

Title: PSI 17/18 Front End - Theoretical Mechanics (Kubiznak)

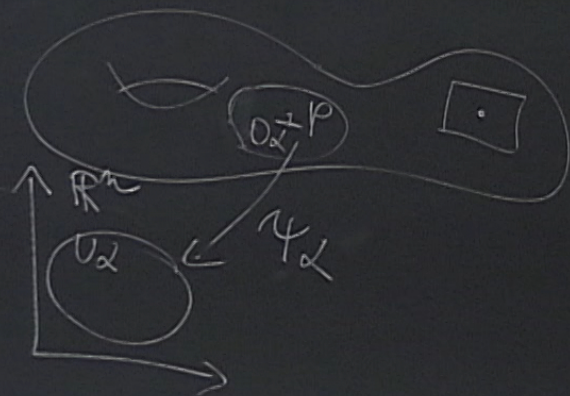
Date: Aug 10, 2017 09:00 AM

URL: <http://pirsa.org/17080043>

Abstract:

2) INTRODUCTION TO DIFFERENTIAL GEOMETRY

a) MANIFOLDS & TENSORS



DEF: AN n -DIMENSIONAL MANIFOLD M WITH A COLLECTION OF SUBSETS

- i) EACH $p \in M$ LIES IN AT LEAST
- ii) FOR EACH α , THERE IS U_α ONTO IS AN OPEN SUBSET OF $\mathbb{R}^n$

DIFFERENTIAL GEOMETRY

DEF: AN m -DIMENSIONAL MANIFOLD M IS A "SET OF POINTS" TOGETHER WITH A COLLECTION OF SUBSETS $\{O_\alpha\}$ SATISFYING:

- i) EACH $p \in M$ LIES IN AT LEAST ONE O_α : IE. $\{O_\alpha\}$ COVER M .
- ii) FOR EACH α , THERE IS $1-1$, ONTO, MAP $\psi_\alpha: O_\alpha \rightarrow U_\alpha$, WHERE U_α IS AN OPEN SUBSET OF $\mathbb{R}^m$.
- iii) IF ANY TWO SETS O_α AND O_β OVERLAP, $O_\alpha \cap O_\beta \neq \emptyset$, THEN THE MAP $\psi_\beta \circ \psi_\alpha^{-1}$ IS SMOOTH.

2) INTRODUCTION TO DIFFERENTIAL GEO

a) MANIFOLDS & TENSORS

DEF: AN
WIT

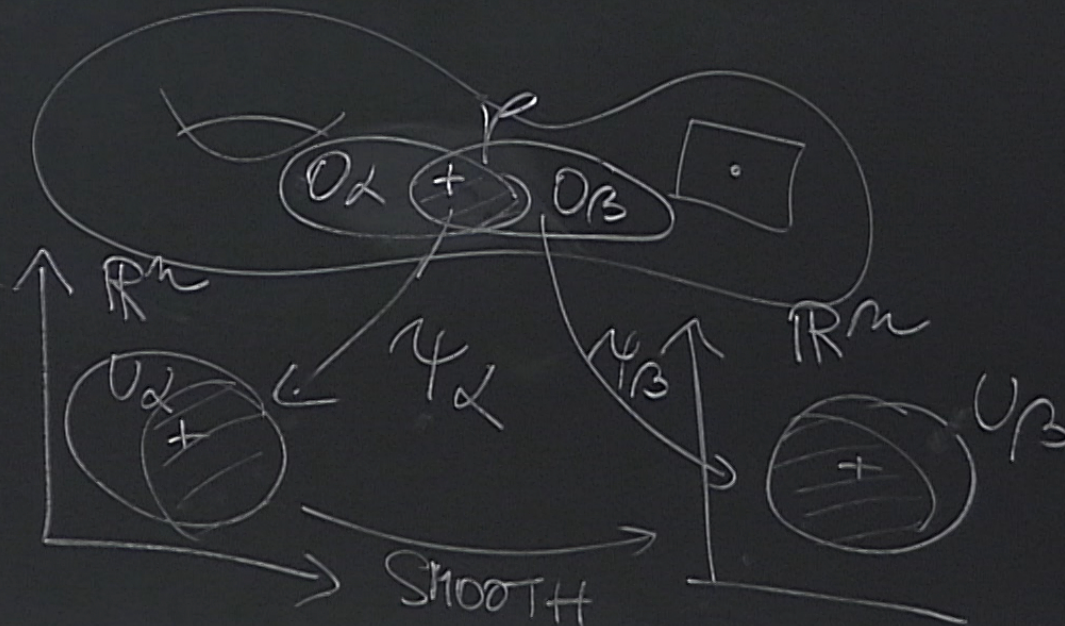
i) FAC

ii) FOR

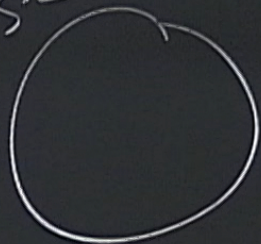
IS A

iii) IF

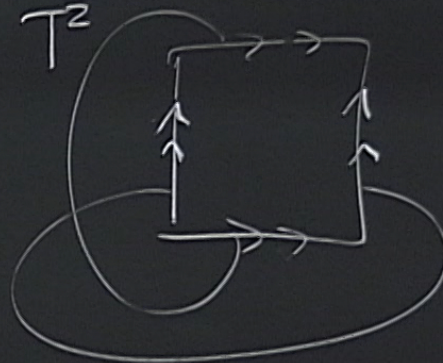
TH



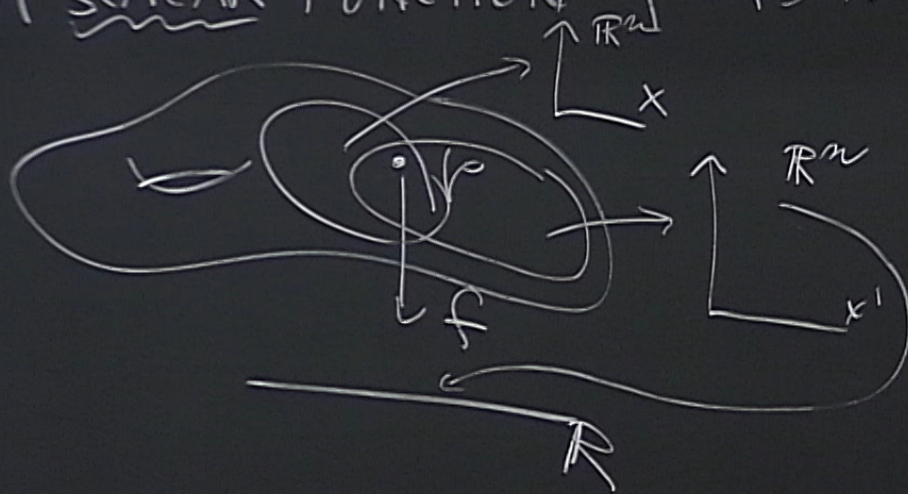
EXAMPLES: S^2



T^2



• A SCALAR FUNCTION f IS A MAP $f: M \rightarrow \mathbb{R}$



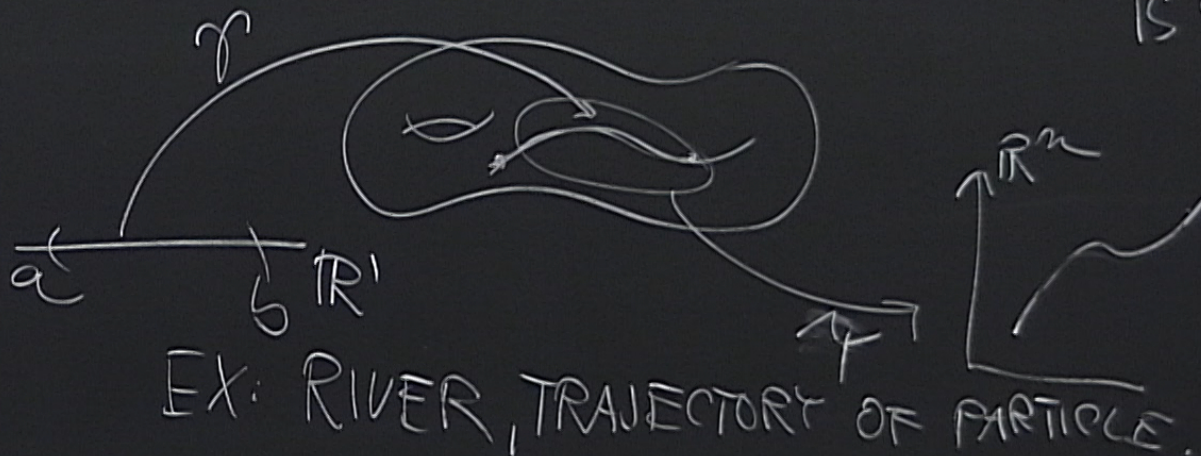
$$f(p) = f(x(p)) = f'(x'(p))$$

$$f(x) = f'(x')$$

$$f(p) = f'(x'(p))$$

$$f'(x')$$

A CURVE γ ON M IS A MAP $\gamma: I \subset \mathbb{R} \rightarrow M$
 SUCH THAT $(\psi \circ \gamma)(t) = [x^1(t), x^2(t), \dots, x^n(t)]$
 IS SMOOTH

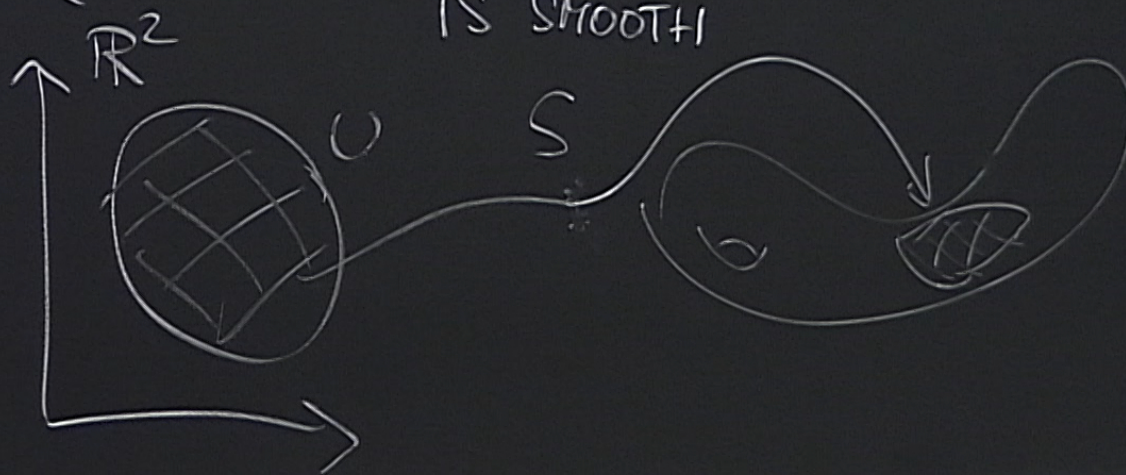


A SURFACE S IS A MAP $S: U \subset \mathbb{R}^2 \rightarrow M$

$$(\psi \circ S)(\tau, \sigma) = [x^1(\tau, \sigma), x^2(\tau, \sigma), \dots, x^n(\tau, \sigma)]$$

IS SMOOTH

EX. LAKE
WORLD SHEET
OF A STRING



IS ASSOCIATED WITH "DIRECTION OF A DERIVATIVE AT A POINT"

$v = (v^1, v^2, \dots, v^m) \iff$ DIRECTIONAL
DERIVATIVE OPERATOR

$$v^m \frac{\partial}{\partial x^m} = \hat{v}$$

$$\hat{v}f = v^m \frac{\partial f}{\partial x^m} \in \mathbb{R} \quad \text{AT A POINT}$$

CHARACTERIZED BY LINEARITY & LEIBNITZ RULE.

$\frac{\partial}{\partial x^2}$

$\frac{\partial}{\partial x^1}$

x^1

DEF: LET $\mathcal{F}$ BE A COLLECTION OF C^∞ SCALAR FUNCTIONS.
A TANGENT VECTOR V AT A POINT $p \in M$ IS A MAP

$V: \mathcal{F} \rightarrow \mathbb{R}$ SO THAT

i) LINEAR: $V(af+bg) = aV(f) + bV(g)$ $a, b \in \mathbb{R}$

ii) LEIBNITZ: $V(fg) = f(p)V(g) + g(p)V(f)$

CTIONS.
IS A MAP

$a, b \in \mathbb{R}$

$g(p) \vee (f)$

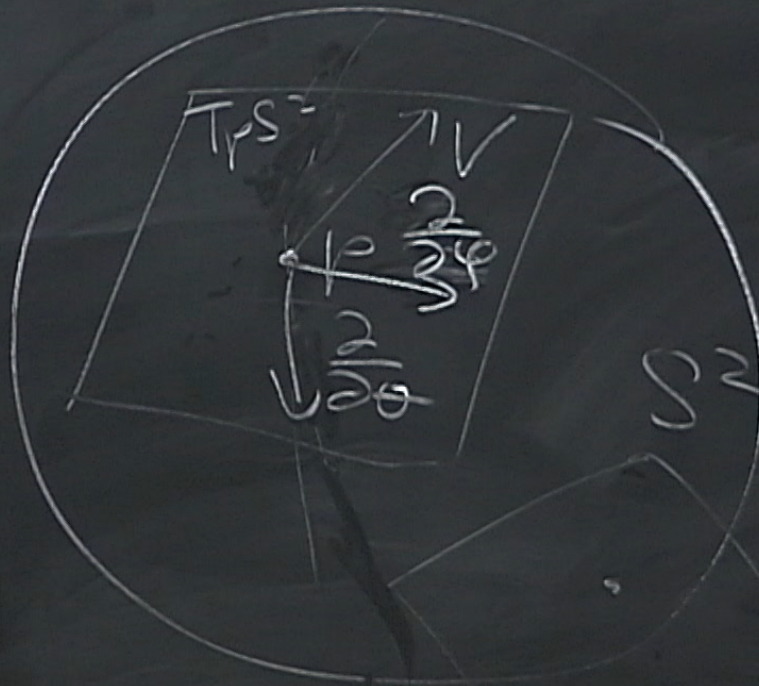
THEOREM. THE SET OF TANGENT VECTORS AT p
FORMS A TANGENT VECTOR SPACE $T_p M$
OF DIM m . (SAME AS MANIFOLD).

USING THE CHAIN RULE, THE COMPONENTS OF THE VECTOR CHANGE
OF COORDINATES AS FOLLOWS:

$$V = V^{\mu}(x) \frac{\partial}{\partial x^{\mu}} = V^{\mu}(x) \frac{\partial x'^{\nu}}{\partial x^{\mu}} \frac{\partial}{\partial x'^{\nu}} = V'^{\nu}(x')$$

$$V'^{\nu}(x') = \frac{\partial x'^{\nu}}{\partial x^{\mu}} V^{\mu}(x)$$

EX:



USING
OF

$V =$

DEF: A TANGENT VECTOR V IS DEFINED AS $\sum V|_p \in T_p M$ FOR
A TANGENT BUNDLE $TM = \bigcup_p T_p M$.. LOCAL CO

DEF: A COTANGENT VECTOR (1-FORM) ω AT A POINT $p \in M$
 IS A MAP $\omega: T_p M \rightarrow \mathbb{R}$. HAVE COORDINATE BASIS
 IN COTANGENT VECTOR SPACE $T_p^* M = \Omega^1_x M$

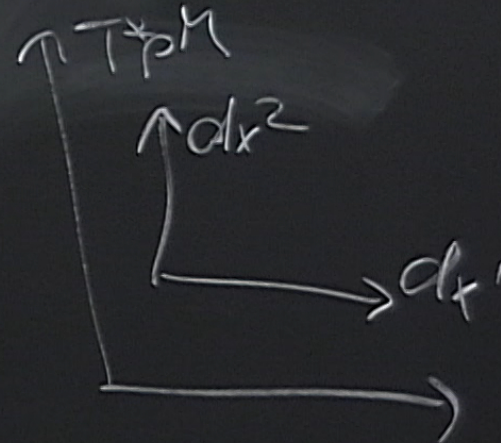
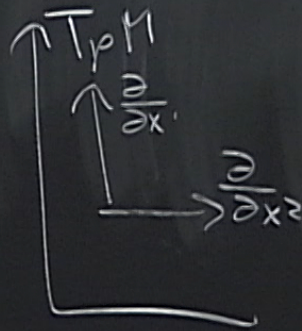
$\{ V|_p \in T_p M \text{ FOR ALL } p \in M. \quad V(f) \text{ IS SMOOTH} \}$

$M \quad \dots \text{ LOCAL COORDINATES } (x^M, V^V)$

AT A POINT $p \in M$

COORDINATE BASIS

M



DEF: A TANGENT VECTOR V IS DEFINED AS $\sum V|_p \in T_p M$
ATANGENT BUNDLE $TM = \bigcup_p T_p M$... LOCAL

DEF: A COTANGENT VECTOR (1-FORM) ω AT A POINT $p \in M$
 IS A MAP $\omega: T_p M \rightarrow \mathbb{R}$. HAVE COORDINATE BASIS
 IN COTANGENT VECTOR SPACE $T_p^* M : dx^i$
 $\left[dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta^i_j \right]$

$$T^*M = \bigcup_p T_p^*M \dots \text{COTANGENT BUNDLE}$$

(EX OF FIBRE BUNDLE)

(x^M, ω_v)

CANONICAL PROJECTION

$V \in T^*M$

(x^M, v^L)

$$\pi: T^*M \rightarrow M$$

$$\pi(V) = p$$

$\omega_M(x)$

CANONICAL PROJECTION

(x^M, ω_V)
 $V \in TM$

$\pi: TM \rightarrow M$

$\pi(V) = p$

$\pi(x^1, \dots, x^n, V^1, \dots, V^n) \rightarrow (x^1, \dots, x^n)$

V. 4 - ID on T.M.

TENSOR $\approx$ PRODUCT OF VECTORS & FORMS

DEF: A TENSOR OF TYPE (k, l) AND RANK $k+l$
IS A MULTILINEAR MAP:

$$T: \underbrace{T_p^* \times T_p^* \times \dots \times T_p^*}_k \times \underbrace{T_p \times \dots \times T_p}_l \rightarrow \mathbb{R}$$

$$T^{\alpha \dots}_{\mu \dots}(x') = \underbrace{\frac{\partial x'^{\alpha}}{\partial x^{\sigma}} \dots}_{k} \underbrace{\frac{\partial x^{\sigma}}{\partial x'^{\mu}} \dots}_{l} T^{\sigma \dots}_{\sigma \dots}(x)$$

EX: METRIC g - $(0,2)$ TENSOR
 $g(V, W) \rightarrow \mathbb{R}$

TENSOR ALGEBRA

i) CAN ADD TWO TENSORS OF THE SAME TYPE

ii) TENSOR PRODUCT $T \otimes S$ $(t+s)$ RANK TENSOR

iii) CONTRACTION

$T^{\alpha\beta}$

$\rightarrow$ CONTR.

$T^{\alpha\beta}$

$\alpha = \sqrt{\beta}$

DUMMY INDICES