Title: Time Evolution of Complexity in Abelian Gauge Theories - And Playing Quantum Othello Game

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Abstract: <p>Quantum complexity is conjectured to probe inside of black hole horizons (or wormhole) via gauge gravity correspondence. In order to have a better understanding of this correspondence, we study time evolutions of complexities for generic Abelian pure gauge theories. For this purpose, we discretize U(1) gauge group as Z_N and also continuum spacetime as lattice spacetime, and this enables us to define a universal gate set for these gauge theories, and evaluate time evolutions of the complexities explicitly. We find that to achieve a large complexity âˆ¼exp(entropy), which is one of the conjectured criteria necessary to have a dual black hole, the Abelian gauge theory needs to be maximally nonlocal. (Based on collaboration with Norihiro Iizuka and Sotaro Sugishita, https://arxiv.org/abs/1707.03840)</p>

Perimeter Institute, 11 August, 2017

Time Evolution of Complexity in Abelian Gauge Theories
- and Playing Quantum Othello Game-

ArXiv:1707.03840(hep-th)

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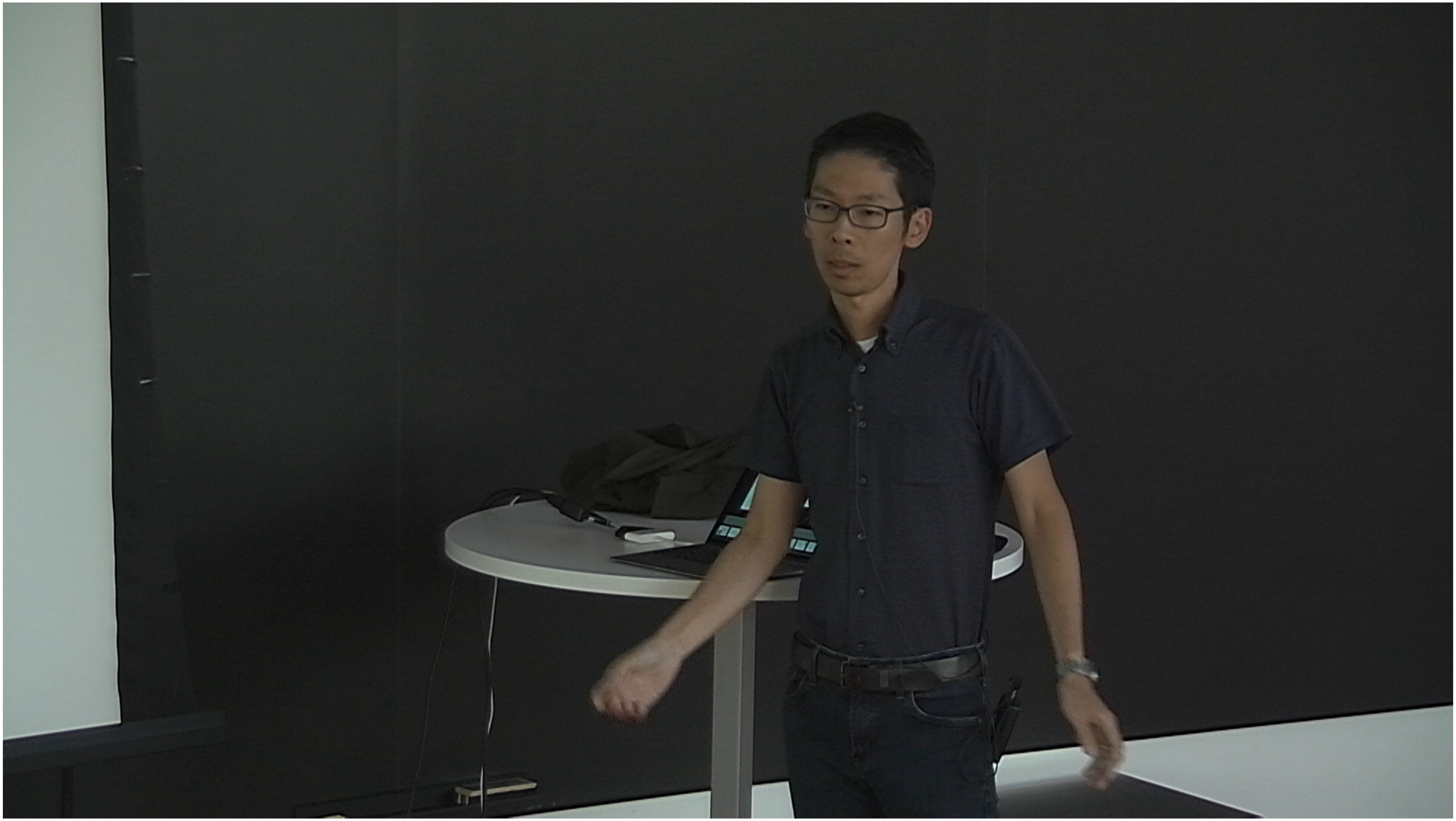
w/ Norihiro Iizuka,
Sotaro Sugishita

Abelian gauge theories with

Complexity = exp[Entropy]

is maximally nonlocal.



Abelian gauge theories with

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1 Review

Complexity ∼ exp[Entropy] in gravity dual

6pp.

2 Review

Abelian gauge theory ∼ Qubits on lattice

3pp.

3

Calculating Complexity

7pp.

For k-local Hamiltonians,
$$\begin{cases} k \ll S \; : \; \mathcal{C} \sim S^k \\ k \sim S \; : \; \mathcal{C} \sim e^S \end{cases}$$

Review

1

Complexity ~ exp[Entropy] in gravity dual

Eternal BH has growing extremal surface.

Susskind: Dual is Complexity.

How to calculate Complexity in QFTs.

Complexity = # gates for reconstructing e^{-iHt} .

Universal gate set.

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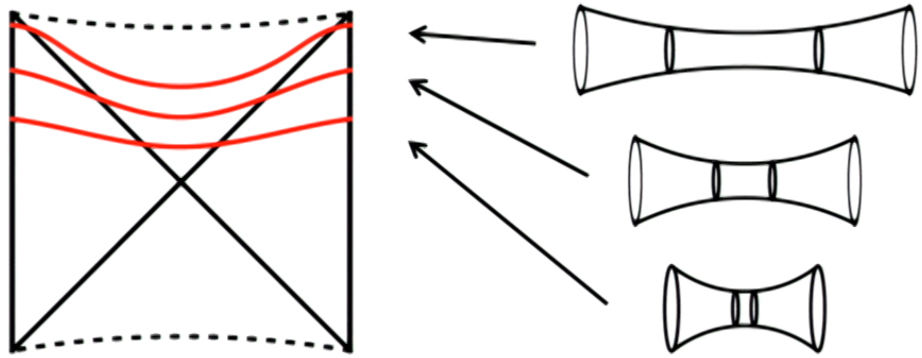
Eternal BH has growing extremal surface.



Eternal black hole

Extremal surfaces
[Hartman, Maldacena]

Dual

Dual

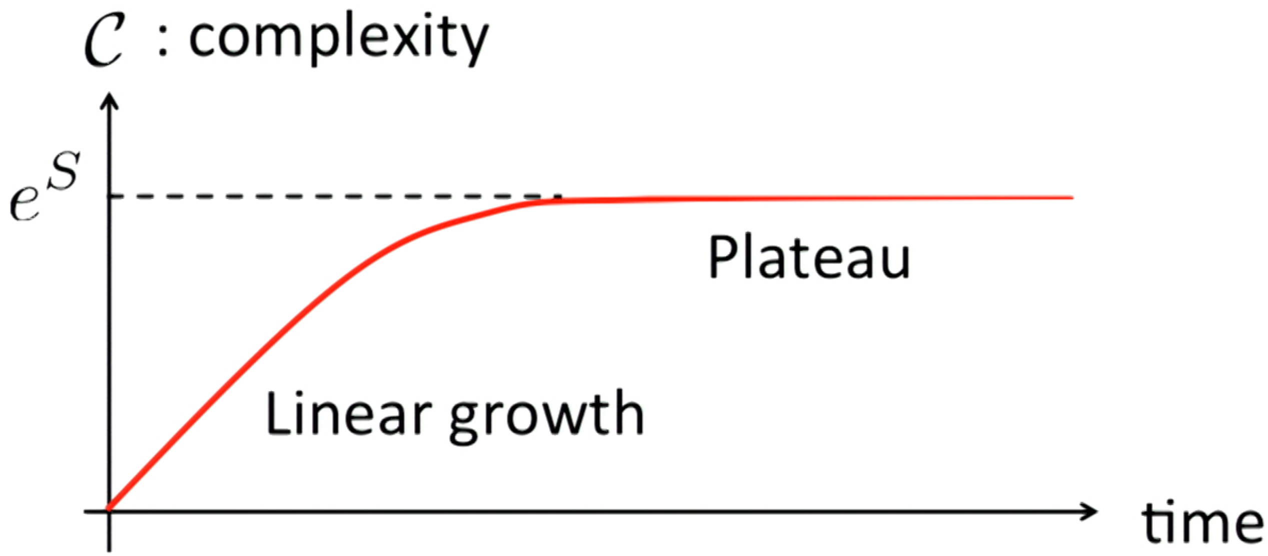
Finite temperature CFT

Complexity [Susskind]

[Susskind] [Stanford, Susskind][Brown, Roberts, Susskind, Swingle, Zhao]
and [Lehner, Myers, Poisson, Sorkin] [Couch, Fishler, Nguyen]
[Ben-Ami, Carmi] [Carmi, Myers, Rath], ...

Susskind: Dual is Complexity.

[Susskind "Entanglement is not enough"]



- Entanglement entropy grows till S
  (Ryu-Takayanagi surface, not go into horizon)
- Complexity grows till exp[S]

How to calculate Complexity in QFTs.

Our strategy: Discretize gauge theories.

$$\left.\begin{array}{l}\text{U(1) gauge}\\ \text{theory in 3d}\end{array}\right\{\begin{array}{l} U(1) \rightarrow Z_N \rightarrow Z_2 \\ \text{2d sp.} \rightarrow L \times L \text{ lattice}\end{array}\left.\vphantom{\begin{array}{l}a\\b\end{array}}\right\} \begin{array}{l}L^2\text{-qubit}\\ \text{system}\end{array}$$

Cf) Z_2 case includes Kitaev's toric code. [Kitaev]

Cf) Free scalar theories

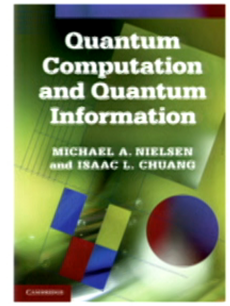
[Chapman, Heller, Morrochio, Pastawski]

[Jefferson, Myers] (Myers @ Strings2017)

Cf) 1+1d information metric.

[Miyaji, Numasawa, Shiba, Takayanagi, Watanabe]

C = # gates for reconstructing e^{-iHt}.



Def Complexity $\mathcal{C}(U)$:

For unitary operator U , $\mathcal{C}(U)$ is a minimum # of gates U_i to satisfy

$$||U - U_1 U_2 \cdots U_{\mathcal{C}}|| < \epsilon$$

- $U_i \in$ Universal gate set
- Trace norm :

$$\Big|\Big| \; U - V \; \Big|\Big|^2 \equiv \frac{1}{\mathrm{Tr}(1)} \mathrm{Tr}\Big[(U-V)^\dagger (U-V)\Big]$$

- $\epsilon(\ll 1)$: Regularization cut-off

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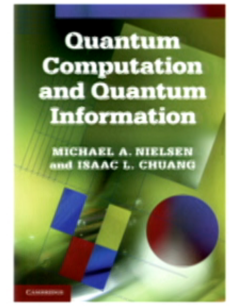
Universal gate set.



Def Universal gate set :

a set of unitary operators which can reconstruct any unitary operator

Ex) n-qubit system (2^n x 2^n unitary matrix)

Basis of states: $|0\rangle \otimes |1\rangle \otimes \cdots \otimes |0\rangle, \cdots$

Universal gate set: $\{U_i, U_{i,j}\}$

U_i : single qubit gate acting on i-th qubit

$U_{i,j}$: gate entangling i-th and j-th qubits

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Universal gate set.

U_i : single qubit gate acting on i-th qubit

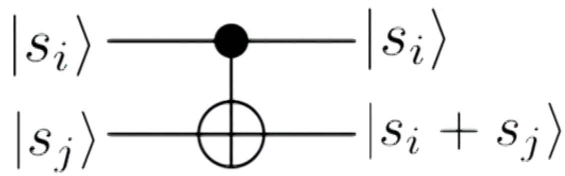
$\alpha|0\rangle + \beta|1\rangle$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow U_i \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$U_{i,j}$: gate entangling i-th and j-th qubits

$\{|00\rangle\, , |01\rangle\, , |10\rangle\, , |11\rangle\}$

ex) CNOT gate



$$U^{\mathrm{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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Review

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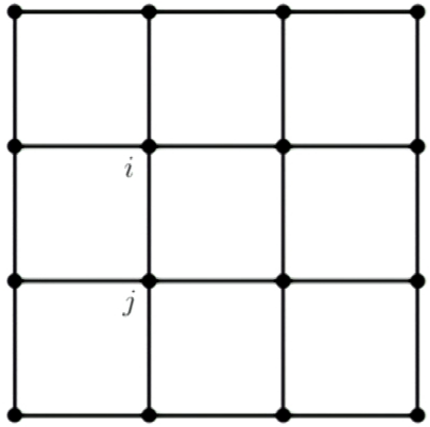
Universal gate set.

Discretization: Z_N, spatial lattice.

Z_2 : [Fradkin, Shenker '79]



- States in Z_2 gauge theory
  $|0\rangle\rangle$ or $|1\rangle\rangle$ living on every link.

  General states spanned by

$$\otimes_{\text{all links}}|n_{ij}\rangle\rangle_{ij} \quad (n_{ij} = 0, 1)$$

- Gauge symmetry
  acting on each vertex independently,

$$g_i = \bigotimes_{j \text{ adjacent to } i} \sigma_1^{(ij)}, \quad \sigma_1^{(ij)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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Physical Hilbert space by fluxes.

Introduce a new basis $\sigma_1^{(ij)}|\pm\rangle\rangle_{ij} = \pm|\pm\rangle\rangle_{ij}$

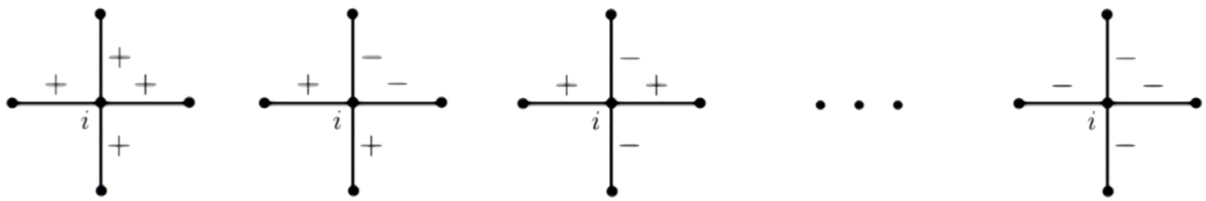
Gauge transformation :

$$g_i\left(\otimes_{\text{all links}}|\beta_{ij}\rangle\rangle_{ij}\right) = \left(\prod_{j \text{ adjacent to } i} \beta^{(ij)}\right)\left(\otimes_{\text{all links}}|\beta_{ij}\rangle\rangle_{ij}\right)$$

$$\beta_{ij} = \pm 1$$

Gauge invariant state :

$$\prod_{j \text{ adjacent to } i} \beta^{(ij)} = 1 \quad \text{for all vertices } i$$



Physical state = "Flux" conserved at each vertex

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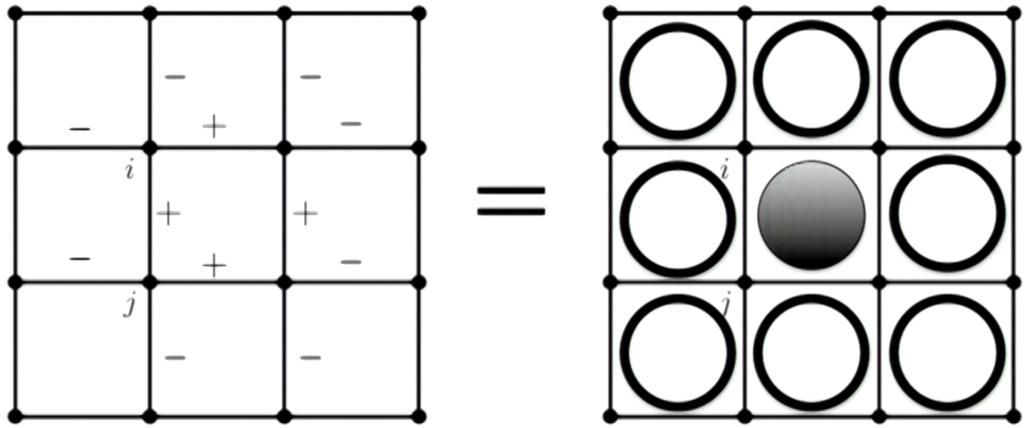
Othello game board.



Flux conserved at each vertex
→ Gauge invariant states labeled by "disk" at each plaquette

Gauge invariant states of Z_2 gauge theory
= States of Ising model (qubits) on plaquettes

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Calculating Complexity

For k-local Hamiltonians,
$$\begin{cases} k \ll S \ : \ \mathcal{C} \sim S^k \\ k \sim S \ : \ \mathcal{C} \sim e^S \end{cases}$$

[Classical] Random disk-flip model.

[Classical] Evolution of Complexity.

[Quantum, 2-site] Hamiltonian.

[Quantum, 2-site] Evolution of Complexity.

[Quantum, 2-site] Fastest Hamiltonian.

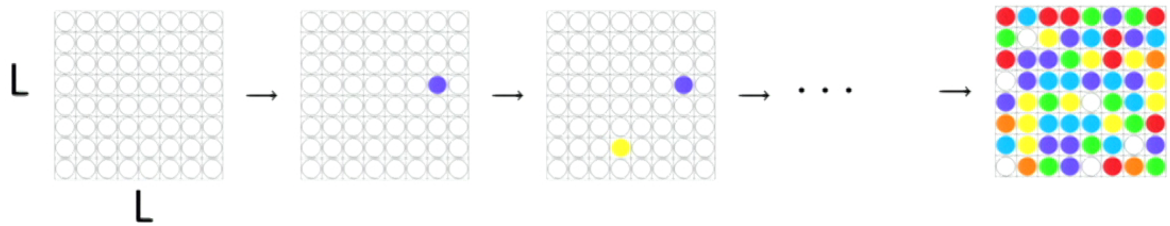
[Quantum, general] Z_2, L^2 sites.

[Quantum, general] Z_N, L^2 sites.

[Classical] Random disk-flip model.

Classical model: random disk-flip at every step

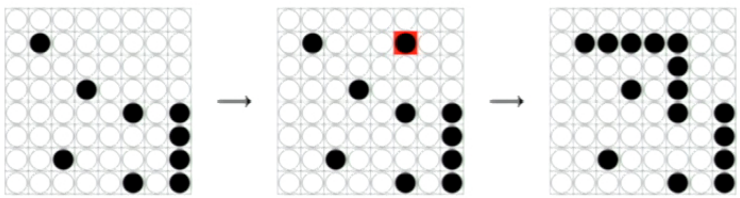
Time evolution for $\mathbf{Z}_N$, L x L



Faster Complexity evolution?

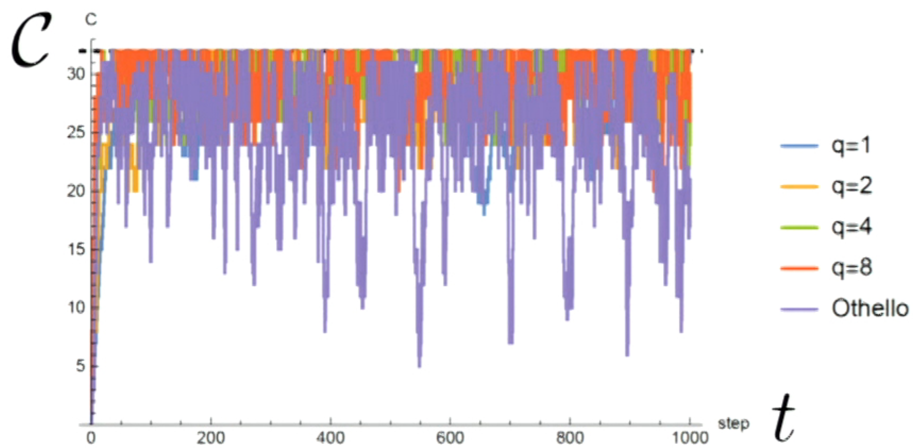
1) q flips at every step (q>1)

2) Othello rule! (Non-local)   Cf) Toffoli gate

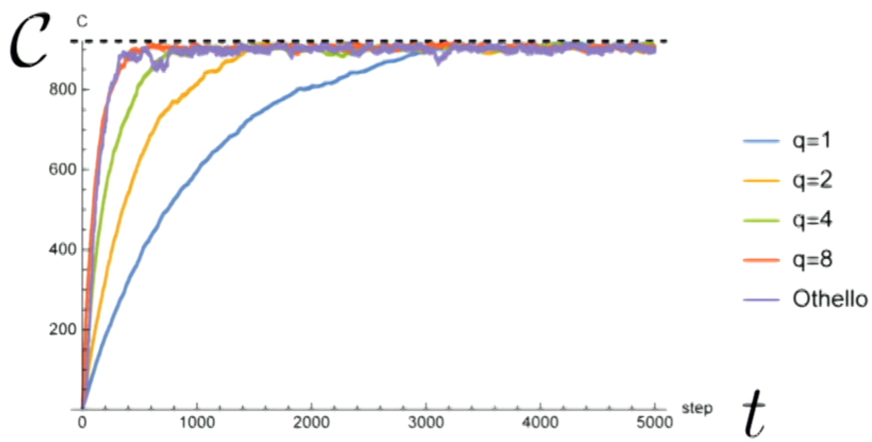


[Classical] Evolution of Complexity.

Z_2 , 8 x 8



Z_{10} , 32 x 32



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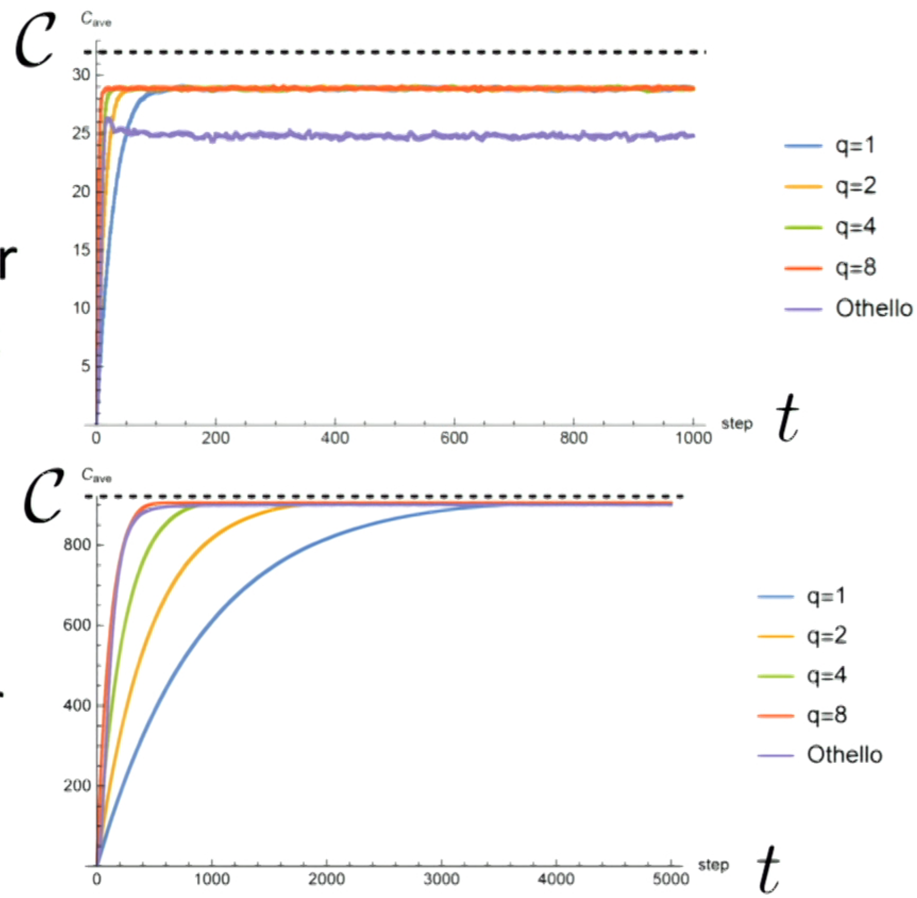
[Classical] Evolution of Complexity.

Z_2 , 8 x 8

averaged over 1000 samples

Z_{10} , 32 x 32

averaged over 1000 samples



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[Quantum, 2-site] Hamiltonian.

Generic Hamiltonian

$$H = \cancel{a_{00}\,(\mathbf{1}\otimes\mathbf{1})} + \sum_{i=1}^{3} a_{0i}\,(\mathbf{1}\otimes\sigma_i) + \sum_{i=1}^{3} a_{i0}\,(\sigma_i\otimes\mathbf{1}) + \sum_{i,j} a_{ij}\,(\sigma_i\otimes\sigma_j)$$

Strategy: Time dependence of complexity of e^{-iHt} come from diagonalized eigenvalues

$$\exp[-iHt] = U_0^\dagger \exp[-i\Lambda t] U_0$$

$$U_0 H U_0^\dagger = \Lambda \qquad \Lambda \equiv \mathrm{diag}\{e_1, e_2, e_3, e_4\}$$

U_0 ~ State-dependent part

[Quantum, 2-site] Evolution of Complexity.

Decomposition by gates:

$$\exp[-i\Lambda t] = \underline{U_1(a)}\,\underline{U_2(b)}\,\underline{U_{\rm ent}(c)}$$

Single-qubit gates   Entangling gate

$$\begin{cases} U_1(a) \equiv \exp[-iat(\sigma_3 \otimes \mathbf{1})]\,, \\ U_2(b) \equiv \exp[-ibt(\mathbf{1} \otimes \sigma_3)]\,, \\ U_{\rm ent}(c) \equiv \exp[-ict(\sigma_3 \otimes \sigma_3)] = U_{12}^{\rm CNOT} U_2(c) U_{12}^{\rm CNOT} \end{cases}$$

$$\begin{pmatrix} e_1 = a + b + c\,, & e_2 = a - b - c\,, \\ e_3 = -a + b - c\,, & e_4 = -a - b + c \end{pmatrix}$$

Time-dependence of complexity :

$$\mathcal{C}(t) = \theta(|\sin at| - \epsilon) + \theta(|\sin bt| - \epsilon) + 3\theta(|\sin ct| - \epsilon)$$

[Quantum, 2-site] Fastest Hamiltonian.

Time to reach the maximum complexity

$$t_{\max} = \epsilon \times \max\{|a|^{-1}, |b|^{-1}, |c|^{-1}\}$$

Complexity growth rate

$$v \equiv \mathcal{C}_{\max}/t_{\max} = \frac{5}{\epsilon}\min\{|a|, |b|, |c|\}$$

Fastest = All entangling, all equal weight

Fixing variance of eigenvalues, $\sigma^2 = \frac{1}{4}\sum_{k=1}^{4} e_k^2 = a^2 + b^2 + c^2$

the fastest Hamiltonian is with $|a| = |b| = |c| = \frac{\sigma}{\sqrt{3}}$,

$$H = U_0^\dagger \left[J(\sigma_3 \otimes \mathbf{1}) + J(\mathbf{1} \otimes \sigma_3) \pm J(\sigma_3 \otimes \sigma_3)\right] U_0$$

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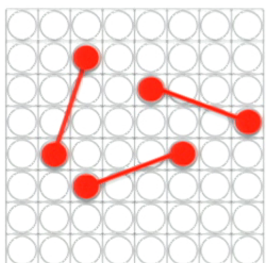
[Quantum, general] Z_2 , L^2 sites.

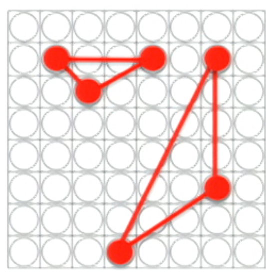
Hamiltonian	1-local	2-local	k-local	maximally nonlocal
Maximum complexity $\mathcal{C}_{\max}$	L^2	L^4	L^{2k}	$2^{L^2} L^2$
Fastest speed v	$\frac{JL^2}{\epsilon}$	$\frac{\sigma}{\epsilon} L^2$	$\frac{\sigma}{\epsilon} L^k$	$\frac{\sigma}{\epsilon} 2^{L^2/2} L^2$

Def "k-local Hamiltonian" has terms entangling at most k qubits.

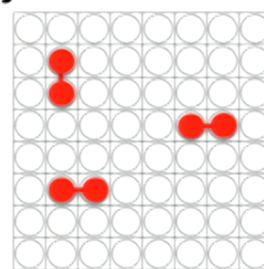
Ex) 2-local



3-local



Adjacent 2-local



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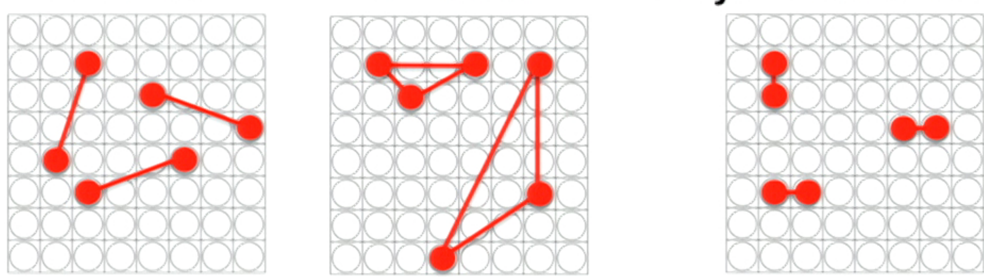
Ex) 2-local 3-local Adjacent 2-local



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Abelian gauge theories with

Complexity = exp[Entropy]

is maximally nonlocal.


General result for k-local Hamiltonian
$k \ll S : \mathcal{C} \sim S^k$
$k \sim S : \mathcal{C} \sim e^S$


maximally
nonlocal
iltonian