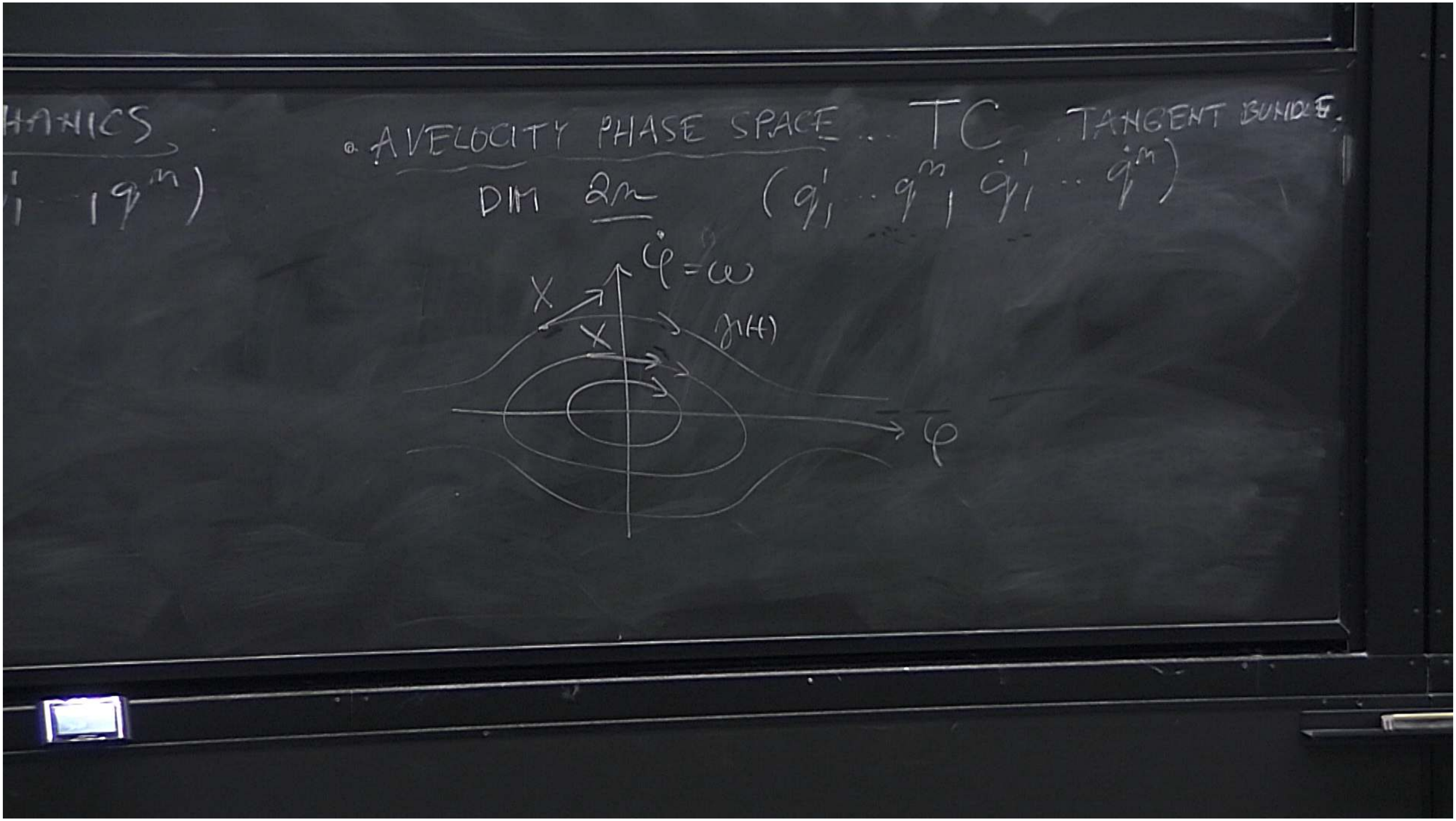

$\omega = \frac{1}{p!} \omega_{\alpha_1 \dots \alpha_p} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p}$
d) HINTS ON GEOMETRIC FORMULATION OF LAGR MECHANICS
- A CONFIGURATION SPACE .. n-DOF .. M^n $(q^1 \dots q^n)$
$q(t)$
$q(t_0)$

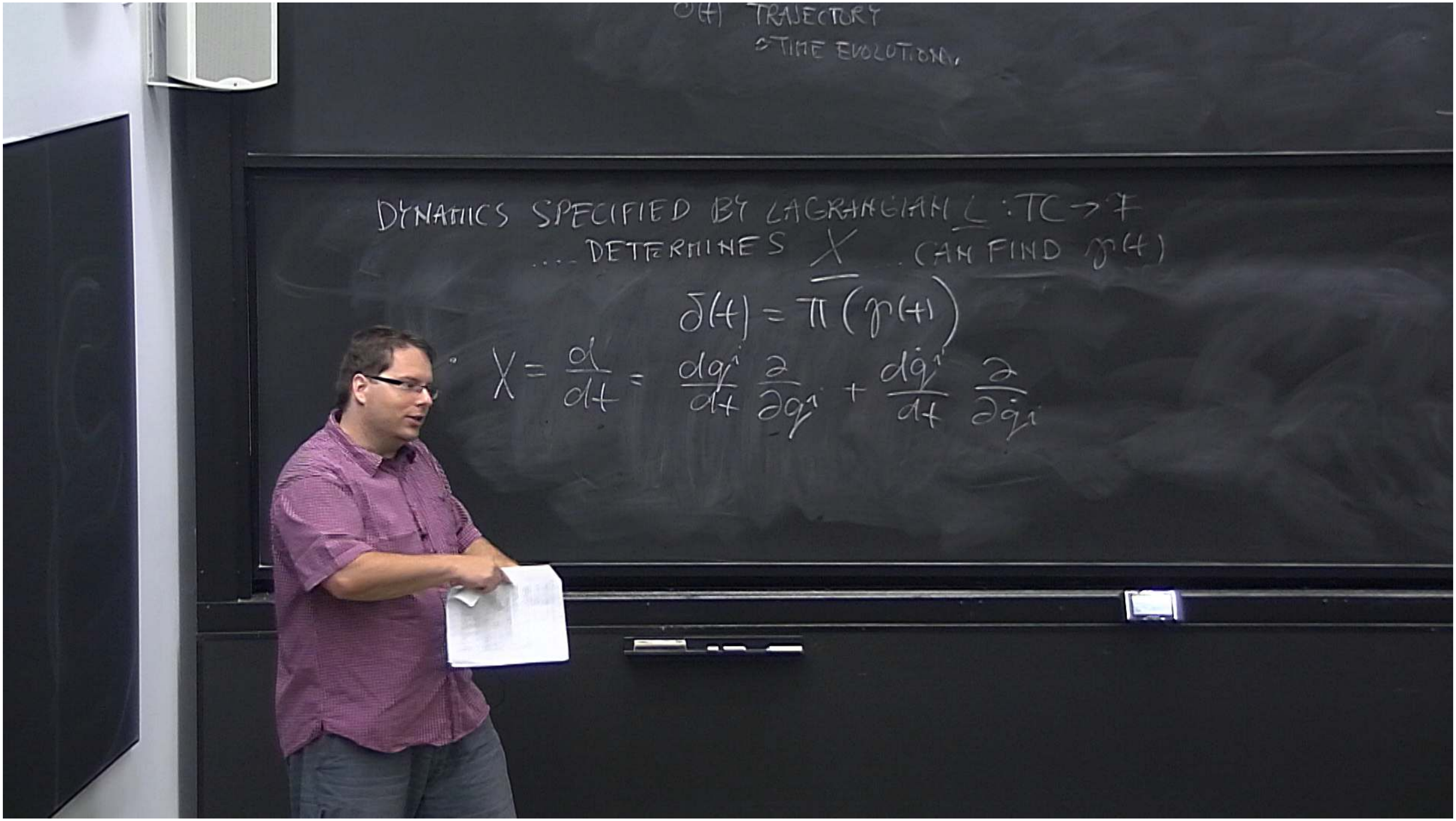
d) HINTS ON GEOMETRIC FORMULATION OF LAGR MECHANICS

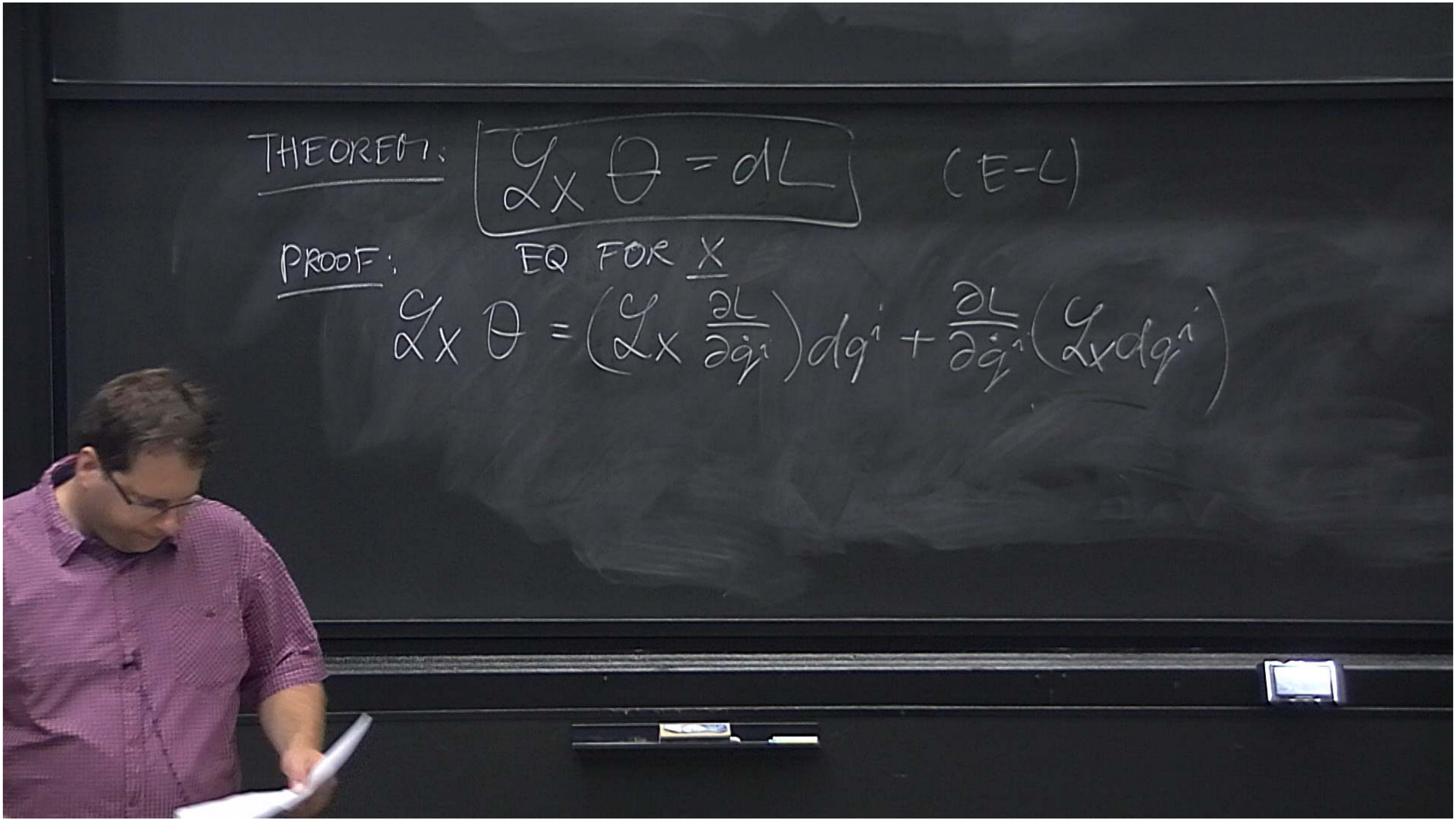
- A CONFIGURATION SPACE ... n-DOF ... M^n ... $(q^1, \dots, q^n)$

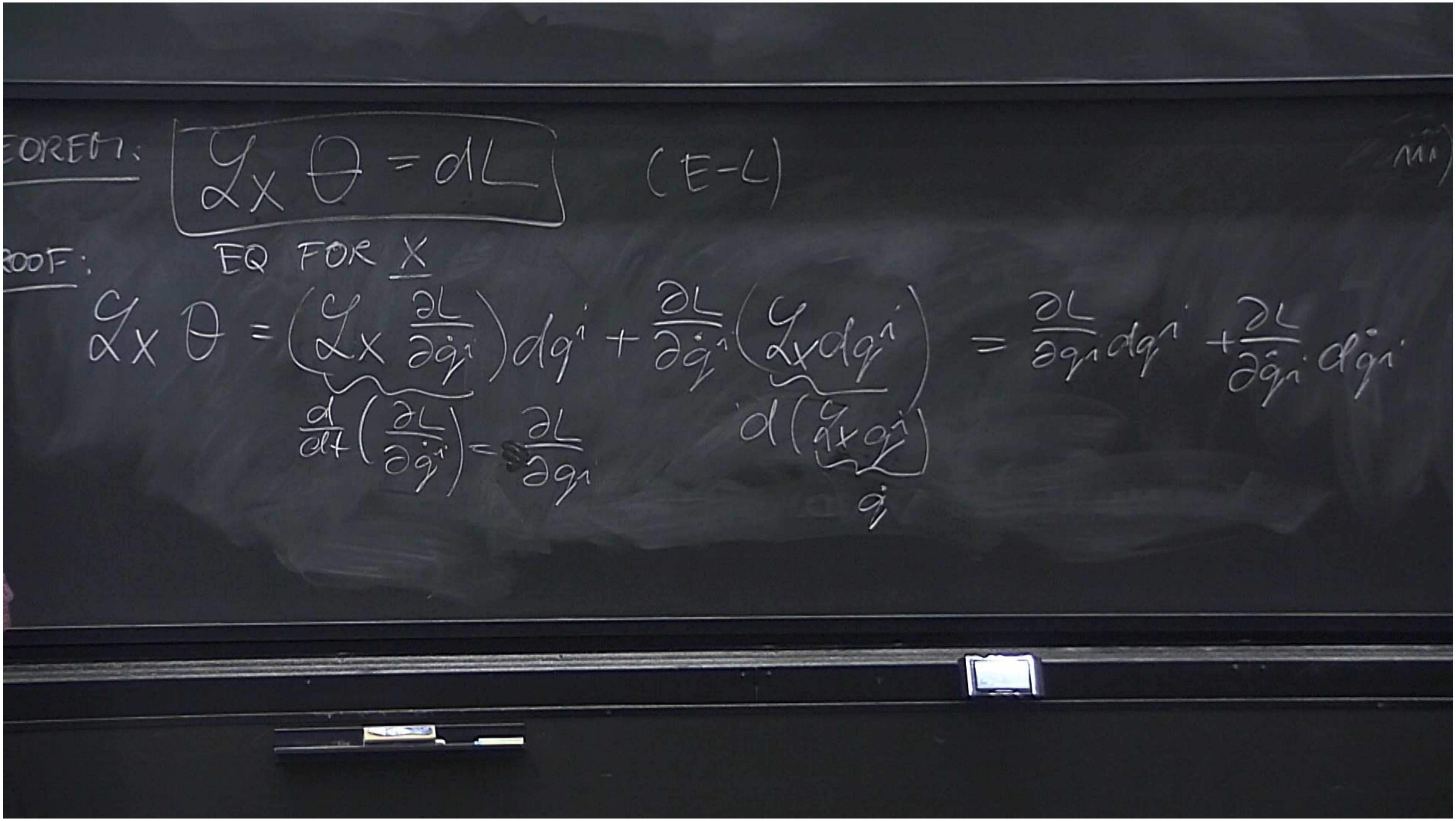
$\gamma(t)$ TRAJECTORY

= TIME EVOLUTION.


HANICS
VELOCITY PHASE SPACE ... TC
TANGENT BUNDLE
DIM 2n
$(q^1 \cdots q^n, \dot{q}^1 \cdots \dot{q}^n)$
$\dot{\varphi} = \omega$
X
X
$\gamma(t)$
φ


σ(t) TRAJECTORY
= TIME EVOLUTION.
DYNAMICS SPECIFIED BY LAGRANGIAN L : TC → ℝ
.... DETERMINES X .. CAN FIND γ(t)
σ(t) = π(γ(t))
X = d/dt = dq^i/dt ∂/∂q^i + dq̇^i/dt ∂/∂q̇^i


THEOREM: $\boxed{\mathcal{L}_X \Theta = dL}$ (E-L)
PROOF: EQ FOR X
$\mathcal{L}_X \Theta = \left(\mathcal{L}_X \frac{\partial L}{\partial \dot{q}^i}\right) dq^i + \frac{\partial L}{\partial \dot{q}^i}\left(\mathcal{L}_X dq^i\right)$


EOREM: $\boxed{\mathcal{L}_X \Theta = dL}$ (E-L)
ROOF: EQ FOR $\underline{X}$
$\mathcal{L}_X \Theta = \underbrace{\left(\mathcal{L}_X \frac{\partial L}{\partial \dot q^i}\right)}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right) = \frac{\partial L}{\partial q^i}} dq^i + \frac{\partial L}{\partial \dot q^i}\left(\underbrace{\mathcal{L}_X dq^i}_{d(\underbrace{\mathcal{L}_X q^i}_{\dot q^i})}\right) = \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial \dot q^i} d\dot q^i$