

Title: Symmetry-enriched topological order in tensor networks: Gauging and anyon condensation

Date: Aug 01, 2017 04:30 PM

URL: <http://pirsa.org/17080013>

Abstract: I will describe a framework for the study of symmetry-enriched topological order using graded matrix product operator algebras. The approach is based upon an explicit construction of the extrinsic symmetry defects, which facilitates the extraction of their physical properties. This allows for a simple analysis of dual phase transitions, induced by gauging a global symmetry, and condensation of a bosonic subtheory.

SET order in tensor networks: Gauging and anyon condensation

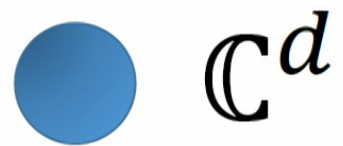
Dominic Williamson
Verstraete Group
University of Vienna

arXiv:1412.5604
arXiv:1511.08090
arXiv:1708.?????

SET order in tensor networks: Gauging and anyon condensation

Dominic Williamson
Verstraete Group
University of Vienna

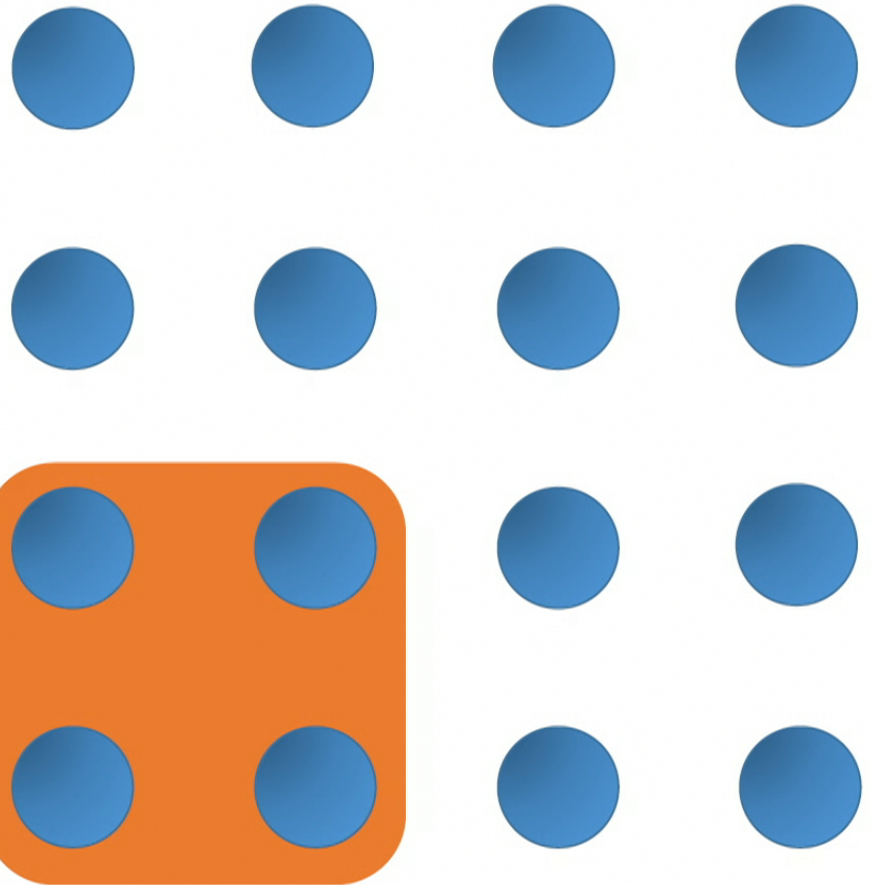
arXiv:1412.5604
arXiv:1511.08090
arXiv:1708.?????

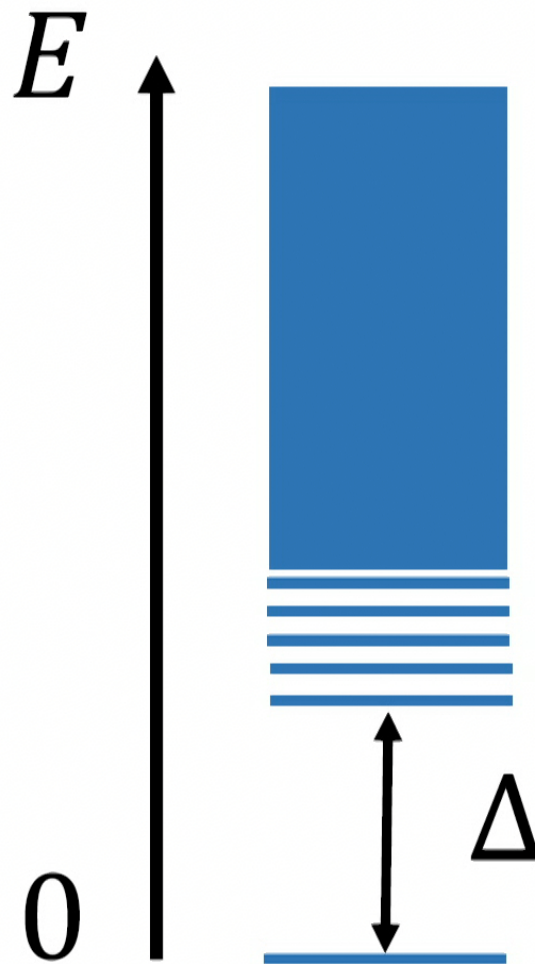


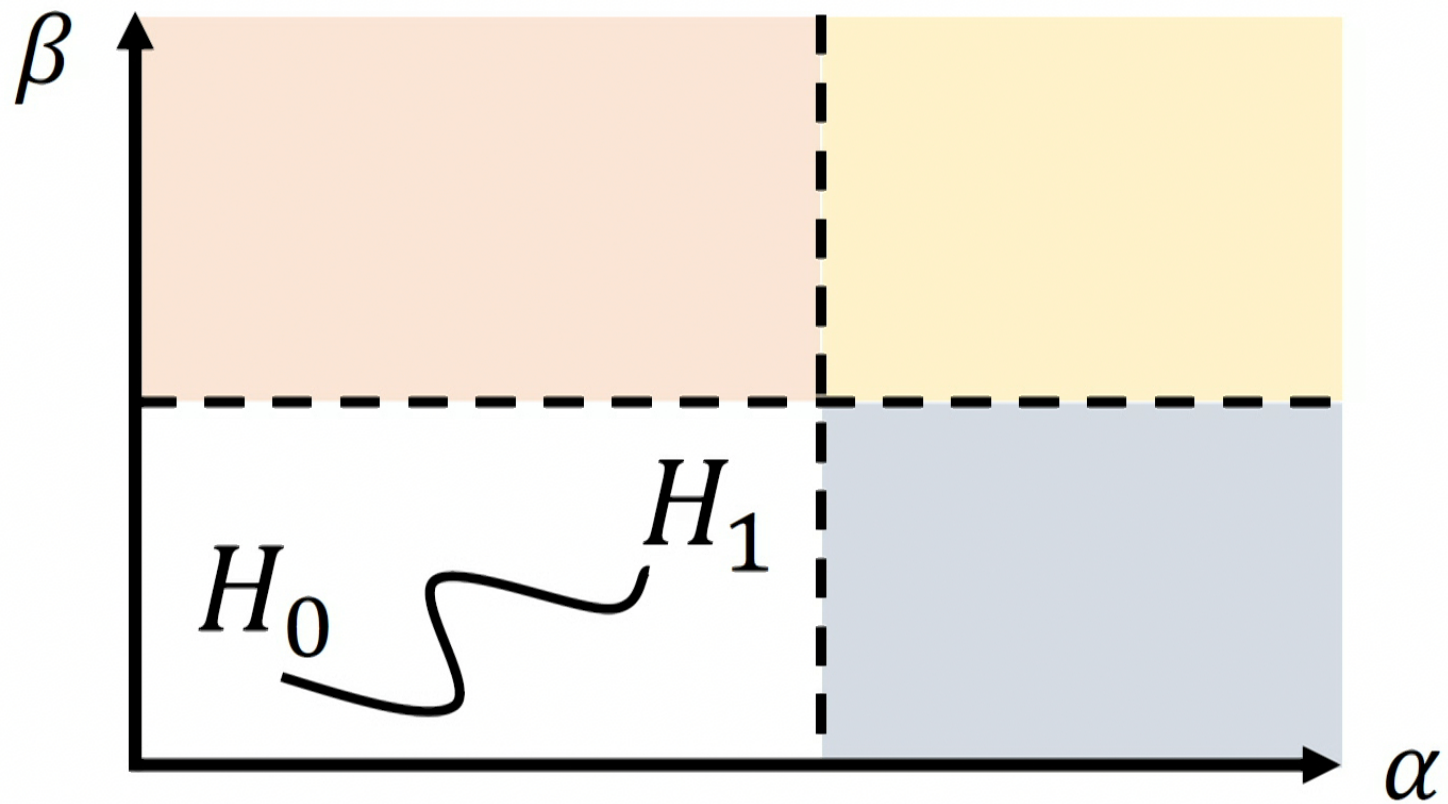
$$\mathbb{H} = \bigotimes_v \mathbb{C}^d$$



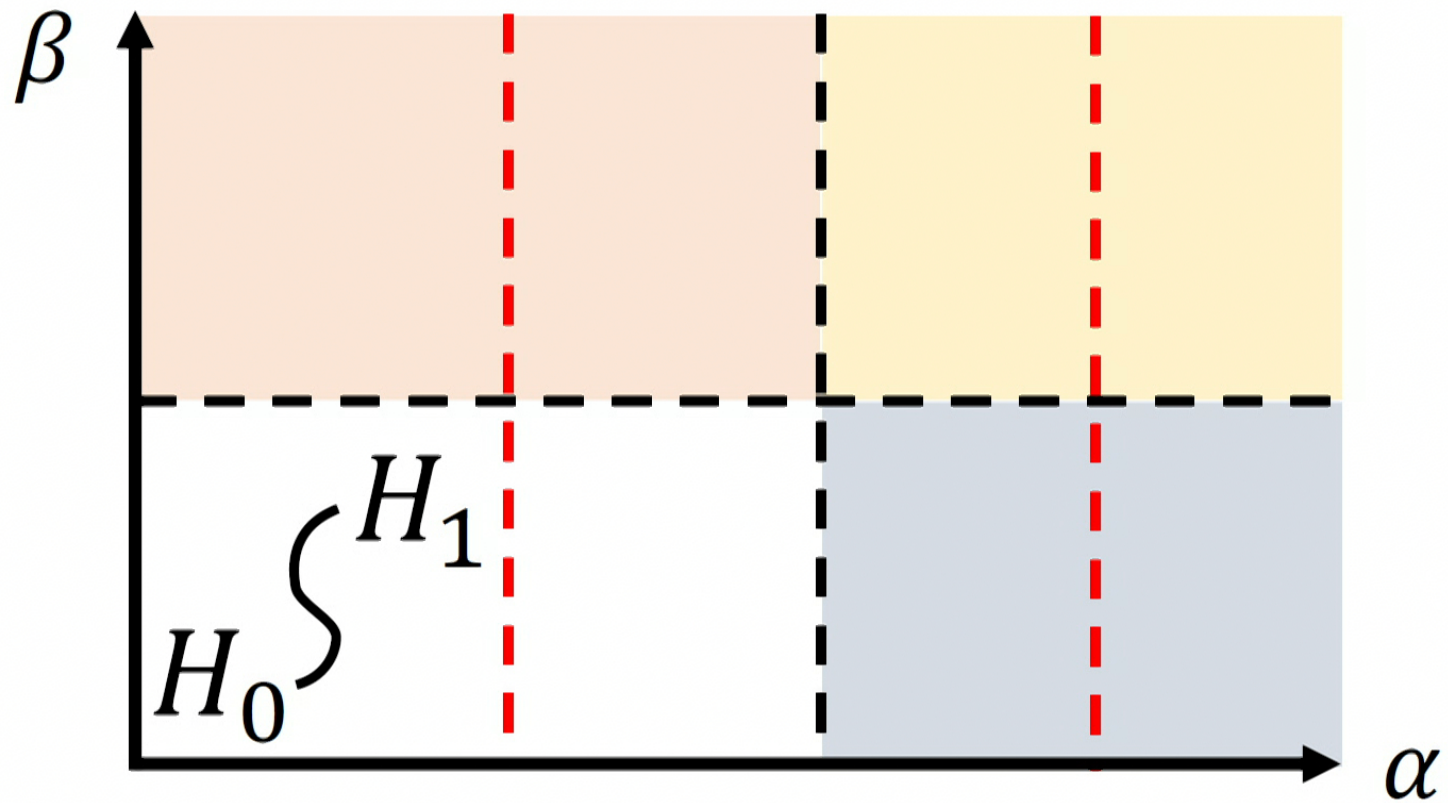
$$H = \sum_v h_v$$







$$H_S = \sum_v h_v^S,$$



$$H_s = \sum_v h_v^s, \quad U_g^{\otimes N}, g \in G, \quad [h_v^s, U_g^{\otimes N}] = 0$$

A Motivating Example

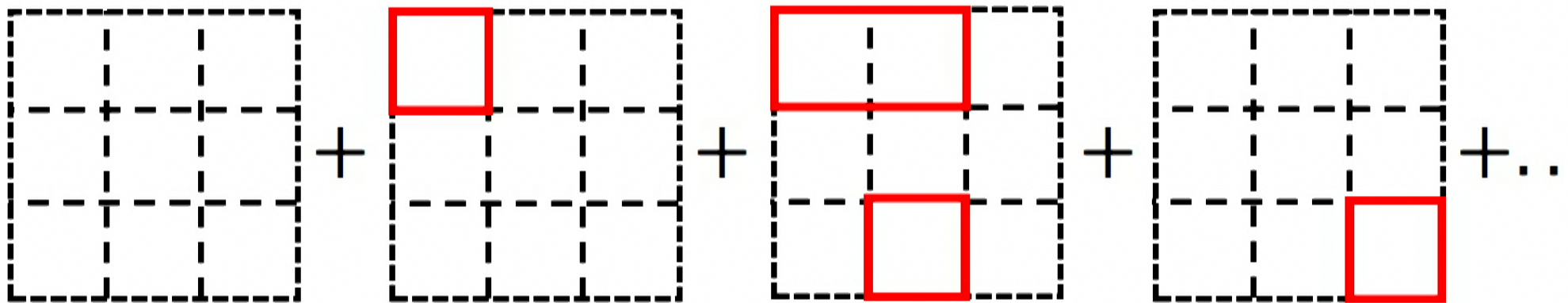
Toric code

$$--- = |+\rangle$$

$$\text{---} = |-\rangle$$

$$H_{TC} = -\sum_p \begin{bmatrix} Z & & \\ & Z & \\ & & Z \end{bmatrix} - \sum_v \begin{bmatrix} & X & \\ X & + & X \\ & X & \end{bmatrix}$$

$$|\psi_0\rangle =$$



Toric code anyons: $Z(\text{Vec}_{\mathbb{Z}_2})$

$$\text{simples} = \{1, e, m, em\}$$

$$e \times m = em$$

$$e \times e = m \times m = 1$$

F symbols trivial

$$\begin{aligned} R_{em}^{e,m} &= -1 \\ &= R_m^{e,em} \\ &= R_e^{em,m} \end{aligned}$$

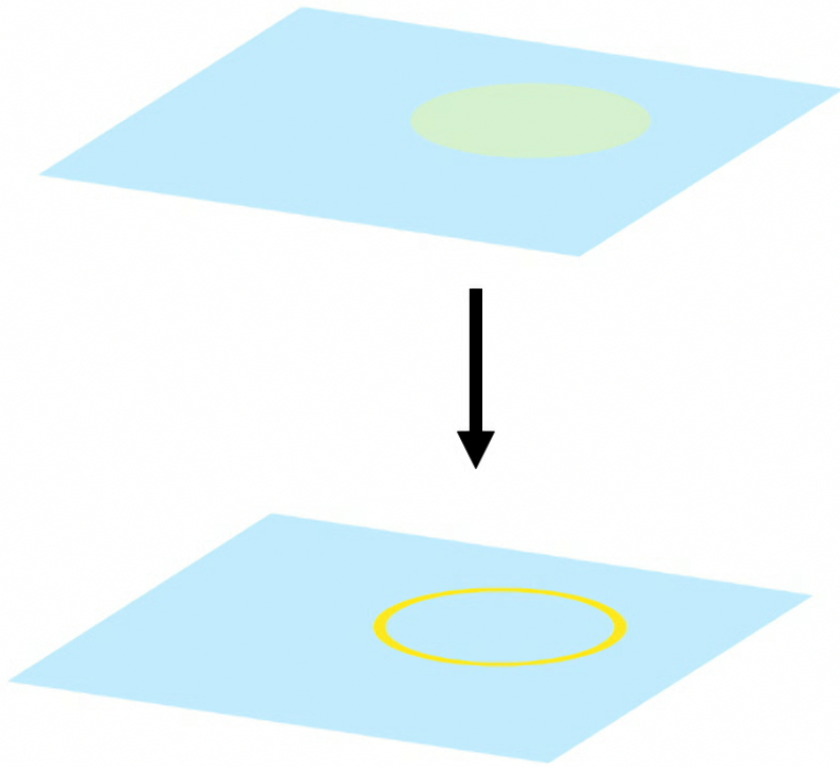
$$U: \quad \begin{array}{l} e \leftrightarrow m \\ em \rightarrow em \end{array}$$

$$U: \begin{array}{l} e \leftrightarrow m \\ em \rightarrow em \end{array}$$

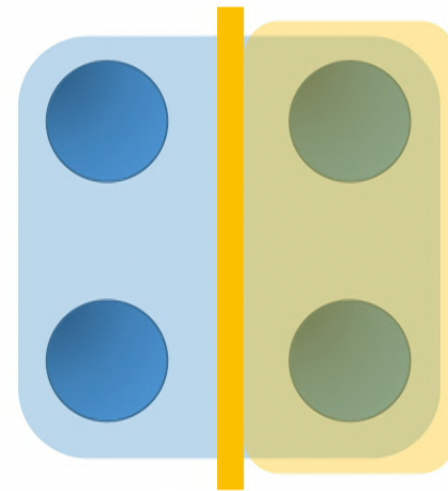
$$UH_{TC}U^\dagger = H_{TC},$$

$$U: \begin{array}{|c|} \hline Z \\ \hline Z \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline X \\ \hline X \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline X \\ \hline X + X \\ \hline X \\ \vdots \\ \hline \end{array}$$

Symmetry Domain walls

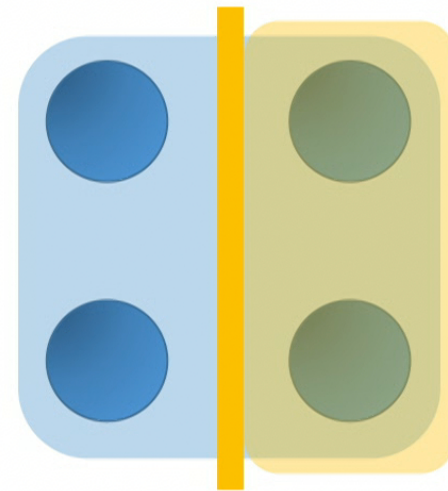
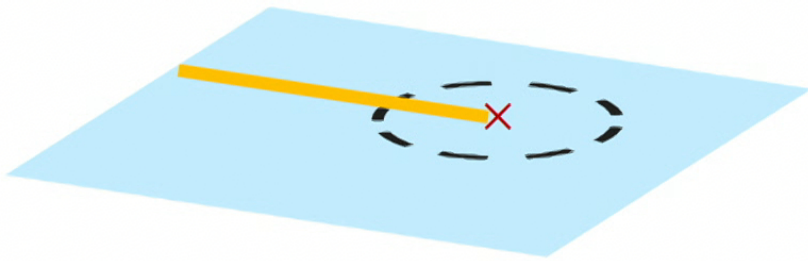


$$h_{|} \rightarrow \pi_g h_{|} \pi_g^{\dagger}$$



Symmetry Defects

$$h| \rightarrow \pi_g h| \pi_g^\dagger$$



arXiv:cond-mat/0506438 Kitaev
arXiv:1004.1838 Bombin

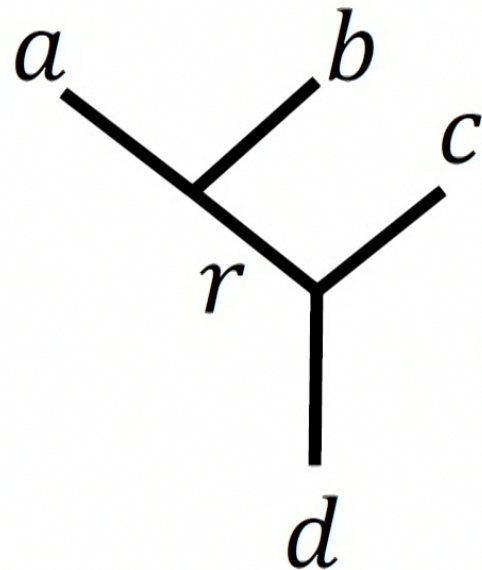
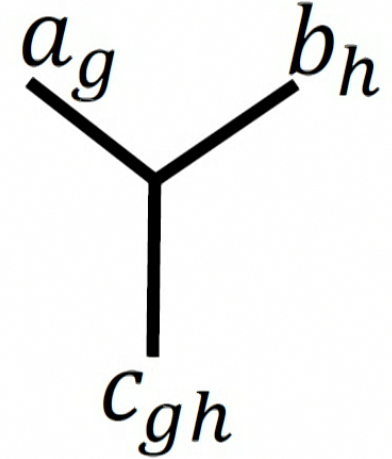
Defects in a topological phase

(Unitary G -Crossed Braided
Fusion Categories)

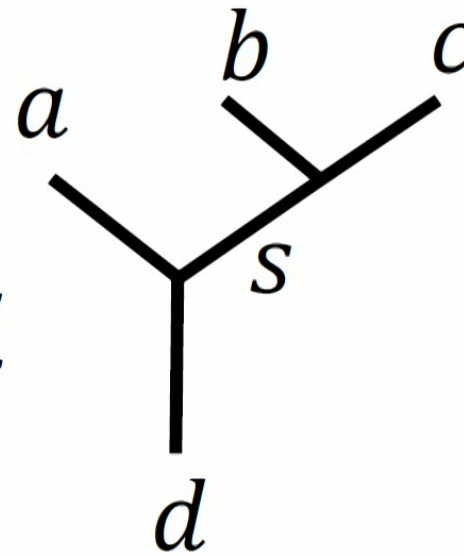
arXiv:1410.4540 Barkeshli, Bonderson, Cheng, Wang

$$a_g \times b_h = c_{gh} + d_{gh} + \dots$$

$$= \sum N_{ab}^c c_{gh}$$

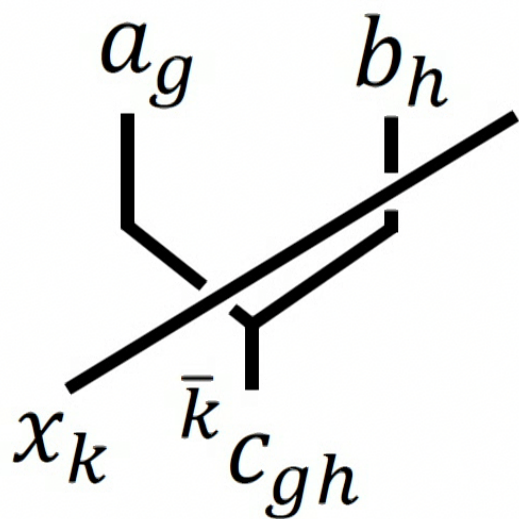


$$= F_{d;rs}^{abc}$$

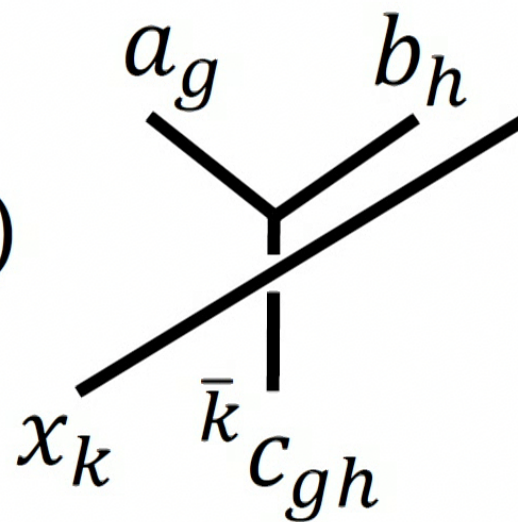


$$R^{ab} = \begin{array}{c} a_g \quad b_h \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \bar{h} a_g \end{array}$$

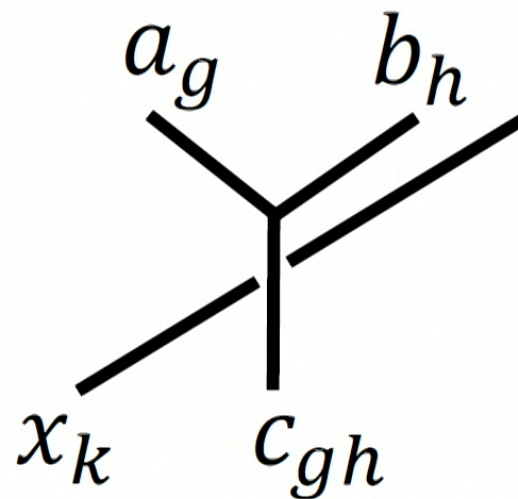
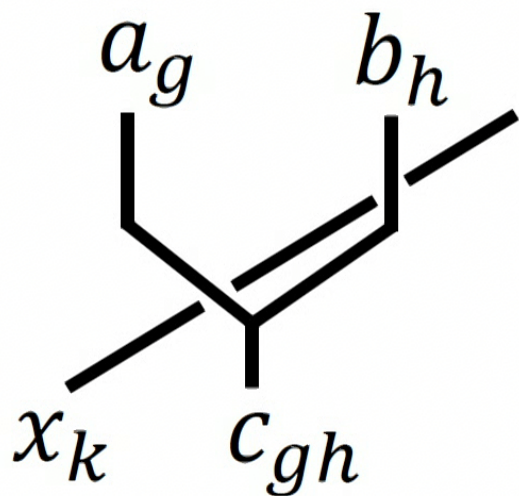
$$\begin{array}{c} a_g \quad b_h \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \bar{h} a_g \\ | \\ c_{gh} \end{array} = R_c^{ab} \begin{array}{c} a_g \quad b_h \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ | \\ c_{gh} \end{array}$$



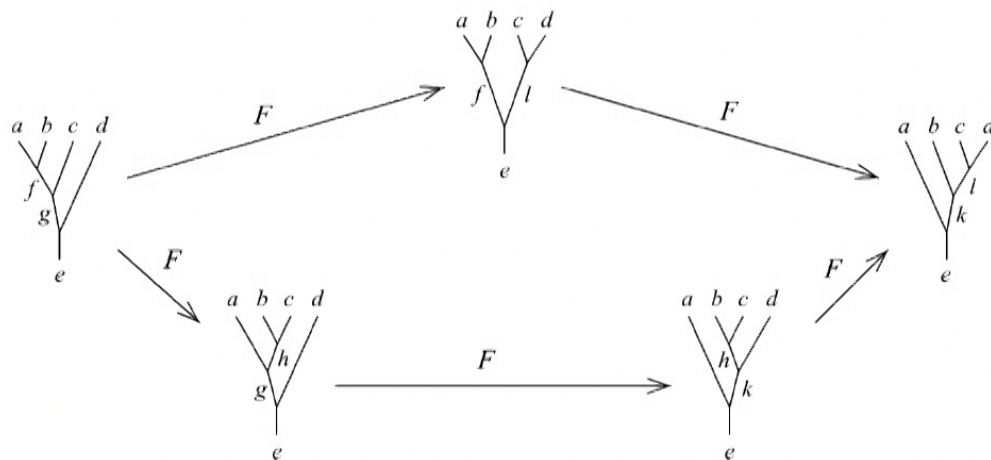
$$= U_k(a, b; c)$$



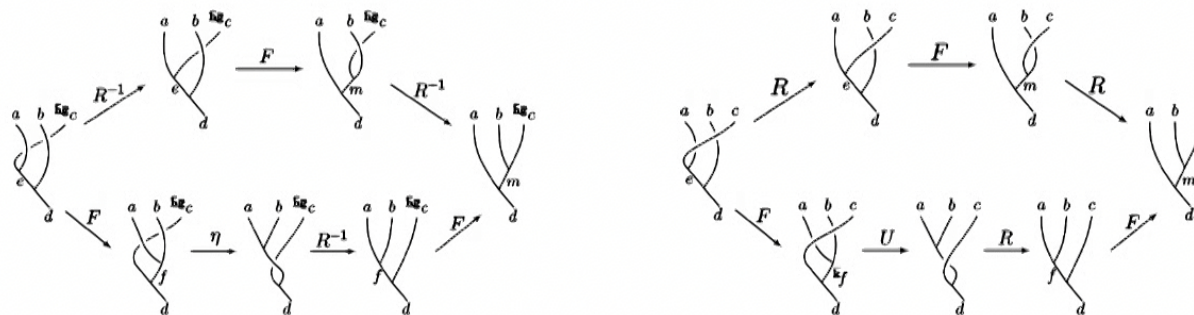
$$= \eta_x(g, h)$$



Pentagon



Heptagons



Toric code with twists $Z(\text{Vec}_{\mathbb{Z}_2})_{\mathbb{Z}_2}$

$$\text{simples} = \{1, e, m, em\} \oplus \{\sigma_+, \sigma_-\}$$

$$\sigma_{\pm} \times e = \sigma_{\mp} = \sigma_{\pm} \times m$$

$$\sigma_{\pm} \times \sigma_{\mp} = e + m$$

$$\sigma_{\pm} \times \sigma_{\pm} = 1 + em$$

$$\sigma_{\pm} \times em = \sigma_{\pm}$$

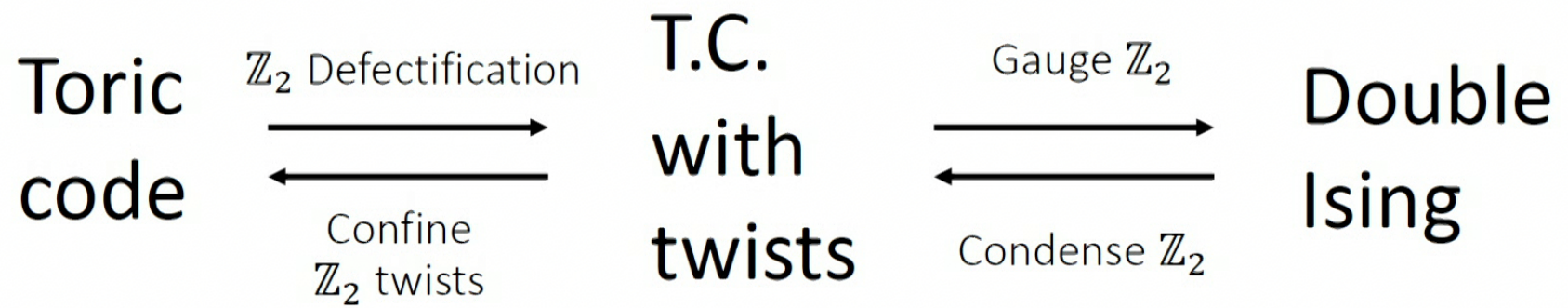
Double Ising: $Z(\text{Ising})$

$$\text{simples} = \{1, \psi, \sigma\} \boxtimes \{1, \bar{\psi}, \bar{\sigma}\}$$

$$\psi \times \psi = 1$$

$$\sigma \times \sigma = 1 + \psi$$

$$\sigma \times \psi = \sigma$$



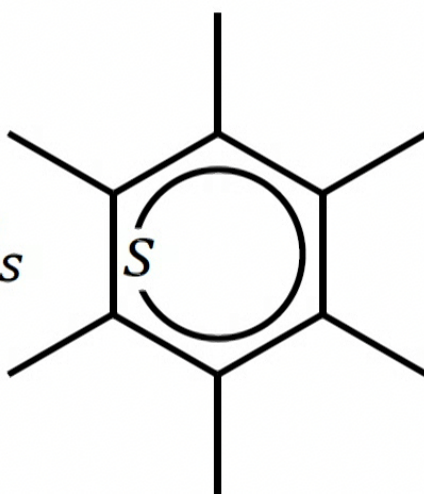
$$\begin{array}{ccccc}
 \mathcal{C} & \begin{array}{c} \xrightarrow{\text{Defectification}} \\ \xleftarrow{\text{Confine Defects}} \end{array} & \mathcal{C}_G^\times & \begin{array}{c} \xrightarrow{\text{Gauge } G} \\ \xleftarrow{\text{Condense Rep}(G)} \end{array} & \mathcal{C}/G
 \end{array}$$

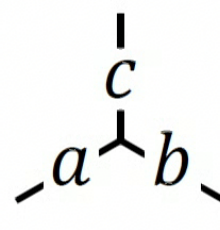
Back to the lattice

String-net models

$$-s- = |s\rangle \quad s \in \mathcal{C}$$

$$\begin{aligned} \text{e.g. } \mathcal{C} &= \text{Vec}_{\mathbb{Z}_2} \\ &= \{1, \psi, \sigma\} \end{aligned}$$

$$H = - \sum_{p,s} d_s \text{ (diagram) }$$


$$- \sum_v \delta_{ab}^c \text{ (diagram) }$$


arXiv:cond-mat/0404617 Levin Wen

Input:

 $\mathcal{C}$

Unitary
Fusion
Category

String-net
Construction

Output:

$$H = -\sum A_v - \sum B_p$$

Commuting
Projector
Hamiltonian

Input:

$\mathcal{C}$

Unitary
Fusion
Category

String-net
Construction

Output:

$$H = -\sum A_v - \sum B_p$$

Commuting
Projector
Hamiltonian

Drinfeld
Center

Sectors
Superselection

Emergent Anyons:

$Z(\mathcal{C})$

Modular
Tensor
Category

G -graded SET String-net models

$$-s_g- = |s_g\rangle \quad s \in \mathcal{C}_G \quad \text{e.g. } \mathcal{C}_{\mathbb{Z}_2} = \{1, \psi\} \oplus \{\sigma\}$$

$$\begin{array}{|c|} \hline h \\ \hline \end{array} = |h\rangle \quad h \in G$$

$$-\sum_{p, s_g} d_{s_g} \text{ (diagram: a hexagon with a central circle labeled } s_g \text{ and } L_g h \text{, and six external lines)} -$$

$$-\sum_v \delta_{ab}^c \text{ (diagram: a vertex with three lines labeled } a, b, c \text{)} -$$

$$-\sum_v \delta_{hk^{-1}}^g \text{ (diagram: a vertex with three lines labeled } h, c_g, k \text{)} -$$

arXiv:1606.08482, arXiv:1606.07816

Input:

Hamiltonian:

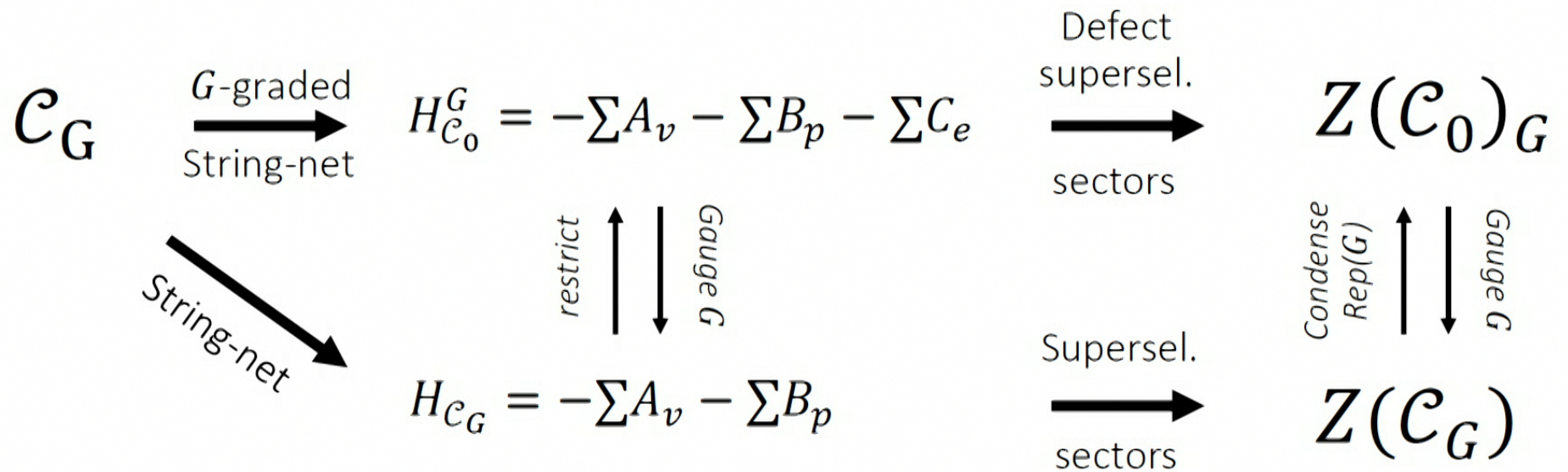
Emergent theory:

$$\mathcal{C}_G \xrightarrow[\text{String-net}]{G\text{-graded}} H_{\mathcal{C}_0}^G = -\sum A_v - \sum B_p - \sum C_e \xrightarrow[\text{sectors}]{\text{Defect supersel.}} Z(\mathcal{C}_0)_G$$

Input:

Hamiltonian:

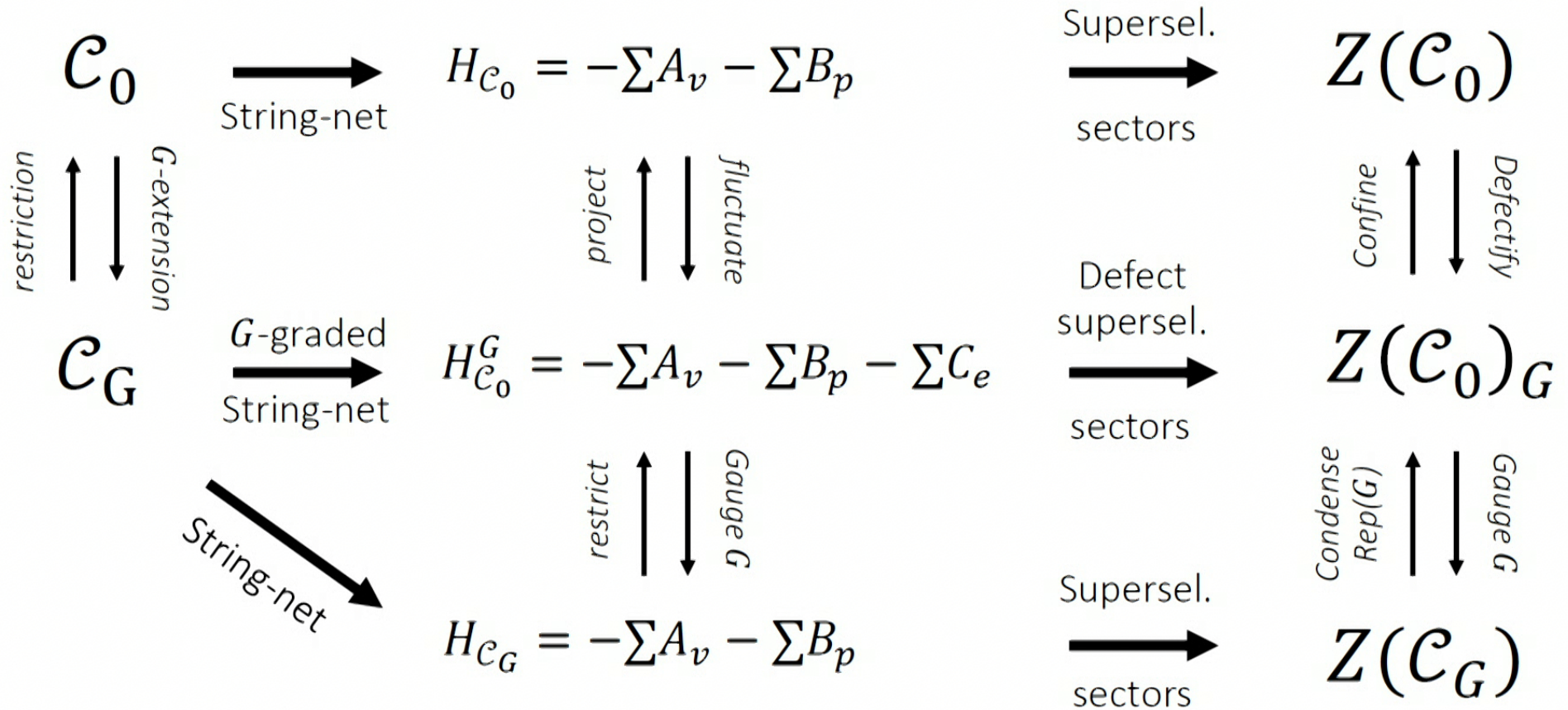
Emergent theory:



Input:

Hamiltonian:

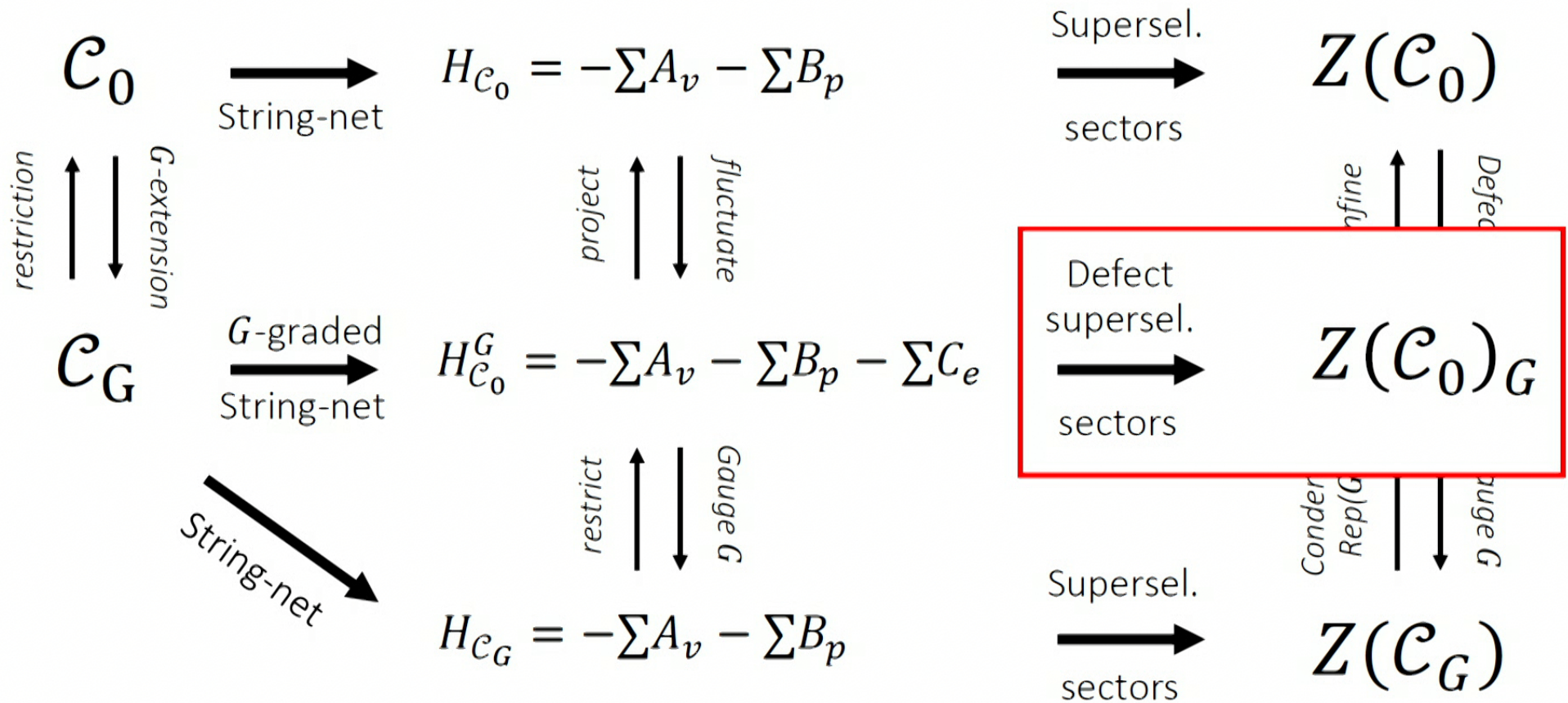
Emergent theory:



Input:

Hamiltonian:

Emergent theory:



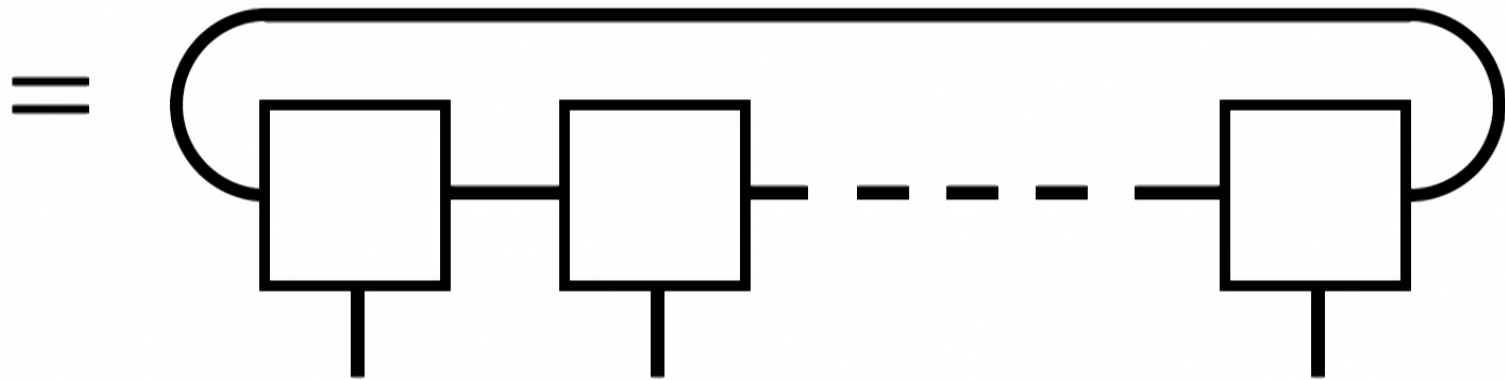
$$\alpha \text{ --- } \boxed{A} \text{ --- } \beta = (A^i)_{\alpha\beta} \in \mathbb{C}$$

i

$$\alpha \text{---} \boxed{A} \text{---} \boxed{B} \text{---} \gamma = \sum_{\beta} (A^i)_{\alpha\beta} (B^j)_{\beta\gamma}$$

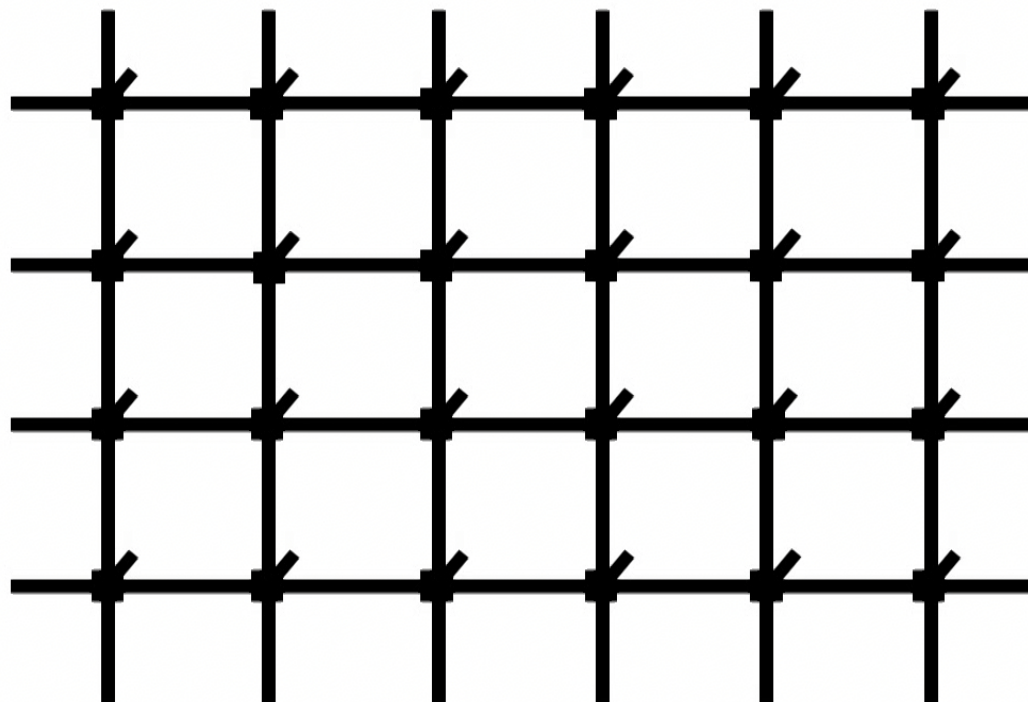
i
 j

$$|MPS_N(A)\rangle = \sum_{i_1 \dots i_N} \text{Tr}[A^{i_1} \dots A^{i_N}] |i_1 \dots i_N\rangle$$

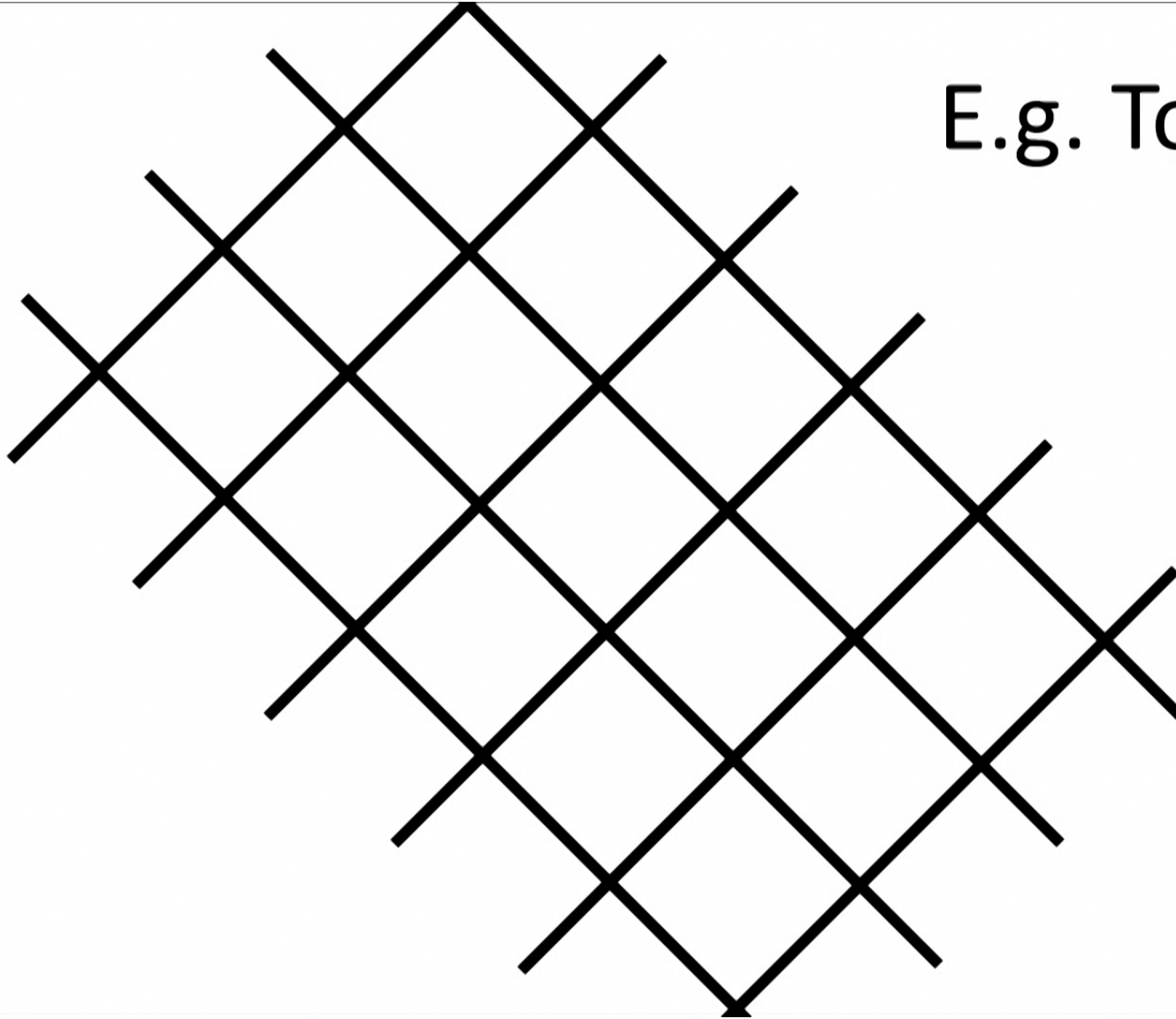


$$|PEPS(A)\rangle =$$

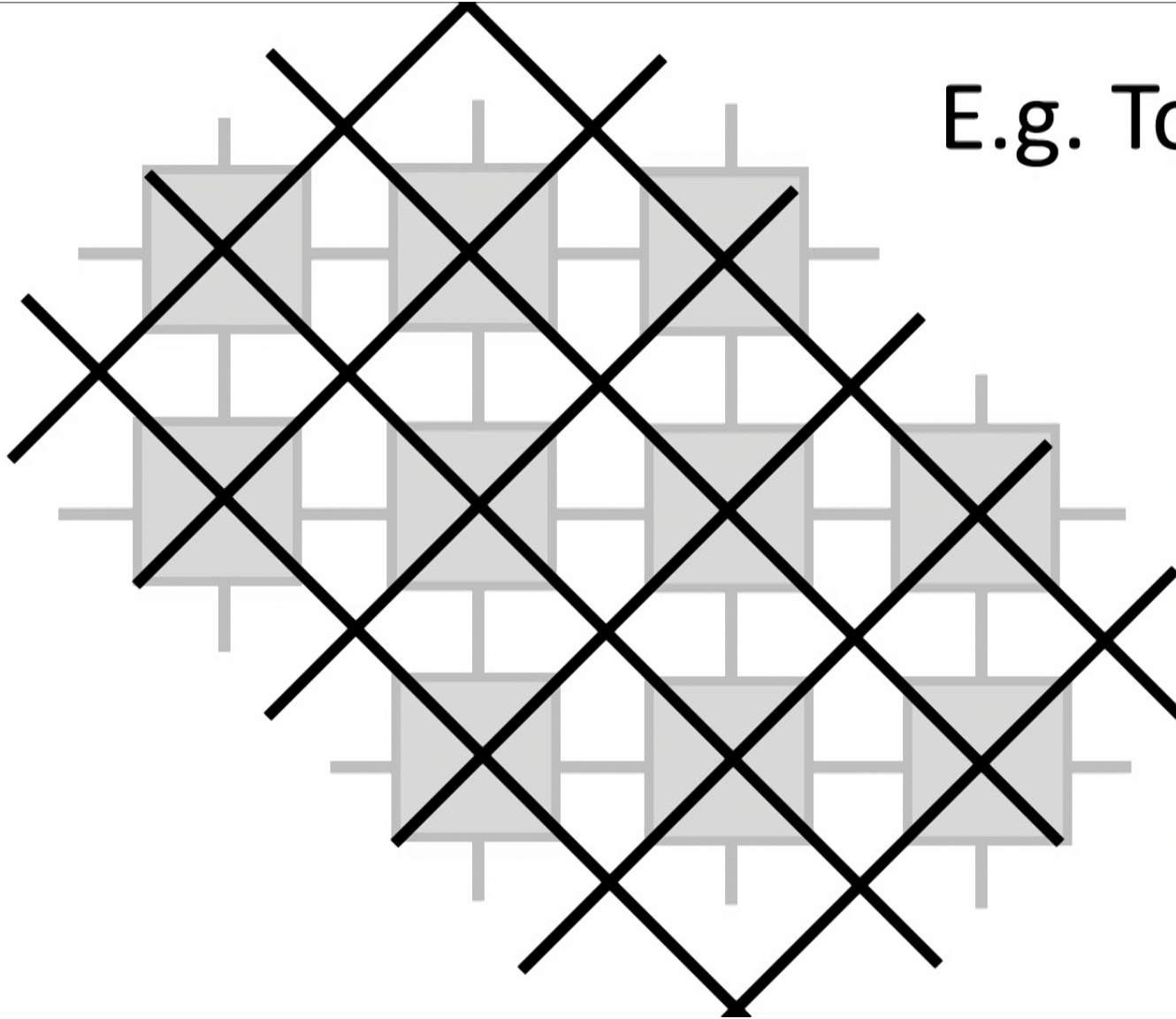
$$= \begin{array}{c} \beta \\ \swarrow \\ \alpha \text{---} \text{---} \gamma \\ \downarrow \\ \delta \end{array}^i$$



E.g. Toric code

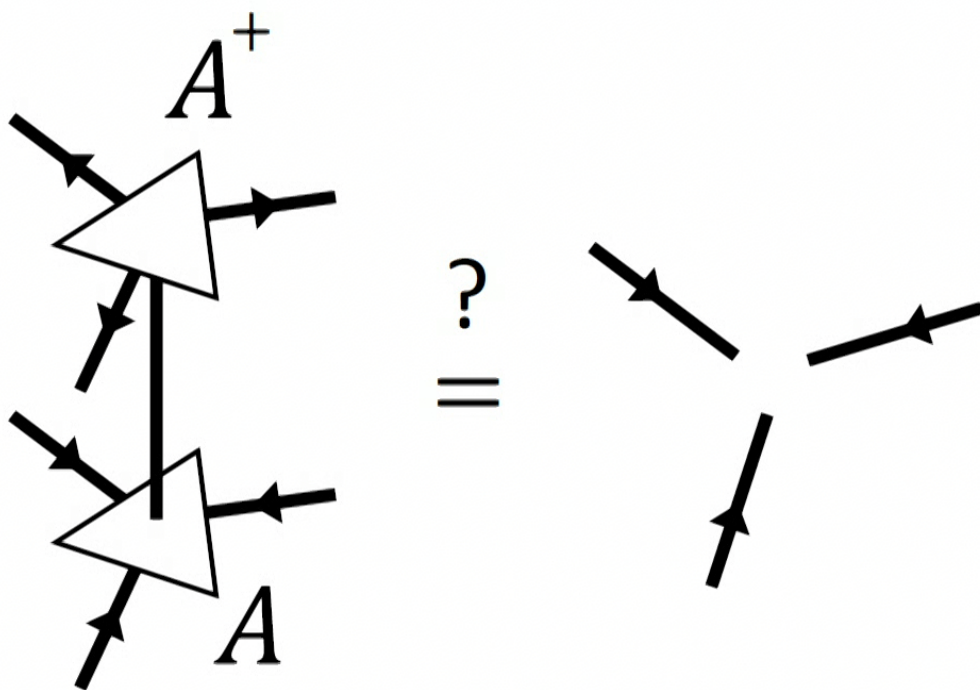


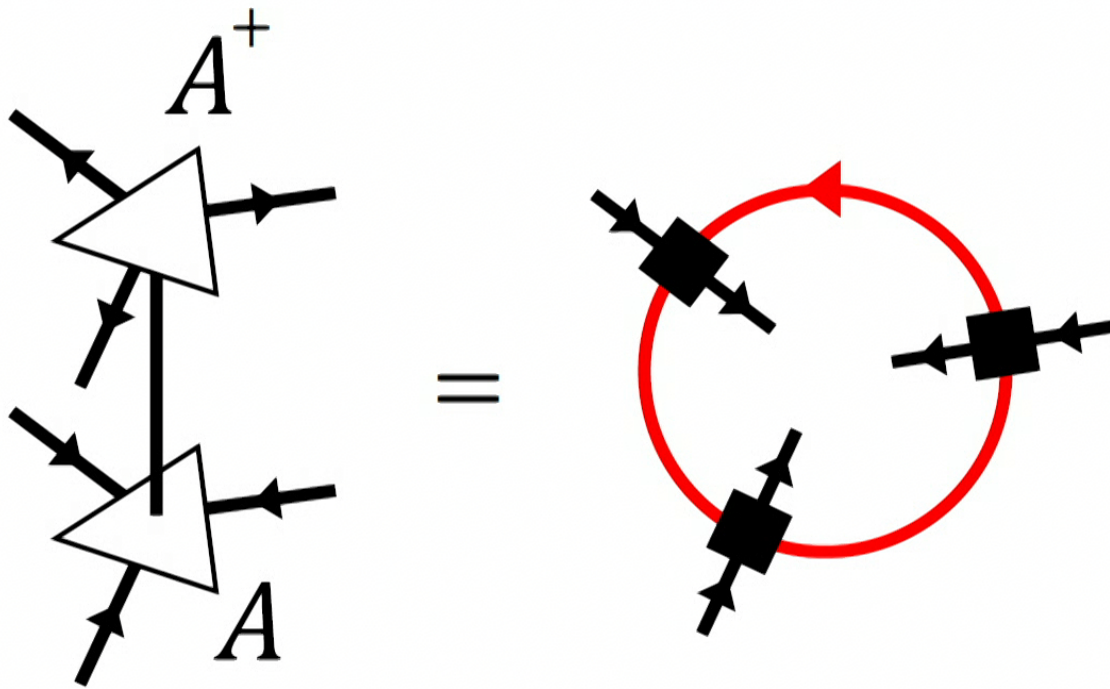
E.g. Toric code



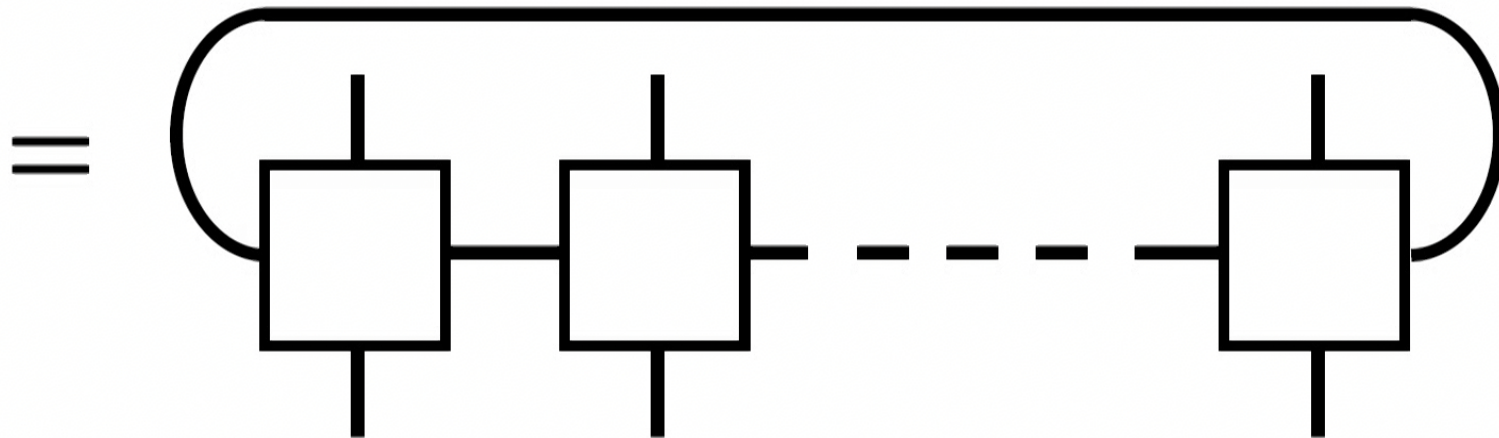
T.C.

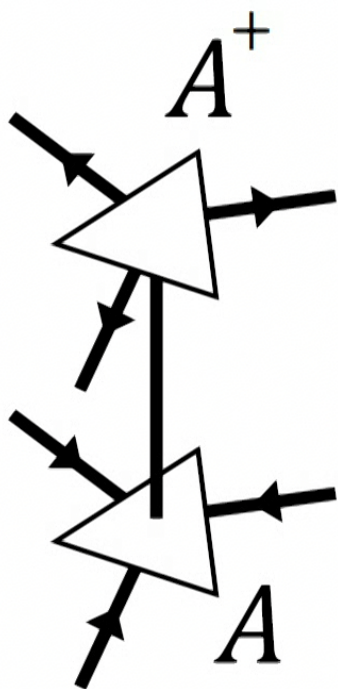
$$\begin{array}{c} \text{---} \square \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} = \sum_{ijkl} i + k \begin{array}{c} i + j \\ \text{---} \square \text{---} \\ \begin{array}{cc} i & j \\ k & l \end{array} \\ \text{---} \\ k + l \end{array} j + l$$



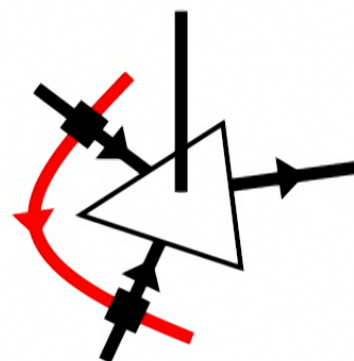
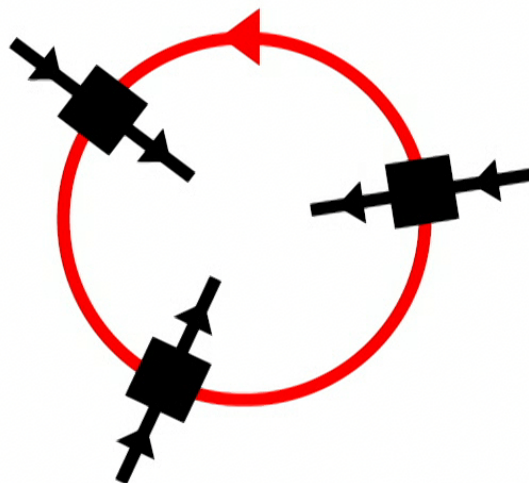


$$MPO_N(A) = \sum_{\substack{i_1 \dots i_N \\ j_1 \dots j_N}} \text{Tr} [A^{i_1 j_1} \dots A^{i_N j_N}] |i_1 \dots i_N\rangle \langle j_1 \dots j_N|$$

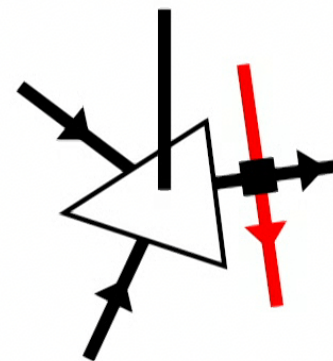




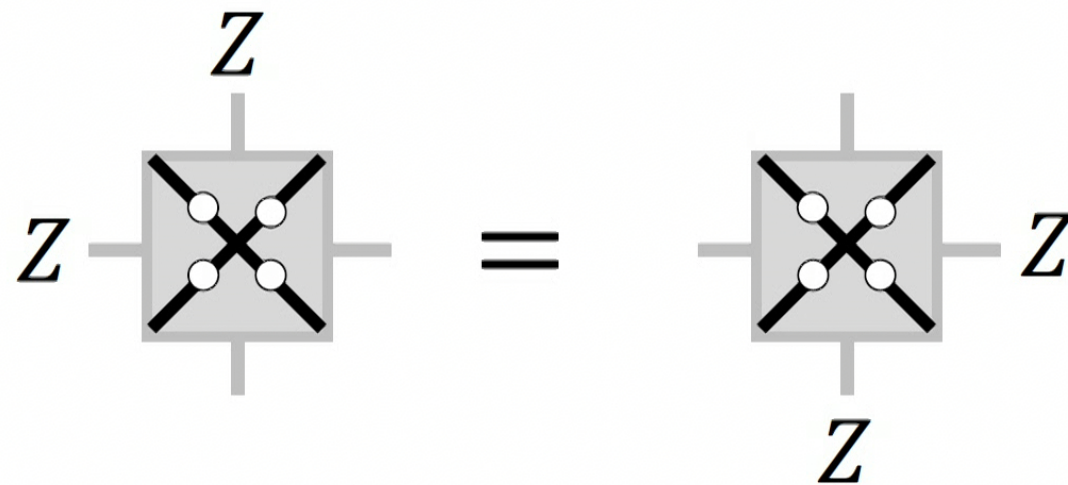
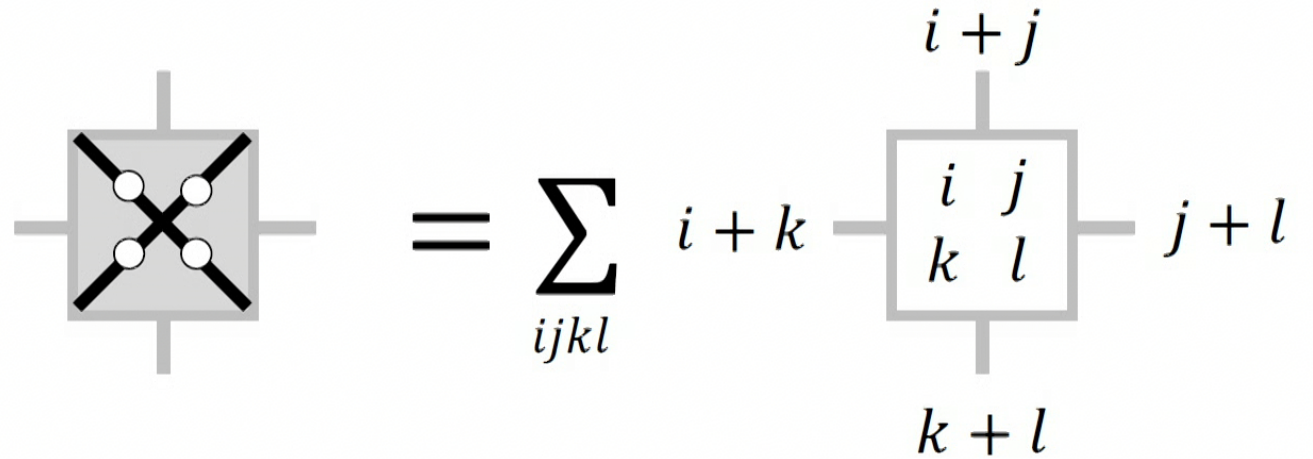
=



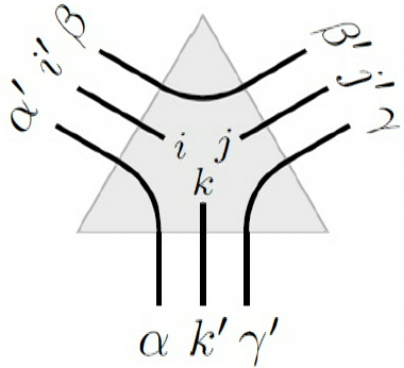
=



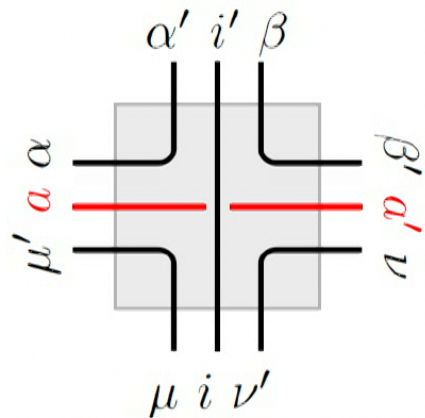
T.C.



E.g. String-net



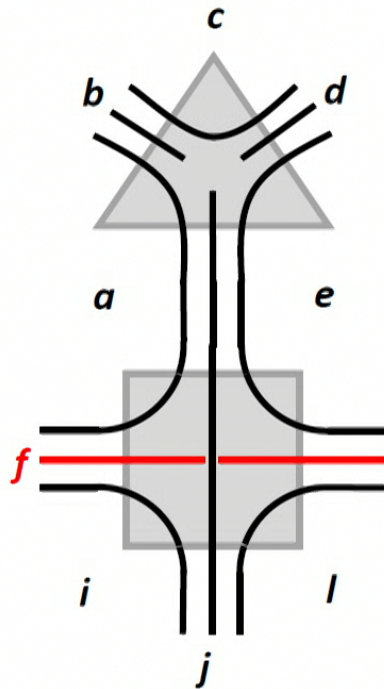
$$= \delta_{ii'} \delta_{jj'} \delta_{kk'} \delta_{\alpha\alpha'} \delta_{\beta\beta'} \delta_{\gamma\gamma'} F_{\gamma\beta k}^{\alpha ij},$$



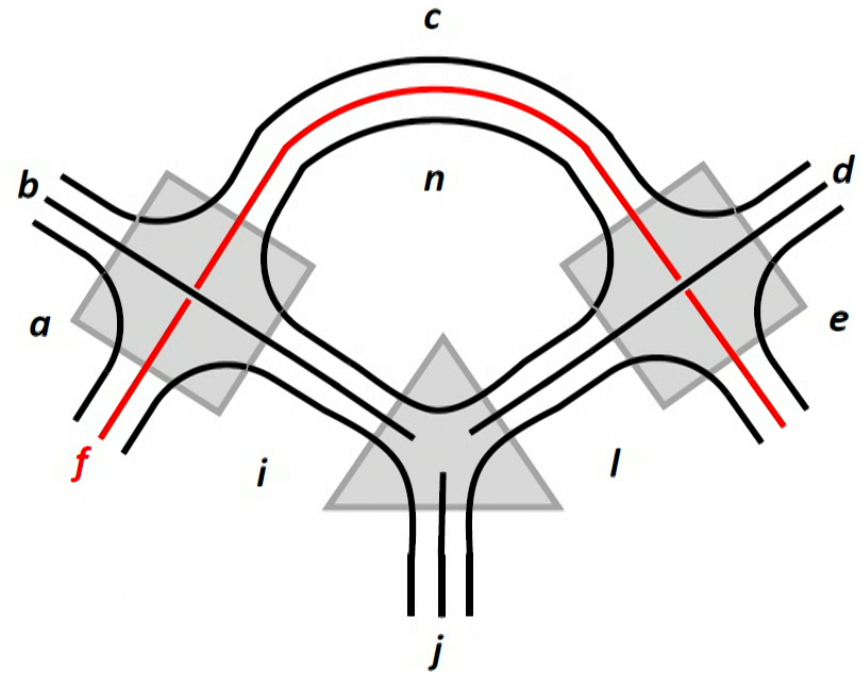
$$= \delta_{\alpha\alpha'} \delta_{\beta\beta'} \delta_{\mu\mu'} \delta_{\nu\nu'} \delta_{ii'} \delta_{aa'} F_{\beta\alpha\nu}^{a\mu i},$$

$F_{j;be}^{acd}$

E.g. String-net

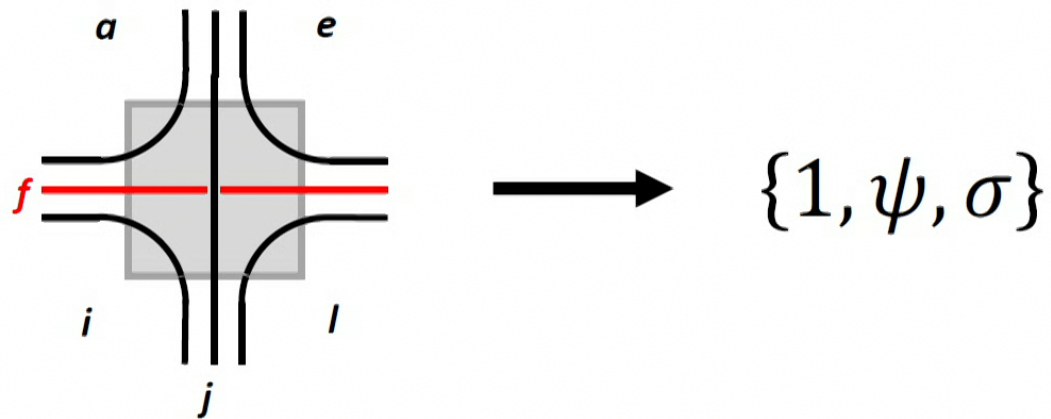


=


 $F_{l;ei}^{jaf}$

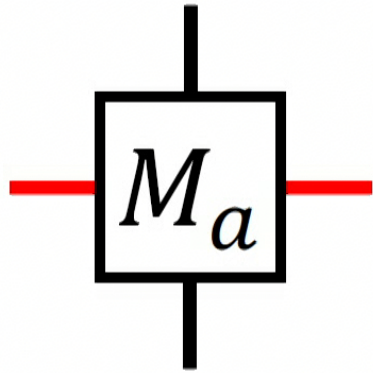
$$FF = \sum FFF$$

Double Ising



$F_{l;ei}^{jaf}$

Form a matrix product operator algebra



Objects

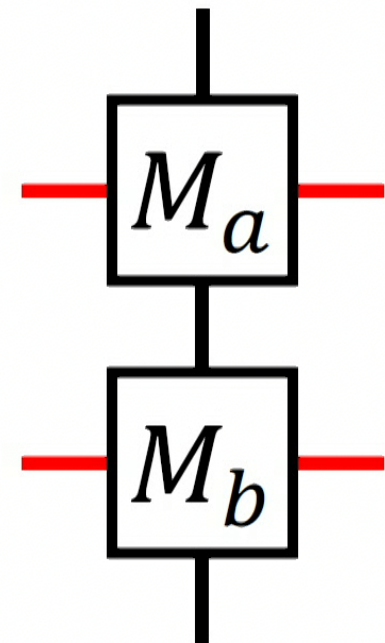


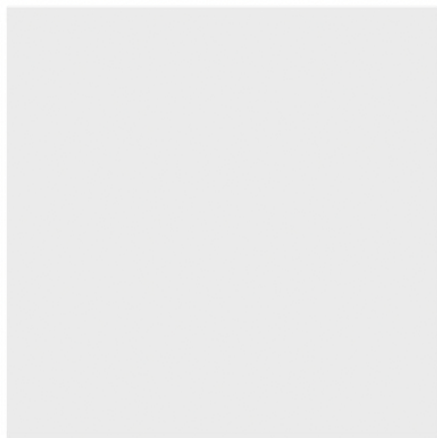
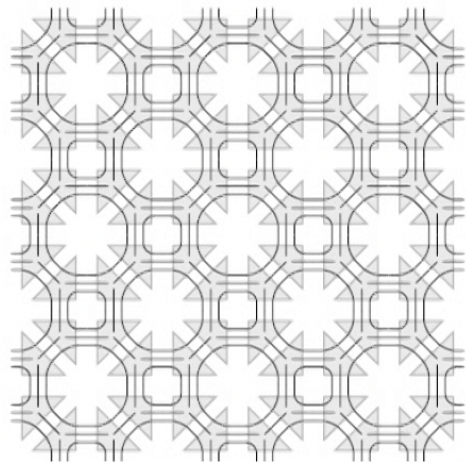
morphisms

Fusion category $\mathcal{C}$

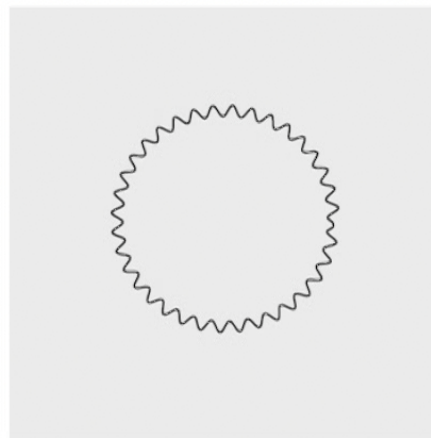
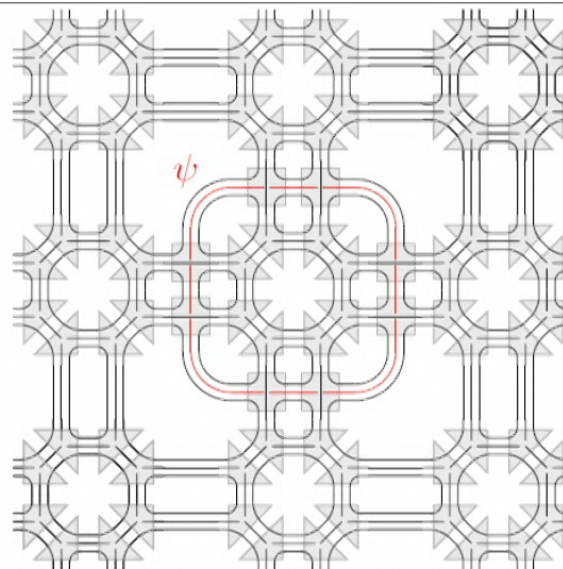
Tensor product

$$a \otimes b$$

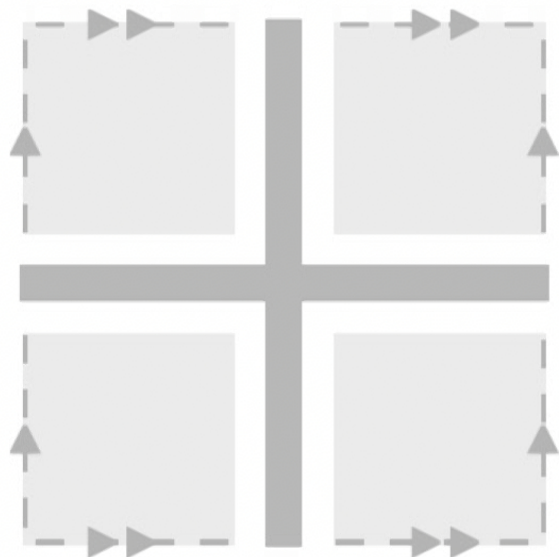




$=$

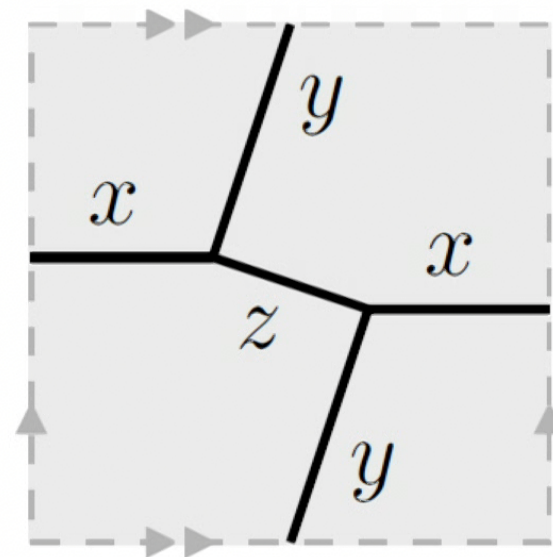


$\sim$

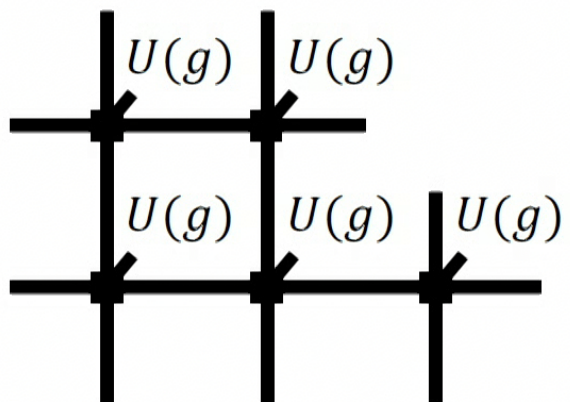


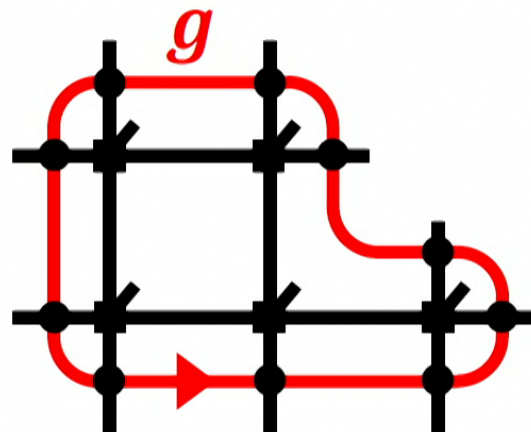
B

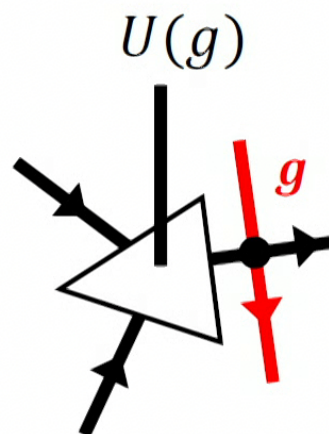
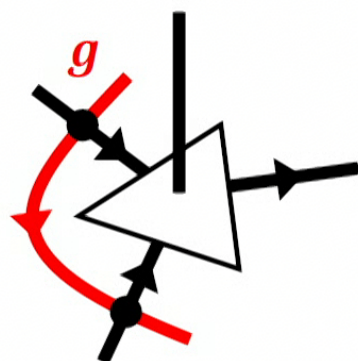
$$= \sum_{xyz} c_{xy}^z$$



With a global symmetry



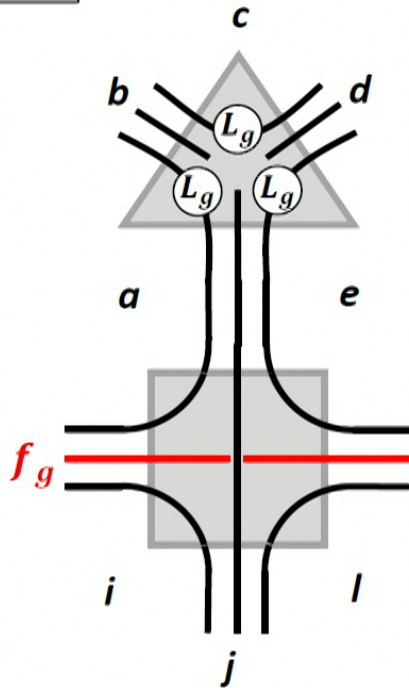




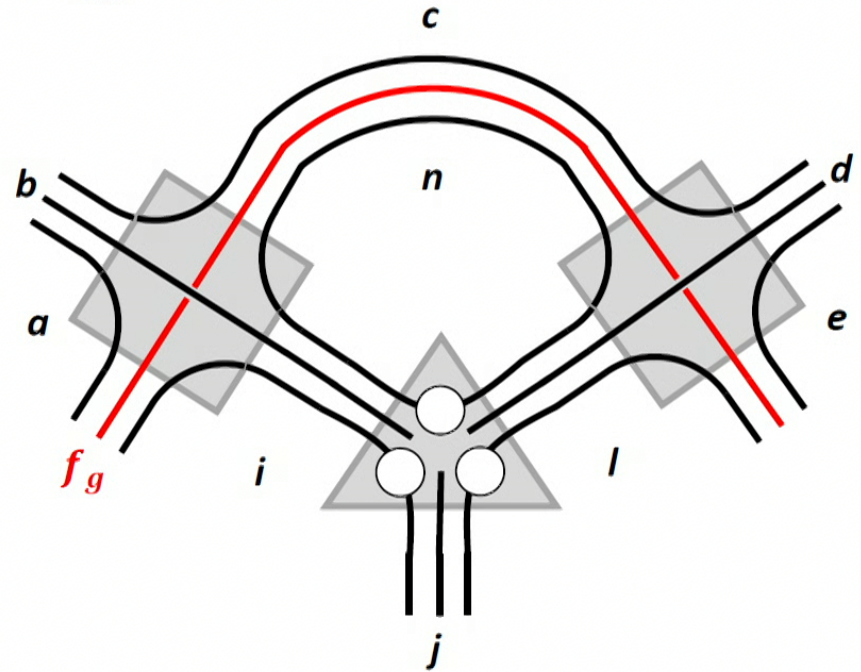
$$a_h - \textcircled{h} - a_h$$

G -Graded String-net

$F_{j;be}^{acd}$



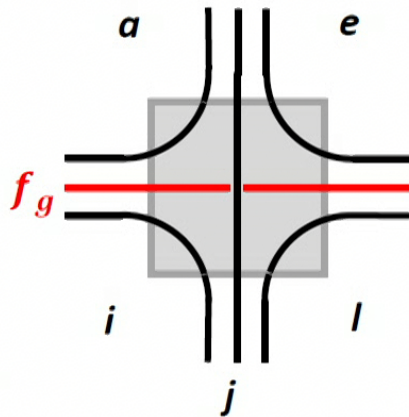
=



$F_{l;ei}^{jaf}$

$$FF = \sum FFF$$

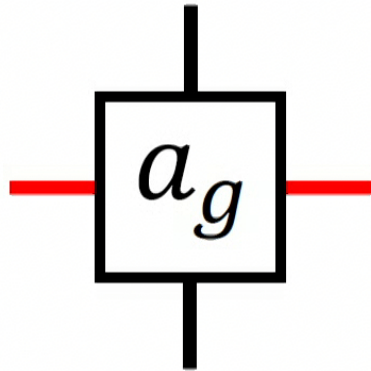
Toric code w/ twists



$$\{1, \psi\} \oplus \{\sigma\}$$

$$F_{l;ei}^{jaf}$$

Form a $\mathbb{Z}_2$ -graded matrix product operator algebra



Objects

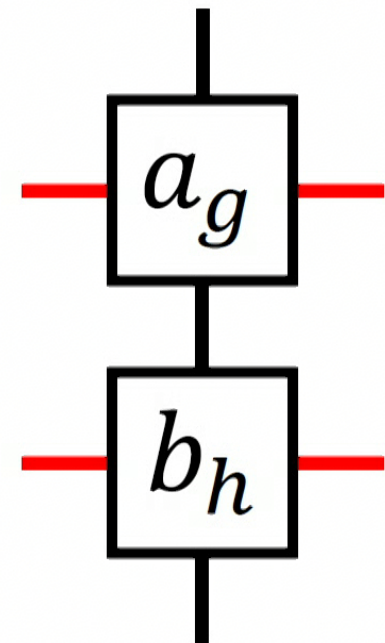


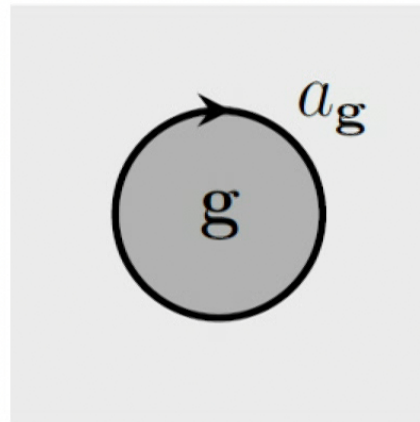
morphisms

G -graded fusion category $\mathcal{C}_G$

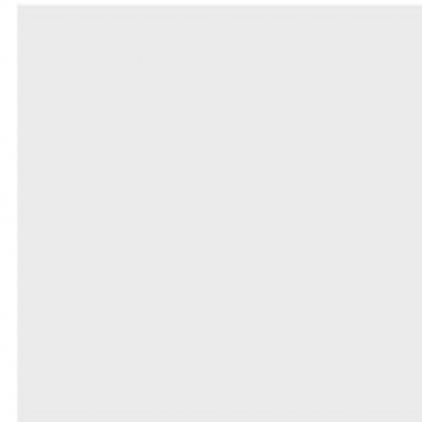
G -graded
tensor product

$$a_g \otimes b_h$$

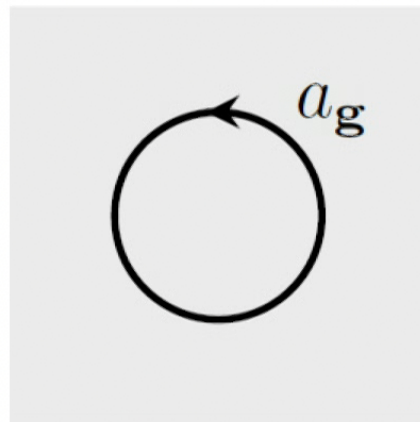




$= d_a$



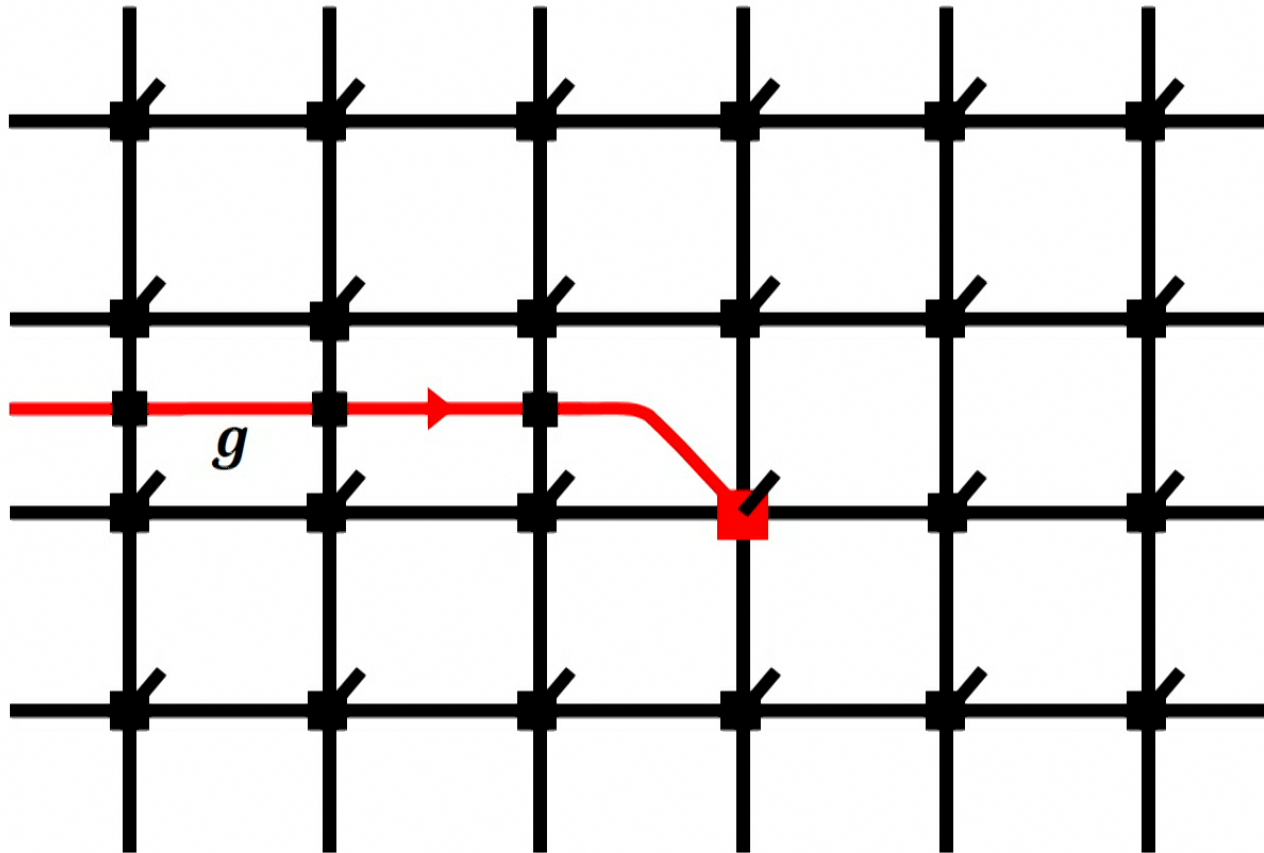
,

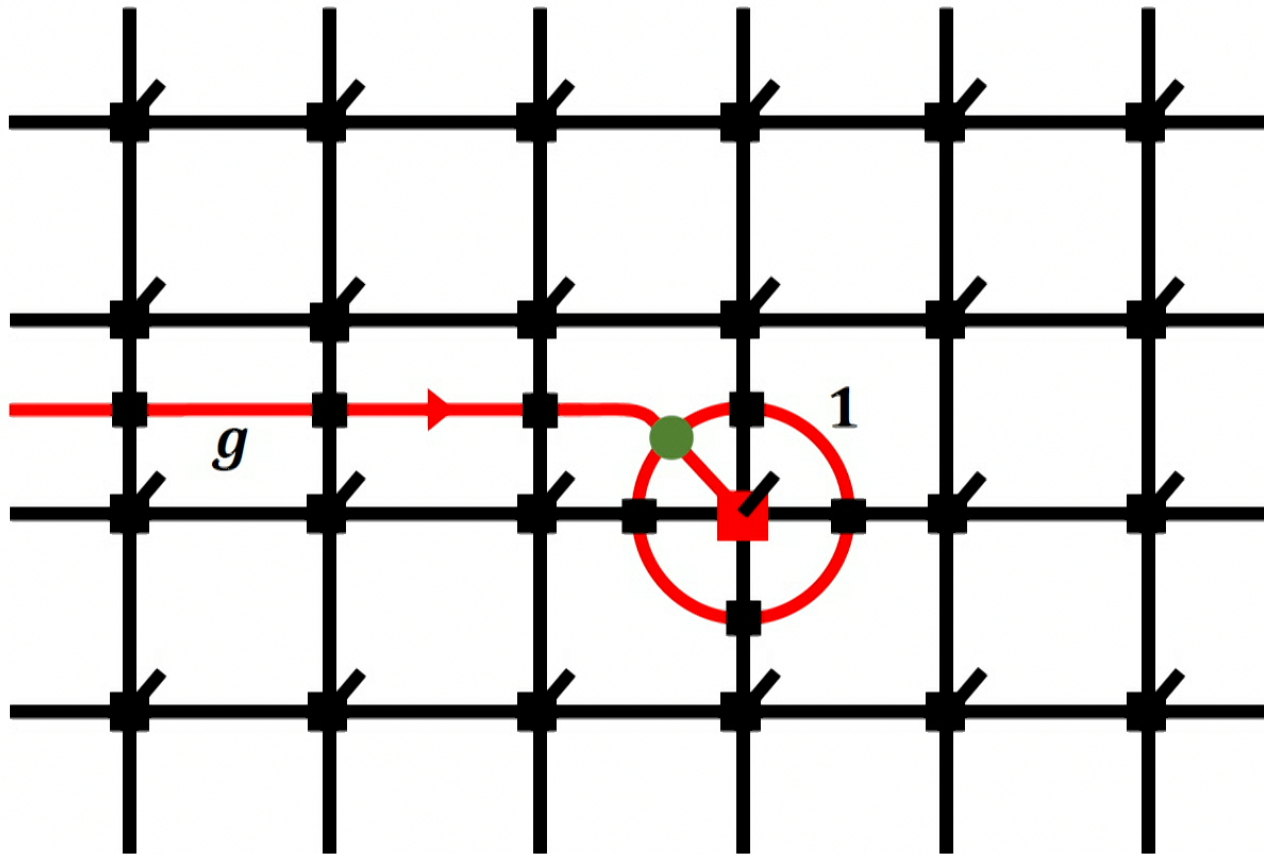


$= d_a$



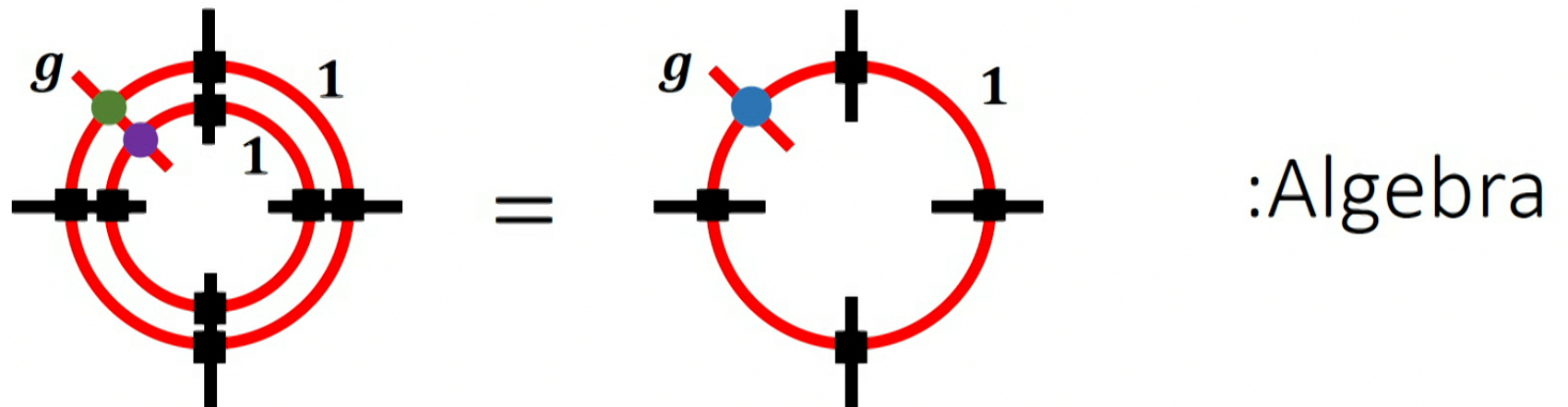
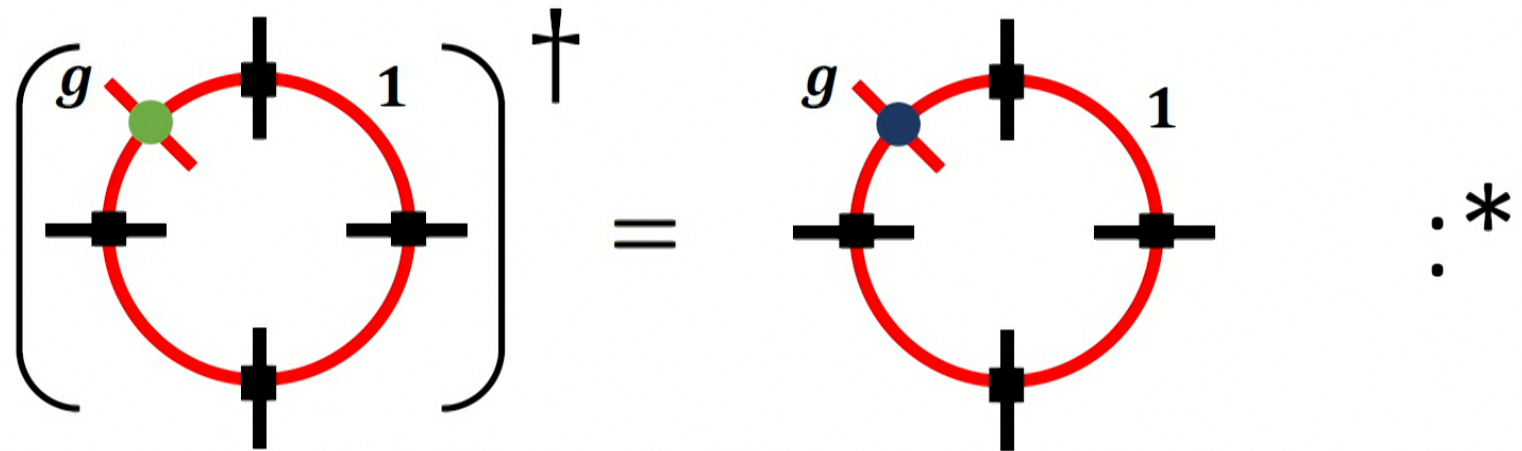
What about anyons and
defects?





Diagrammatic equation showing a circle with four external legs (top, bottom, left, right) and a green dot on the top-left leg labeled g . The circle is labeled 1 on the top-right leg. This is equated to a sum over a blue diamond, $\sum_{\text{blue diamond}} C_{\text{green dot, blue diamond}}$, followed by a similar circle diagram with a blue diamond on the top-left leg labeled g .

Diagrammatic equation showing a four-point vertex (blue square) with external legs labeled g (top), 1 (bottom), and a_g (left), b_g (right). This is equated to a diagram with two boxes, X_μ and X_ν , connected by a line. The external legs are labeled a_g , b_g , c_g , and d_1 .



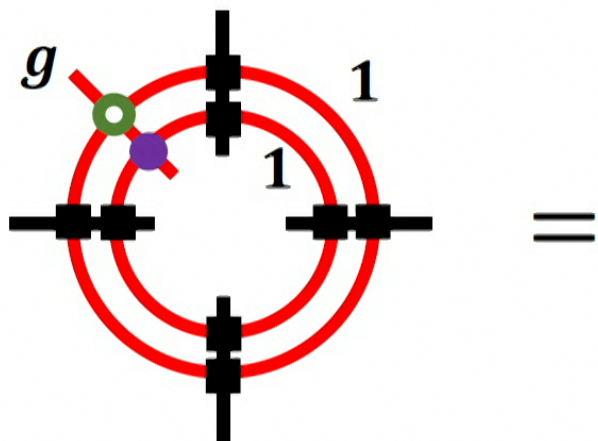
Defect tube algebra w/ $|G|$ -Sectors

$$\left(\begin{array}{c} \text{Diagram with green dot} \end{array} \right)^{\dagger} = \begin{array}{c} \text{Diagram with blue dot} \end{array} \quad \cdot^*$$

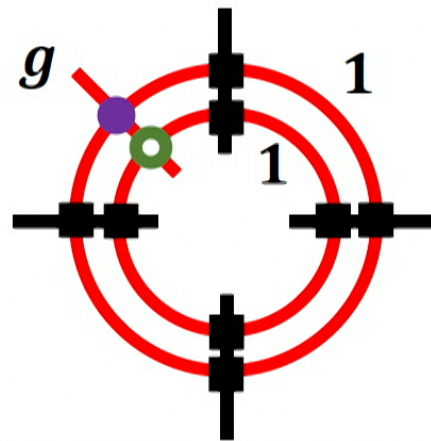
$$\begin{array}{c} \text{Diagram with green and purple dots} \end{array} = \begin{array}{c} \text{Diagram with blue dot} \end{array} \quad \text{:Algebra}$$

Defect tube algebra w/ $|G|$ -Sectors

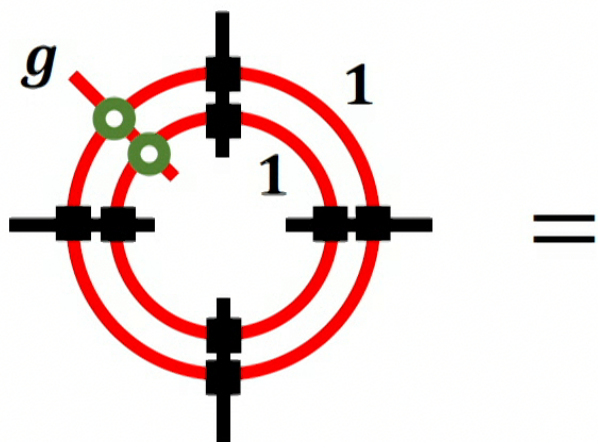
Dube algebra w/ $|G|$ -Sectors



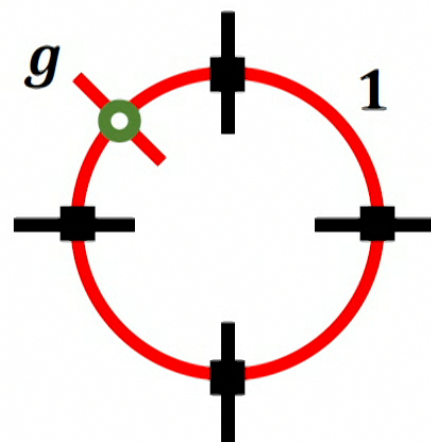
=



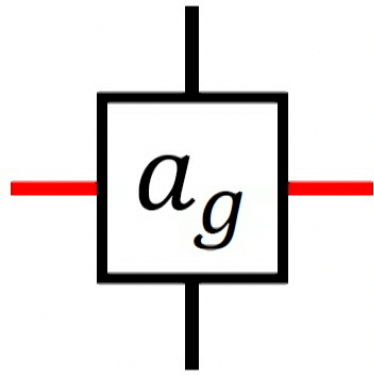
Central



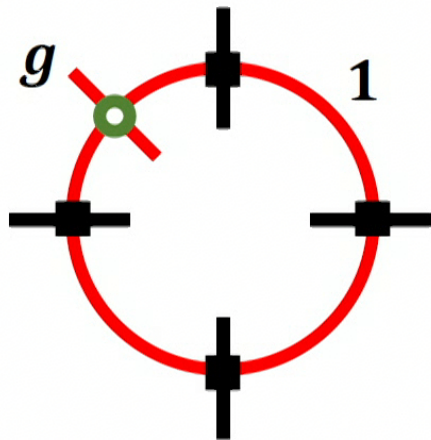
=



Idempotent



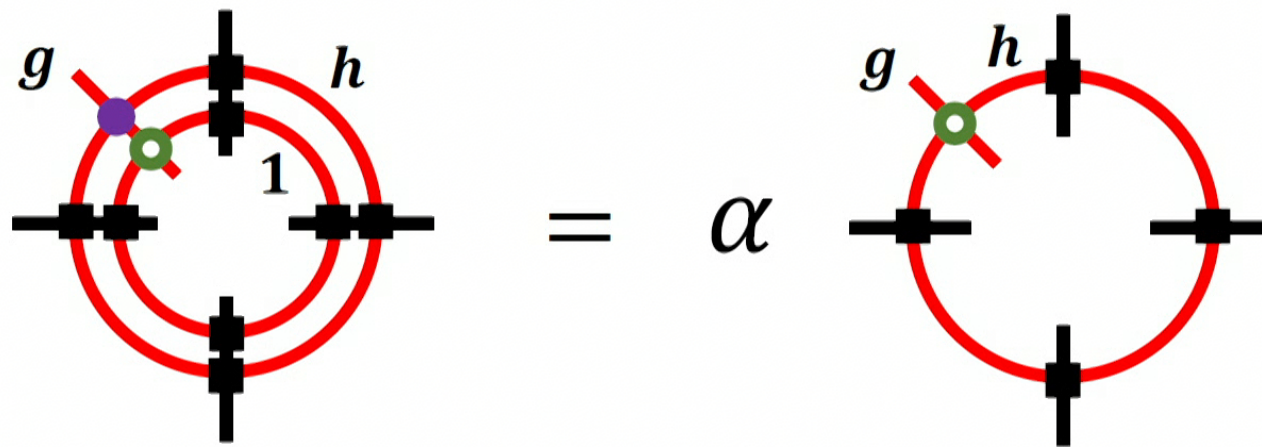
$:\mathcal{C}_G$



$:Z(\mathcal{C}_0)_G$

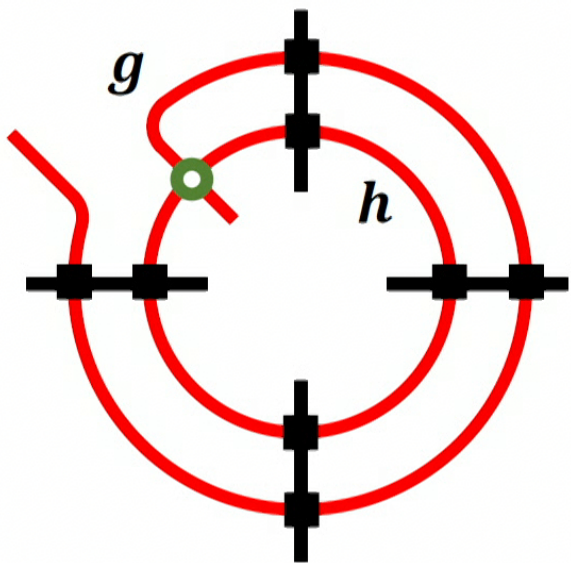
Finding the emergent SET (GxMTC)

Domain Wall projective actions

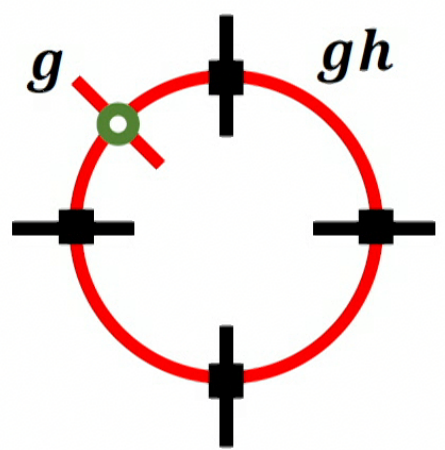


G -crossed modular matrices

$$\begin{array}{c} \text{green circle} \\ \text{left: } g, \text{ top: } h, \text{ right: } h, \text{ bottom: } g \end{array} \xrightarrow{S} \begin{array}{c} \text{green circle} \\ \text{left: } h, \text{ top: } g, \text{ right: } h, \text{ bottom: } h \end{array} = \sum_{\text{blue circle}} S_{\text{green circle}, \text{blue circle}} \begin{array}{c} \text{blue circle} \\ \text{left: } h, \text{ top: } h, \text{ right: } h, \text{ bottom: } g^{-1} \end{array}$$



$$= \theta_a$$

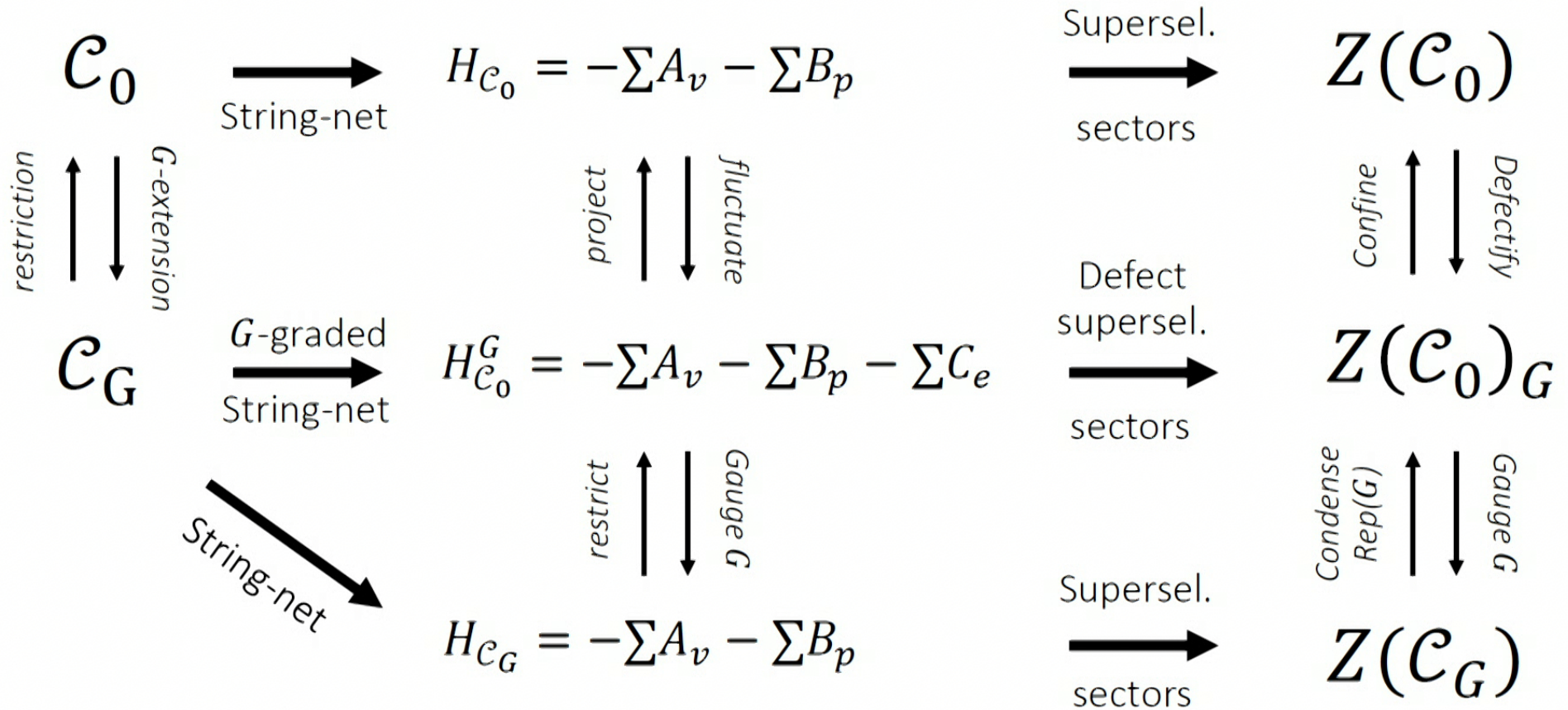


Gauging and $\text{Rep}(G)$ -condensation

Input:

Hamiltonian:

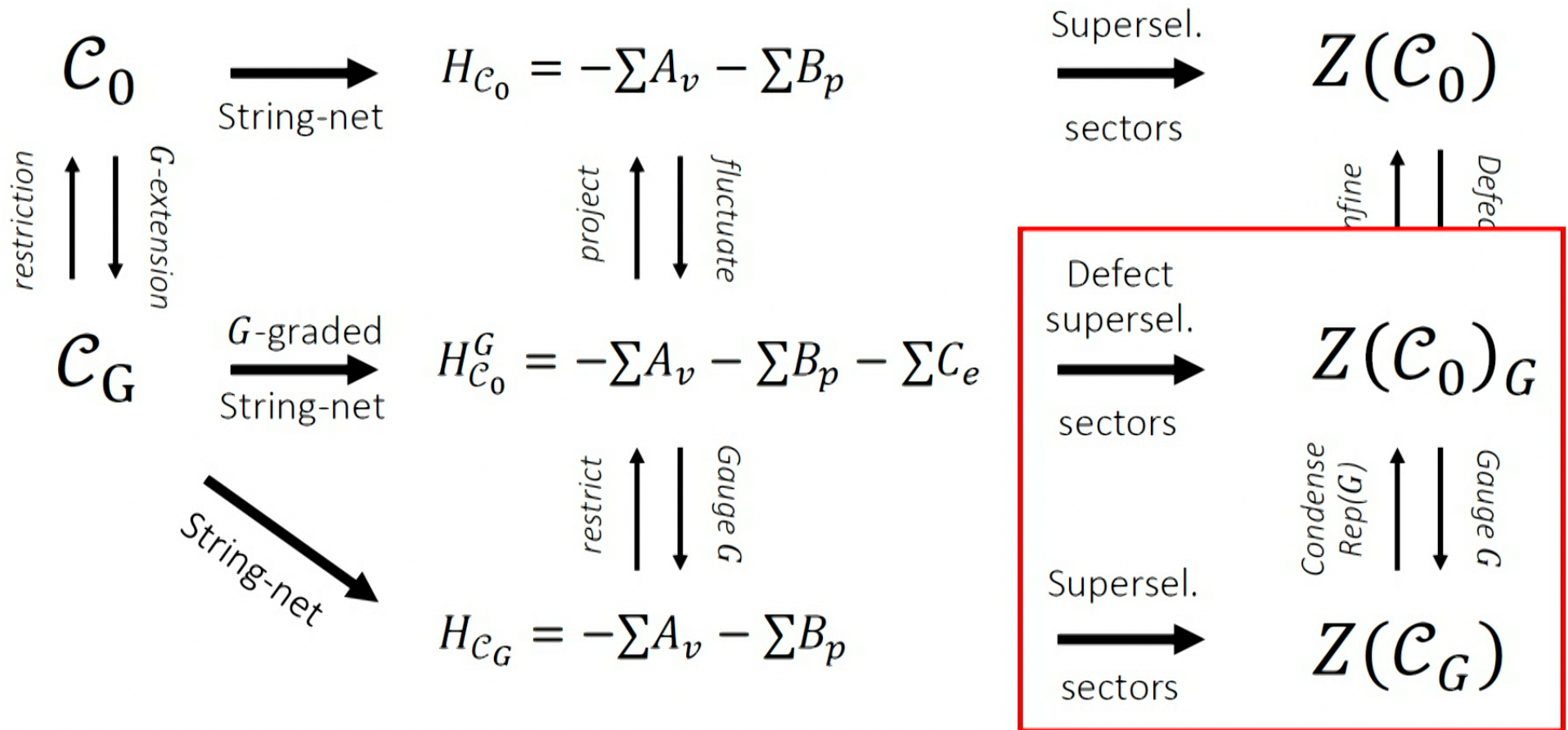
Emergent theory:



Input:

Hamiltonian:

Emergent theory:



Gauging an SPT - $\text{Vec}_G^\alpha \rightarrow Z(\text{Vec}_G^\alpha)$

$$\underline{[\mathbf{g}, \mu]} = \sum_{\mathbf{k} \in \mathcal{G}/Z_{\mathbf{g}}} \underline{(\mathbf{k}\mathbf{g}, \mathbf{k}\mu)}, \quad \underline{(\mathbf{g}, \mu)} := \frac{d_\mu}{|Z_{\mathbf{g}}|} \sum_{\mathbf{h} \in Z_{\mathbf{g}}} \bar{\chi}_{\mathbf{g}}^\mu(\mathbf{h}) \mathcal{B}_{\mathbf{g}}^{\mathbf{h}},$$

Gauging an SPT - $\text{Vec}_G^\alpha \rightarrow Z(\text{Vec}_G^\alpha)$

$$\underline{[\mathbf{g}, \mu]} = \sum_{\mathbf{k} \in \mathcal{G}/Z_{\mathbf{g}}} \underline{(\mathbf{k}\mathbf{g}, \mathbf{k}\mu)}, \quad \underline{(\mathbf{g}, \mu)} := \frac{d_\mu}{|Z_{\mathbf{g}}|} \sum_{\mathbf{h} \in Z_{\mathbf{g}}} \bar{\chi}_{\mathbf{g}}^\mu(\mathbf{h}) \mathcal{B}_{\mathbf{g}}^{\mathbf{h}},$$

Gauging $Z(\mathcal{C}_0)_G \rightarrow Z(\mathcal{C}_G)$

$$\underline{[a_{\mathbf{g}}, \mu]} = \sum_{\mathbf{k} \in \mathcal{G}/Z_{a_{\mathbf{g}}}} \underline{(\mathbf{k}a_{\mathbf{g}}, \mathbf{k}\mu)}, \quad \underline{(a_{\mathbf{g}}, \mu)} := \frac{d_\mu}{|Z_{a_{\mathbf{g}}}|} \sum_{\mathbf{h} \in Z_{a_{\mathbf{g}}}} \bar{\chi}_{a_{\mathbf{g}}}^\mu(\mathbf{h}) \mathcal{B}_{a_{\mathbf{g}}}^{\mathbf{h}}.$$

$$S_{[a_{\mathbf{g}}, \mu][b_{\mathbf{h}}, \nu]}^{\mathcal{Z}(\mathcal{C}_{\mathcal{G}})} = \frac{1}{|\mathcal{G}|} \sum_{\substack{\mathbf{r} \in \mathcal{G}/Z_a \\ \mathbf{s} \in \mathcal{G}/Z_b}} \mathbf{r} \overline{\chi}_a^\mu(\mathbf{s} \mathbf{h}) \mathbf{s} \chi_b^\nu(\mathbf{r} \overline{\mathbf{g}}) S_{\mathbf{r} a_{\mathbf{g}} \mathbf{s} b_{\mathbf{h}}} ,$$

$$T_{[a_{\mathbf{g}}, \mu][b_{\mathbf{h}}, \nu]}^{\mathcal{Z}(\mathcal{C}_{\mathcal{G}})} = \delta_{[a_{\mathbf{g}}, \mu][b_{\mathbf{h}}, \nu]} \frac{\chi_a^\mu(\mathbf{g})}{\chi_a^\mu(\mathbf{1})} \theta_a .$$

$$S_{[a_{\mathbf{g}}, \mu][b_{\mathbf{h}}, \nu]}^{\mathcal{Z}(\mathcal{C}_{\mathcal{G}})} = \frac{1}{|\mathcal{G}|} \sum_{\substack{\mathbf{r} \in \mathcal{G}/Z_a \\ \mathbf{s} \in \mathcal{G}/Z_b}} \mathbf{r} \bar{\chi}_a^\mu(\mathbf{s} \mathbf{h}) \mathbf{s} \chi_b^\nu(\mathbf{r} \bar{\mathbf{g}}) S_{\mathbf{r} a_{\mathbf{g}} \mathbf{s} b_{\mathbf{h}}} ,$$

$$T_{[a_{\mathbf{g}}, \mu][b_{\mathbf{h}}, \nu]}^{\mathcal{Z}(\mathcal{C}_{\mathcal{G}})} = \delta_{[a_{\mathbf{g}}, \mu][b_{\mathbf{h}}, \nu]} \frac{\chi_a^\mu(\mathbf{g})}{\chi_a^\mu(\mathbf{1})} \theta_a .$$

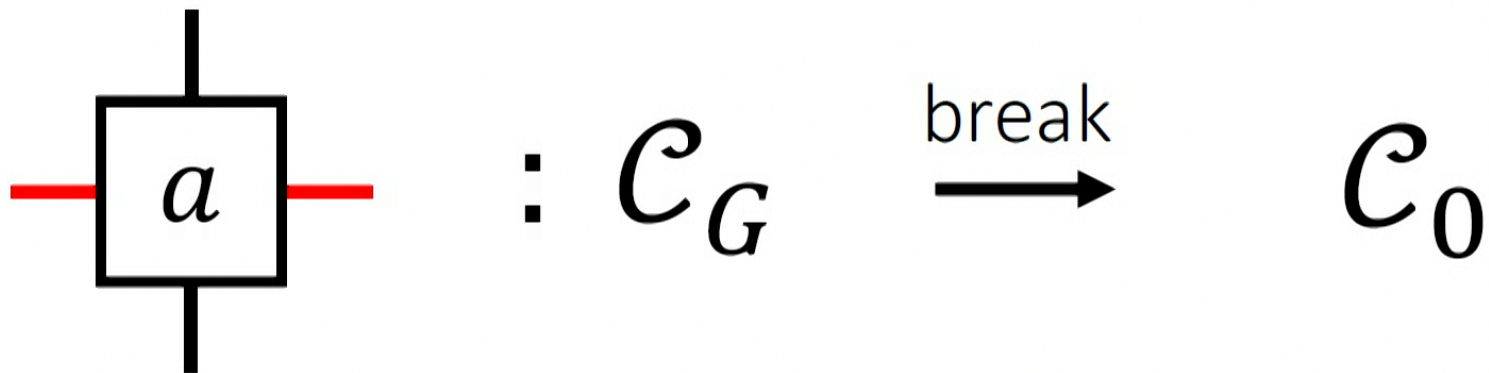
$$T(\underline{[a_{\mathbf{g}}, \mu]}) = T \frac{d_\mu}{|Z_{a_{\mathbf{g}}}|} \left(\bar{\chi}_{a_{\mathbf{g}}}^\mu(\mathbf{1}) \underline{a_{\mathbf{g}}} + \dots \right)$$

$$S_{[a_{\mathbf{g}}, \mu][b_{\mathbf{h}}, \nu]}^{\mathcal{Z}(\mathcal{C}_{\mathcal{G}})} = \frac{1}{|\mathcal{G}|} \sum_{\substack{\mathbf{r} \in \mathcal{G}/Z_a \\ \mathbf{s} \in \mathcal{G}/Z_b}} \mathbf{r} \bar{\chi}_a^{\mu}(\mathbf{s} \mathbf{h}) \mathbf{s} \chi_b^{\nu}(\mathbf{r} \bar{\mathbf{g}}) S_{\mathbf{r} a_{\mathbf{g}} \mathbf{s} b_{\mathbf{h}}} ,$$

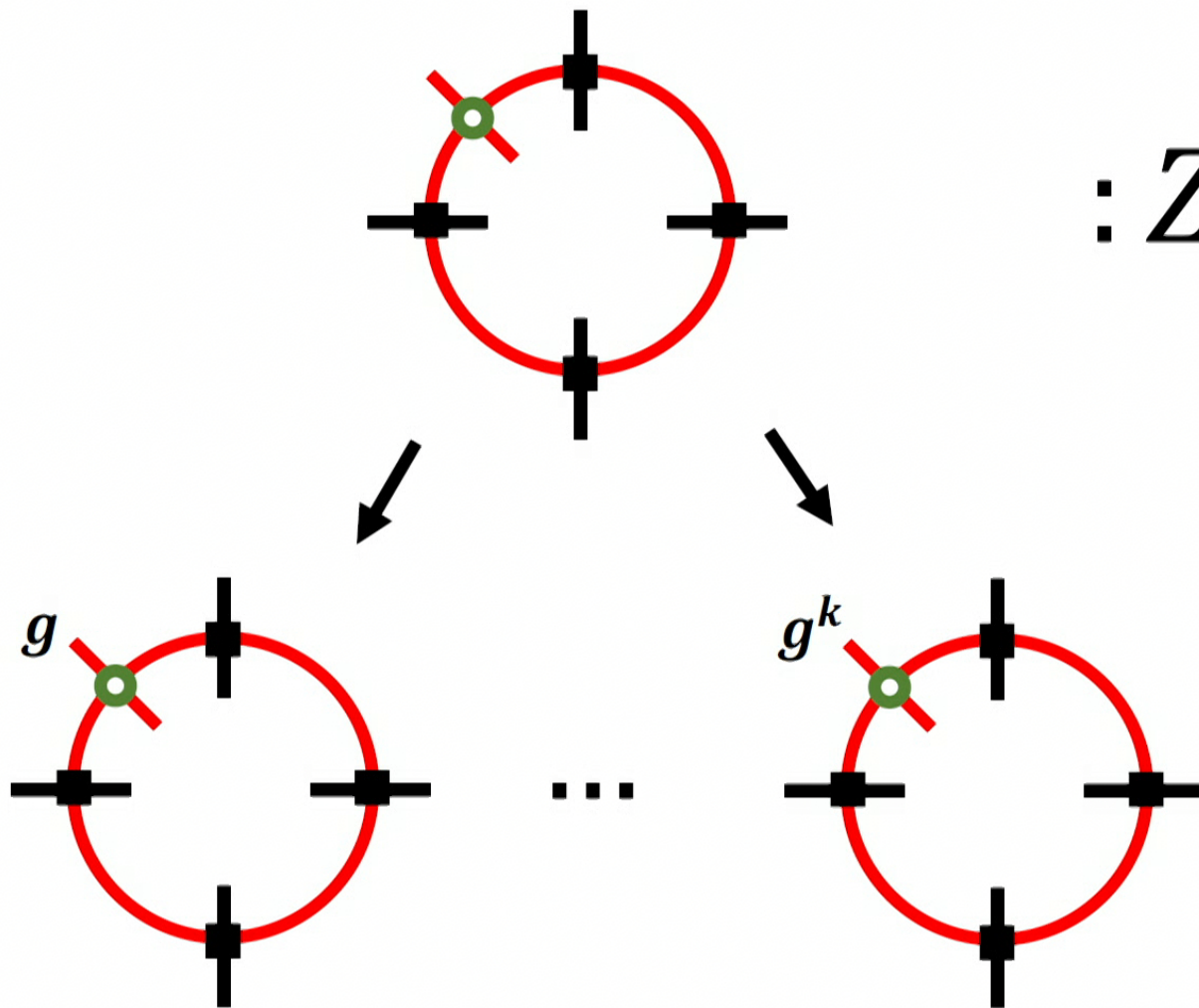
$$T_{[a_{\mathbf{g}}, \mu][b_{\mathbf{h}}, \nu]}^{\mathcal{Z}(\mathcal{C}_{\mathcal{G}})} = \delta_{[a_{\mathbf{g}}, \mu][b_{\mathbf{h}}, \nu]} \frac{\chi_a^{\mu}(\mathbf{g})}{\chi_a^{\mu}(\mathbf{1})} \theta_a .$$

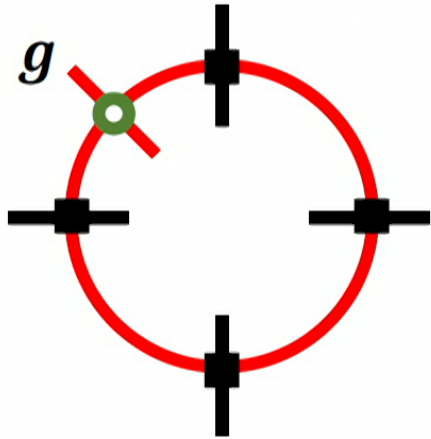
$$T(\underline{[a_{\mathbf{g}}, \mu]}) = \frac{d_{\mu}}{|Z_{a_{\mathbf{g}}}|} \left(\bar{\chi}_{a_{\mathbf{g}}}^{\mu}(\mathbf{1}) \frac{\chi_{a_{\mathbf{g}}}^{\mu}(\mathbf{g})}{\chi_{a_{\mathbf{g}}}^{\mu}(\mathbf{1})} \left(\frac{\chi_{a_{\mathbf{g}}}^{\mu}(\mathbf{g})}{\chi_{a_{\mathbf{g}}}^{\mu}(\mathbf{1})} \right)^* \theta_{a_{\mathbf{g}}} \mathcal{B}_{a_{\mathbf{g}}}^{\mathbf{g}} + \dots \right)$$

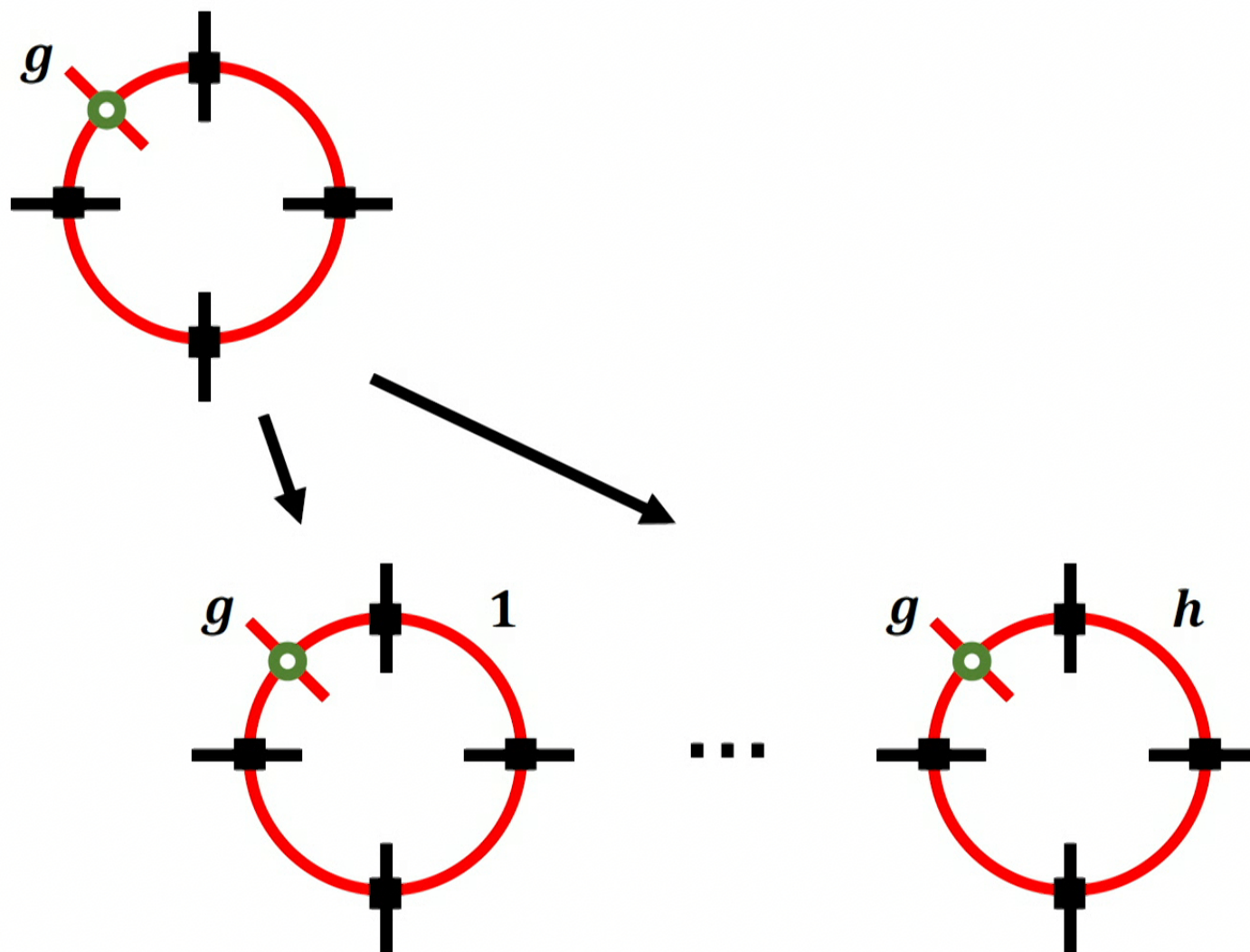
Rep(G)-condensation via MPO symmetry breaking

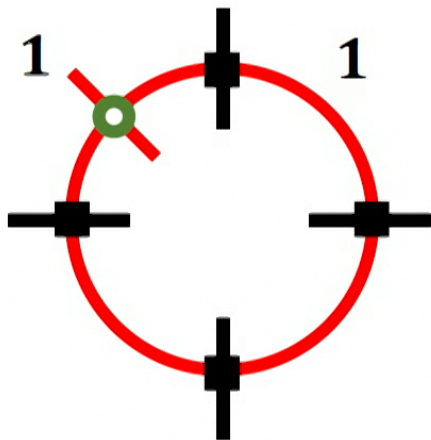


$:Z(\mathcal{C}_G)$





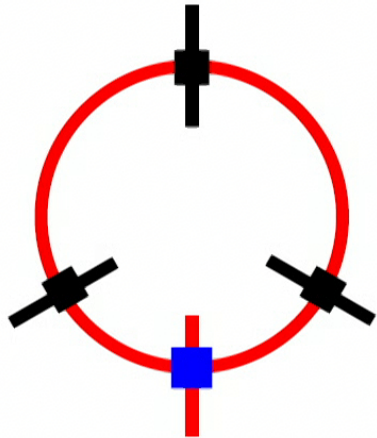




$: Z(\mathcal{C}_0)$

Examples

Toric Code

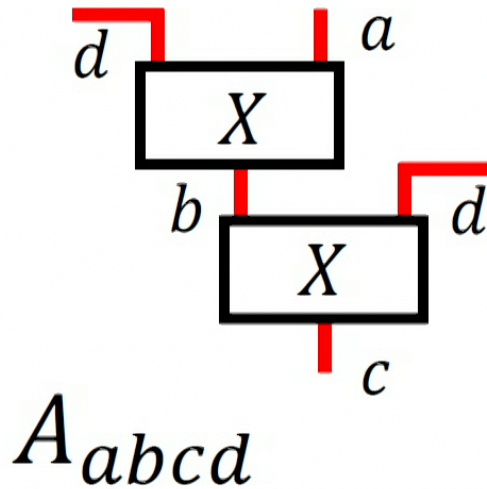


$$\begin{array}{c}
 0 \text{ --- } \text{---} \text{---} 0 \\
 \text{---} \text{---} \text{---} \\
 1 \text{ --- } \text{---} \text{---} 1
 \end{array}
 = I$$

$$\begin{array}{c}
 1 \text{ --- } \text{---} \text{---} 1 \\
 \text{---} \text{---} \text{---} \\
 1 \text{ --- } \text{---} \text{---} 1
 \end{array}
 = Z$$

1	$ \begin{array}{c} 0 \\ \text{---} \text{---} \text{---} \\ 0 \text{ --- } \text{---} \text{---} 0 = 1 \\ 0 \end{array} \quad \begin{array}{c} 0 \\ \text{---} \text{---} \text{---} \\ 1 \text{ --- } \text{---} \text{---} 1 = 1 \\ 0 \end{array} $
e	$ \begin{array}{c} 0 \\ \text{---} \text{---} \text{---} \\ 0 \text{ --- } \text{---} \text{---} 0 = 1 \\ 0 \end{array} \quad \begin{array}{c} 0 \\ \text{---} \text{---} \text{---} \\ 1 \text{ --- } \text{---} \text{---} 1 = -1 \\ 0 \end{array} $
m	$ \begin{array}{c} 1 \\ \text{---} \text{---} \text{---} \\ 0 \text{ --- } \text{---} \text{---} 0 = 1 \\ 1 \end{array} \quad \begin{array}{c} 1 \\ \text{---} \text{---} \text{---} \\ 1 \text{ --- } \text{---} \text{---} 1 = 1 \\ 1 \end{array} $
em	$ \begin{array}{c} 1 \\ \text{---} \text{---} \text{---} \\ 0 \text{ --- } \text{---} \text{---} 0 = 1 \\ 1 \end{array} \quad \begin{array}{c} 1 \\ \text{---} \text{---} \text{---} \\ 1 \text{ --- } \text{---} \text{---} 1 = -1 \\ 1 \end{array} $

Double Ising



$$(1, 1) = \frac{1}{4} (A_{1111} + 2^{3/4} A_{1\sigma 1\sigma} + A_{1\psi 1\psi})$$

$$(\sigma, 1) = \frac{1}{4} \left(A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{\pi i}{8}} A_{\sigma 1\sigma\sigma} + 2^{1/4} e^{-\frac{3\pi i}{8}} A_{\sigma\psi\sigma\sigma} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi} \right)$$

$$(1, \bar{\sigma}) = \frac{1}{4} \left(A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{\pi i}{8}} A_{\sigma 1\sigma\sigma} + 2^{1/4} e^{\frac{3\pi i}{8}} A_{\sigma\psi\sigma\sigma} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi} \right)$$

$$(\psi, 1) = \frac{1}{4} \left(A_{\psi\psi\psi 1} + 2^{3/4} e^{\frac{\pi i}{2}} A_{\psi\sigma\psi\sigma} - A_{\psi 1\psi\psi} \right)$$

$$(1, \psi) = \frac{1}{4} \left(A_{\psi\psi\psi 1} + 2^{3/4} e^{-\frac{\pi i}{2}} A_{\psi\sigma\psi\sigma} - A_{\psi 1\psi\psi} \right)$$

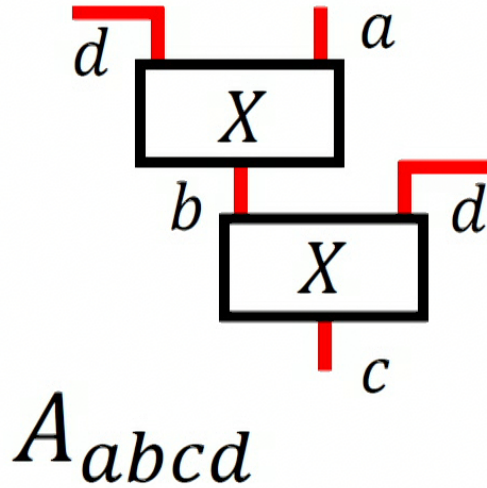
$$(\sigma, \bar{\psi}) = \frac{1}{4} \left(A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{7\pi i}{8}} A_{\sigma 1\sigma\sigma} + 2^{1/4} e^{\frac{5\pi i}{8}} A_{\sigma\psi\sigma\sigma} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi} \right)$$

$$(\psi, \bar{\sigma}) = \frac{1}{4} \left(A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{7\pi i}{8}} A_{\sigma 1\sigma\sigma} + 2^{1/4} e^{-\frac{5\pi i}{8}} A_{\sigma\psi\sigma\sigma} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi} \right)$$

$$(\psi, \bar{\psi}) = \frac{1}{4} (A_{1111} - 2^{3/4} A_{1\sigma 1\sigma} + A_{1\psi 1\psi})$$

$$(\sigma, \bar{\sigma}) = \frac{1}{2} (A_{1111} + A_{\psi\psi\psi 1} - A_{1\psi 1\psi} + A_{\psi 1\psi\psi})$$

Condensing $\mathbb{Z}_2$



$$(1, 1) = \frac{1}{4} (A_{1111} + 2^{3/4} \overline{A_{1\sigma 1\sigma}} + A_{1\psi 1\psi})$$

$$(\sigma, 1) = \frac{1}{4} \left(A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{\pi i}{8}} \overline{A_{\sigma 1\sigma\sigma}} + 2^{1/4} e^{-\frac{3\pi i}{8}} \overline{A_{\sigma\psi\sigma\sigma}} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi} \right)$$

$$(1, \bar{\sigma}) = \frac{1}{4} \left(A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{\pi i}{8}} \overline{A_{\sigma 1\sigma\sigma}} + 2^{1/4} e^{\frac{3\pi i}{8}} \overline{A_{\sigma\psi\sigma\sigma}} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi} \right)$$

$$(\psi, 1) = \frac{1}{4} \left(A_{\psi\psi\psi 1} + 2^{3/4} e^{\frac{\pi i}{2}} \overline{A_{\psi\sigma\psi\sigma}} - A_{\psi 1\psi\psi} \right)$$

$$(1, \psi) = \frac{1}{4} \left(A_{\psi\psi\psi 1} + 2^{3/4} e^{-\frac{\pi i}{2}} \overline{A_{\psi\sigma\psi\sigma}} - A_{\psi 1\psi\psi} \right)$$

$$(\sigma, \bar{\psi}) = \frac{1}{4} \left(A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{7\pi i}{8}} \overline{A_{\sigma 1\sigma\sigma}} + 2^{1/4} e^{\frac{5\pi i}{8}} \overline{A_{\sigma\psi\sigma\sigma}} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi} \right)$$

$$(\psi, \bar{\sigma}) = \frac{1}{4} \left(A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{7\pi i}{8}} \overline{A_{\sigma 1\sigma\sigma}} + 2^{1/4} e^{-\frac{5\pi i}{8}} \overline{A_{\sigma\psi\sigma\sigma}} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi} \right)$$

$$(\psi, \bar{\psi}) = \frac{1}{4} (A_{1111} - 2^{3/4} \overline{A_{1\sigma 1\sigma}} + A_{1\psi 1\psi})$$

$$(\sigma, \bar{\sigma}) = \frac{1}{2} (A_{1111} + A_{\psi\psi\psi 1} - A_{1\psi 1\psi} + A_{\psi 1\psi\psi})$$

Condensing $\mathbb{Z}_2$

1	$(1, 1) \sim (\psi, \bar{\psi})$
e, m	$(\sigma, \bar{\sigma}) \rightarrow \text{Splits}$
em	$(\psi, 1) \sim (1, \psi)$
σ_+	$(\sigma, 1) \sim (\sigma, \bar{\psi})$
σ_-	$(1, \bar{\sigma}) \sim (\psi, \bar{\sigma})$



$$(1, 1) = \frac{1}{4} (A_{1111} + 2^{3/4} \cancel{A_{1\sigma 1\sigma}} + A_{1\psi 1\psi})$$

$$(\sigma, 1) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{\pi i}{8}} \cancel{A_{\sigma 1 \sigma \sigma}} + 2^{1/4} e^{-\frac{3\pi i}{8}} \cancel{A_{\sigma \psi \sigma \sigma}} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(1, \bar{\sigma}) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{\pi i}{8}} \cancel{A_{\sigma 1 \sigma \sigma}} + 2^{1/4} e^{\frac{3\pi i}{8}} \cancel{A_{\sigma \psi \sigma \sigma}} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(\psi, 1) = \frac{1}{4} (A_{\psi\psi\psi 1} + 2^{3/4} e^{\frac{\pi i}{2}} \cancel{A_{\psi\sigma\psi\sigma}} - A_{\psi 1 \psi \psi})$$

$$(1, \psi) = \frac{1}{4} (A_{\psi\psi\psi 1} + 2^{3/4} e^{-\frac{\pi i}{2}} \cancel{A_{\psi\sigma\psi\sigma}} - A_{\psi 1 \psi \psi})$$

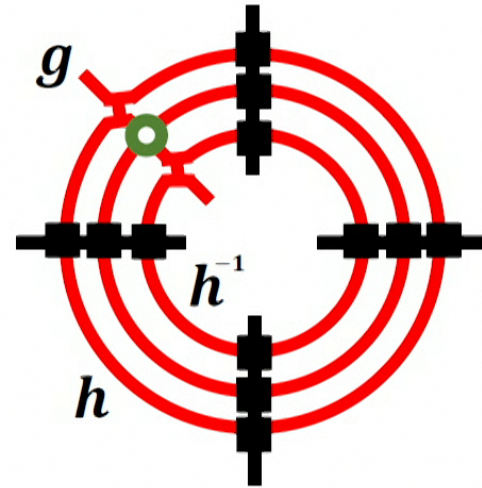
$$(\sigma, \bar{\psi}) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{7\pi i}{8}} \cancel{A_{\sigma 1 \sigma \sigma}} + 2^{1/4} e^{\frac{5\pi i}{8}} \cancel{A_{\sigma \psi \sigma \sigma}} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(\psi, \bar{\sigma}) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{7\pi i}{8}} \cancel{A_{\sigma 1 \sigma \sigma}} + 2^{1/4} e^{-\frac{5\pi i}{8}} \cancel{A_{\sigma \psi \sigma \sigma}} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(\psi, \bar{\psi}) = \frac{1}{4} (A_{1111} - 2^{3/4} \cancel{A_{1\sigma 1\sigma}} + A_{1\psi 1\psi})$$

$$(\sigma, \bar{\sigma}) = \frac{1}{2} (\underline{A_{1111}} + \underline{A_{\psi\psi\psi 1}} - \underline{A_{1\psi 1\psi}} + \underline{A_{\psi 1 \psi \psi}})$$

$U_g:$



$$A_{11111}$$

$$A_{\psi\psi\psi 1}$$

$$A_{1\psi 1\psi}$$

$$A_{\psi 1\psi\psi}$$



$$A_{11111} + A_{\psi\psi\psi 1} + A_{1\psi 1\psi} + A_{\psi 1\psi\psi}$$

$$A_{11111} + A_{\psi\psi\psi 1} - A_{1\psi 1\psi} - A_{\psi 1\psi\psi}$$

$$A_{11111} - A_{\psi\psi\psi 1} + A_{1\psi 1\psi} - A_{\psi 1\psi\psi}$$

$$A_{11111} - A_{\psi\psi\psi 1} - A_{1\psi 1\psi} + A_{\psi 1\psi\psi}$$

$U:$

e	m
$A_{1111} - A_{1\psi 1\psi}$	$A_{\psi\psi\psi 1} + A_{\psi 1\psi\psi}$

A_{1111}		$A_{1111} + A_{\psi\psi\psi 1} + A_{1\psi 1\psi} + A_{\psi 1\psi\psi}$
$A_{\psi\psi\psi 1}$		$A_{1111} + A_{\psi\psi\psi 1} - A_{1\psi 1\psi} - A_{\psi 1\psi\psi}$
$A_{1\psi 1\psi}$	$\longrightarrow$	$A_{1111} - A_{\psi\psi\psi 1} + A_{1\psi 1\psi} - A_{\psi 1\psi\psi}$
$A_{\psi 1\psi\psi}$		$A_{1111} - A_{\psi\psi\psi 1} - A_{1\psi 1\psi} + A_{\psi 1\psi\psi}$

Looking forward

- Constructing transversal gates in topological codes
- More general boson condensation transitions

- Tensor networks describing SET states have matrix product operator symmetries
- By studying these MPOs we extract a graded fusion algebra
- From this we can construct a dube algebra to extract (all) information about the SET order in the system

Questions?

"Dad whys my sisters name rose"

"Because ur mother loves roses"

"Thanks dad!"

"No problem, Ocneanu's tube algebra "

