Title: Hopf algebras and parafermionic lattice models

Date: Aug 04, 2017 09:30 AM

URL: http://pirsa.org/17080012

Abstract: Ground state degeneracy is an important characteristic of topological order. It is a natural question under what conditions such topological degeneracy extends to higher energy states or even to the full energy spectrum of a model, in such a way that the degeneracy is preserved when the Hamiltonian of the system is perturbed. It appears that Ising/Majorana wires have this property due to the presence of robust edge zero modes. Generalized wire models based on parafermions also have edge zero modes and topological degeneracy at special points in parameter space, but the stability of these modes is a much more intricate question. These models are related to Hopf algebras or tensor categories in several ways. In particular they are "golden

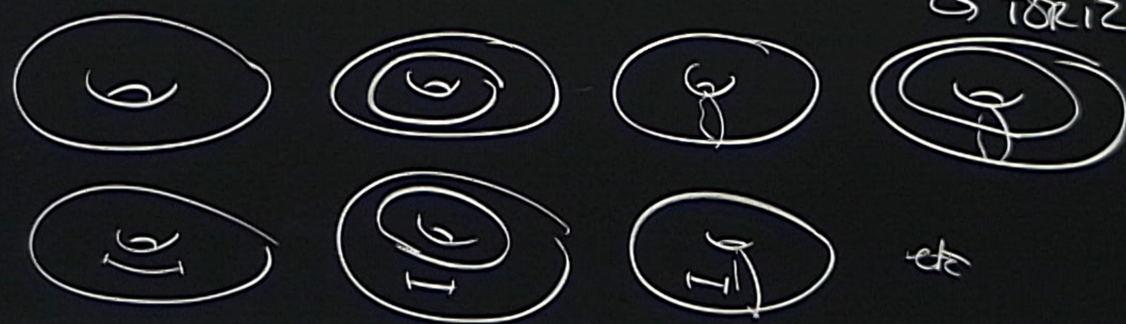
chain" type models based on fusion categories for boundary defects of Abelian TQFTs. As such they are part of a much larger class of Hopf algebra based chain models with edge modes. It is natural to ask which of these have stable edge zero modes and/or full spectrum degeneracy.

HOPF ALGEBRAS + PARAFERMIONS

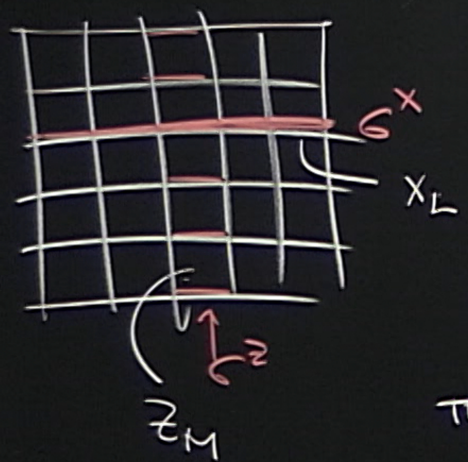
↳ REALLY TOPOLOGICAL ZERO MODES FOR FULL SPECTRUM (DEGENERACY)

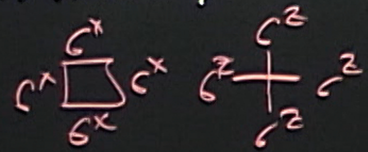
THIS TALK. FIND (1D) MODELS
WHERE TOP. DEGEN EXTENDS TO ALL ENERGY LEVELS, STABLY

F. S. DEGEN FAMILIAR, EG IN STRING NET MODELS
↳ TORIC CODE



GET DEGEN FROM SYMMETRY





HAVE NONCOMMUTING SYMMETRIES

$$X_L Z_M = -Z_M X_L \qquad [X_L, H] = [Z_M, H] = 0$$

THESE OPERATORS GUARANTEE DEGEN FOR ALL STATES

WHAT IF H_{TC} IS PERTURBED?

$$H = H_{TC} - h \sum_{\ell} \sigma^z_\ell$$ ← STRING TENSION (LONG STRINGS COST)

TOPOLOGICAL PROTECTION IS OK FOR STATES WITH FEW EXC. BUT WHAT ABOUT HIGH E? → GENERALLY SHOULD BREAK DEGEN

NOTE HAVE 1D MODELS WITH PROTECTED DEGEN/ZERO MODES

KITAEV WIRE:

$$H = -\mu \sum_{i=1}^{L} \left(c_i^\dagger c_i - \tfrac{1}{2}\right) + \sum_{i=1}^{L} -t\, c_i^\dagger c_{i+1} + \Delta e^{i\theta} c_i^\dagger c_{i+1}^\dagger + h.c.$$

1 2 · · · · · · L

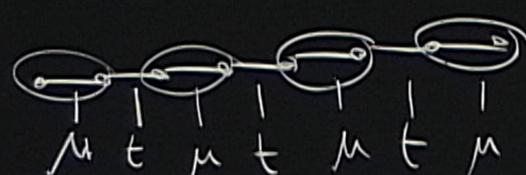
$$\gamma_{2j-1} = \tfrac{1}{2}\left(e^{-i\theta/2} c_j + e^{i\theta/2} c_j^\dagger\right)$$

$$\gamma_{2j} = \tfrac{1}{2}\left(-i e^{-i\theta/2} c_j + i e^{i\theta/2} c_j^\dagger\right)$$

KITAEV: REWRITE USING MAJORANAS

$\gamma_j^\dagger = \gamma_j$ $\quad \{\gamma_i, \gamma_j\} = \delta_{ij}$

$$H = -\mu \sum_{i=1}^{L} \gamma_{2i-1}\gamma_{2i} - t \sum_{i=1}^{L-1} \gamma_{2i}\gamma_{2i+1} \quad (t = \Delta)$$



IF $\mu \to 0$ THEN γ_1, γ_{2L} NO LONGER APPEAR IN H

BUT $\{\gamma_1, \gamma_{2L}\} = 0$, $[\gamma_1, H] = [\gamma_{2L}, H] = 0$

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1 2 L

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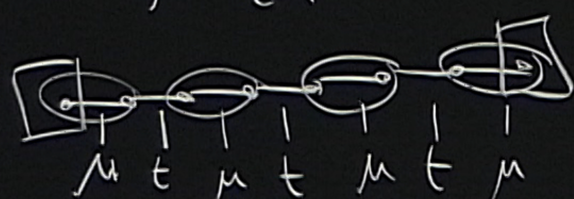


IF $\mu \to 0$ THEN γ_1, γ_{2L} NO LONGER APPEAR IN H

BUT $\{\gamma_1, \gamma_{2L}\} = 0$, $[\gamma_1, H] = [\gamma_{2L}, H) = 0$

$\to$ 2-FOLD DEGEN.

HOPF ALGEBRAS + PARAFERMIONS

NOTE THIS IS ALSO THE TRANSVERSE ISING MODEL

JORDAN WIGNER : $H = -t \sum \sigma_i^z \sigma_{i+1}^z - \mu \sum \sigma_i^x$

$\hookrightarrow J \qquad \hookrightarrow f$

SYMMETRY

$(-1)^F \equiv \prod_{i=1}^{L} \sigma_i^x$

ITERATIVELY CONSTRUCT ZERO MODES ALSO FOR $\mu \neq 0$ $(\mu < t)$

WRITE $\quad \Psi_\ell = \gamma_1 + \left(\frac{\mu}{t}\right) \Psi_\ell^{(1)} + \left(\frac{\mu}{t}\right)^2 \Psi_\ell^{(2)} + \text{etc}$

$\hookrightarrow$ SUPPORTED FURTHER INTO CHAIN.

WANT $\quad \left[\sum_{\mu=1}^{p} \Psi_\ell^{(j)}, H \right] = O\left(\left(\frac{\mu}{t}\right)^{p+1}\right)$

FIND $\quad [\gamma_1, H] = [\gamma_1, H_t] + [\gamma_1, H_\mu] = [\gamma_1, -\mu \gamma_1 \gamma_2] = -2\mu \gamma_2$

$[\gamma_1, H_t] = 0$

$$\left[\gamma_1 + \left(\tfrac{\mu}{t}\right)\gamma_\ell^{(1)}, H\right] = \underbrace{\left[\gamma_1, H\right]}_{-2\mu\gamma_2} + \frac{\mu}{t}\left[\gamma_\ell^{(1)}, H_t\right] + \frac{\mu}{t}\left[\gamma_\ell^{(1)}, H_\mu\right] \checkmark$$

WANT $\frac{\mu}{t}\left[\gamma_\ell^{(1)}, H_t\right] = +2\mu\gamma_2$ $\Rightarrow$ TAKE $\gamma_\ell^{(1)} = \gamma_3$ WORKS!

$H_t \downarrow -t\gamma_2\gamma_3$

KEEP GOING: $\gamma_\ell = \gamma_1 + \left(\frac{\mu}{t}\right)\gamma_3 + \left(\frac{\mu}{t}\right)^2\gamma_5 + \left(\frac{\mu}{t}\right)^3\gamma_7$ etc...

SIMILAR ON RIGHT

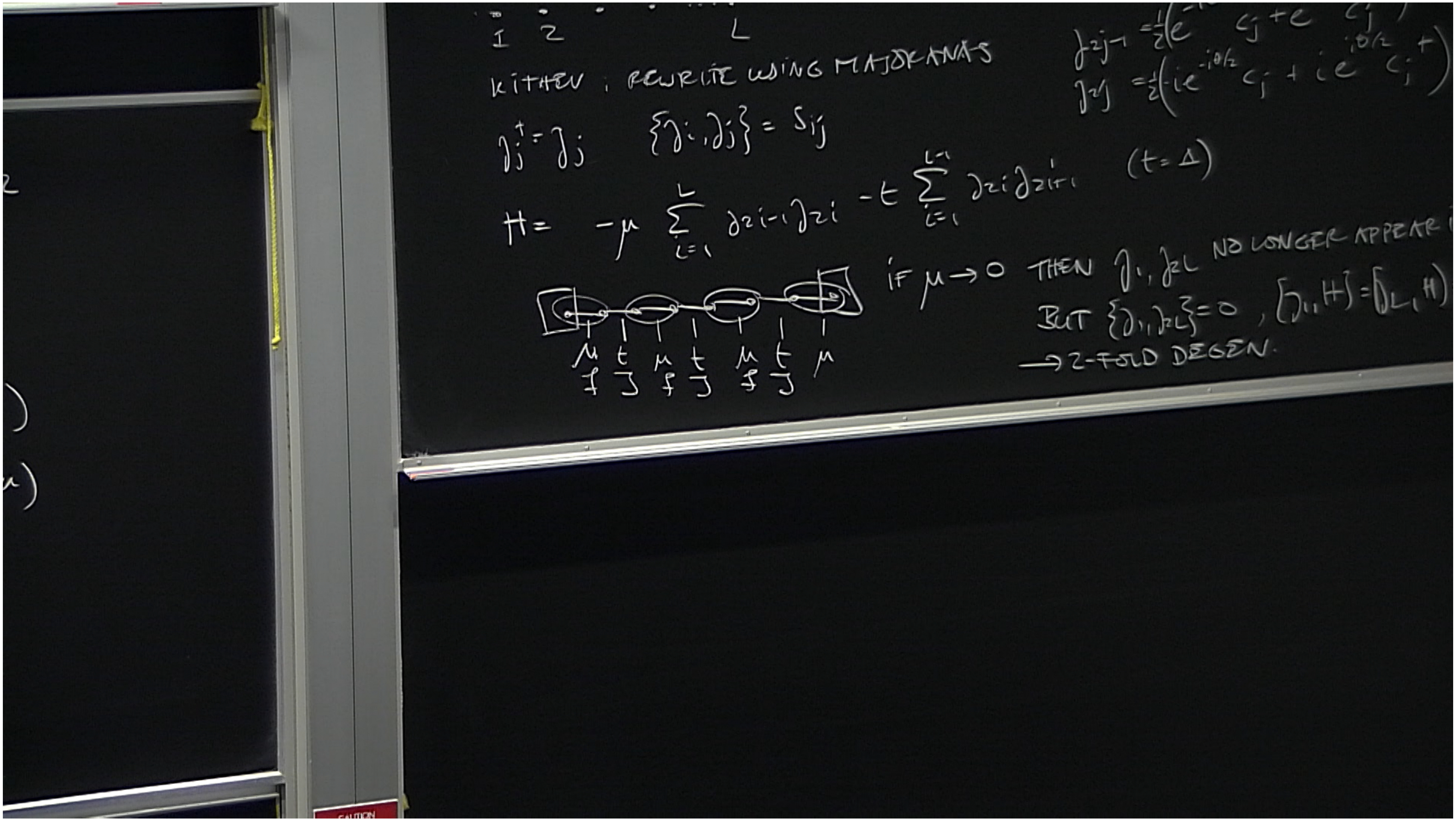
$\rightarrow \{\gamma_\ell, \gamma_R\} = O\left(\left(\frac{\mu}{t}\right)^L\right)$

HAVE DEGEN. ALL STATES UP TO $O\left(\left(\frac{\mu}{t}\right)^L\right)$

ARB. PERT λV to H_t

SOLVE

$$\left[H_t, \gamma^{(n+1)}\right] = -\left[V, \gamma^{(n)}\right] + O(\lambda^{n+2})$$


KITAEV : REWRITE USING MAJORANAS
(t = Δ)
IF μ → 0 THEN
NO LONGER APPEAR
BUT
→ 2-FOLD DEGEN.

GENERALIZE TO OTHER MODELS

PARAFERMIONS

FENDLEY 1209.0472

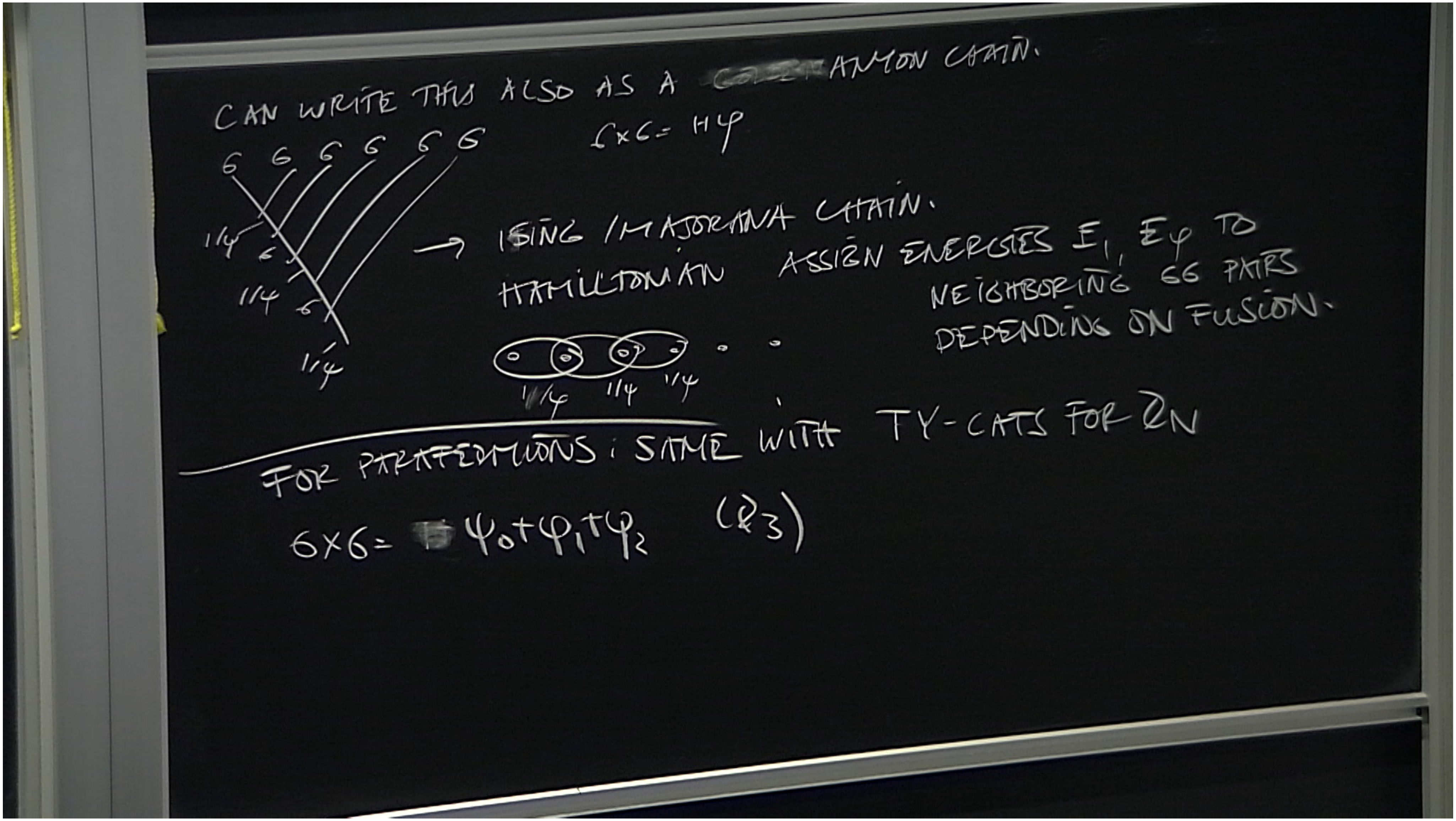
$$\gamma_i \gamma_j = \omega \gamma_j \gamma_i \quad (i>j) \qquad \omega = e^{2\pi i/N}$$

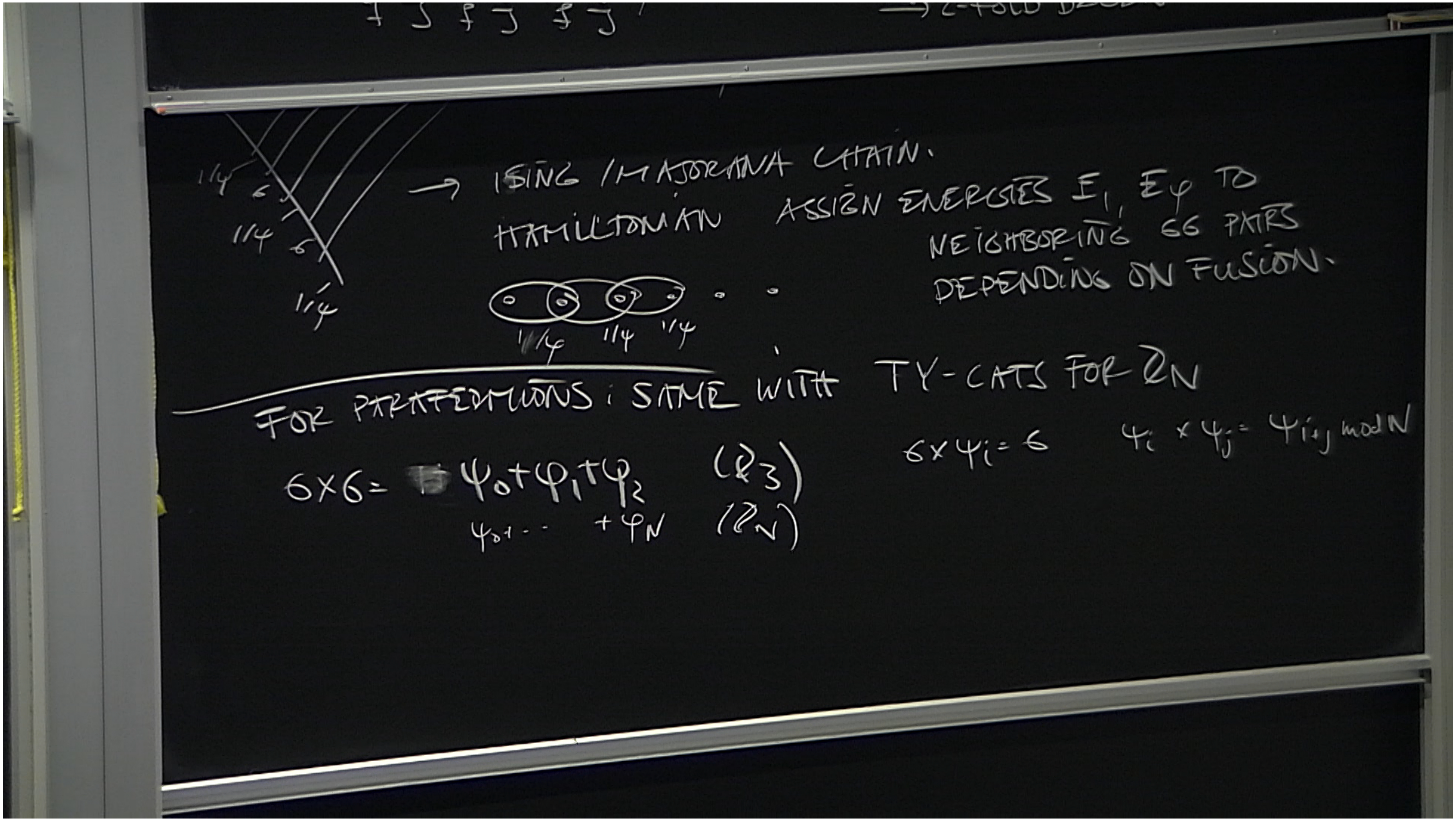
. . . . . .

$$H = -J\,\omega^{(N-1)/2} e^{i\theta} \sum \gamma^\dagger_{2x}\gamma_{2x+1} + h.c. \qquad (J \text{ or } t)$$

$$\quad - f\,\omega^{(N-1)/2} e^{i\varphi} \sum \gamma^\dagger_{2x+1}\gamma_{2x} + h.c. \qquad (f \text{ or } \mu)$$

NAIVELY: ZERO MODES γ_1, γ_L → N-FOLD DEGEN IF $f=0$


CAN WRITE THIS ALSO AS A ANYON CHAIN.
σ×σ = 1+ψ
σ σ σ σ σ σ
1/ψ σ
1/ψ σ
1/ψ
→ ISING / MAJORANA CHAIN.
HAMILTONIAN ASSIGN ENERGIES E1, Eψ TO NEIGHBORING σσ PAIRS DEPENDING ON FUSION.
1/ψ 1/ψ 1/ψ
FOR PARAFERMIONS: SAME WITH TY-CATS FOR ZN
σ×σ = ψ0+ψ1+ψ2 (Z3)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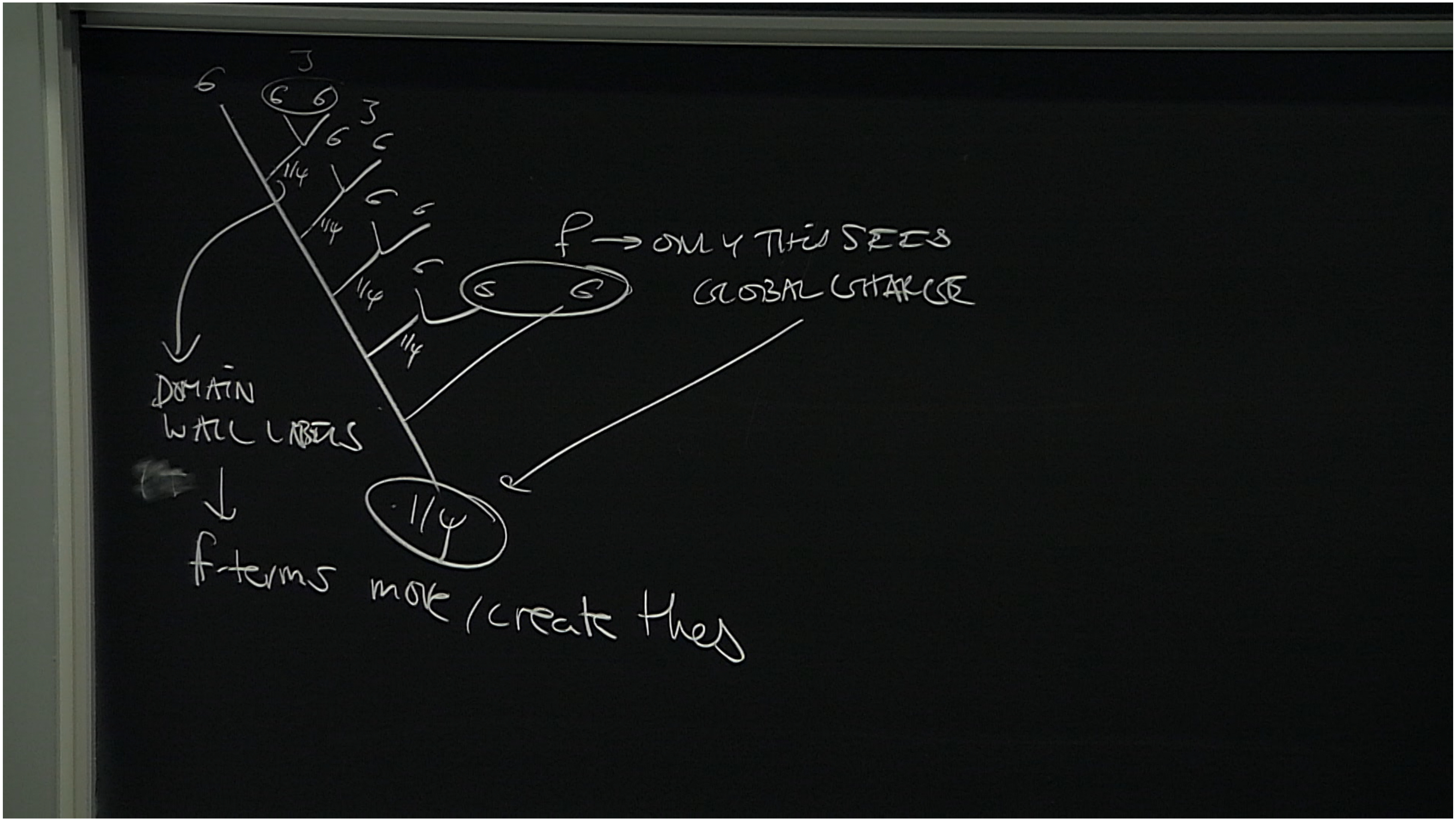
1/ψ
1/ψ
1/ψ
FOR PARAFERMIONS: SAME WITH TY-CATS FOR ZN
σ×σ = ψ0+ψ1+ψ2 (Z3)
ψ0+ ... +ψN (ZN)
σ×ψi = σ
ψi × ψj = ψi+j mod N


6
6 6
1/4
f → ONLY THIS SEES
GLOBAL CHARGE
DOMAIN
WALL LABELS
f-terms move/create them

Hopf Algebras and Parafermions

Some results from Phys. Rev. B 95, 235127 (2017), arxiv:1701.05270

Parafermionic clock models and quantum resonance

Niall Moran, Domenico Pellegrino, Joost Slingerland, Graham Kells
Maynooth University / Dublin Institute for Advance Studies

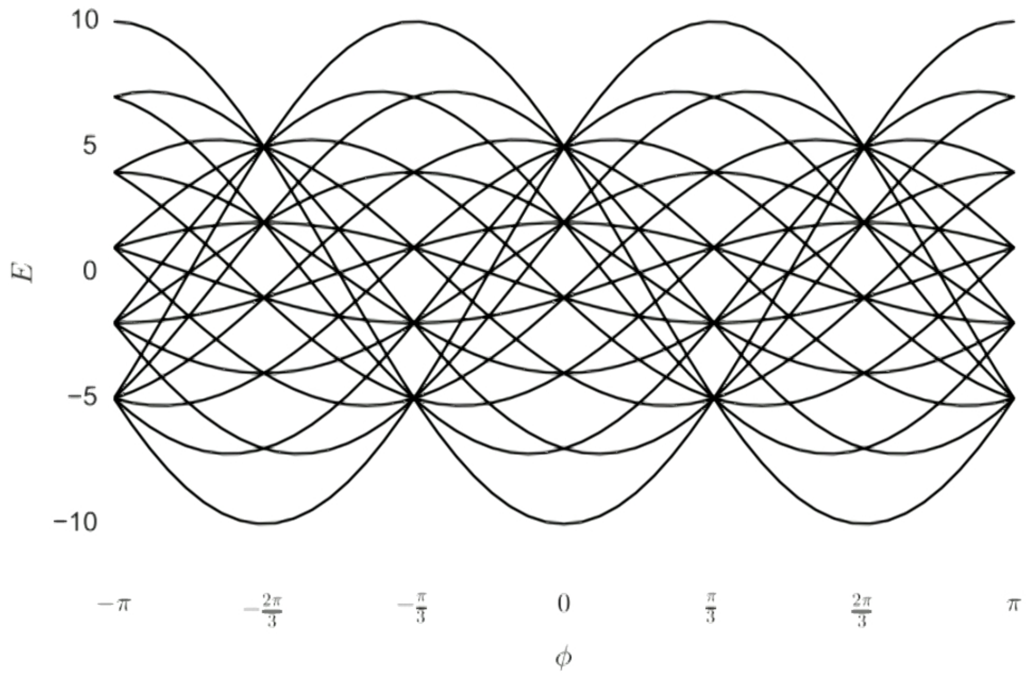
Hopf Algebras in Kitaev's Quantum Double Models....

Perimeter Institute, August 4th, 2017

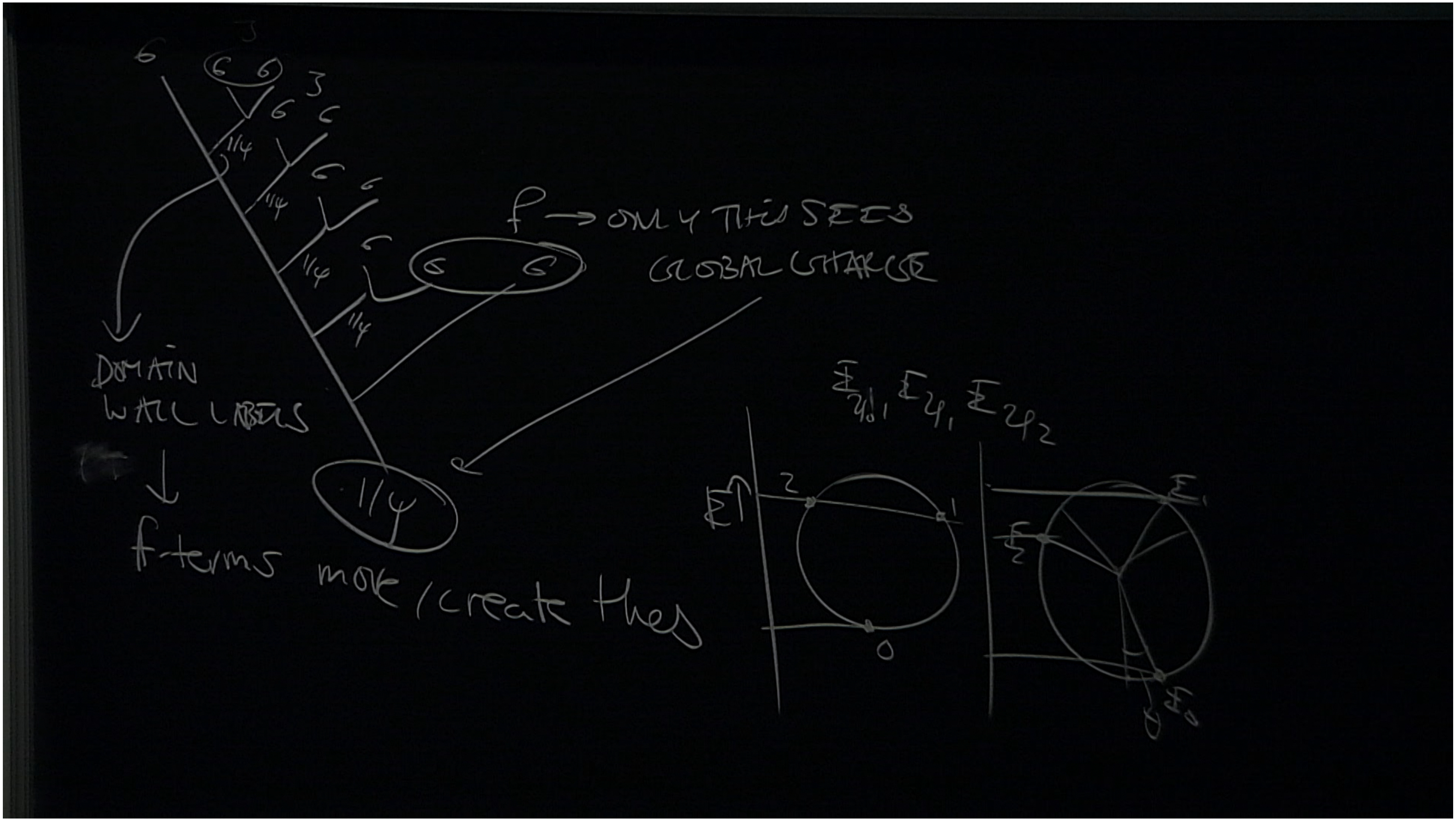
PERIMETER INSTITUTE

Maynooth University National University of Ireland Maynooth

$N = 3$

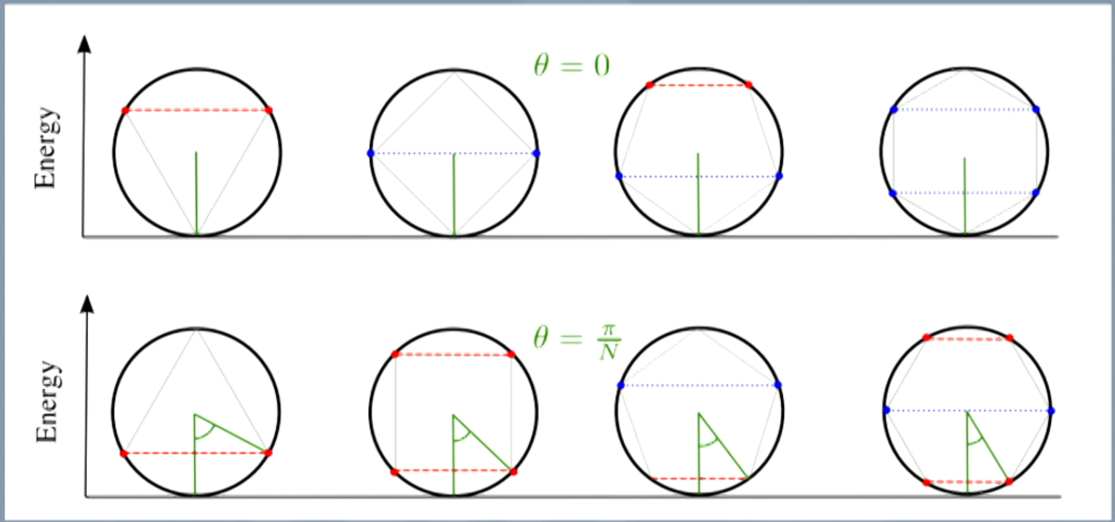


Full spectrum of 6 site chain with $f = 0$



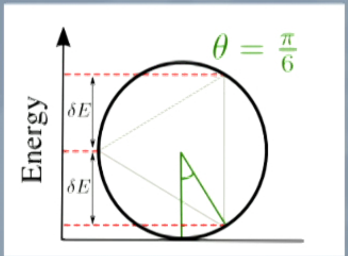
f → ONLY THIS SEES GLOBAL CHARGE
DOMAIN WALL LABELS
f-terms move/create them

Simple resonances



Coincidences between the energies of single domain walls cause simple resonances

Note: Order 1 resonances for odd N, θ=0 (similarly for θ=π/N)
Order 2 resonances for even N, θ=0 (order 1 for θ=π/N)



Slightly less simple. N=3 at θ=π/6
Need to change multiple walls for resonance.

With resonances, have to consider virtual processes between all states in the degenerate bands

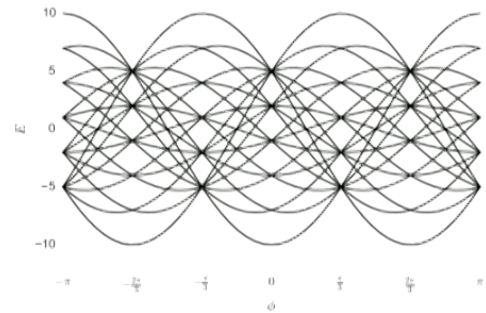
Perturbation theory

At $\phi = 0$:

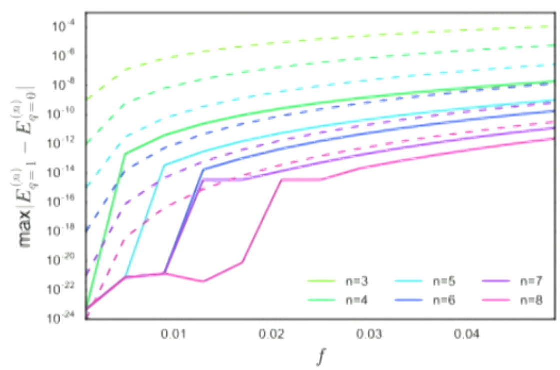
- First order processes
- For large L, splitting $\Delta E \propto \frac{f}{L}$

At $\phi \neq 0$ (off-resonant):

- No first or second order processes
- Numerically no processes to 8th order

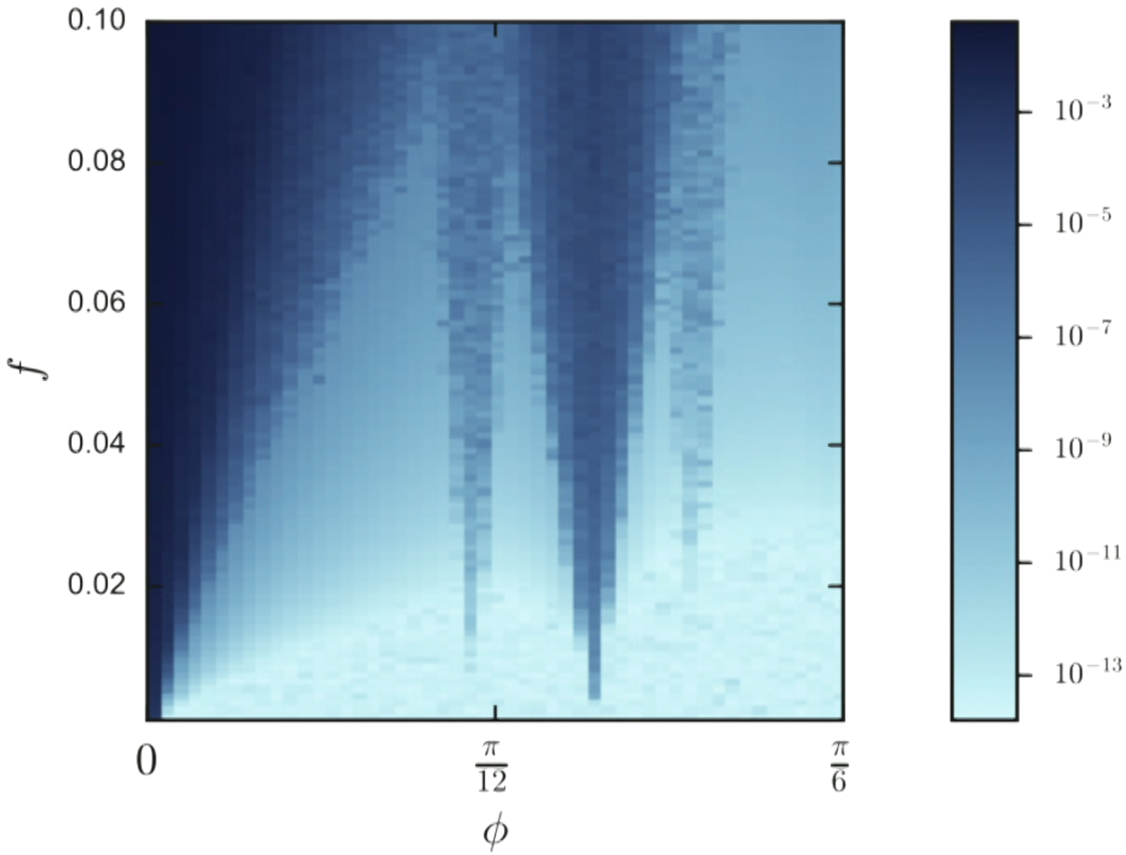


Bands for 6 site system



All orders are consistent
With error in PT expansion

$N = 3$ full spectrum splitting

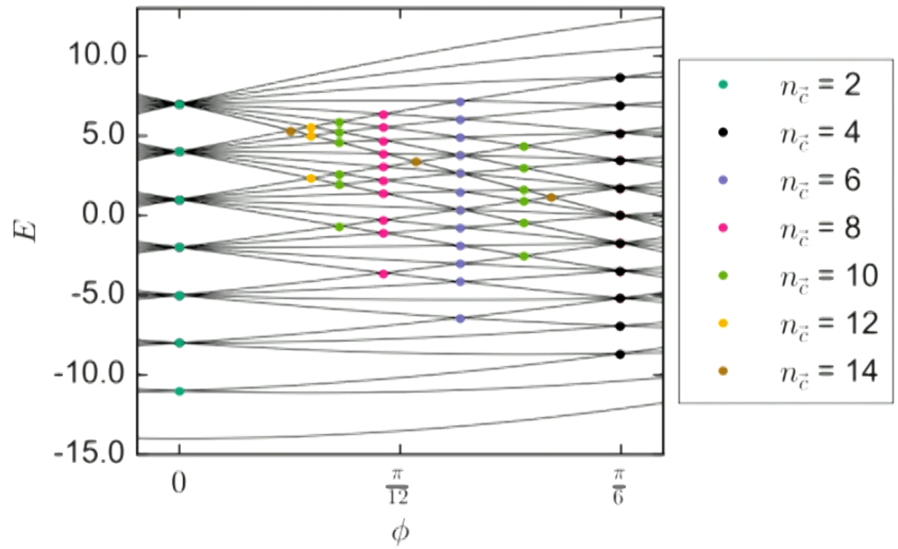


Maximum energy splitting for 9 site chain

Resonance points

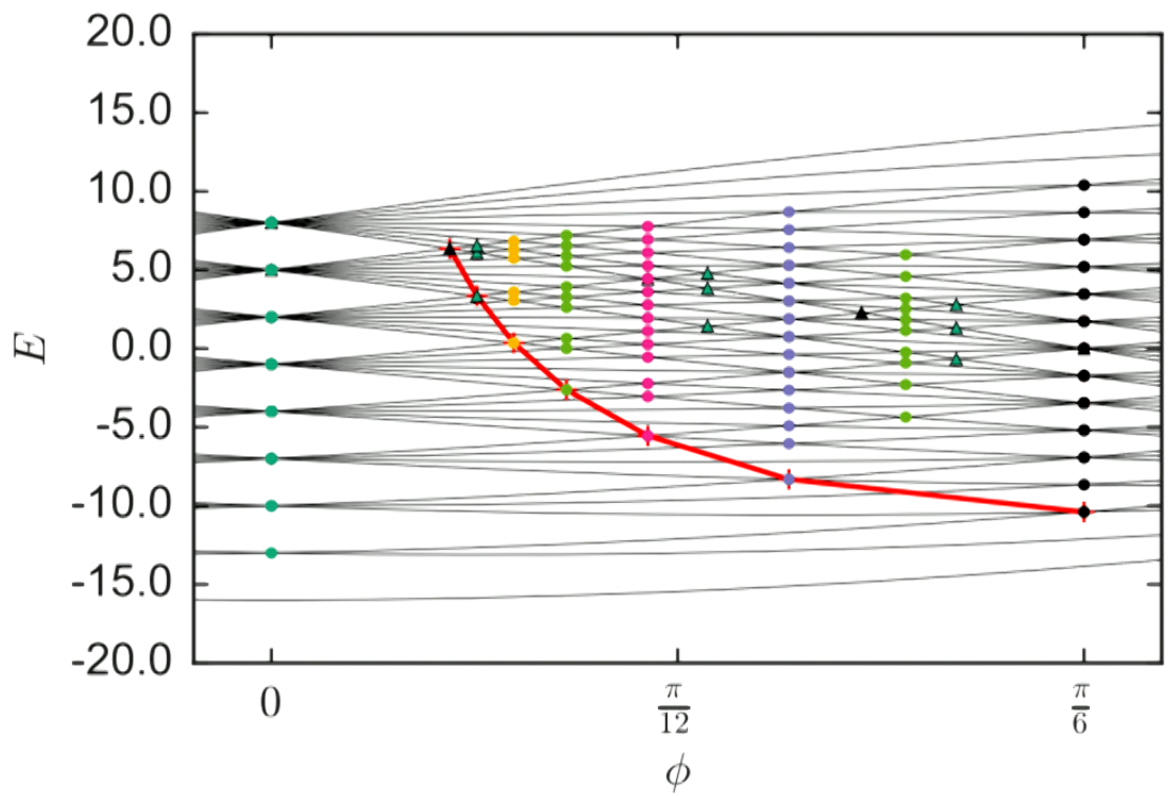
The a and b bands cross at two values of ϕ given by:

$$\tan(\phi) = \frac{\sum_j^{N-1} \cos(\frac{2\pi j}{N})(a_j - b_j)}{\sum_j^{N-1} \sin(\frac{2\pi j}{N})(a_j - b_j)}$$



Resonance points for 8 site chain.

Resonance points



Resonance points of 9 site chain for $f = 0$

Natural Questions about Resonances

Q: Do all resonances cause Q-dependent energy splitting?

Q: Are resonance dense everywhere on the phi axis for large L ? (worst case scenario)

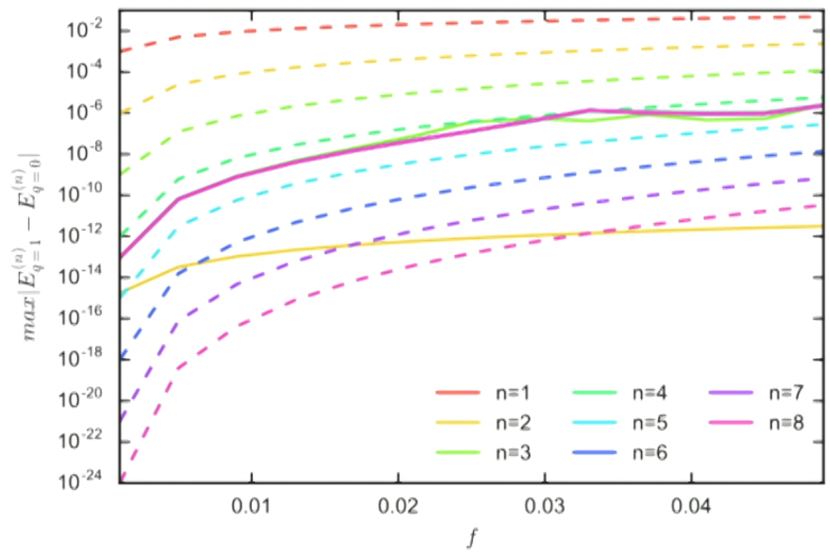
Q: Finite size structure, e.g. Where are the largest gaps?

Q: Energy structure, e.g.
At what energy does the nearest resonance point appear where resonances are dense?

Q: Any dependence on N?

Q: So, are there actually any strong zero modes in this model?

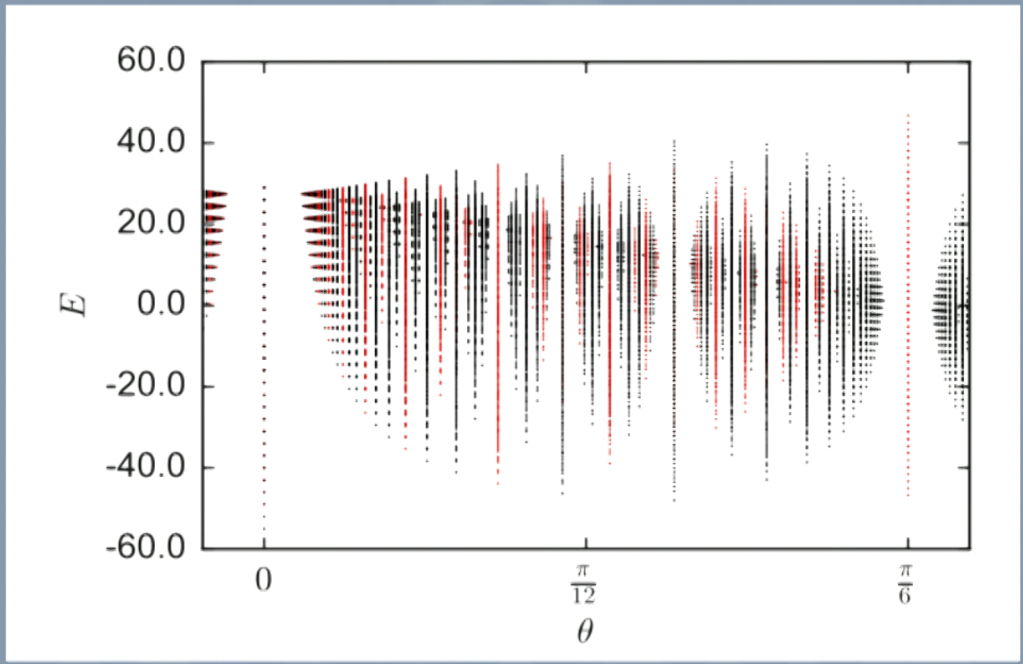
Third order process



Third order splitting at for 8 site chain at $\phi = \tan^{-1}(\frac{\sqrt{3}}{5})$

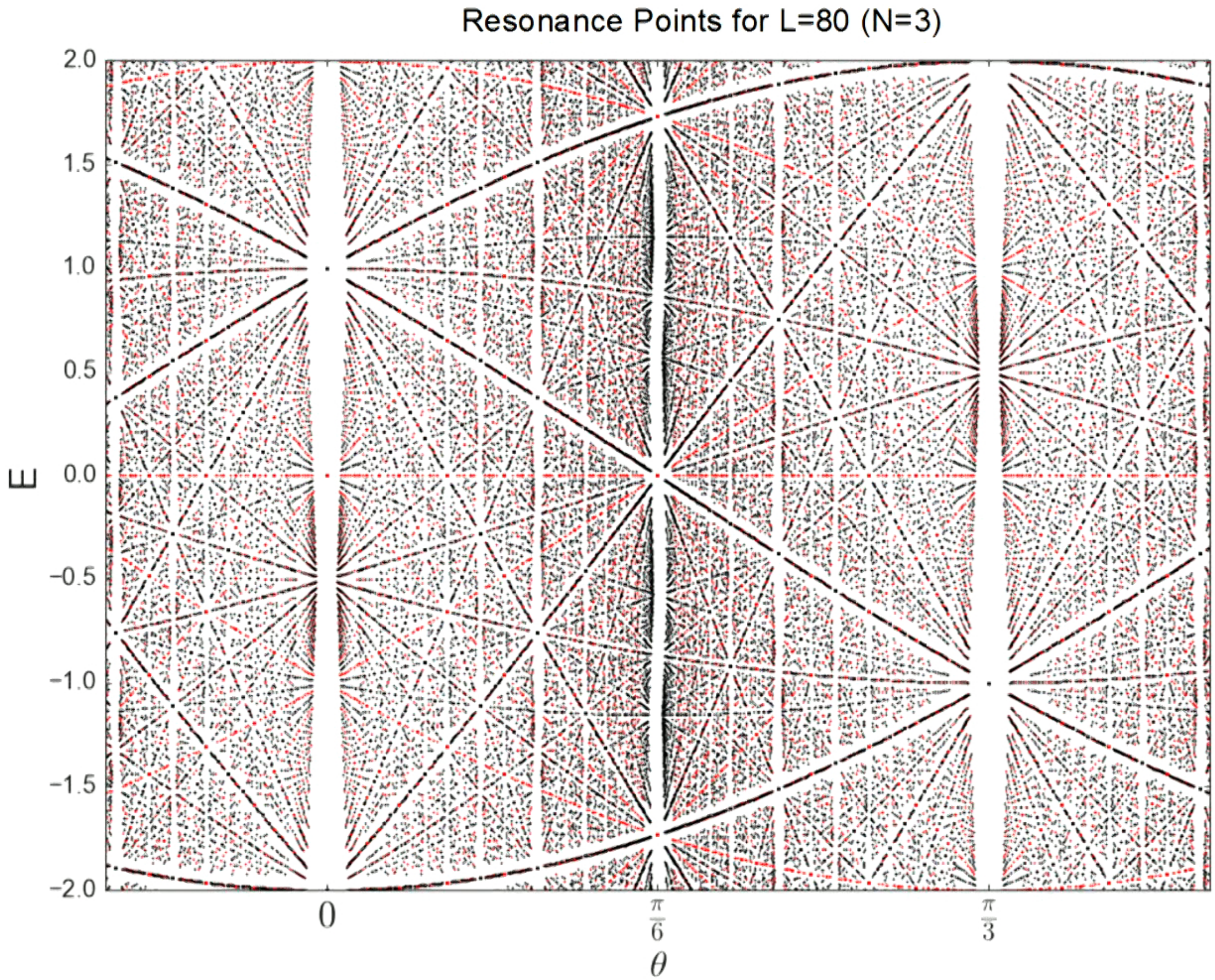
here $\epsilon_0 + 2\epsilon_1 = 3\epsilon_2$

Resonance Points for L=30 (N=3)

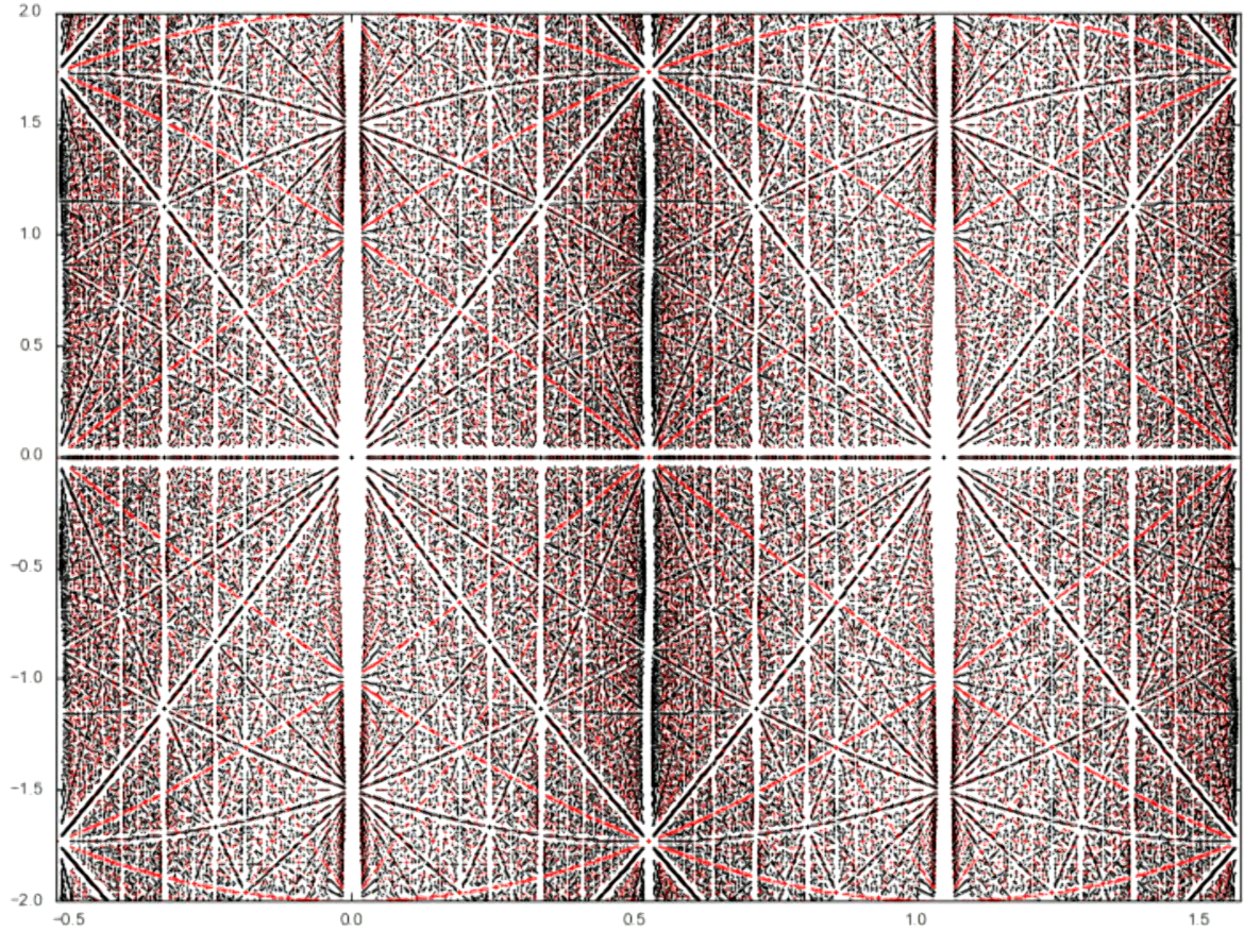


Total domain wall angle of band α: $p_{\vec{a}} = \sum_i i a_i \pmod N$

Red points: $p_a = p_b$


Resonance Points for L=80 (N=3)
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
E
0
π/6
π/3
θ

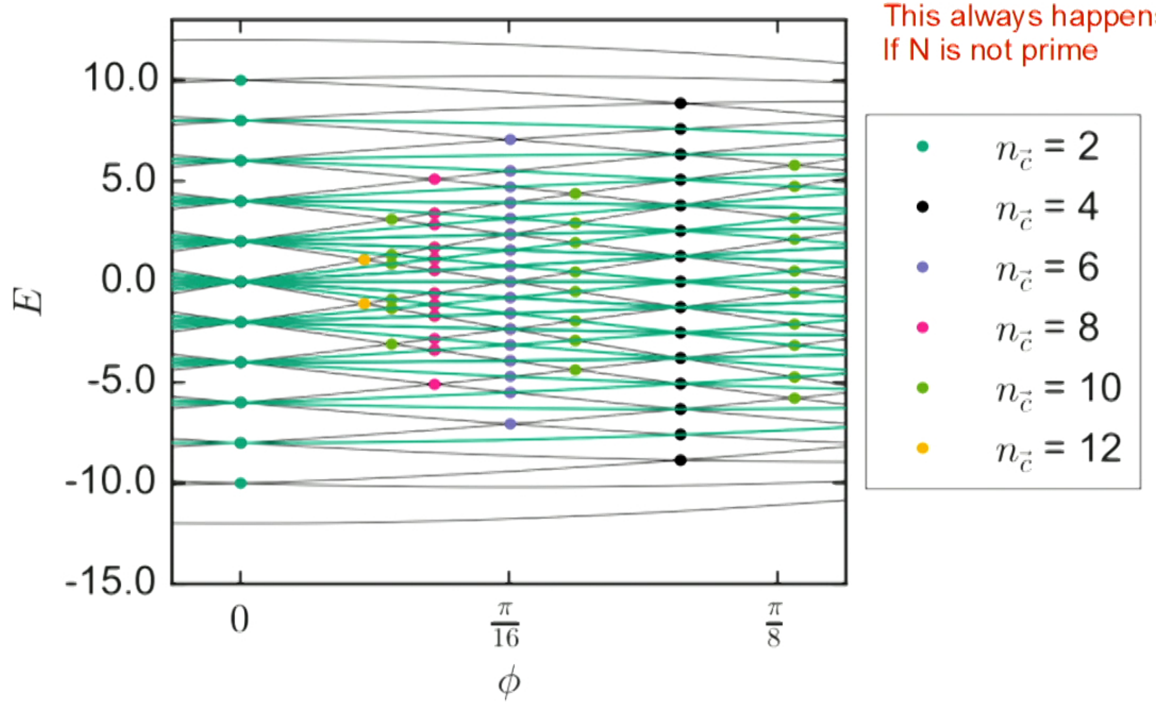
Resonance Points for L=100 (N=3)



$\mathbb{Z}_4$ model

Qualitatively different

Have completely overlapping bands (different sets of domain walls with same energy at all φ)



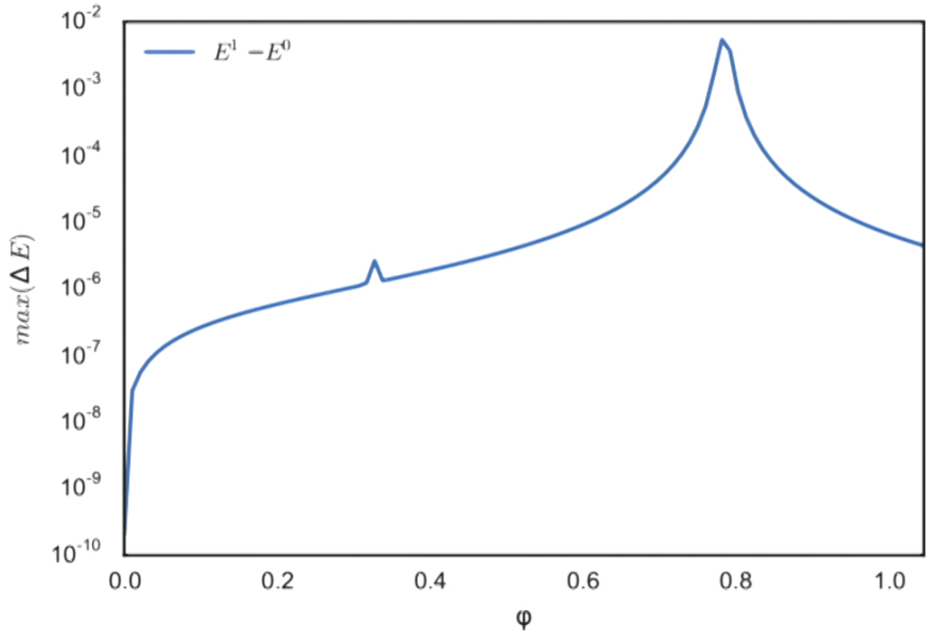
This always happens
If N is not prime

Resonances for 7 site chain

$\mathbb{Z}_4$ model

Qualitatively different

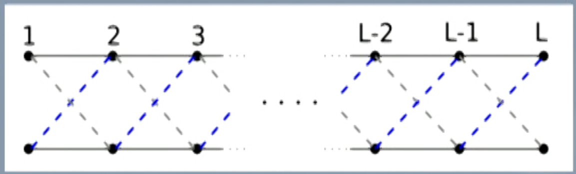
Overlapping bands cause splitting everywhere



Max difference for 5 site chain with $f = 0.01$.

Looking more carefully at N=4

Can rewrite this model in terms of 2 coupled Ising (N=2) chains
These decouple at θ=0, θ=π/2 and the model is solvable there



At solvable points (only) parafermionic zero modes in terms of Majoranas

$$\alpha_L = \frac{1}{\sqrt{2}}(e^{i\frac{\pi}{4}}\alpha_L^u - e^{-i\frac{\pi}{4}}\alpha_L^d)$$
$$\alpha_R = \frac{1}{\sqrt{2}}(\alpha_R^u Q^2 - i\alpha_R^d)T$$

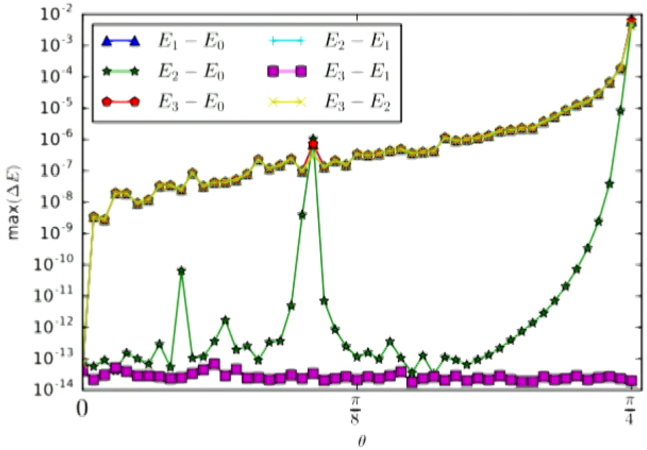


FIG. 13. Maximal energy splitting between q-sectors at $N = 4$, as a function of θ (for $\phi = 0$). The maximum is taken over the entire spectrum (obtained by exact diagonalisation), for an $L = 8$ chain with $f = 0.01$

Elsewhere no parafermionic modes, but:

Bosonic mode retained,
Except at discrete resonance points.

To first order:

$$\Phi = \sigma^z_{1,u}\sigma^z_{1,d} + \frac{f}{J}\frac{\sigma^x_{1,u}}{\cos(2\theta)}\left(\cos(\theta)\sigma^z_{d,1}\sigma^z_{u,2} - \sin(\theta)\sigma^z_{d,1}\sigma^z_{d,2}\right) + \frac{f}{J}\frac{\sigma^x_{1,d}}{\cos(2\theta)}\left(\cos(\theta)\sigma^z_{u,1}\sigma^z_{d,2} + \sin(\theta)\sigma^z_{u,1}\sigma^z_{u,2}\right)$$

Some answers

Q: Do all resonances cause Q-dependent energy splitting?
A: No, it depends on the "total domain wall angle"

Q: Are resonances dense everywhere on the phi axis for large L ? (worst case scenario)
A: Yes

Q: Finite size structure, e.g. Where are the largest gaps?
A: Around resonances! ("rational" points). Gaps there of order 1/L

Q: Energy structure, e.g.
At what energy does the nearest resonance point appear where resonances are dense?
A: Energy of resonance at distance d from another resonance is at least of order 1/d

Q: Any dependence on N?
A: Yes, composite N always has degenerate bands for all θ

Q: So, are there actually any strong zero modes in this model?
A: Bad Question? Maybe for special theta, with slightly relaxed definitions

More questions welcome!