Title: Frobenius algebras, Hopf algebras and 3-categories

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URL: http://pirsa.org/17080011

Abstract: It is well known that commutative Frobenius algebras can be represented as topological surfaces, using the graphical calculus of dualizable objects in monoidal 2-categories. We build on related ideas to show that the interacting Frobenius algebras of Duncan and Dunne, which have a Hopf algebra structure, arise naturally in a similar way, by requiring a single 3-morphism in a 3-category to be invertible. We show that this gives a purely geometrical proof of Mueger's version of Tannakian reconstruction of Hopf algebras from fusion categories equipped with a fibre functor. We also relate our results to the theory of lattice code surgery.

Frobenius algebras, Hopf algebras and 3-categories

David Reutter

University of Oxford

Hopf algebras in Kitaev's quantum double models
Perimeter Institute, Canada

August 3, 2017

The plan

- **Part 1.** Motivation
- **Part 2.** 2-categories
- **Part 3.** 3-categories
- **Part 4.** Hopf algebras
- **Part 5.** Higher linear algebra
- **Part 6.** Lattice models

The plan

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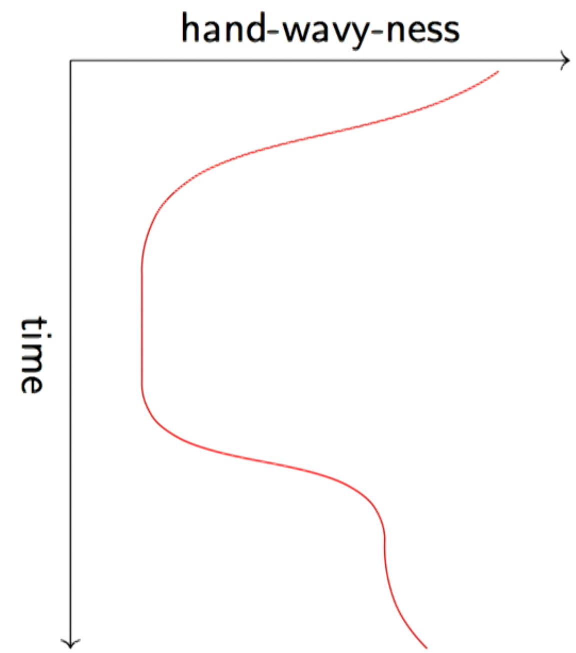
- **Part 2.** 2-categories
- **Part 3.** 3-categories
- **Part 4.** Hopf algebras
- **Part 5.** Higher linear algebra
- **Part 6.** Lattice models



Part 1
Motivation

What is higher algebra?

- Ordinary algebra lets us compose along a line:

$$xy^2zyx^3$$

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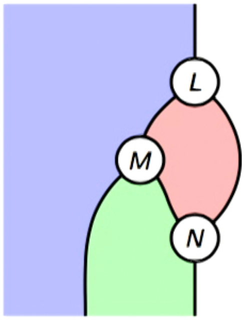
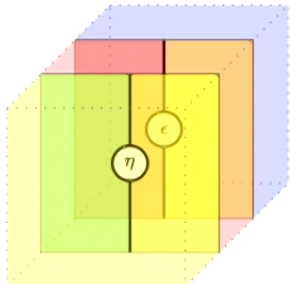
- Ordinary algebra lets us compose along a line:

$$xy^2zyx^3$$

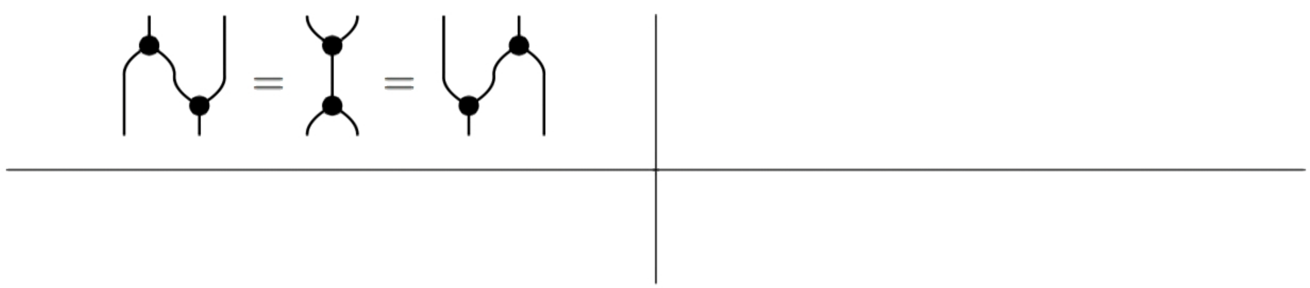
- *Higher algebra* lets us compose in higher dimensions:





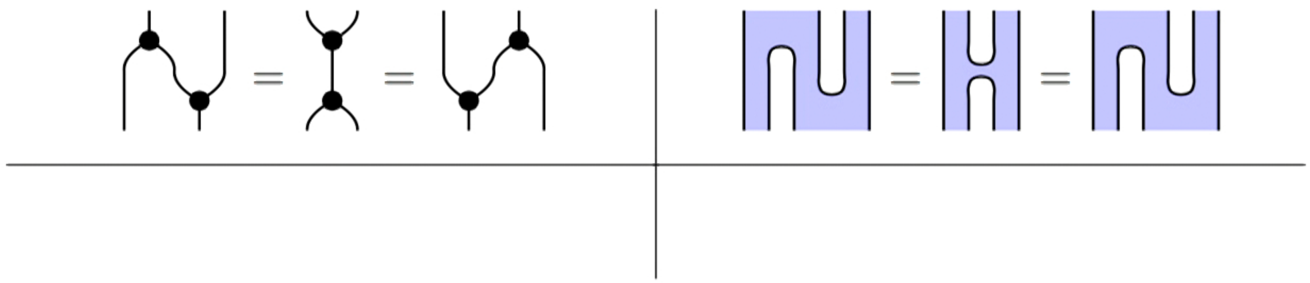
A tradeoff between algebra and topology

Frobenius algebras



A tradeoff between algebra and topology

Frobenius algebras as a 'shadow' of a two dimensional theory.



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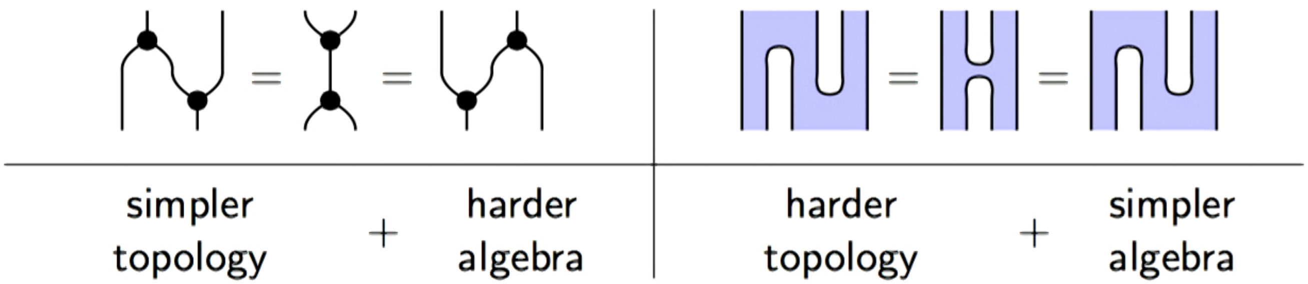

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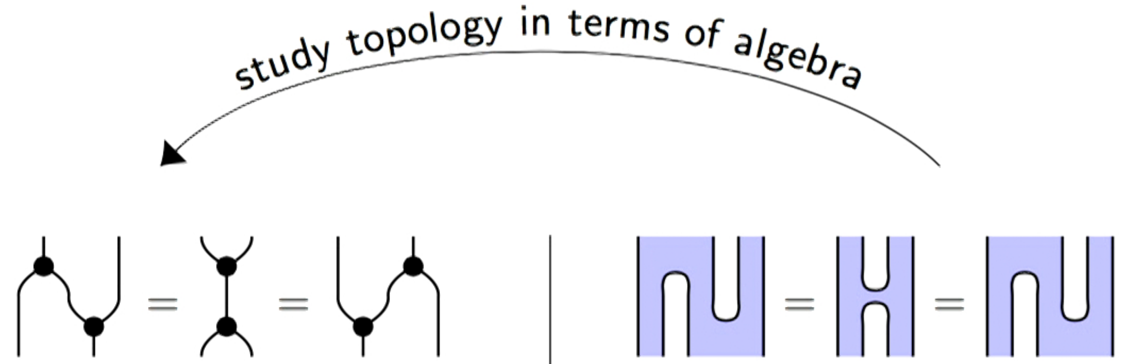

lower dimensional topology + harder algebra	higher dimensional topology + simpler algebra

A tradeoff between algebra and topology

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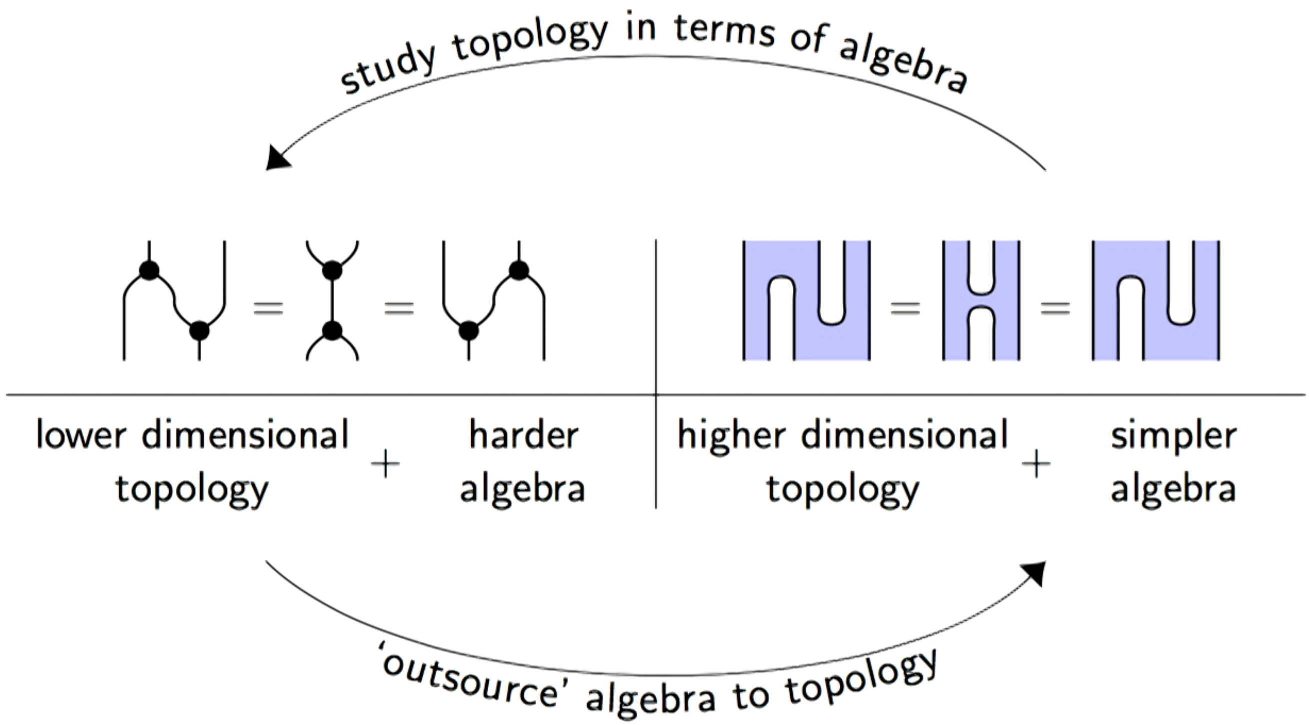

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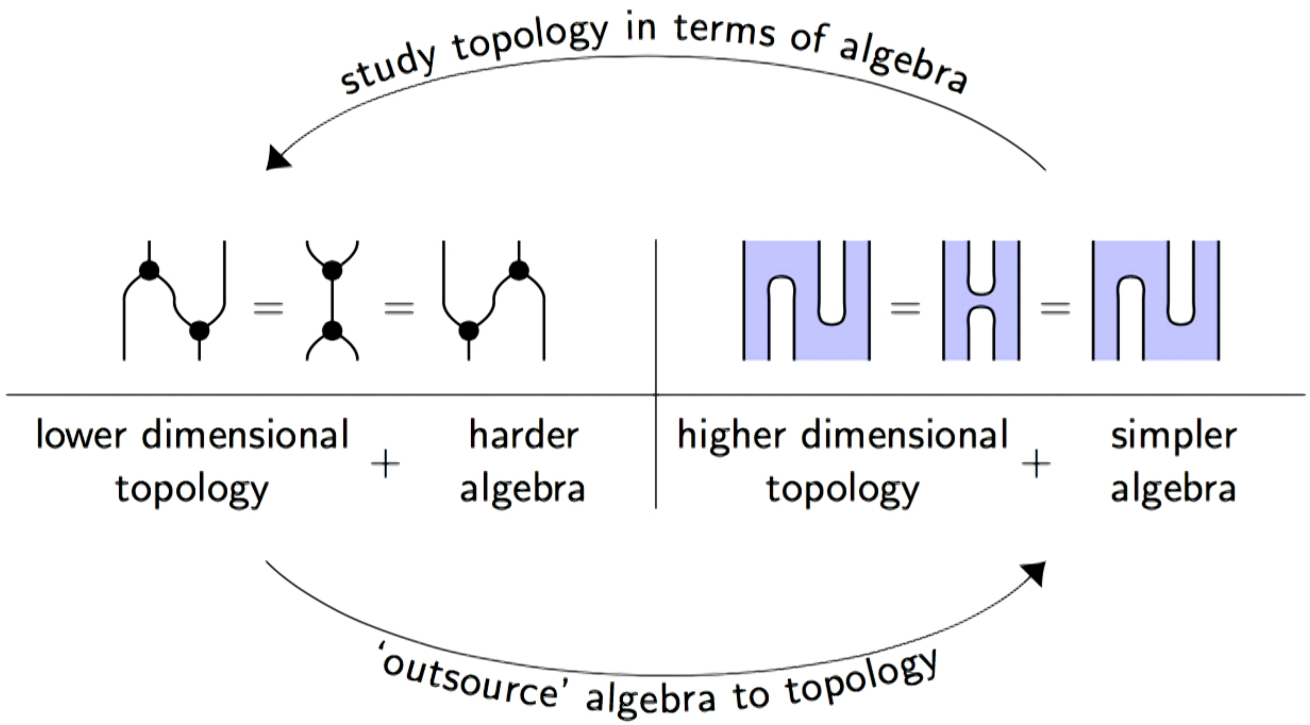

Next hour: Hopf algebras as a 'shadow' of a three dimensional theory.

Part 2
2-categories

Algebra in the plane = 2-category theory

The language describing algebra in the plane is *2-category theory*:

A	$A \xrightarrow{f} B$	$A \underset{f}{\overset{g}{\curvearrowright}} B$, $\Uparrow\eta$
object	1-morphism	2-morphism

Algebra in the plane = 2-category theory

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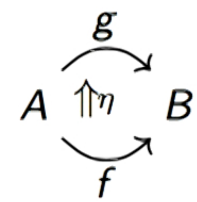
object

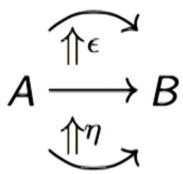
$A \xrightarrow{f} B$

1-morphism

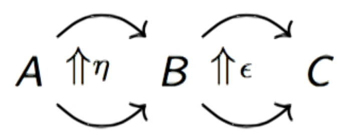


2-morphism

We can compose 2-morphisms like this:



vertical composition



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These are *pasting diagrams*.

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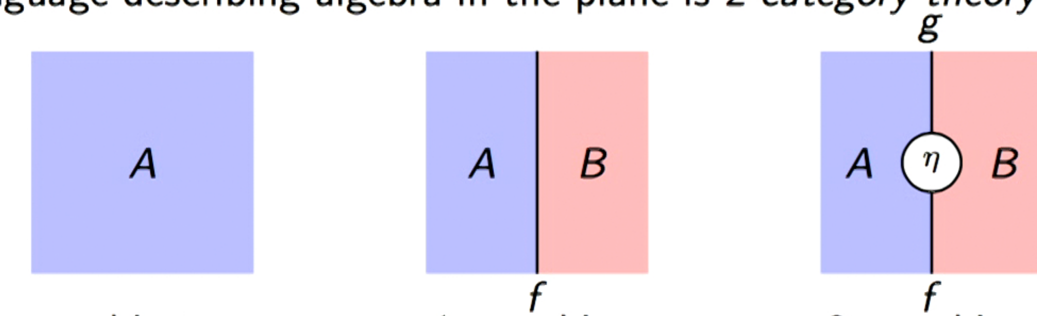

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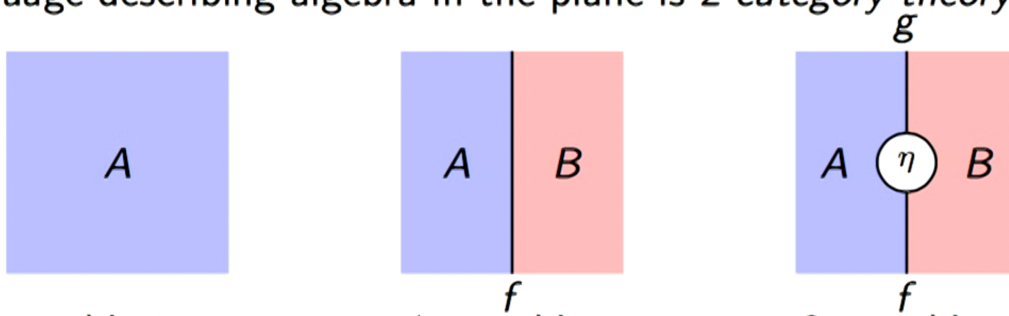

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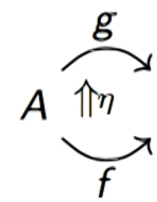

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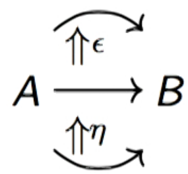

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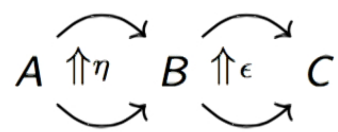

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Seite 18 von 182

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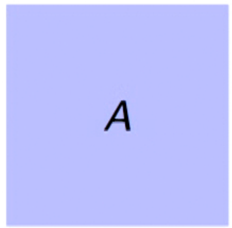
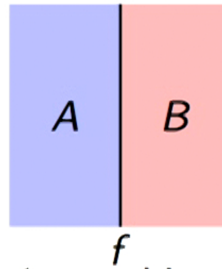
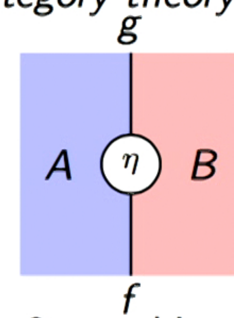


object

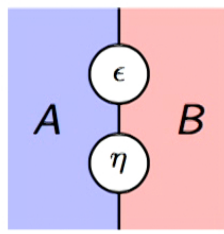


1-morphism



2-morphism

We can compose 2-morphisms like this:



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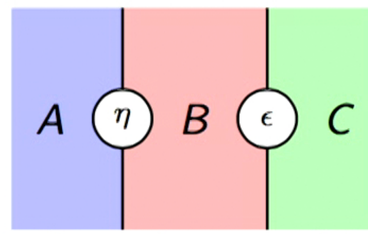

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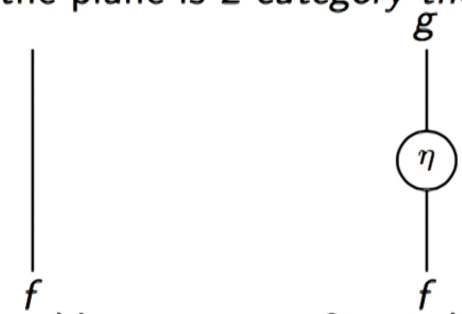

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These are *pasting diagrams*.The *dual* diagrams are the graphical calculus.
A 2-category with one object (the 'empty region') is a *monoidal category*.

Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a *dual* $B \xrightarrow{f^*} A$ if there are 2-morphisms:

Seite 21 von 182

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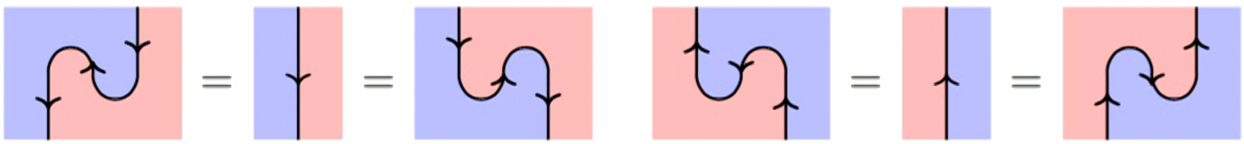

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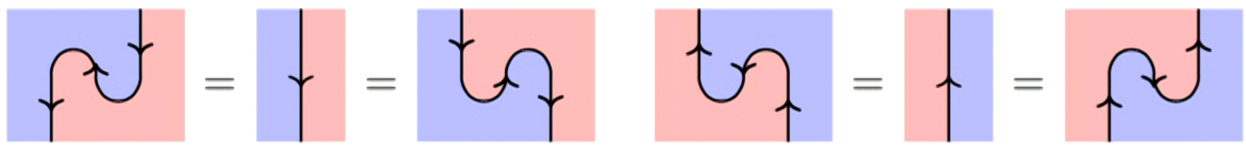

Theorem. Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

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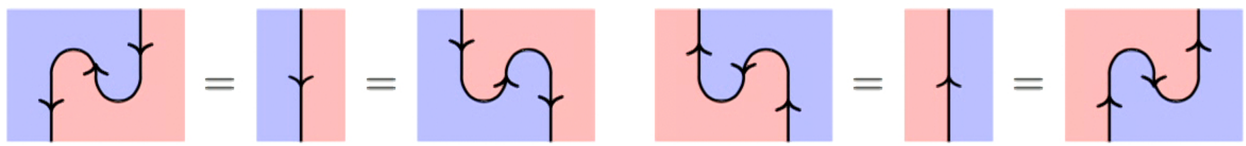

Theorem. Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

Tangle hypothesis.
$\mathbf{Bord}_{1,0}^{2D} \cong$ free monoidal category on a dualizable object

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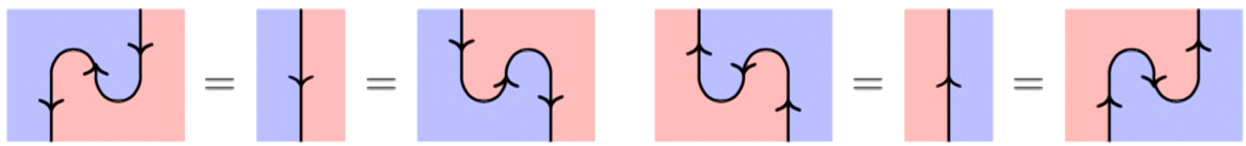

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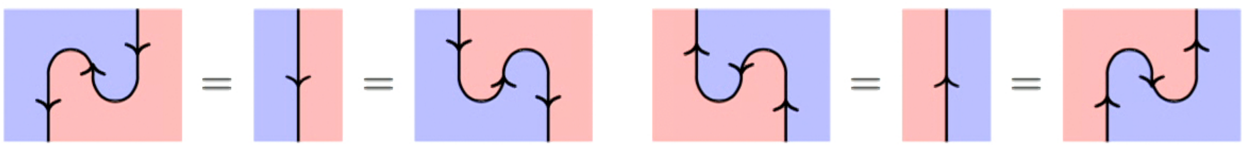

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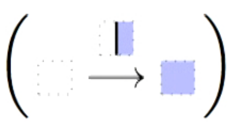
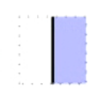
Example. $\mathcal{F}_2$  : free 2-category on dualizable 1-morphism 

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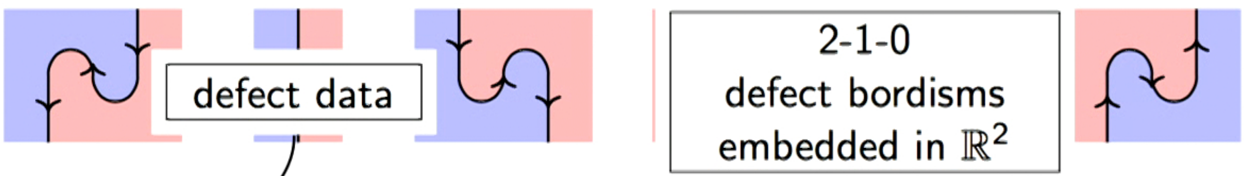

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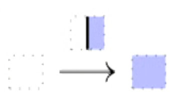
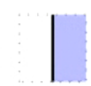
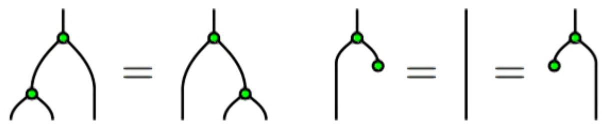
Example. $\mathcal{F}_2$ () : free 2-category on dualizable 1-morphism 

Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:



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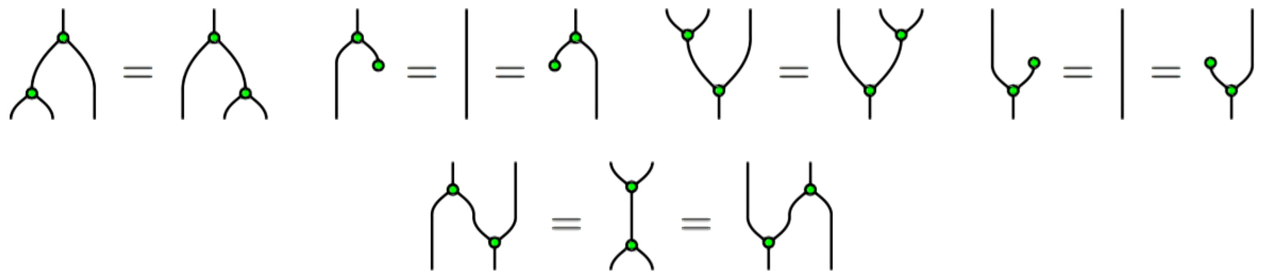

Seite 31 von 182

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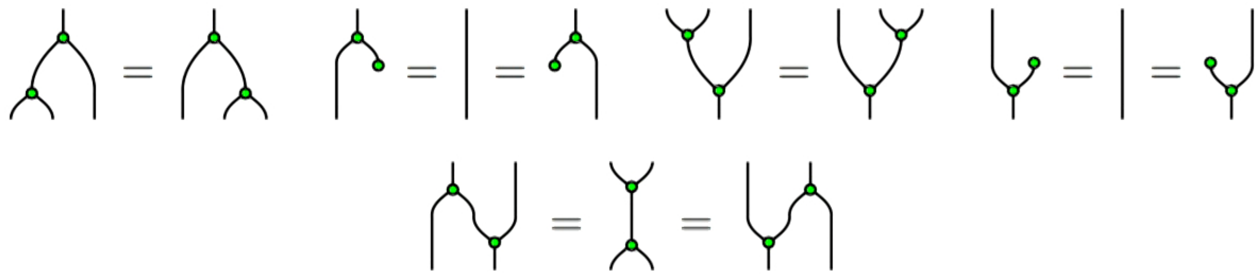

Frob: free monoidal category on a Frobenius algebra

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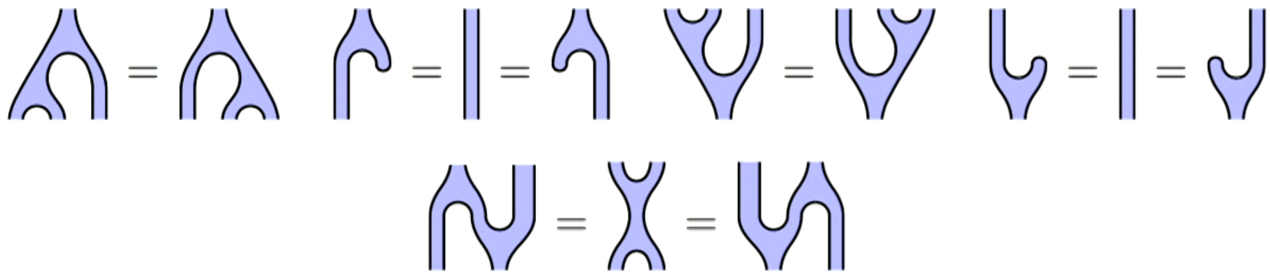

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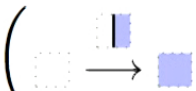
There is a 2-functor **Frob** $\xrightarrow{\text{'thickening'}} \mathrm{F}_2 := \mathcal{F}_\mathbf{2}\Bigg($  $\Bigg)$.

Seite 33 von 182

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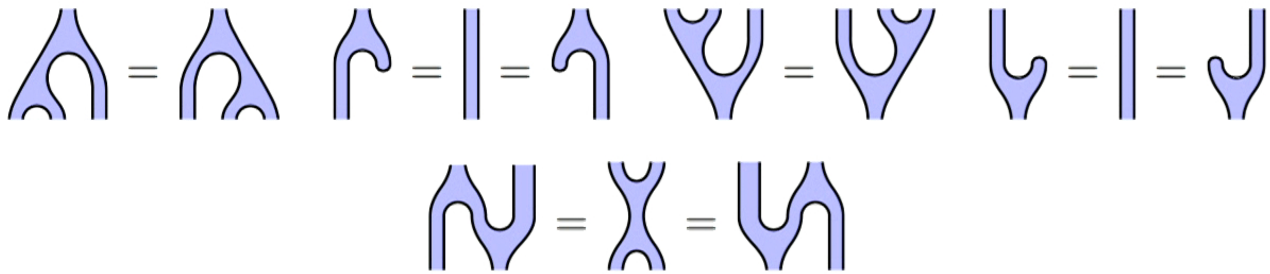

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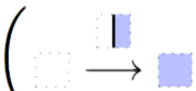
There is a 2-functor **Frob** $\xrightarrow{\text{'thickening'}}$ $\mathrm{F}_2 := \mathcal{F}_{\mathbf{2}}\Big($  $\Big)$.

Seite 33 von 182

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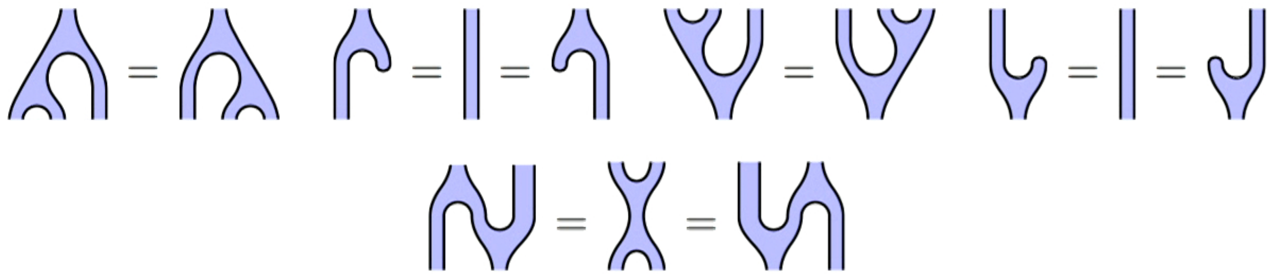

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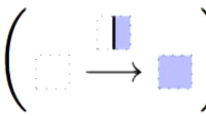
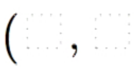
There is a 2-functor **Frob** $\xrightarrow{\text{'thickening'}}$ $\mathrm{F}_2 := \mathcal{F}_2$ .

Theorem. This induces a monoidal equivalence **Frob** $\cong \mathrm{F}_2$ .

Seite 34 von 182

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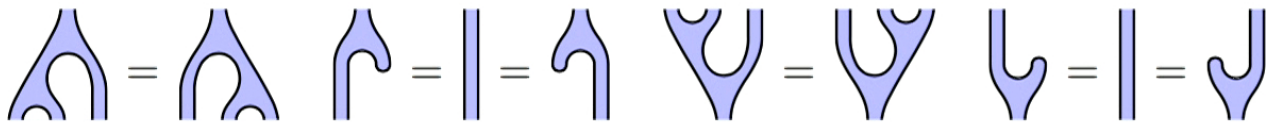
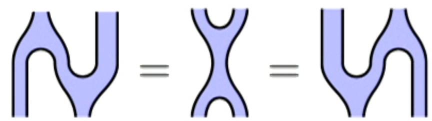
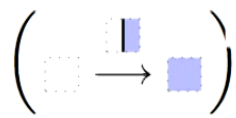




Frob: free monoidal category on a Frobenius a

open strings in the plane

There is a 2-functor **Frob** $\xrightarrow{\text{'thickening'}} \mathrm{F}_2 := \mathcal{F}_{\mathbf{2}}$ .

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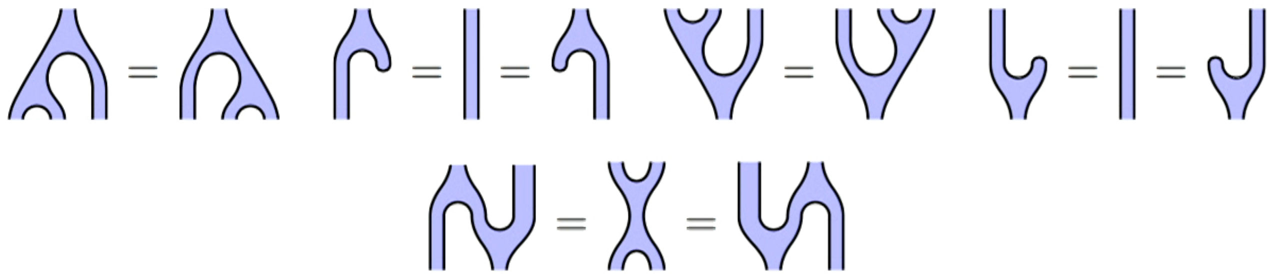

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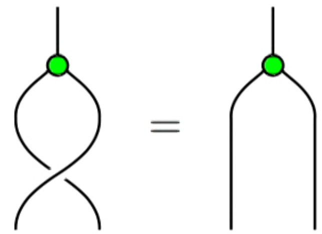
Frob as a 'shadow' of the theory of dualizable 1-morphisms in 2-categories.

Other algebraic theories?

What about *commutative* Frobenius algebras



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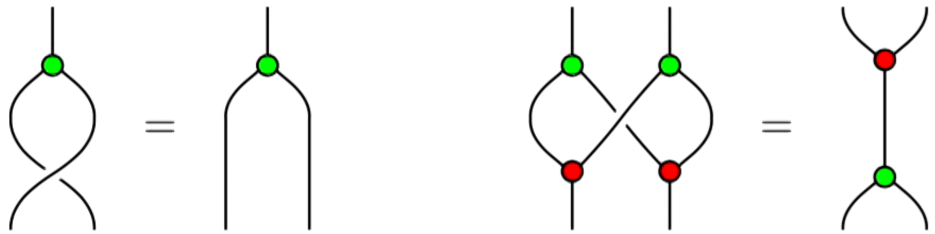

Seite 38 von 182

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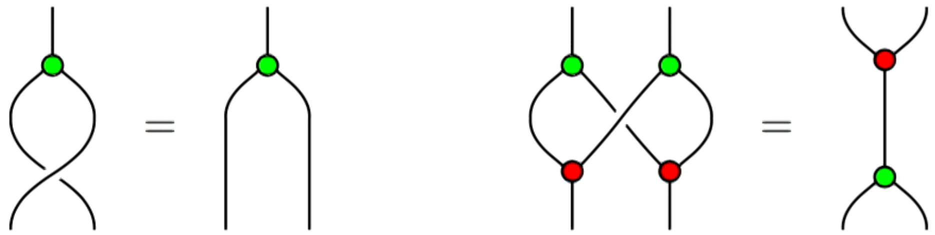

Only make sense in (at least) three dimensional space.

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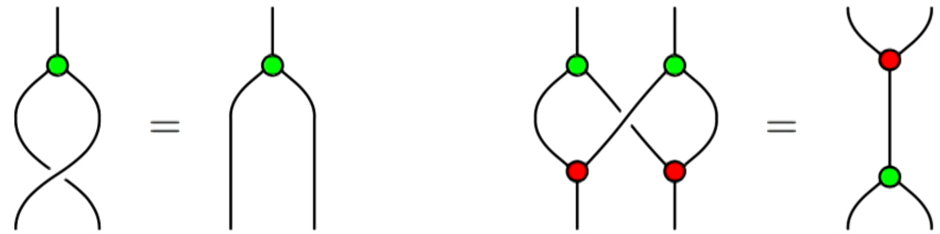

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$\Downarrow$

Shadows of 3D structures?

Part 3
3-categories

Seite 41 von 182

Algebra in three dimensions = 3-category theory

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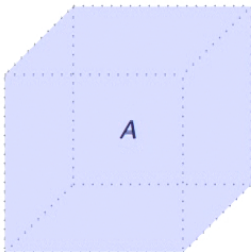
Seite 42 von 182

Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is *3-category theory*:



object

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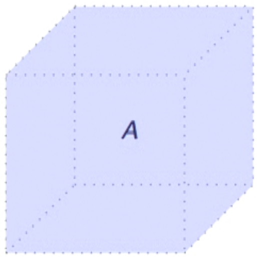

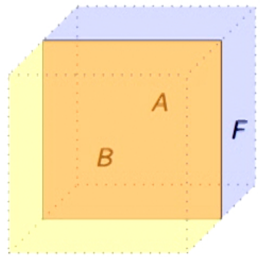

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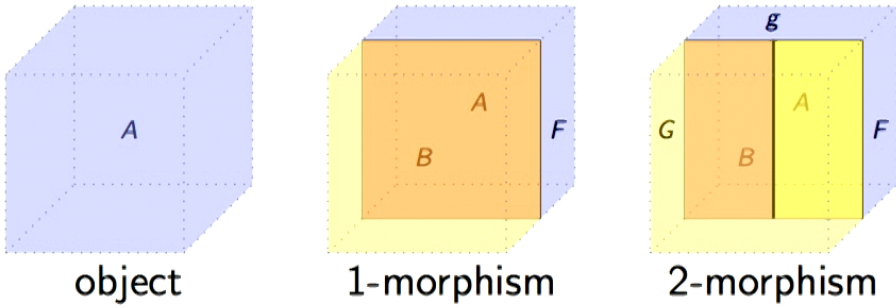

Seite 45 von 182

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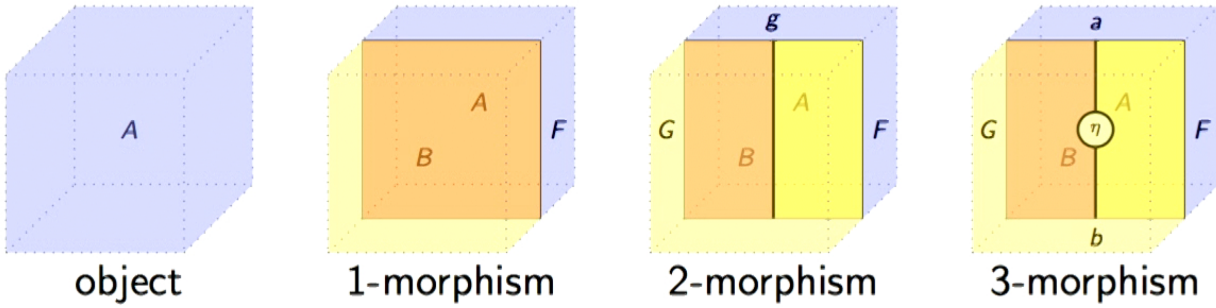

Seite 46 von 182

Algebra in three dimensions = 3-category theory

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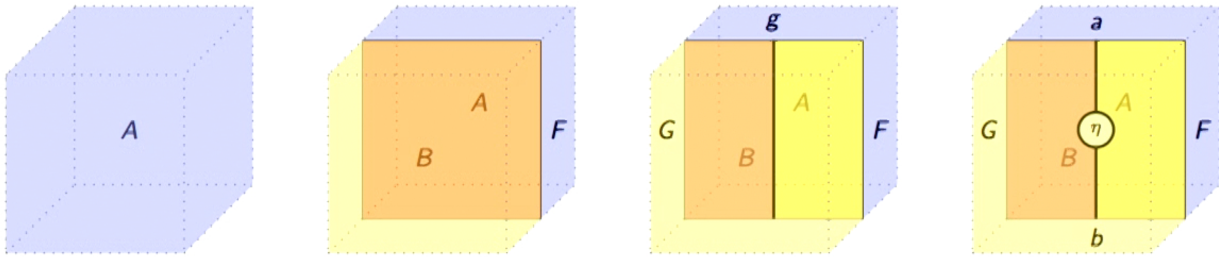

object 1-morphism 2-morphism 3-morphism

We can compose 3-morphisms like this:

Seite 47 von 182

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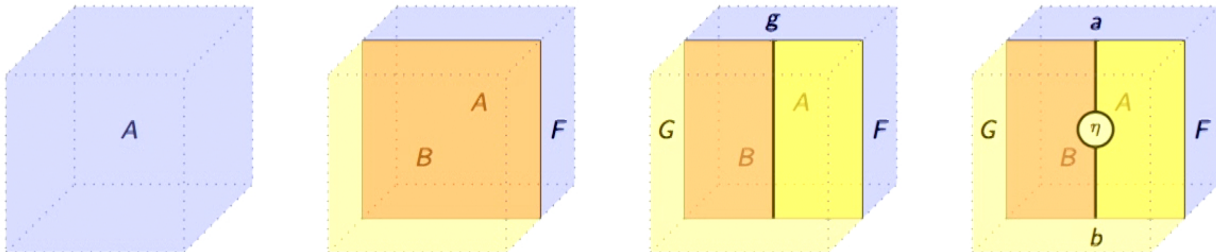

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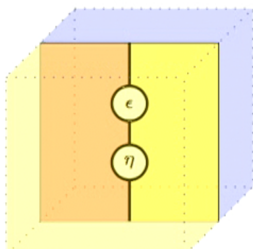

vertical composition

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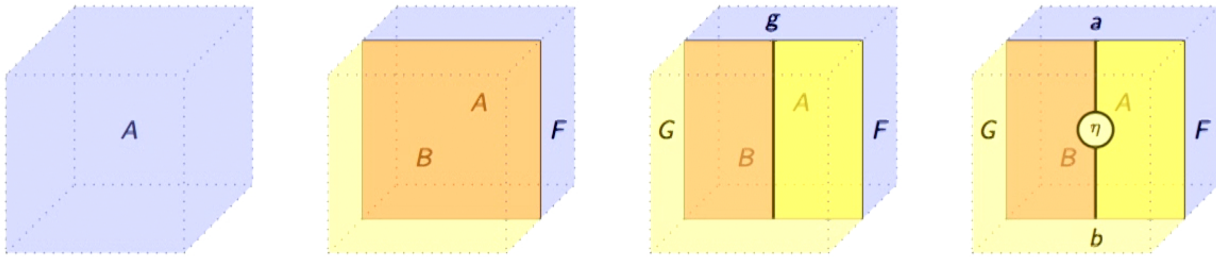

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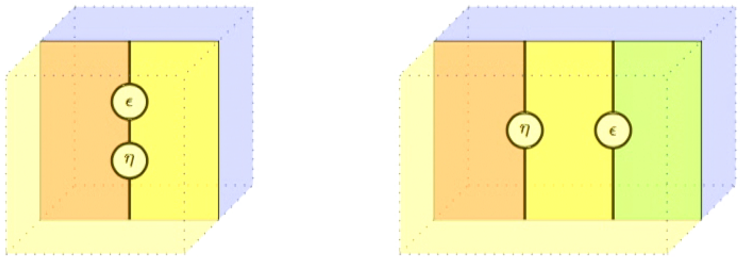

vertical composition horizontal composition

Seite 49 von 182

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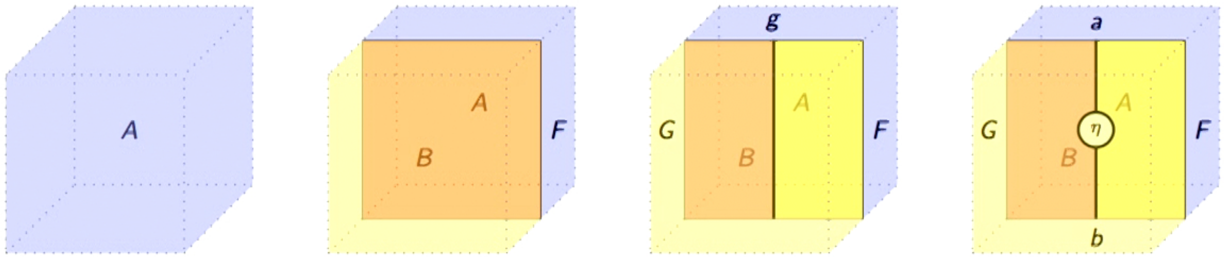

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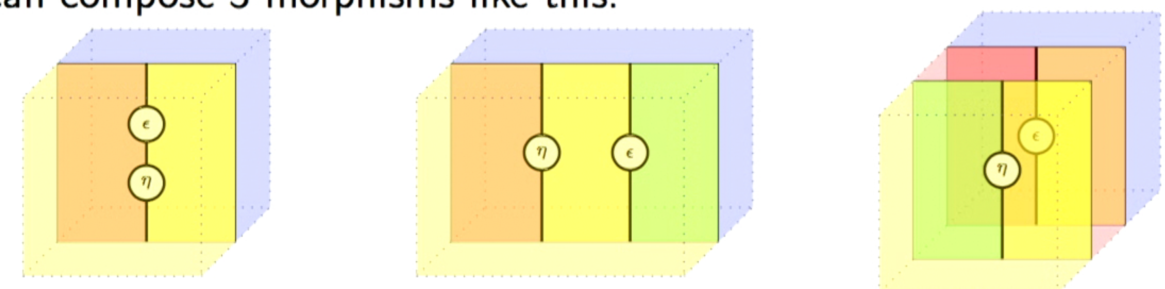

vertical composition horizontal composition layered composition

Seite 50 von 182

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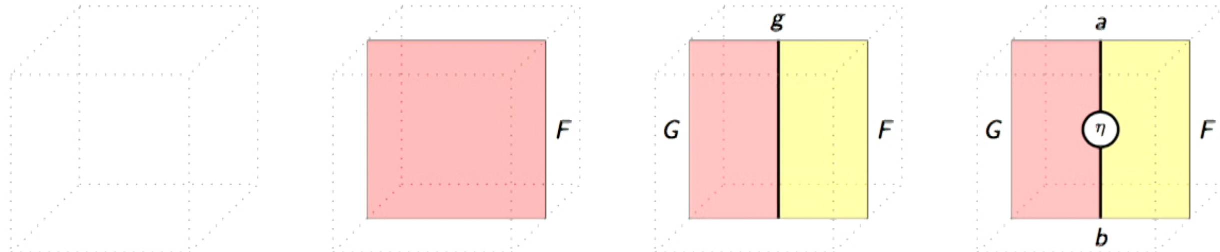

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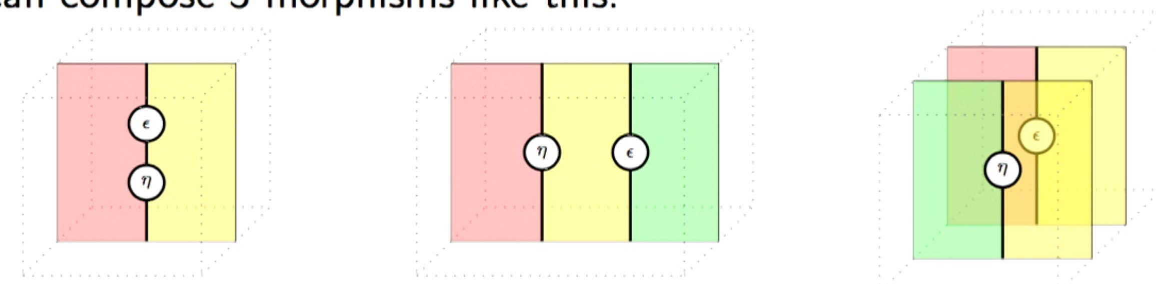

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A one object (the 'empty region') 3-category is a *monoidal 2-category*.

Algebra in three dimensions = 3-category theory

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object | 1-morphism | 2-morphism | 3-morphism

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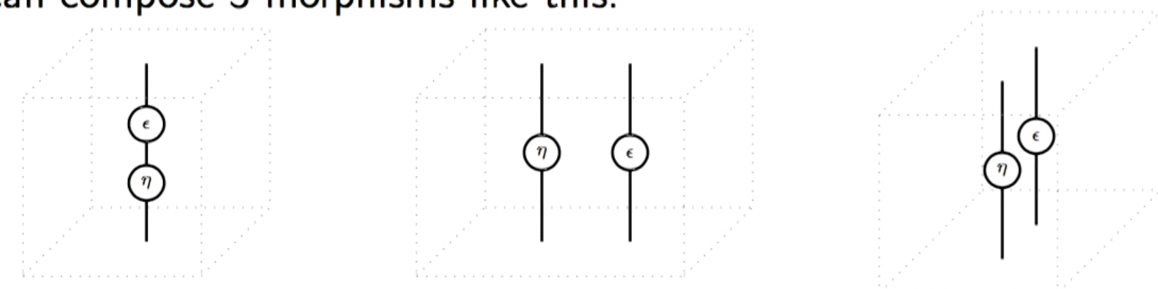
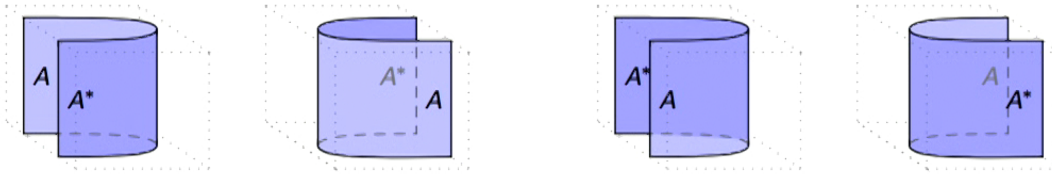


vertical composition | horizontal composition | layered composition

A one object (the ‘empty region’) 3-category is a *monoidal 2-category*.
A one object and one 1-morphism 3-category is a *braided monoidal category*.

Duals in 3-categories

A 1-morphism A has an *oriented dual* A^* if there are 2-morphisms (*folds*):



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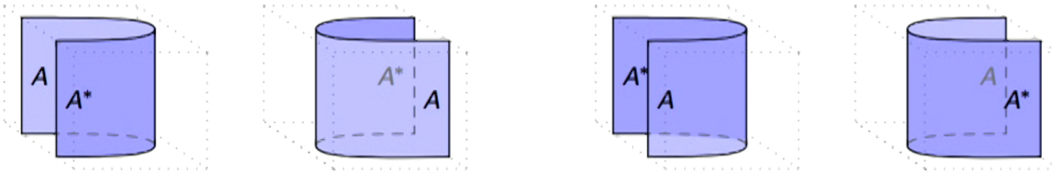
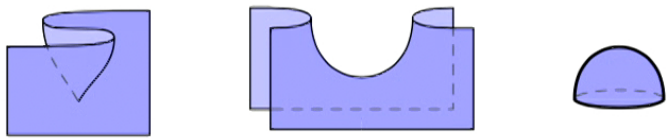


and 3-morphisms (*cusps, saddles and births/deaths of the circle*):



+ horizontal and vertical reflections and opposite orientation

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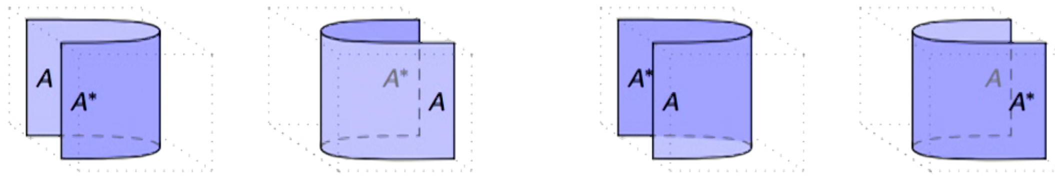

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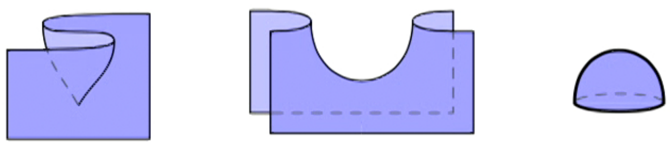
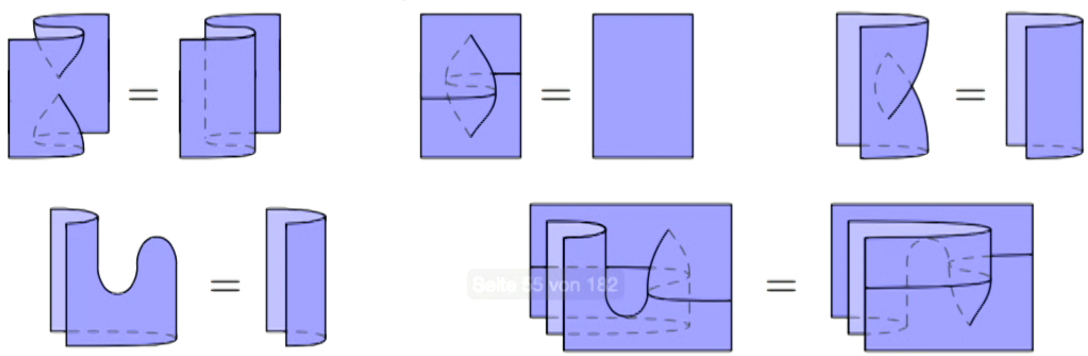


+ horizontal and vertical reflections and opposite orientation

such that the following hold (& reflections and opposite orientation):



Duals in 3-categories

Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Duals in 3-categories

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> **Tangle hypothesis.**
> $\mathbf{Bord}^{3D}_{2,1,0} \cong$ free monoidal 2-category on a dualizable object

Duals in 3-categories

Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathbb{G}$ be a 2-globular set $\mathbb{G} = \left(\text{2-Edges} \rightrightarrows \text{Edges} \rightrightarrows \text{Vertices} \right)$

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Seite 59 von 182

Duals in 3-categories

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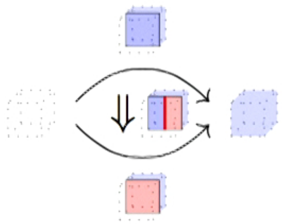


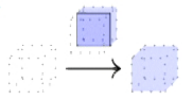
Duals in 3-categories

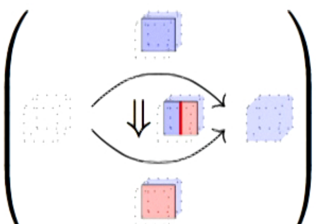
Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathbb{G}$ be a 2-globular set $\mathbb{G} = \left(\text{2-Edges} \rightrightarrows \text{Edges} \rightrightarrows \text{Vertices} \right)$

Def. $\mathcal{F}_3(\mathbb{G})$: free 3-category with duals for 2- and 1-morphisms given in $\mathbb{G}$.

Examples.

$\mathcal{F}_3$ (): free 3-category on a dualizable 1-morphism

$\mathcal{F}_3$ () : free 3-category on $\begin{cases} \text{two dualizable 1-morphisms} \\ \text{one dualizable 2-morphism} \end{cases}$

Seite 61 von 182

Duals in 3-categories

Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

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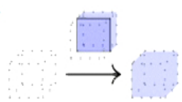
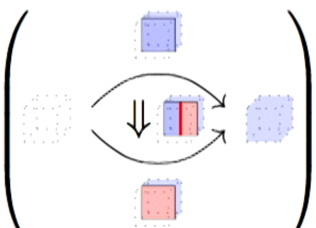
$\mathcal{F}_3$ (): free 3-category on a dualizable 1-morphism

$\mathcal{F}_3$ () : free 3-category on $\begin{cases}\text{two dualizable 1-morphisms} \\ \text{one dualizable 2-morphism}\end{cases}$

Summary. The graphical calculus of $\mathcal{F}_3(\mathbb{G})$ is given by regions, surfaces and wires in three dimensional space.

Duals in 3-categories

Theore defect data culus for duals $\leftrightarrow$ oriented surfaces in 3D space

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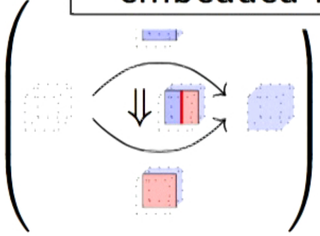
Examples.

$\mathcal{F}_3$ (3-2-1-0 defect bordisms embedded in $\mathbb{R}^3$ gory on a dualizable 1-morphism

$\mathcal{F}_3$  : free 3-category on $\begin{cases} \text{two dualizable 1-morphisms} \\ \text{one dualizable 2-morphism} \end{cases}$

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A *commutative* Frobenius algebra is a Frobenius algebra such that:

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Seite 65 von 182

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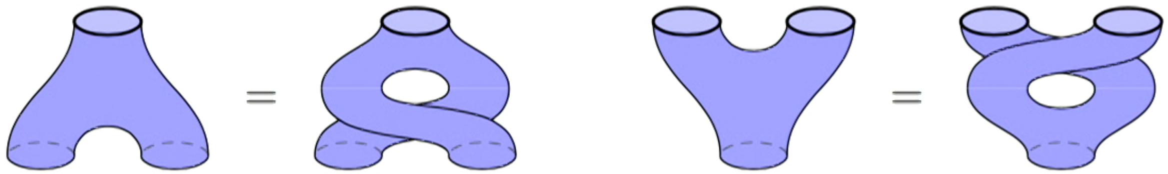
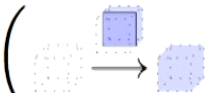


cFrob: free *braided* monoidal category on a commutative Frobenius algebra

There is a 3-functor **cFrob** $\xrightarrow{\text{'thickening'}}$ $\mathrm{F}_3 := \mathcal{F}_3\Big($  $\Big)$.

Seite 66 von 182

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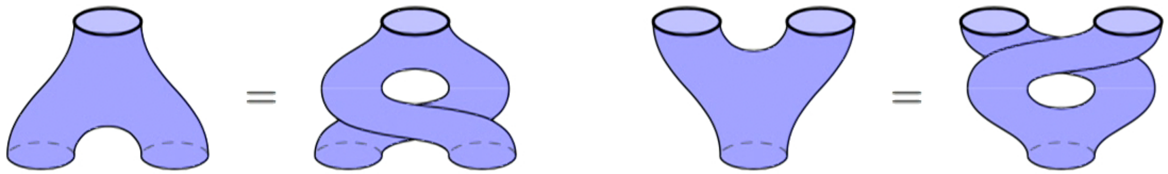

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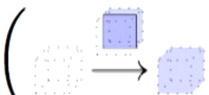
There is a 3-functor **cFrob** $\xrightarrow{\text{'thickening'}} \mathrm{F}_3 := \mathcal{F}_3\Big($  $\Big)$.

Theorem. This induces a braided monoidal equivalence **cFrob** $\cong \mathrm{F}_3\big($  $\big)$.

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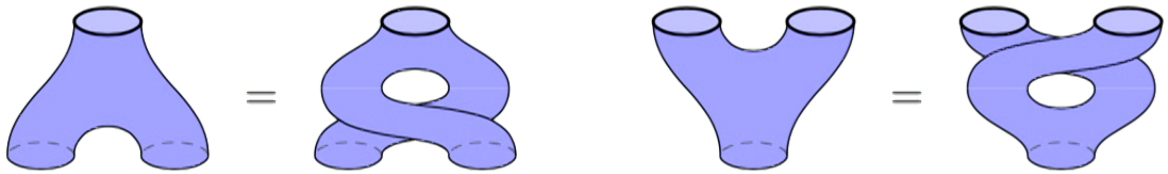

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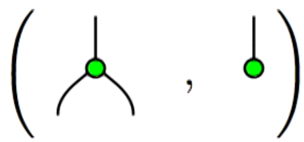
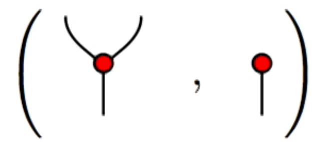
cFrob as a 'shadow' of the theory of dualizable 1-morphisms in 3-categories.

Seite 68 von 182

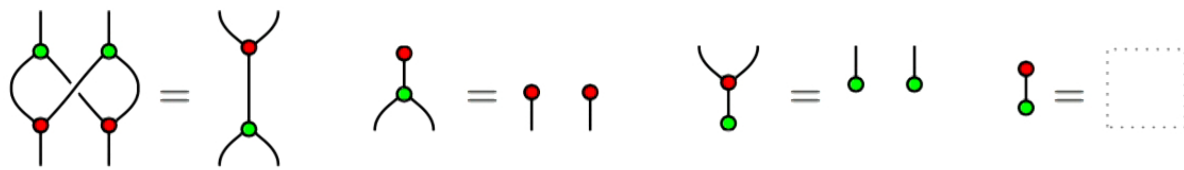
Part 4
Hopf algebras

Hopf algebras

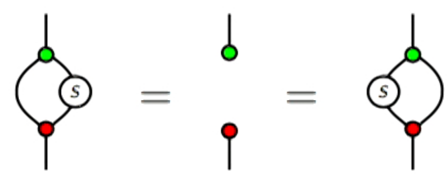
A *Hopf algebra* in a braided monoidal category is a pair of

an algebra  a coalgebra 

that form a *bialgebra*



and have an *antipode*; an endomorphism S fulfilling



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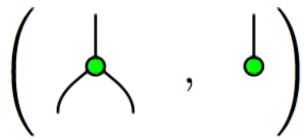
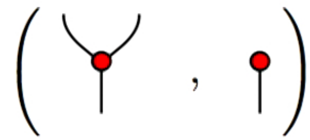
an algebra  a coalgebra 

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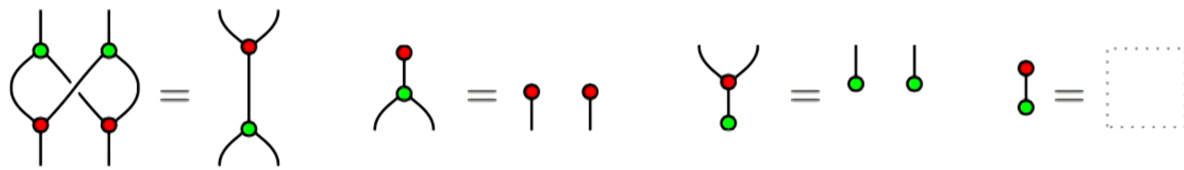

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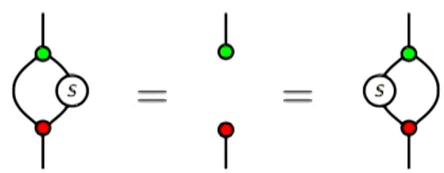

Here, we consider more restrictive algebras.

Unimodular Hopf algebras

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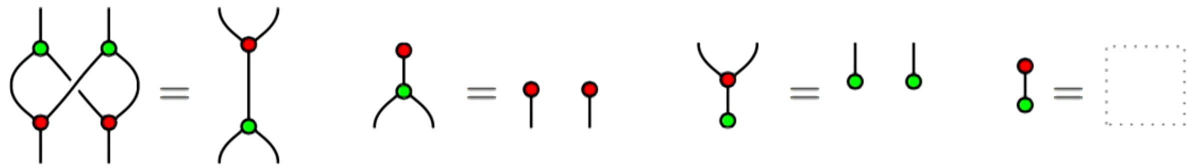

Seite 75 von 182

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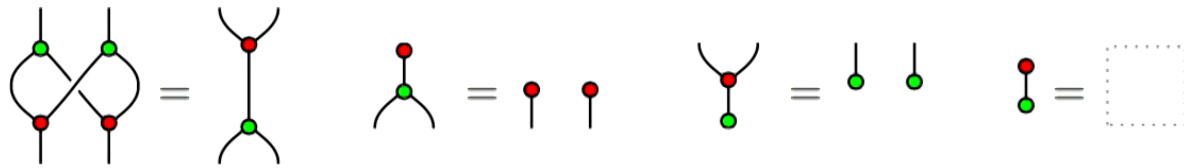

and such that

Seite 76 von 182

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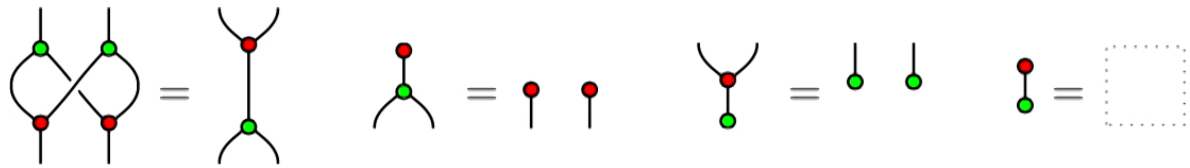

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uHopf: free *braided* monoidal category on a unimodular Hopf algebra

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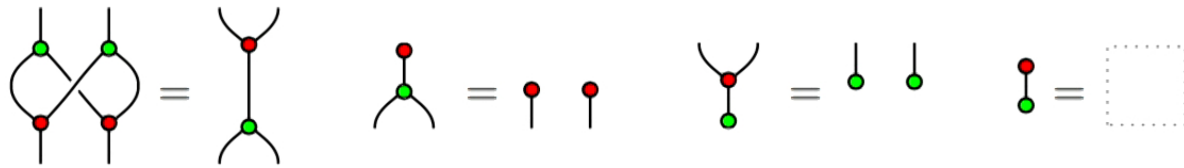

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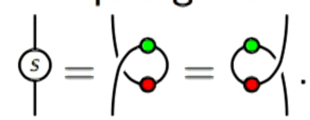
Theorem. The antipode of a unimodular Hopf algebra is 

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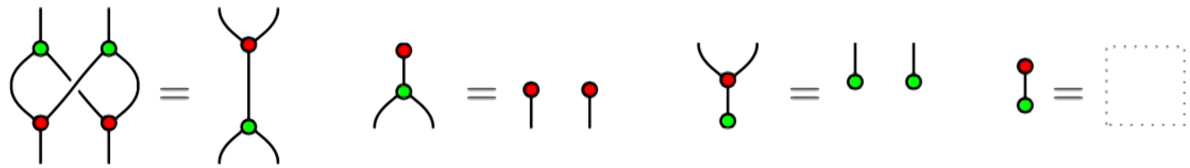

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U. Hopf algebras in $\mathbf{Vect}_k$ are *finite dimensional unimodular Hopf algebras.*

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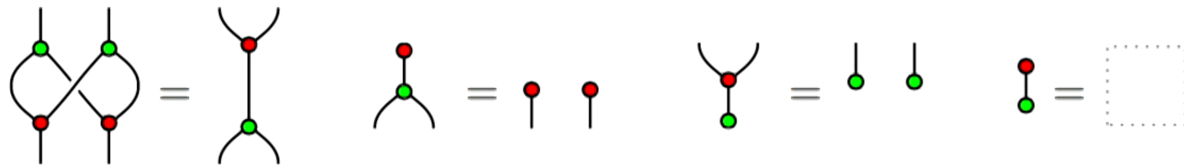

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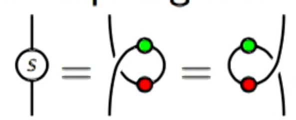
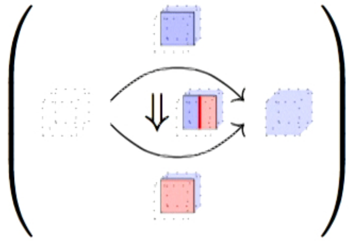
Theorem. The antipode of a unimodular Hopf algebra is 

U. Hopf algebras in $\mathbf{Vect}_k$ are *finite dimensional unimodular Hopf algebras.*

Example. Any finite dimensional semisimple and cosemisimple Hopf algebra.

A topological 3-category

Start with $\mathcal{F}_3$ 

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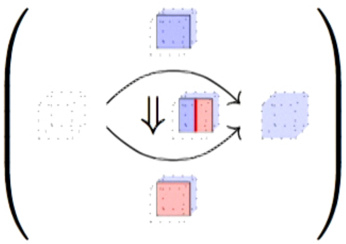
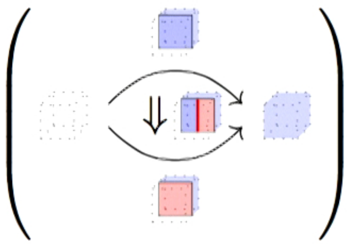
Start with $\mathcal{F}_3$  :

free 'topological' 3-category on

- a blue surface
- a red surface
- a blue-red boundary wire

A topological 3-category

Start with $\mathcal{F}_3$  :

free 'topological' 3-category on

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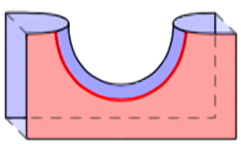
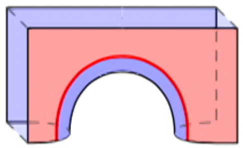
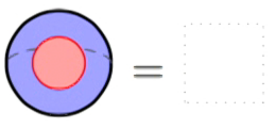
 is inverse to  and 

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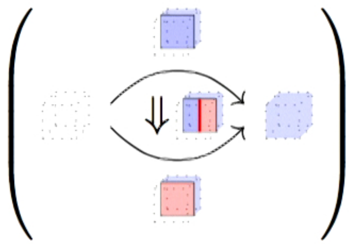
Start with $\mathcal{F}_3$  :

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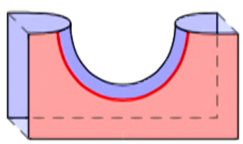
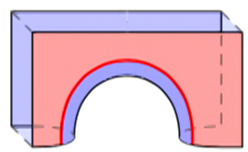
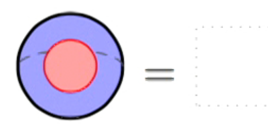
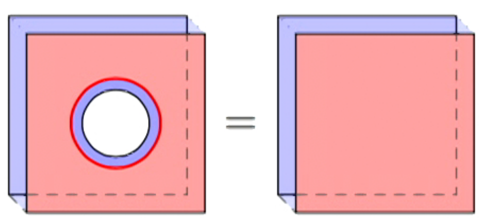
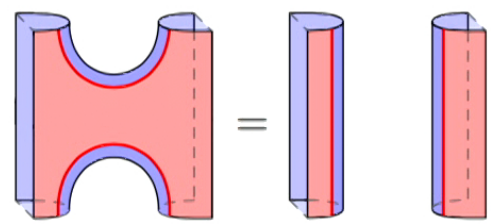
 is inverse to  and 

Explicitly, invertibility of the saddles means:





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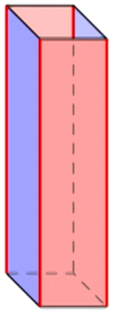

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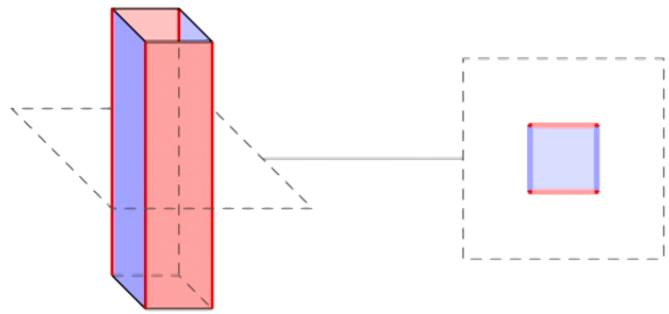

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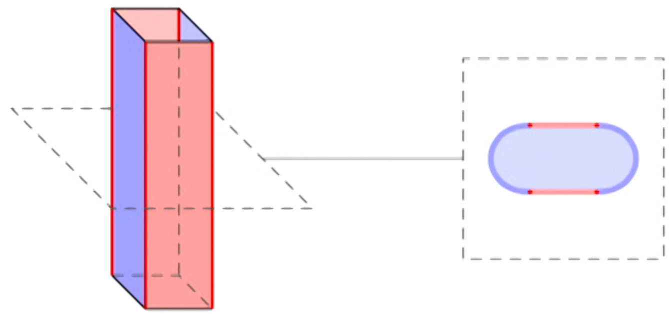

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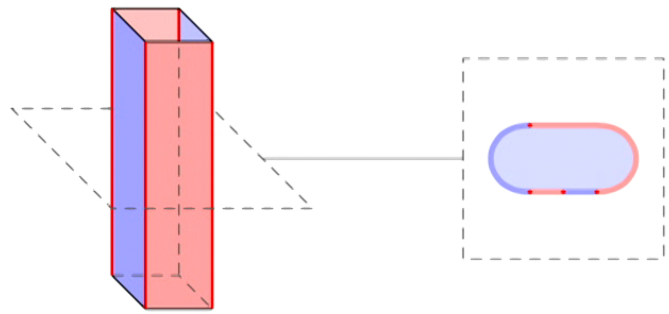

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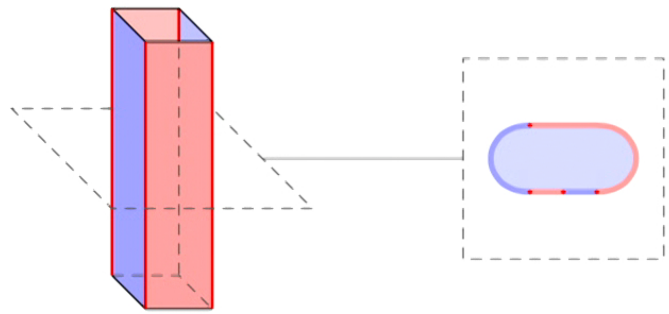

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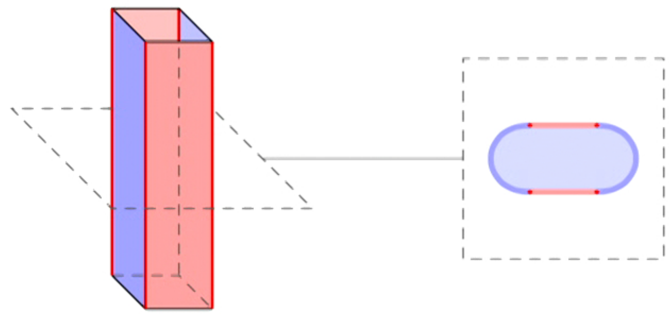

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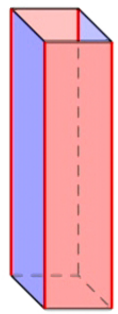

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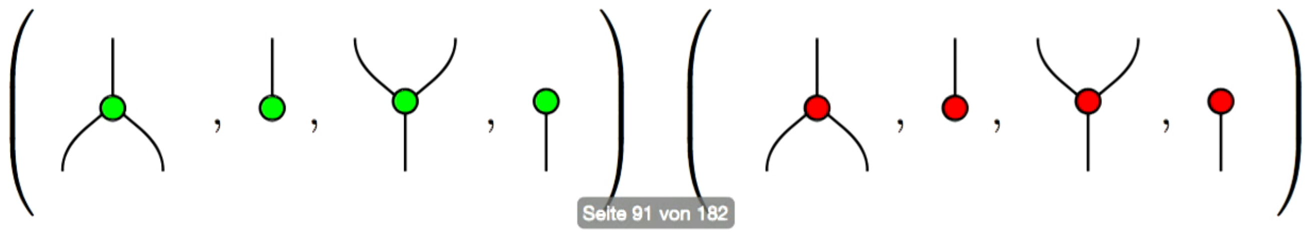

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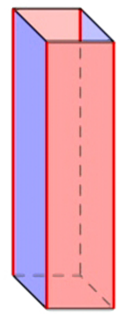

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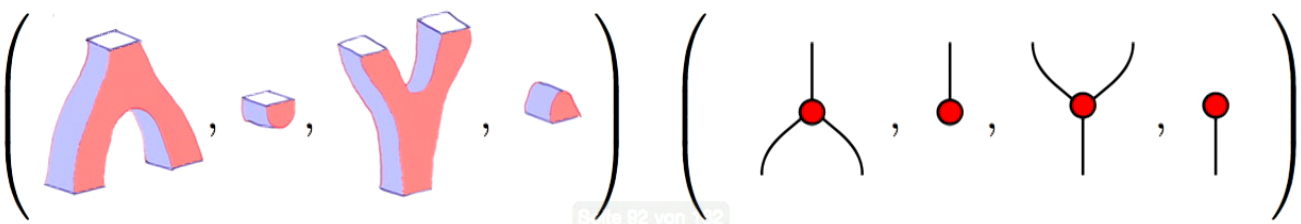

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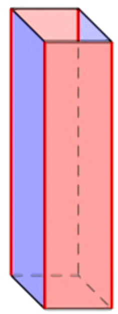

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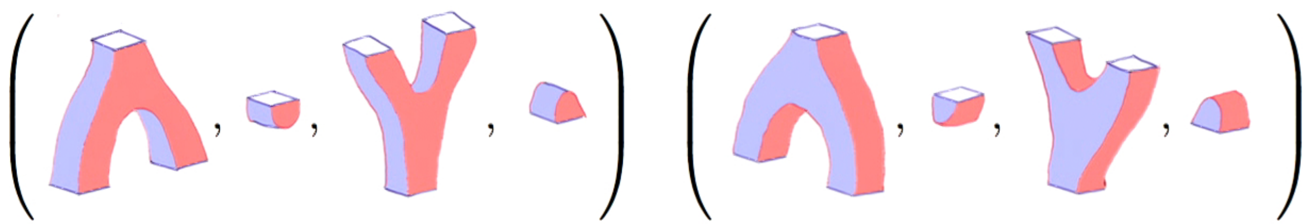

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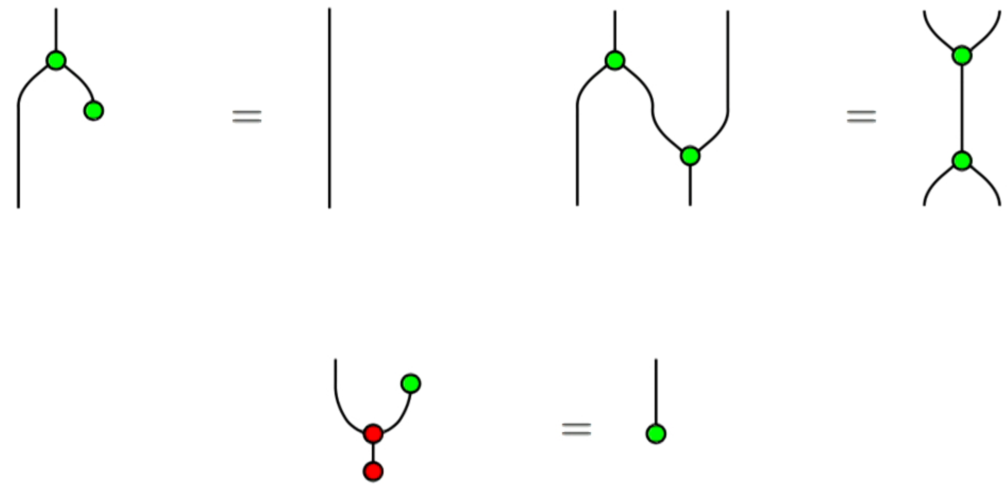

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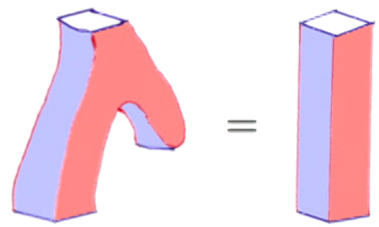

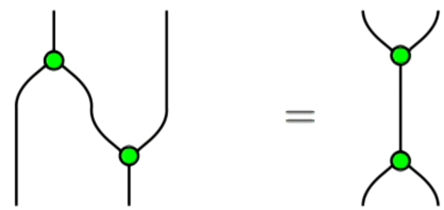

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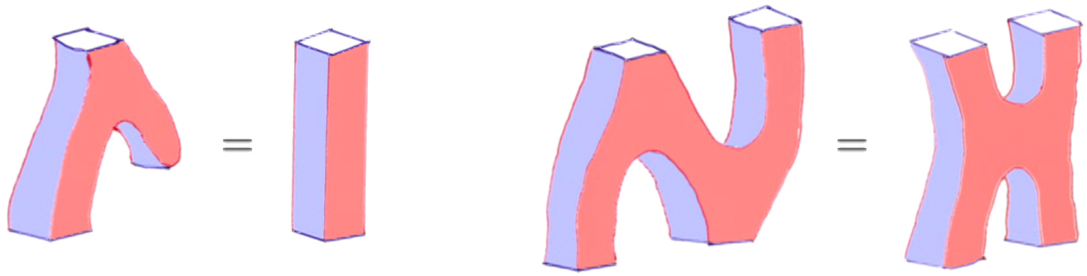

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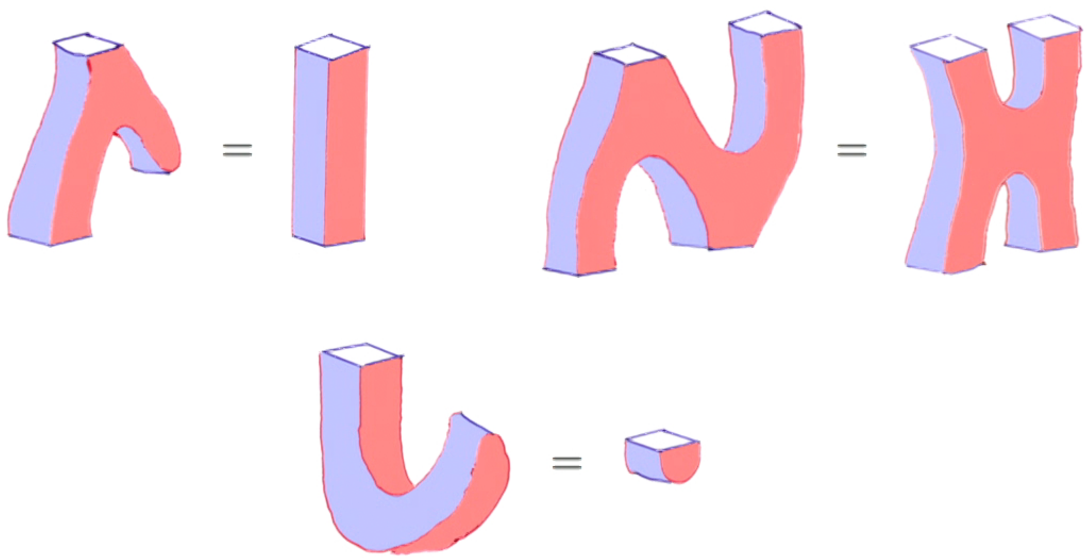

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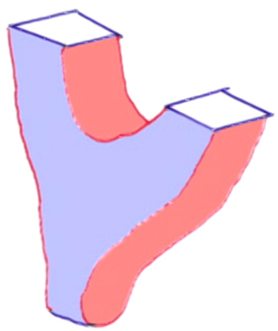

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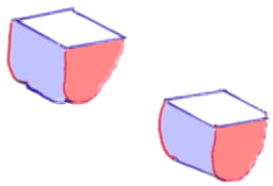

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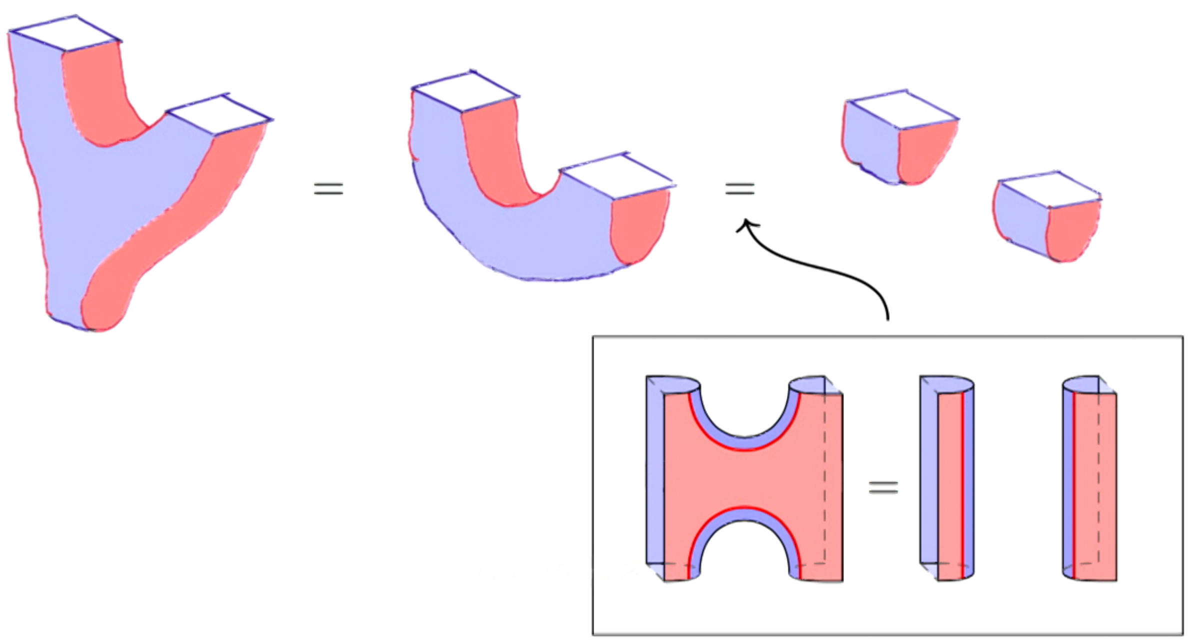

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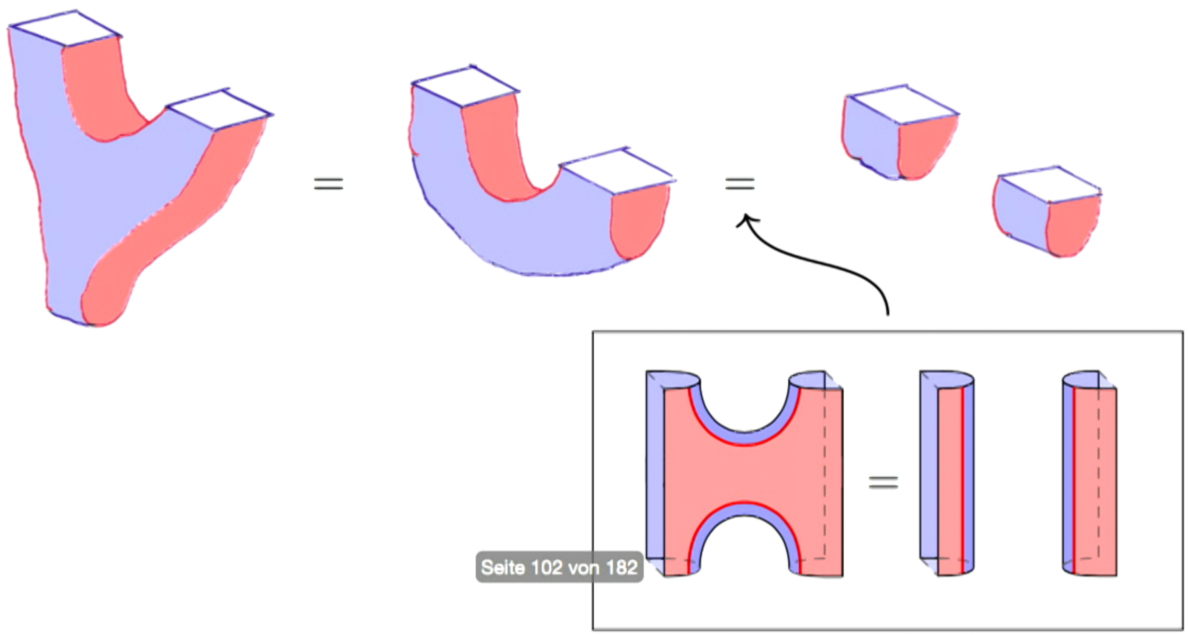

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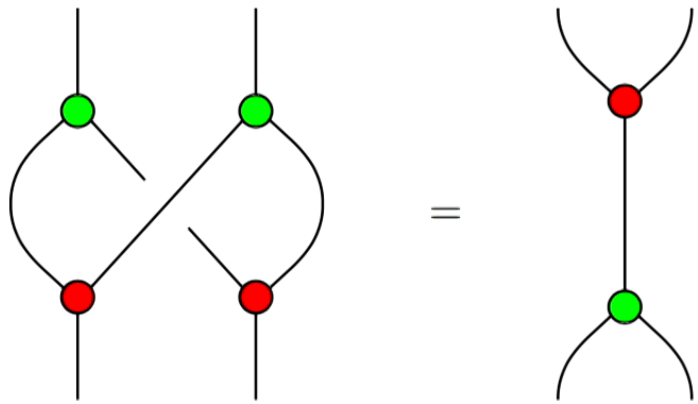

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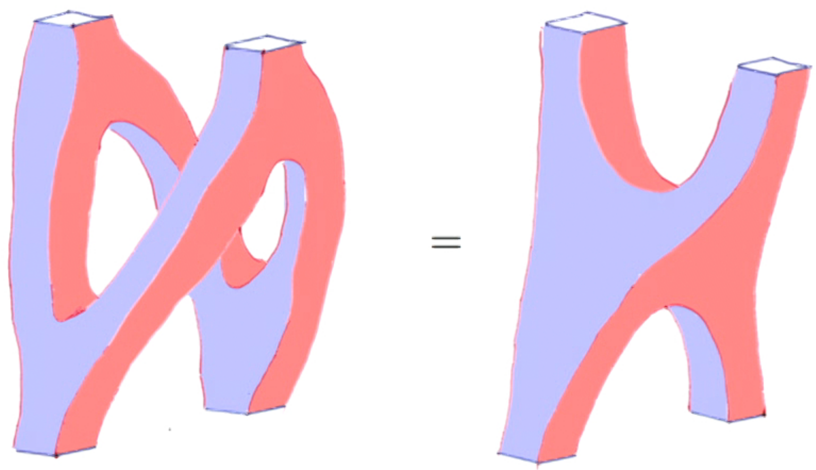

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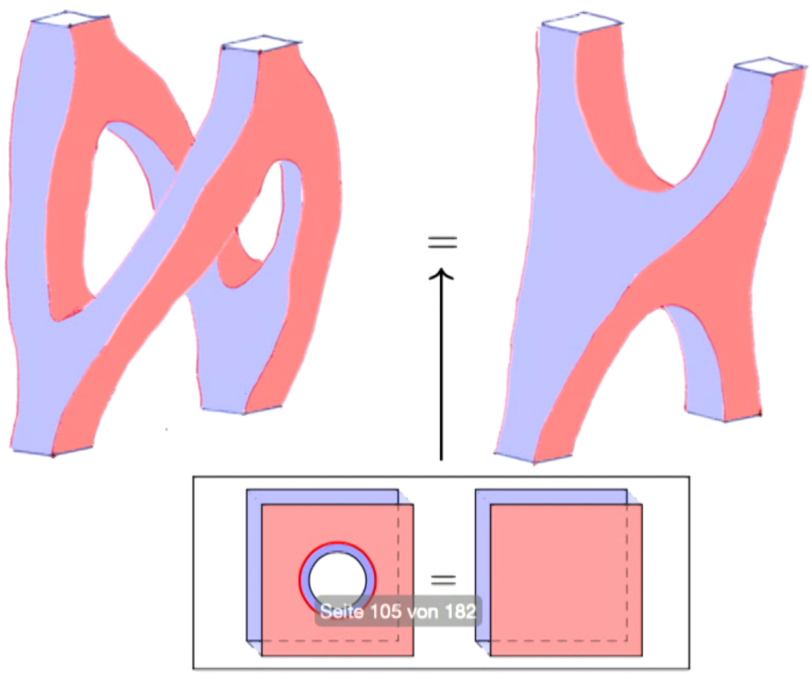

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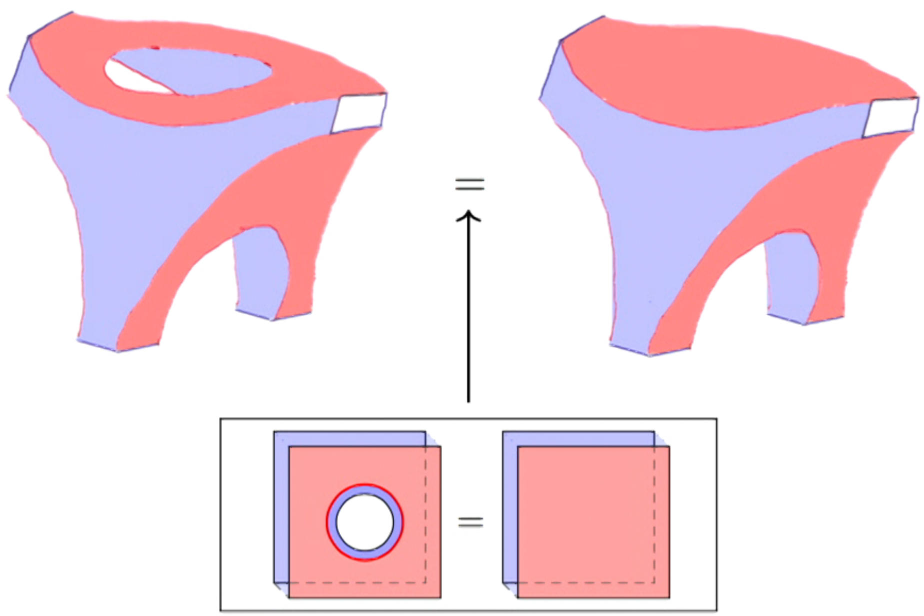

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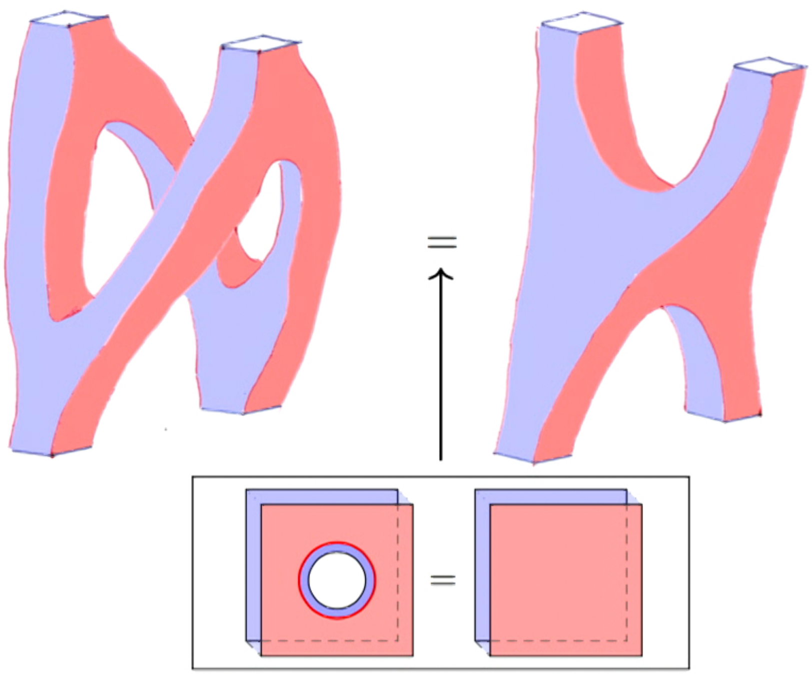

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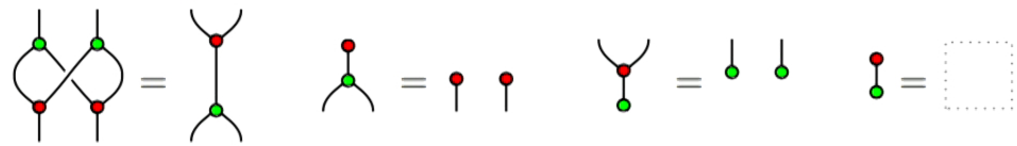

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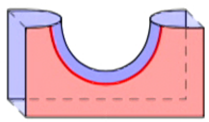
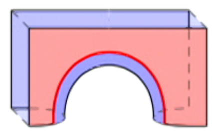
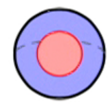
 is inverse to  and  $=$

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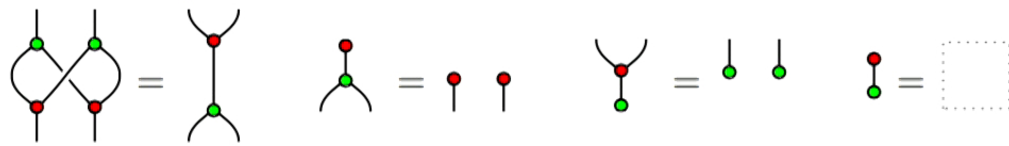

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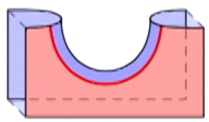
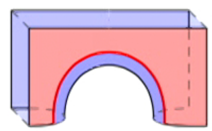
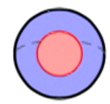
 is inverse to  and  $=$

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Seite 115 von 182

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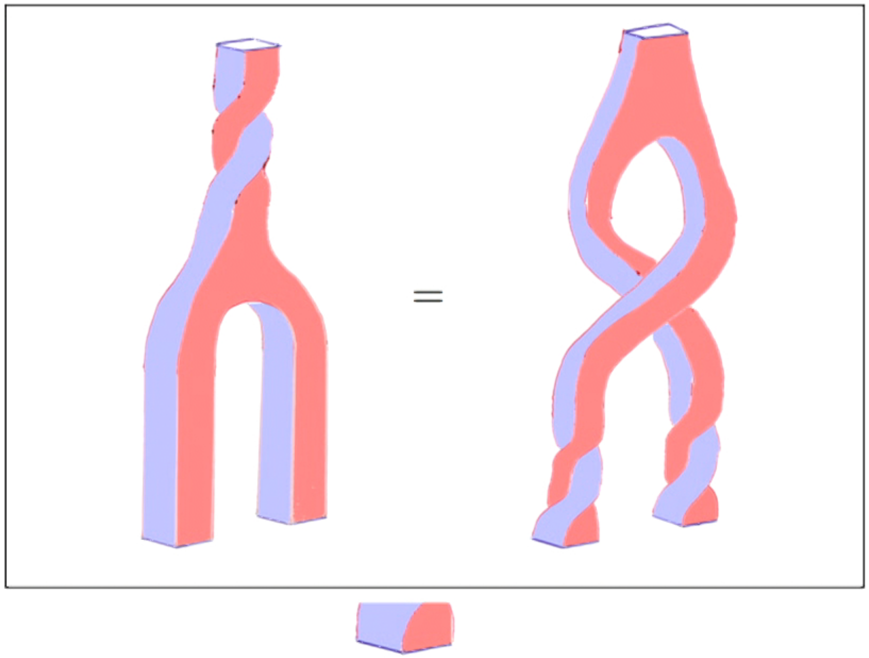

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$\Rightarrow$ The antipode is an algebra antihomomorphism.
$\Rightarrow$ In a unimodular Hopf algebra, the antipode squares to the twist.
In particular, in a symmetric monoidal category, its 4th power is trivial.

Part 5
Higher linear algebra

Representations

So far: algebraic structures in terms of generators & relations

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Now: *representations* - instances of these structures in concrete categories

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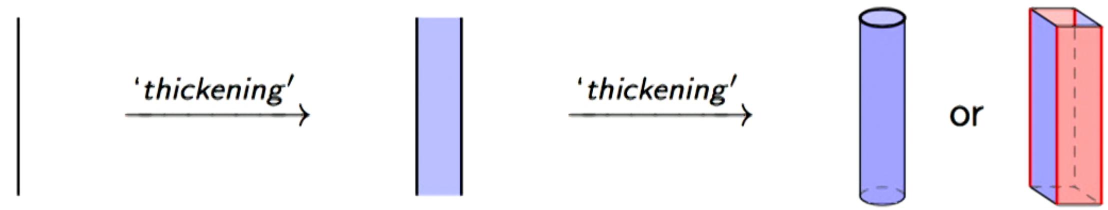
nVect a 'shadow' of $(\mathbf{n}+\mathbf{1})$**Vect** $\leftrightsquigarrow$ $(\mathbf{n}+\mathbf{1})$**Vect** a 'thickening' of **nVect**



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Vect				f.d. vector spaces	linear maps

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Higher linear algebra

	tensor unit	objects	1-morphisms	2-morphisms	3-morphisms
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2Vect	**Vect**	$\mathrm{Rep}(-)$ $\cong$	finite semisimple categories	linear functors	natural transformations
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$\mathbb{C}$	1				complex numbers
Vect	$\mathbb{C}$			f.d. vector spaces	linear maps
2Vect	**Vect**		finite semisimple categories	linear functors	natural transformations
	$\mathbb{C}$	$\mathrm{Rep}(-)$ $\cong$	f.d. semisimple algebras	f.d. bimodules	intertwiners
3Vect	**Vect**	fusion categories	finite semisimple bimodule categories	intertwining functors	natural transformations

Higher linear algebra

	tensor unit	objects	1-morphisms	2-morphisms	3-morphisms
$\mathbb{C}$	1				complex numbers
Vect	$\mathbb{C}$			f.d. vector spaces	linear maps
2Vect	**Vect**	$\mathrm{Rep}(-)$ $\cong$	finite semisimple categories	linear functors	natural transformations
	$\mathbb{C}$		f.d. semisimple algebras	f.d. bimodules	intertwiners
3Vect	**Vect**	fusion categories	finite semisimple bimodule categories	intertwining functors	natural transformations

They are *symmetric monoidal* 1-,2- and 3-categories with duals.

Seite 134 von 182

Higher linear algebra

	tensor unit	objects	1-morphisms	2-morphisms	3-morphisms
$\mathbb{C}$	1				complex numbers
Vect	$\mathbb{C}$			f.d. vector spaces	linear maps
2Vect	**Vect**	$\mathrm{Rep}(-)$ $\cong$	finite semisimple categories	linear functors	natural transformations
	$\mathbb{C}$		f.d. semisimple algebras	f.d. bimodules	intertwiners
3Vect	**Vect**	fusion categories	finite semisimple bimodule categories	intertwining functors	natural transformations

They are *symmetric monoidal* 1-,2- and 3-categories with duals.

$$\mathbf{3Vect}(\mathrm{I},\mathrm{I}) \cong \mathbf{2Vect} \qquad \mathbf{2Vect}(\mathrm{I},\mathrm{I}) \cong \mathbf{Vect} \qquad \mathbf{Vect}(\mathrm{I},\mathrm{I}) = \mathbb{C}$$

Higher linear algebra

	tensor unit	objects	1-morphisms	2-morphisms	3-morphisms
$\mathbb{C}$	1				complex numbers
Vect	$\mathbb{C}$			f.d. vector spaces	linear maps
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They are *symmetric monoidal* 1-,2- and 3-categories with duals.

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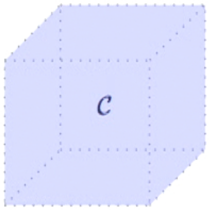
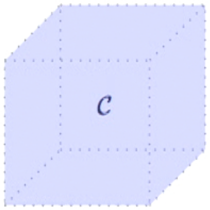
Various generalizations are possible.

The 3-category **3Vect**

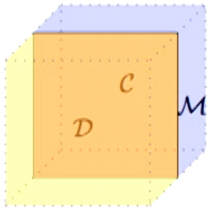


fusion
category

The 3-category **3Vect**

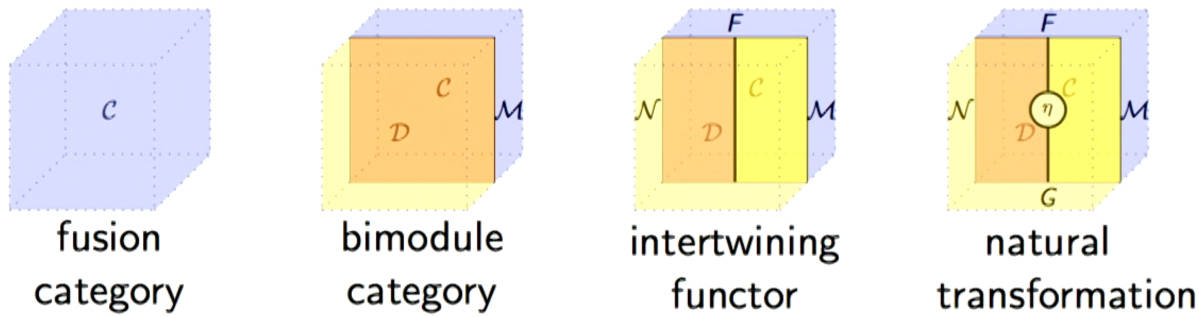


fusion
category



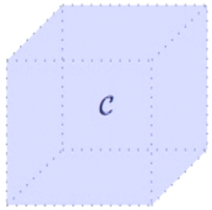
bimodule
category

The 3-category **3Vect**

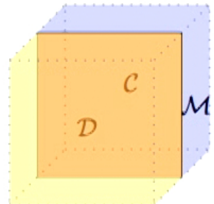


Seite 140 von 182

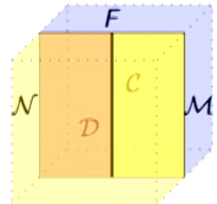
The 3-category **3Vect**



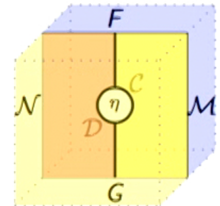
fusion category



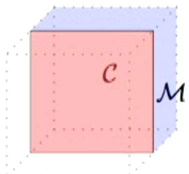
bimodule category



intertwining functor



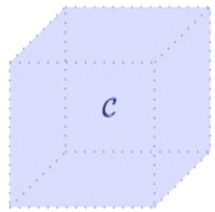
natural transformation



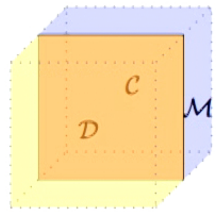
right $\mathcal{C}$-module $\mathcal{M}$

Seite 141 von 182

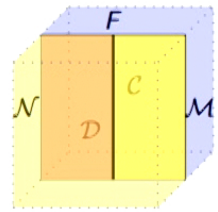
The 3-category **3Vect**



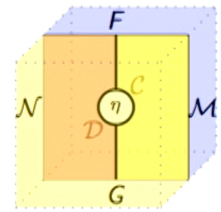
fusion category



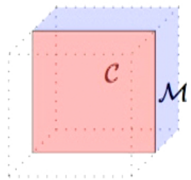
bimodule category



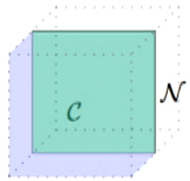
intertwining functor



natural transformation

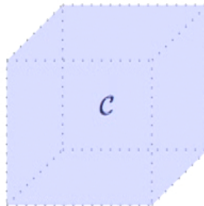


right $\mathcal{C}$-module $\mathcal{M}$

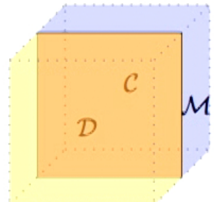


left $\mathcal{C}$-module $\mathcal{N}$

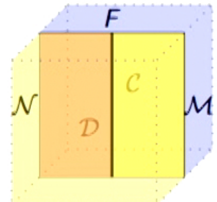
The 3-category **3Vect**



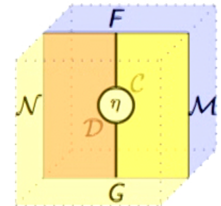
fusion category



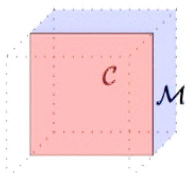
bimodule category



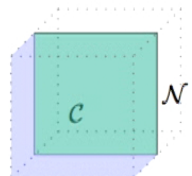
intertwining functor



natural transformation

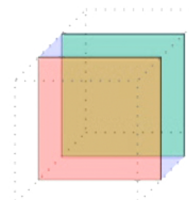


right $\mathcal{C}$-module $\mathcal{M}$



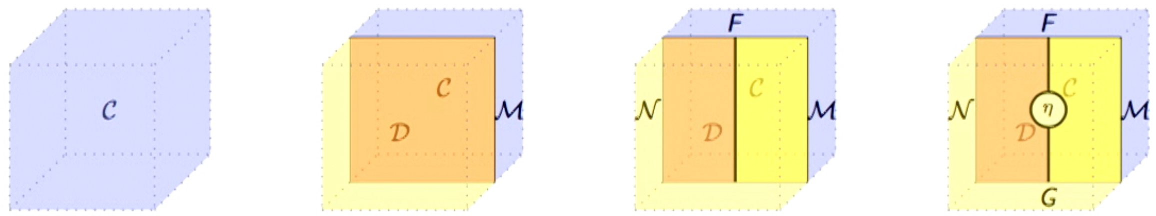
left $\mathcal{C}$-module $\mathcal{N}$

$\Rightarrow$

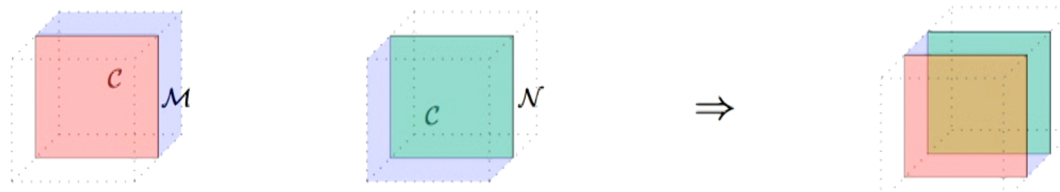


relative Deligne product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$

The 3-category **3Vect**



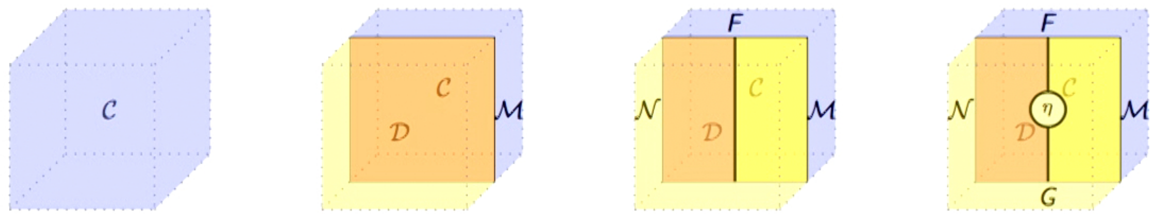
fusion category | bimodule category | intertwining functor | natural transformation



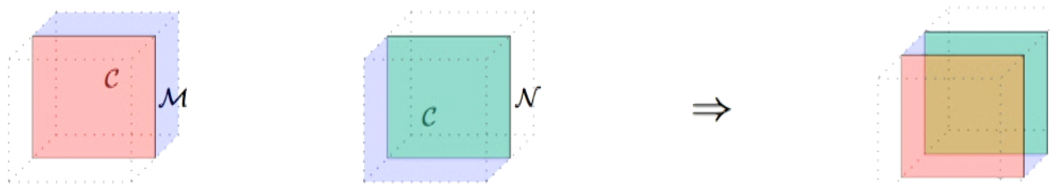
right $\mathcal{C}$-module $\mathcal{M}$ left $\mathcal{C}$-module $\mathcal{N}$ *relative Deligne product* $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$

relative Deligne product: universal for $\mathcal{C}$-bilinear functors out of $\mathcal{M} \times \mathcal{N}$

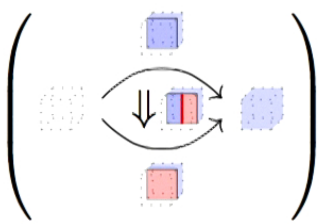
The 3-category **3Vect**



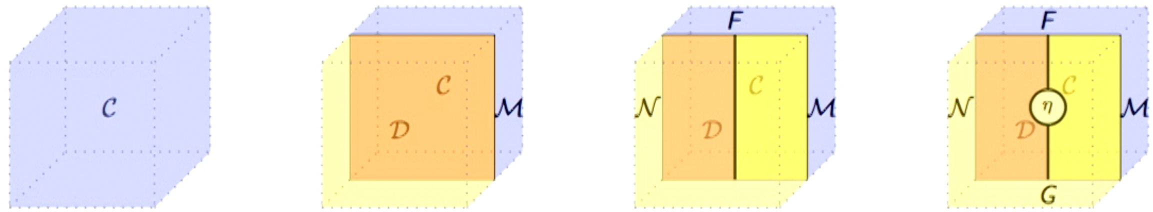
fusion category | bimodule category | intertwining functor | natural transformation



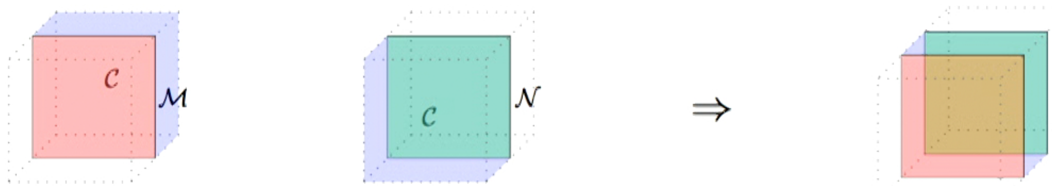
right $\mathcal{C}$-module $\mathcal{M}$ | left $\mathcal{C}$-module $\mathcal{N}$ | *relative Deligne product* $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$

$\mathcal{F}_3$  $\to$ **3Vect**

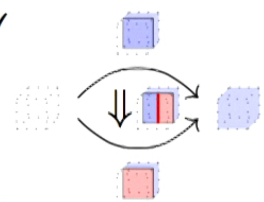
The 3-category **3Vect**



fusion category | bimodule category | intertwining functor | natural transformation



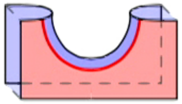
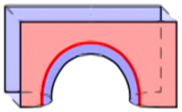
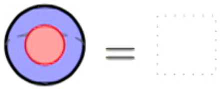
right $\mathcal{C}$-module $\mathcal{M}$ | left $\mathcal{C}$-module $\mathcal{N}$ | *relative Deligne product* $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$

$\mathcal{F}_3$ () $\to$ **3Vect** :

$$\begin{cases} \text{a fusion category } \mathcal{C} \\ \text{two right } \mathcal{C}\text{-module categories } \mathcal{M}, \mathcal{N} \\ \text{an intertwining functor } \mathcal{M} \to \mathcal{N} \end{cases}$$

Hopf algebras and fusion categories - a sketch

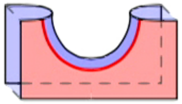
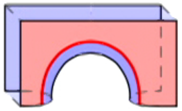
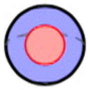
This 3-functor factors through $\mathbb{H}$ if the following hold:

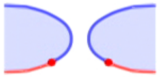
 is inverse to  and 

Seite 147 von 182

Hopf algebras and fusion categories - a sketch

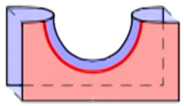
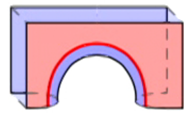
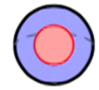
This 3-functor factors through $\mathbb{H}$ if the following hold:

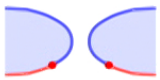
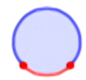
 is inverse to  and  $=$

$\Rightarrow$  is *isomorphic* to 

Hopf algebras and fusion categories - a sketch

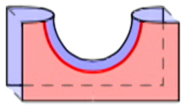
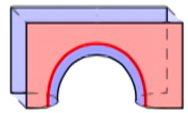
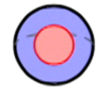
This 3-functor factors through $\mathbb{H}$ if the following hold:

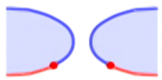
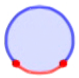
 is inverse to  and  $=$

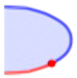
$\Rightarrow$  is *isomorphic* to and  is *isomorphic* to

Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

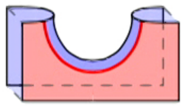
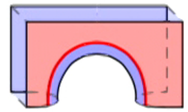
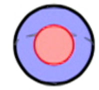
 is inverse to  and  $=$ 

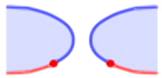
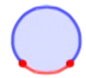
$\Rightarrow$  is *isomorphic* to  and  is *isomorphic* to 

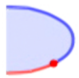
In other words,  : $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to$ **Vect** is an *adjoint equivalence*!

Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

 is inverse to  and  $=$ 

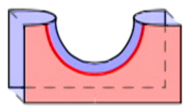
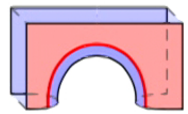
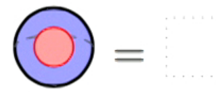
$\Rightarrow$  is *isomorphic* to  and  is *isomorphic* to

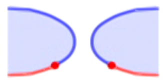
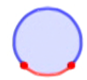
In other words,  : $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect}$ is an *adjoint equivalence*!

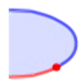
Data of a 3-functor $\mathbb{H} \to \mathbf{3Vect}$:
- a fusion category $\mathcal{C}$
- a left and a right module category $\mathcal{M}, \mathcal{N}$
- an adjoint equivalence $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect}$

Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

 is inverse to  and 

$\Rightarrow$  is *isomorphic* to and  is *isomorphic* to

In other words,  : $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to$ **Vect** is an *adjoint equivalence*!

Data of a 3-functor $\mathbb{H} \to$ **3Vect** :
- a fusion category $\mathcal{C}$
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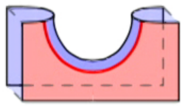
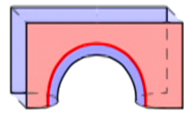
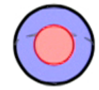
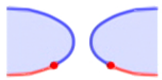
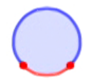
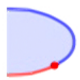
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

 is inverse to  and  $=$

$\Rightarrow$  is *isomorphic* to  and  is *isomorphic* to

In other words,  : $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to$ **Vect** is an *adjoint equivalence*!

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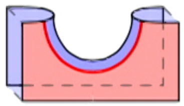
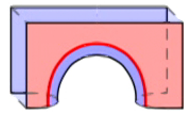
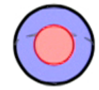
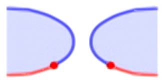
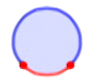
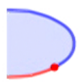
 is inverse to  and  $=$

$\Rightarrow$  is *isomorphic* to and  is *isomorphic* to

In other words,  : $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to$ **Vect** is an *adjoint equivalence*!

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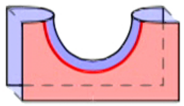
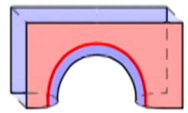
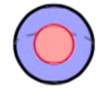
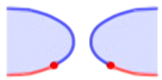
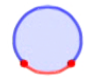
 is inverse to  and  $=$

$\Rightarrow$  is *isomorphic* to and  is *isomorphic* to

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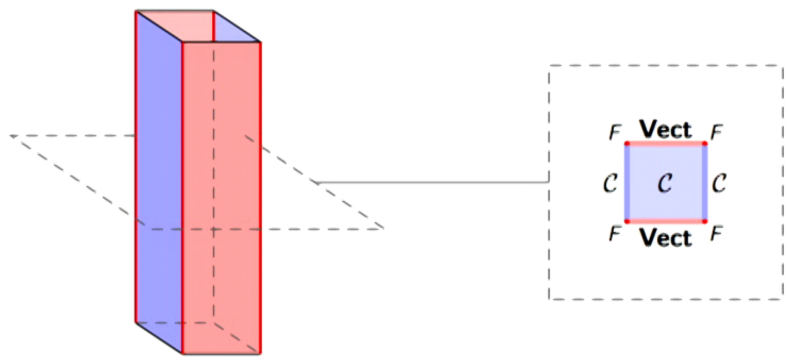
Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \textbf{Vect}$
$\Rightarrow$ The following vector space is a unimodular Hopf algebra:



Seite 157 von 182

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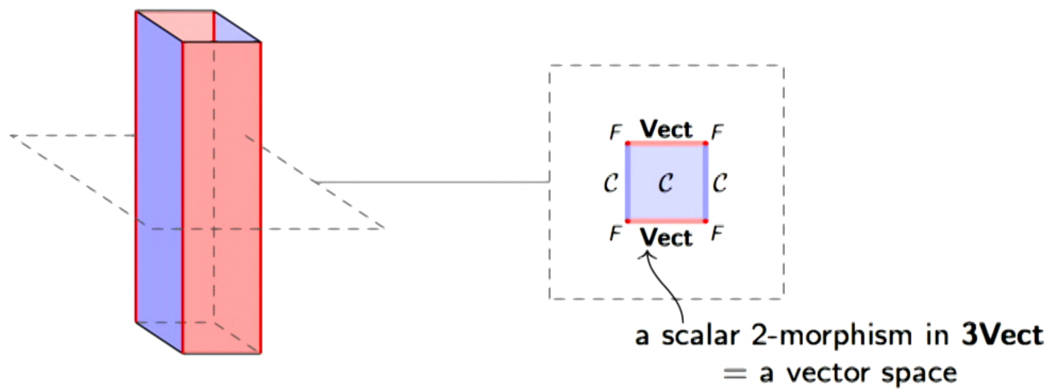

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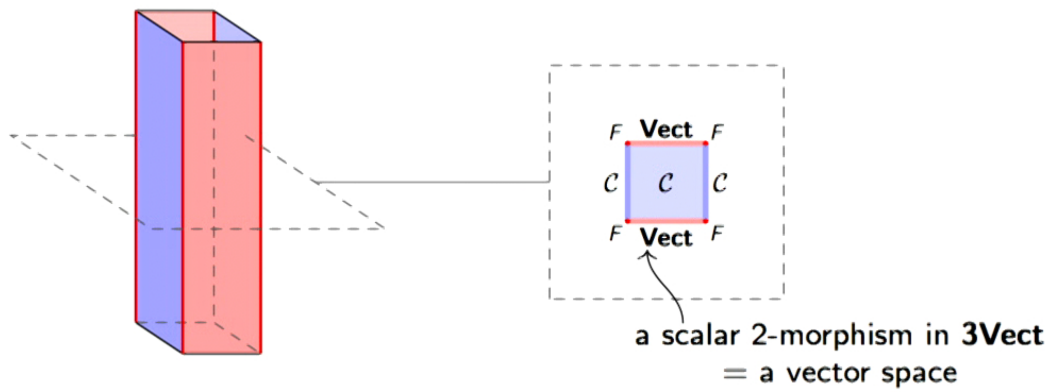

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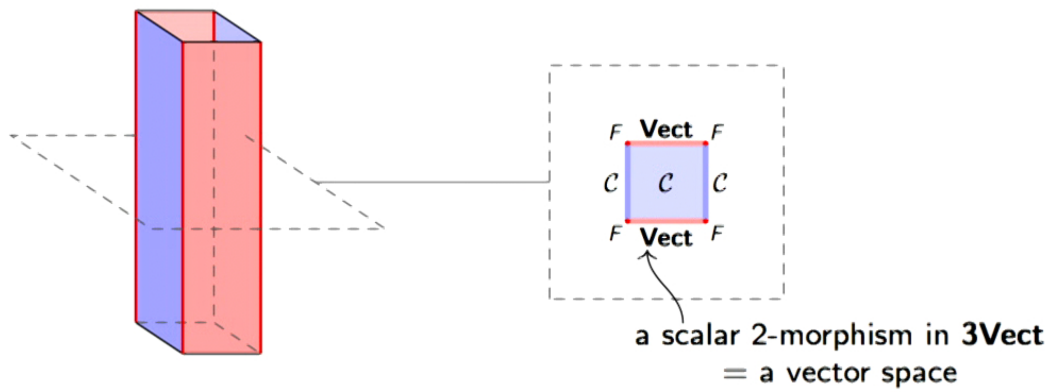

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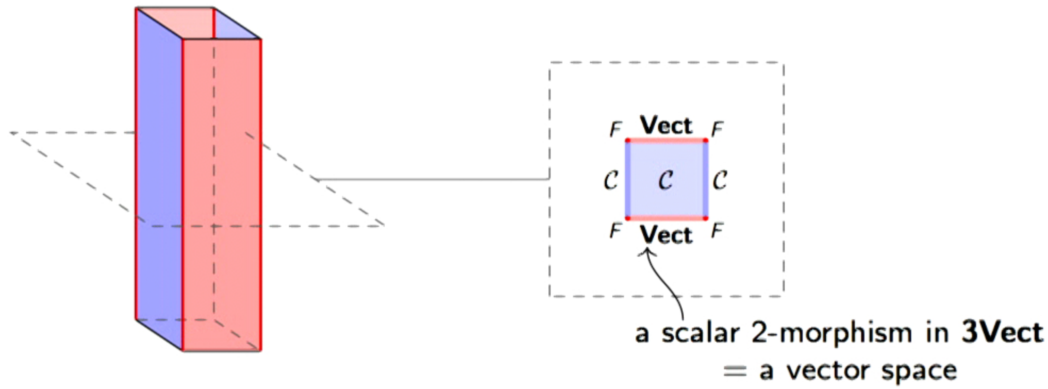

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Proof. Follows from an old result of M. Müger.[1]

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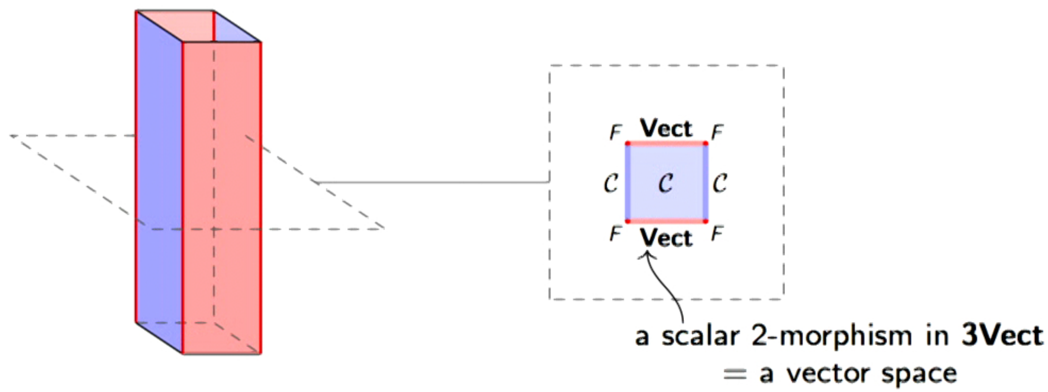

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> **Question.**
> Is there a completely graphical proof, independent of the target **3Vect**?

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Part 6
Lattice models

Seite 163 von 182

Lattice models and **3Vect**

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

Lattice models and **3Vect**

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightsquigarrow$ a fusion category $\mathcal{C}$
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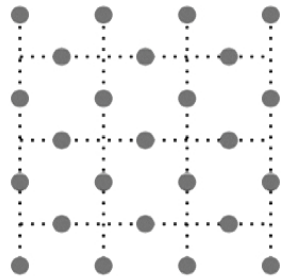

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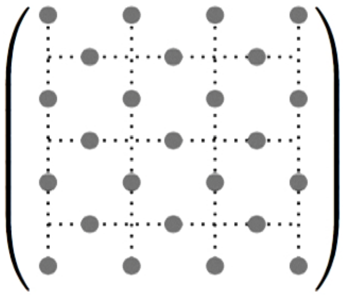
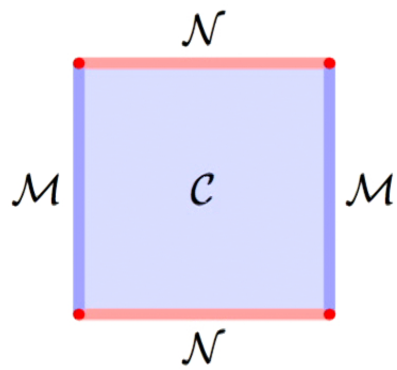
Ground space  $=$ 

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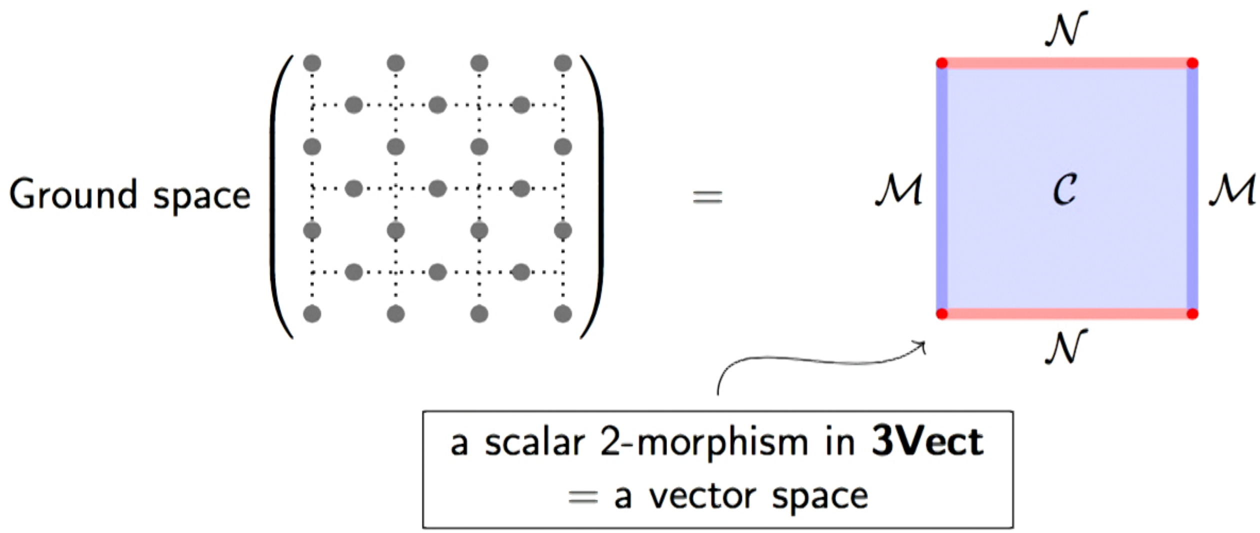

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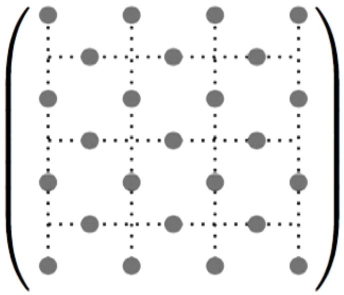
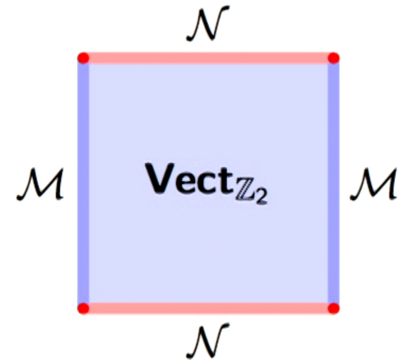
Ground space $\left(\right.$$\left. \right)$ $=$ 

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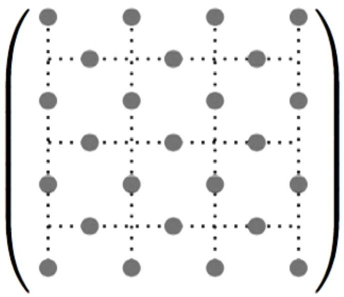
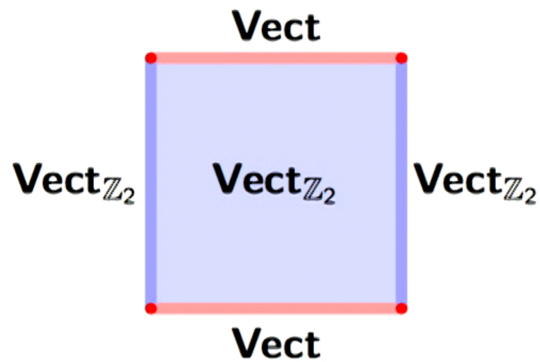
Ground space  $=$ 

surface codes	$\mathcal{C} = \mathbf{Vect}_{\mathbb{Z}_2}$
two possible boundaries: *smooth* and *rough*	two module categories: $\mathbf{Vect}_{\mathbb{Z}_2}$ and **Vect**

Lattice surgery

topologically protected operations on surface codes via
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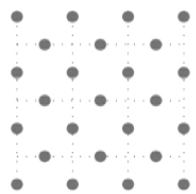

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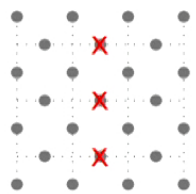

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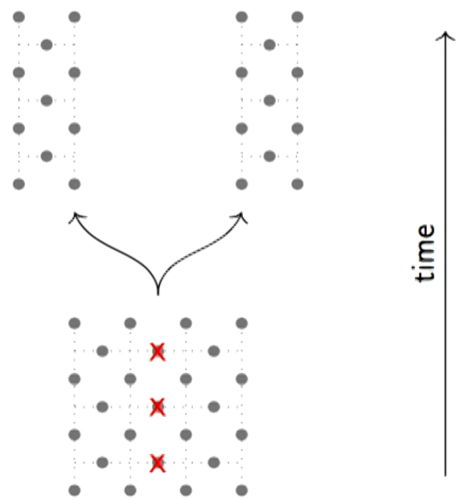

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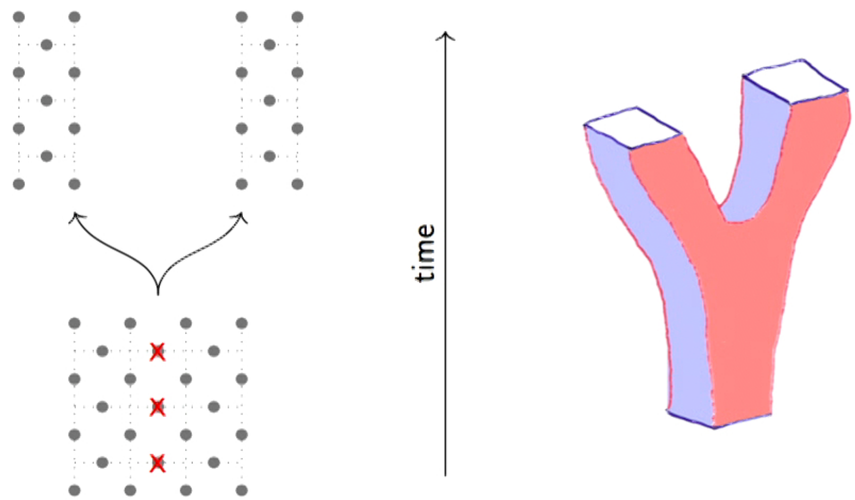

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The End

- Many open questions:
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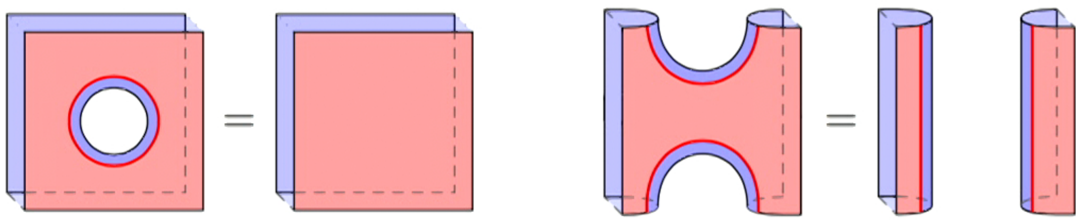

Seite 177 von 182

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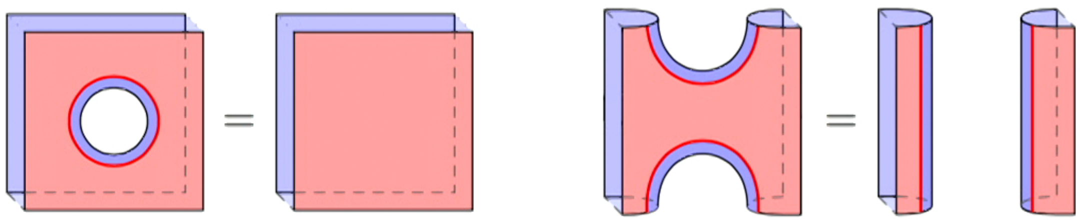
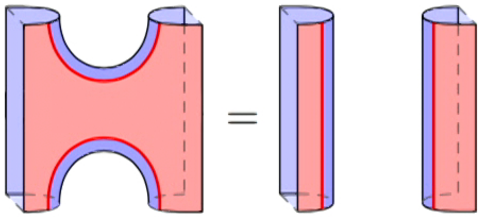


Thanks for listening!

Weak Hopf algebras

If we *drop* the second condition



Seite 179 von 182

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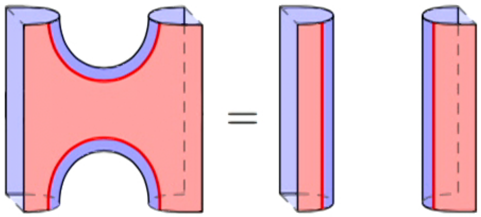
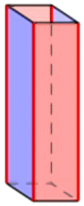


we only obtain a *weak* Hopf algebra on



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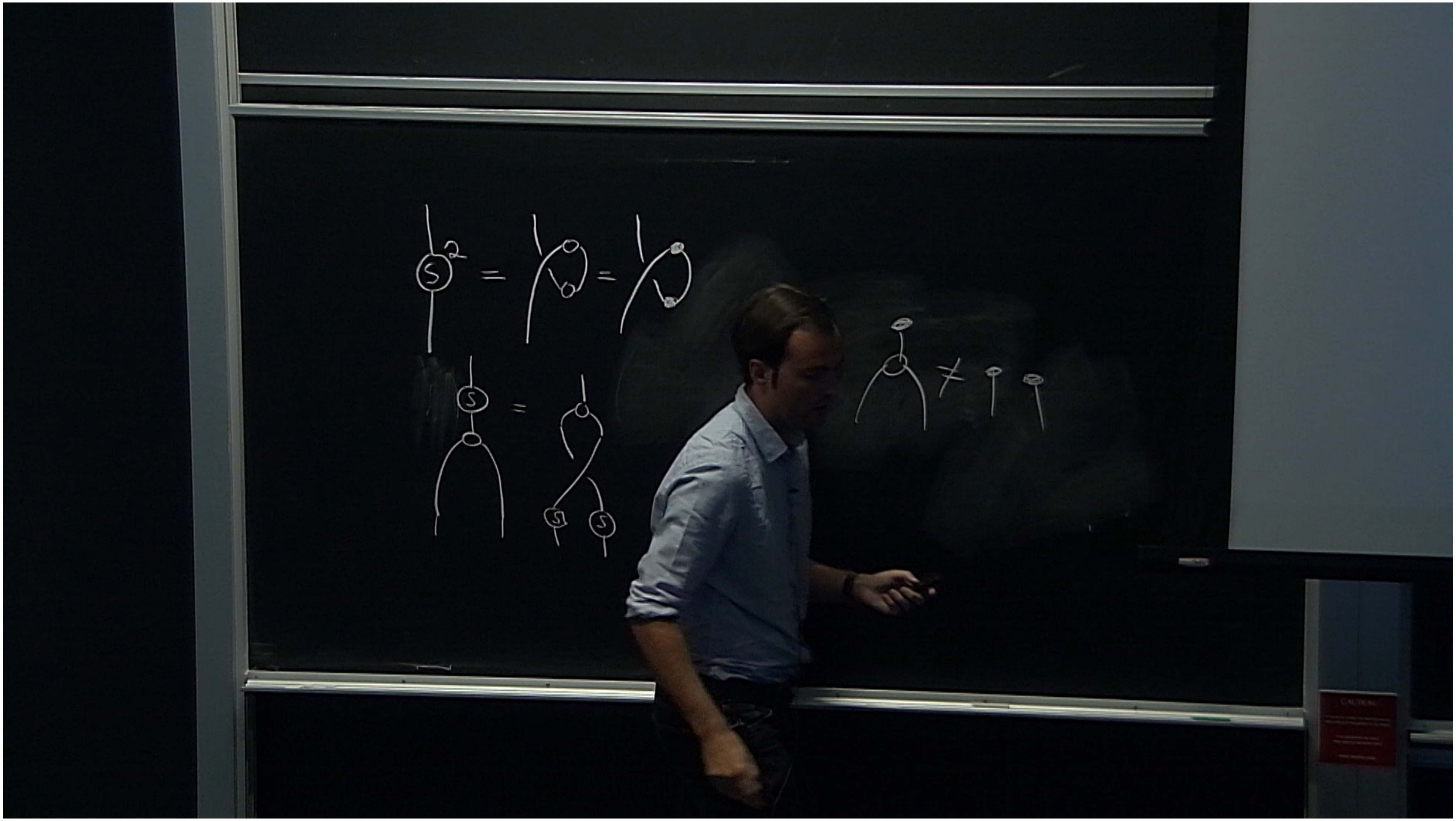
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Weak Hopf algebras

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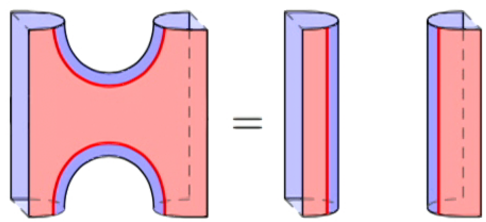

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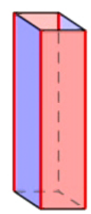

but have more functors $\mathbb{H} \to \mathbf{3Vect}$.
In fact, *every* fusion category induces such a functor. The corresponding Hopf algebra coincides with the Kitaev-Kong construction.

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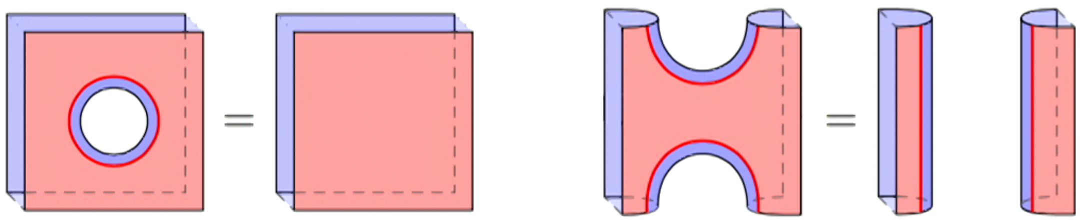

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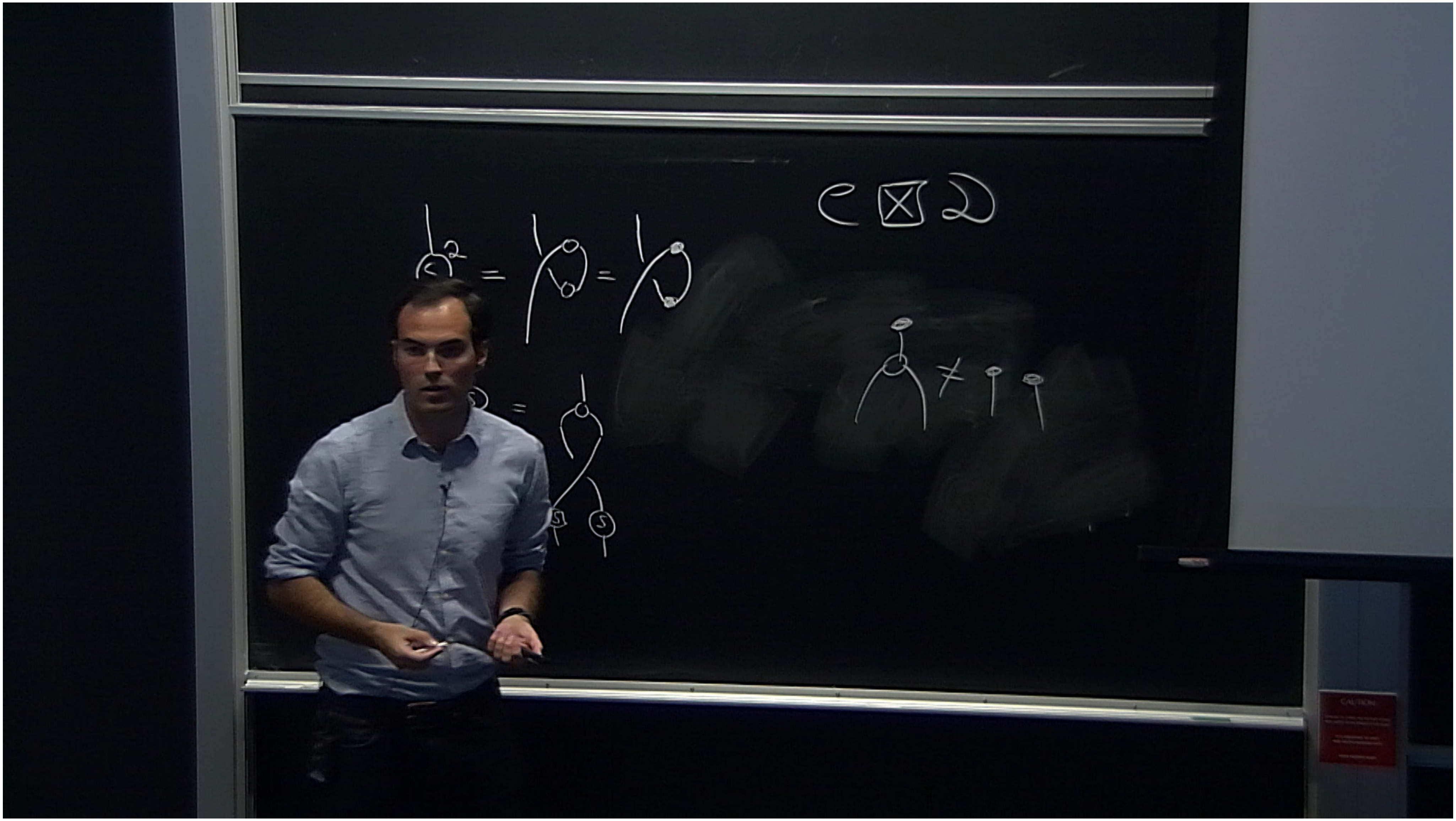

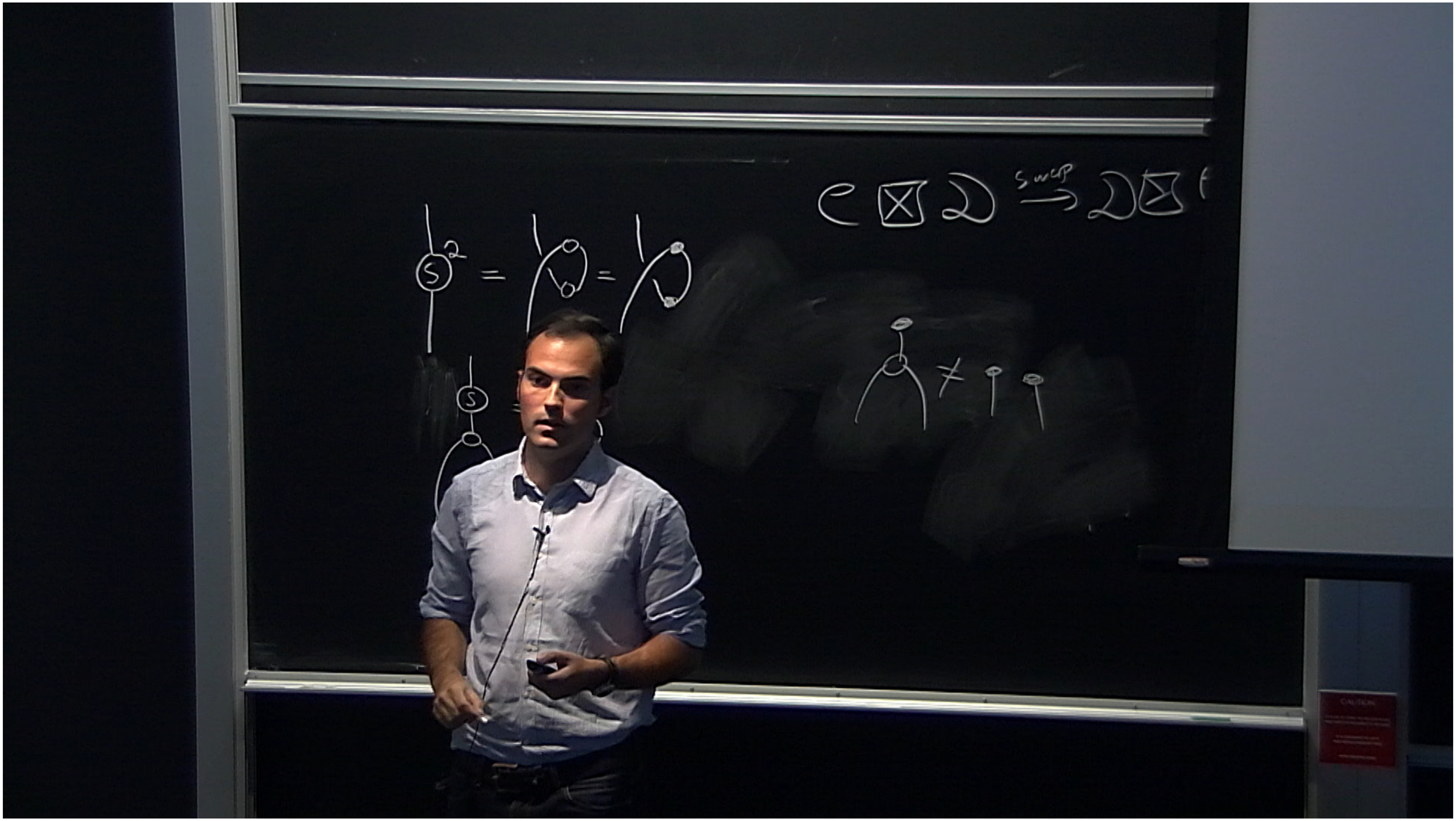
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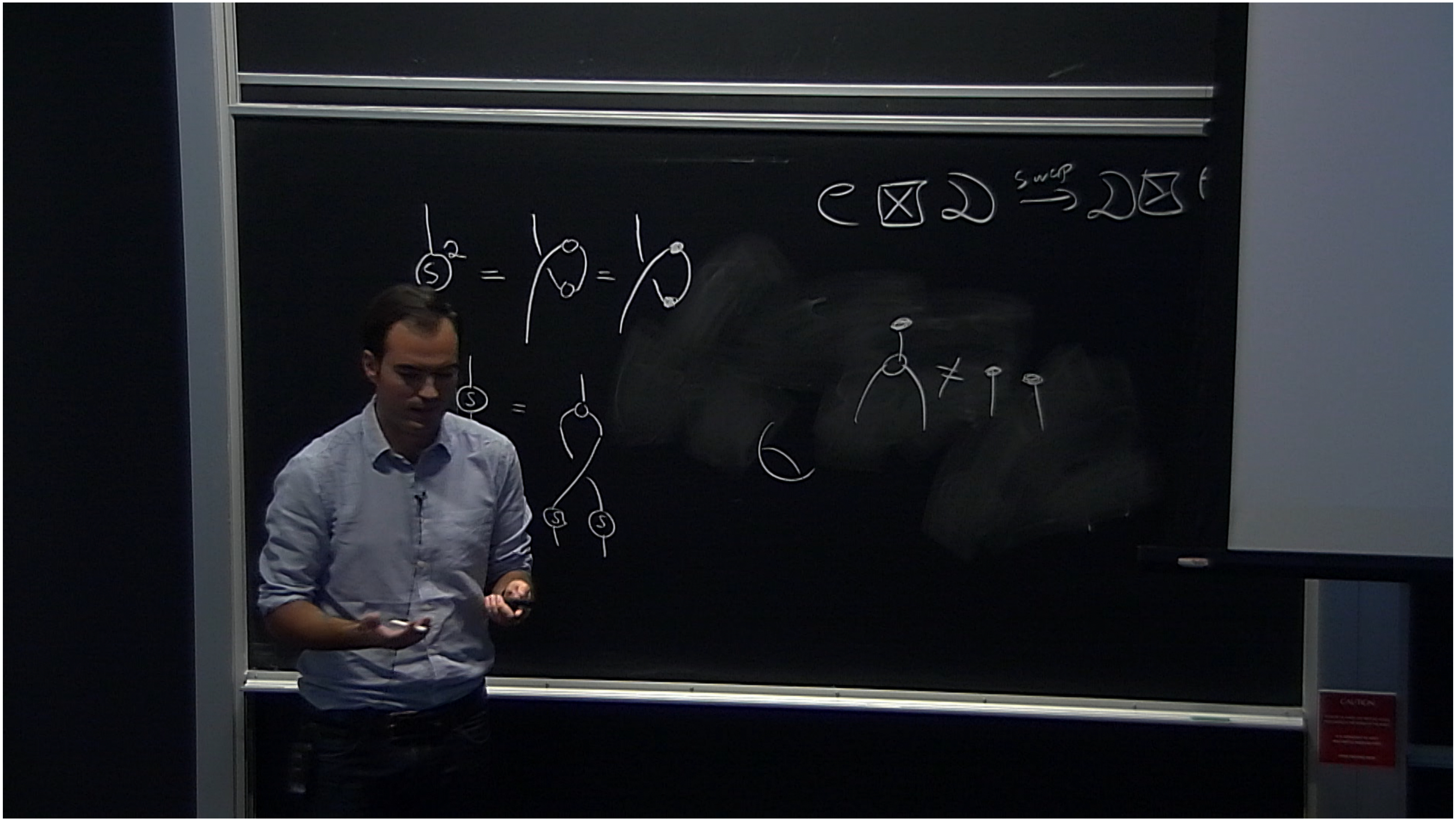
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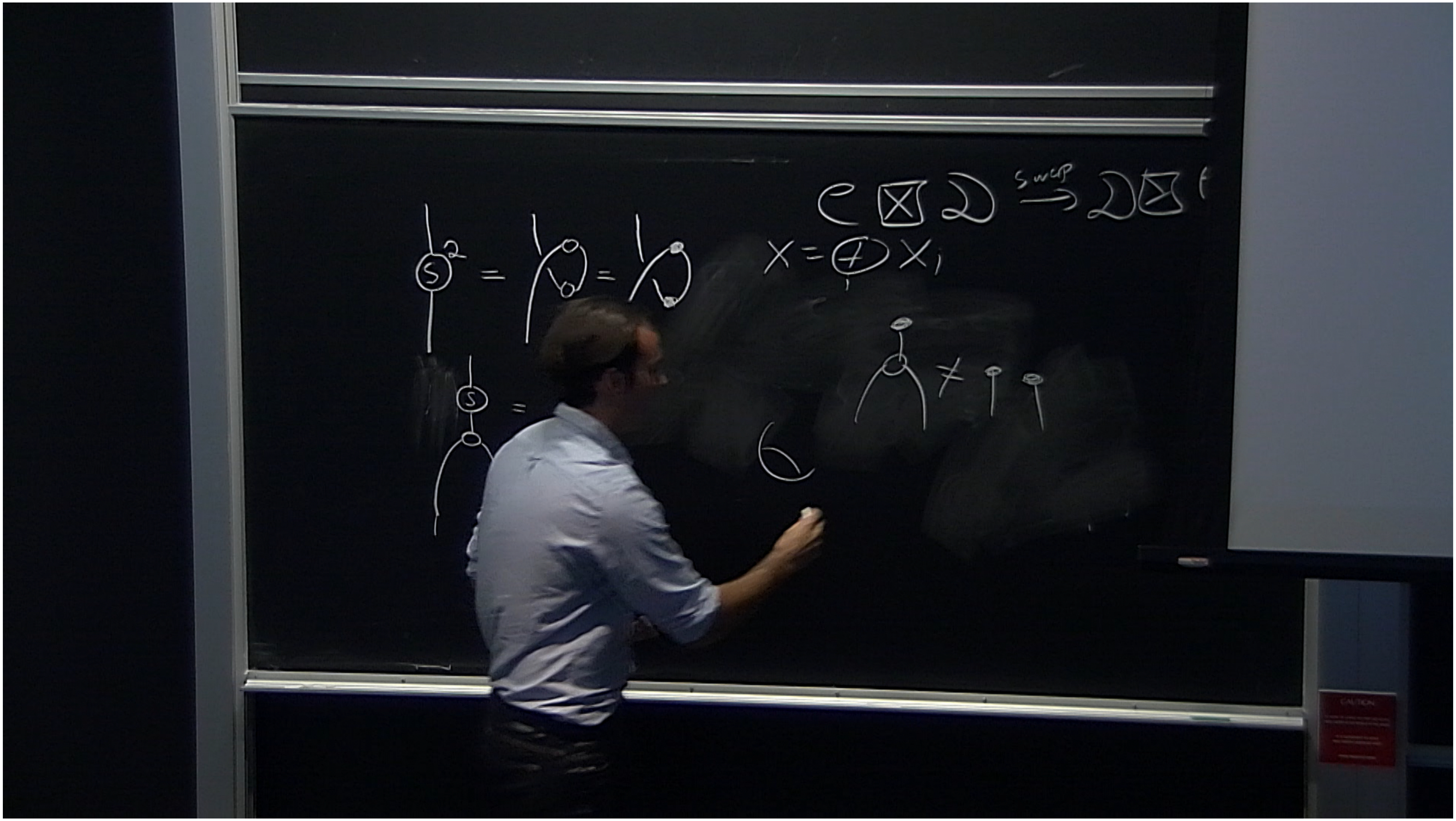
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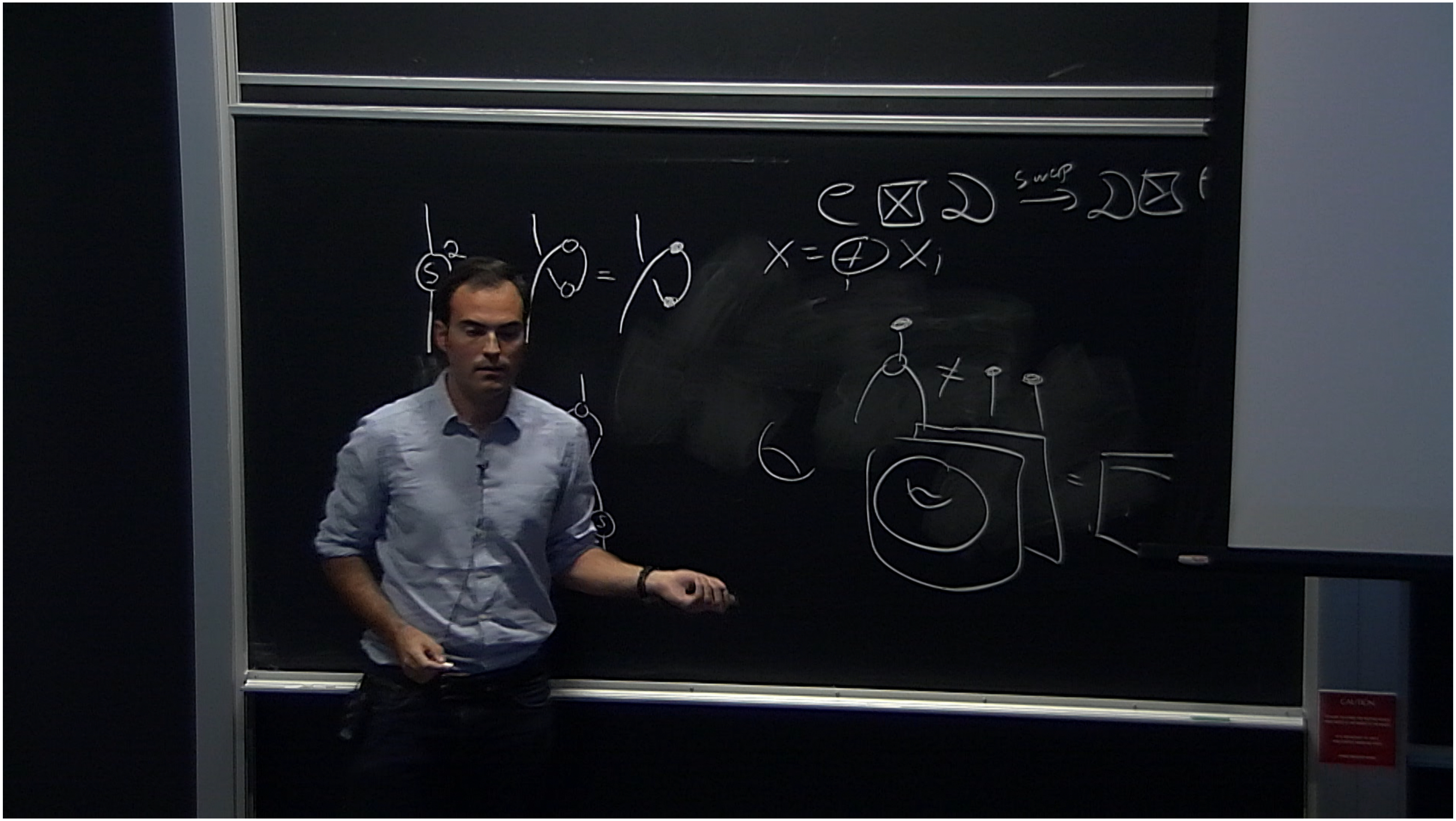
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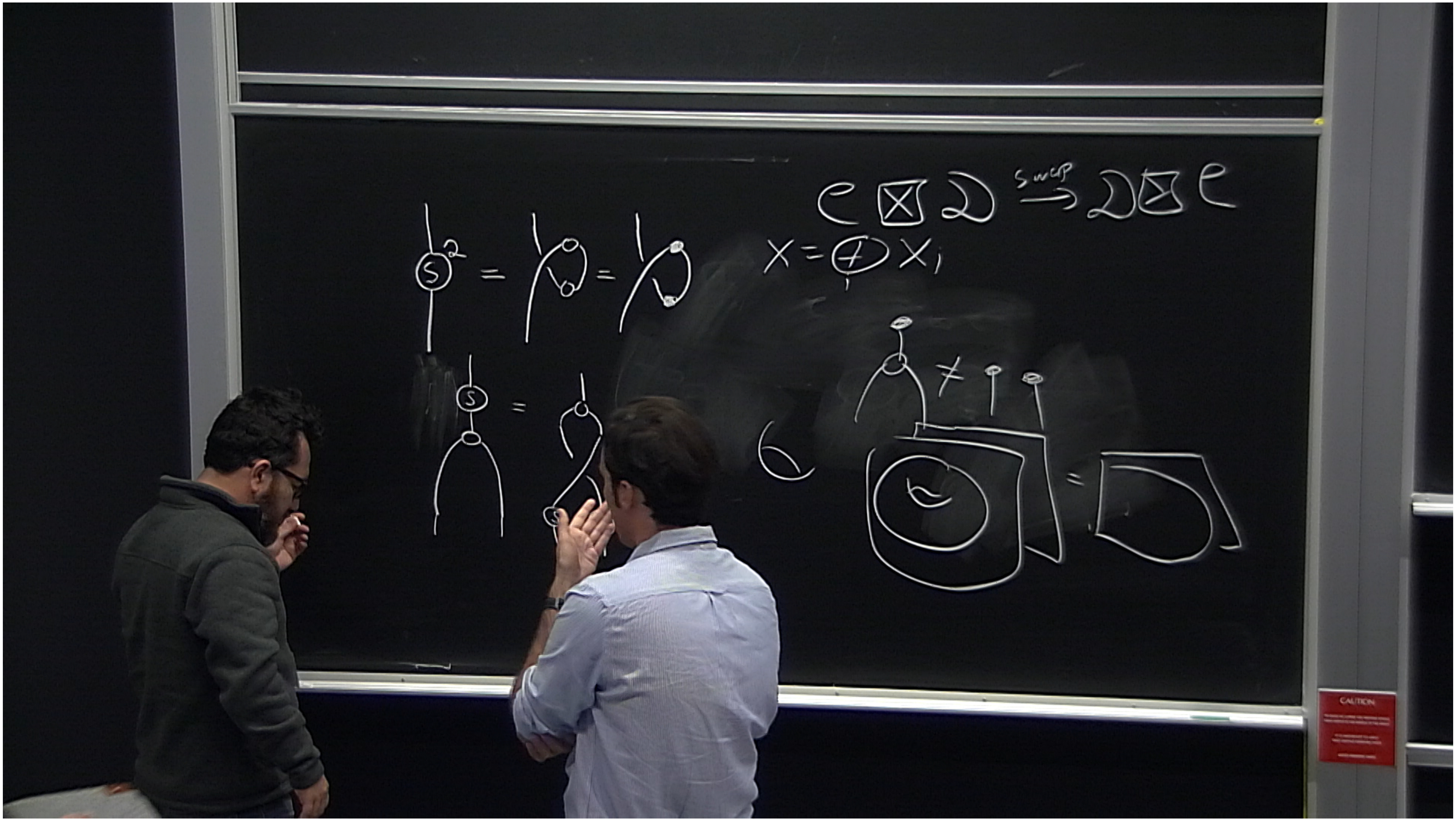
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