Title: Quantum computation with Turaev-Viro codes

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Abstract: The Turaev-Viro invariant for a closed 3-manifold is defined as the contraction of a certain tensor network. The tensors correspond to tetrahedra in a triangulation of the manifold, with values determined by a fixed spherical category. For a manifold with boundary, the tensor network has free indices that can be associated to qudits, and its contraction gives the coefficients of a quantum error-correcting code. The code has local stabilizers determined by Levin and Wen. By studying braid group representations acting on equivalence classes of colored ribbon graphs embedded in a punctured sphere, we identify the anyons, and give a simple recipe for mapping fusion basis states of the doubled category to ribbon graphs. Combined with known universality results for anyonic systems, this provides a large family of schemes for quantum computation based on local deformations of stabilizer codes. These schemes may serve as a starting point for developing fault-tolerance schemes using continuous stabilizer measurements and active error-correction.

This is joint work with Greg Kuperberg and Ben Reichardt.

Quantum computation with Turaev-Viro codes

Robert König

joint work with Greg Kuperberg and Ben Reichardt

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Perimeter Institute,
August 4, 2017

Technische Universität München

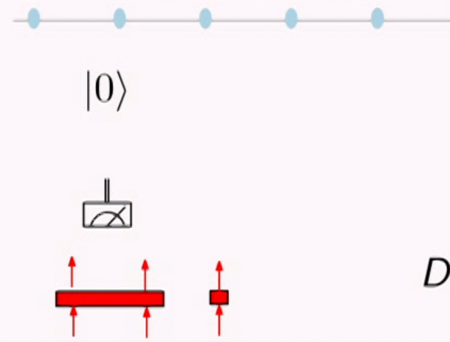
Outline of talk

- Motivation: **quantum fault-tolerance**
- Case study: **Kitaev's toric code**
  - ground states
  - mapping class group representation
  - protected gates
- Our work: The **Turaev-Viro code**
  - relationship to 3-manifold invariants
  - ground states
  - mapping class group representations
  - protected gates

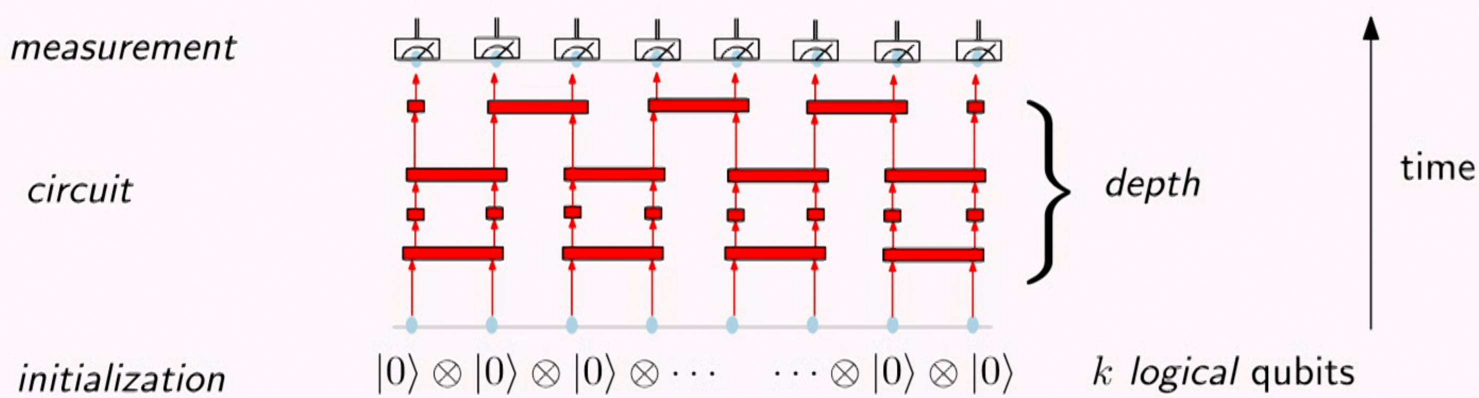
Quantum fault-tolerance: the DiVincenzo criteria

DiVicenzo criteria for fault-tolerant quantum computation

1. scalable physical system with well-characterized qubits
2. ability to initialize fiducial state
3. decoherence times $\gg$ gate operation time
4. qubit-specific measurement capability
5. universal set of quantum gates



David DiVincenzo



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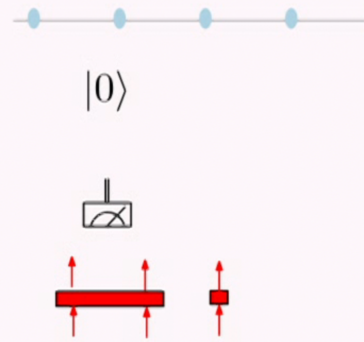


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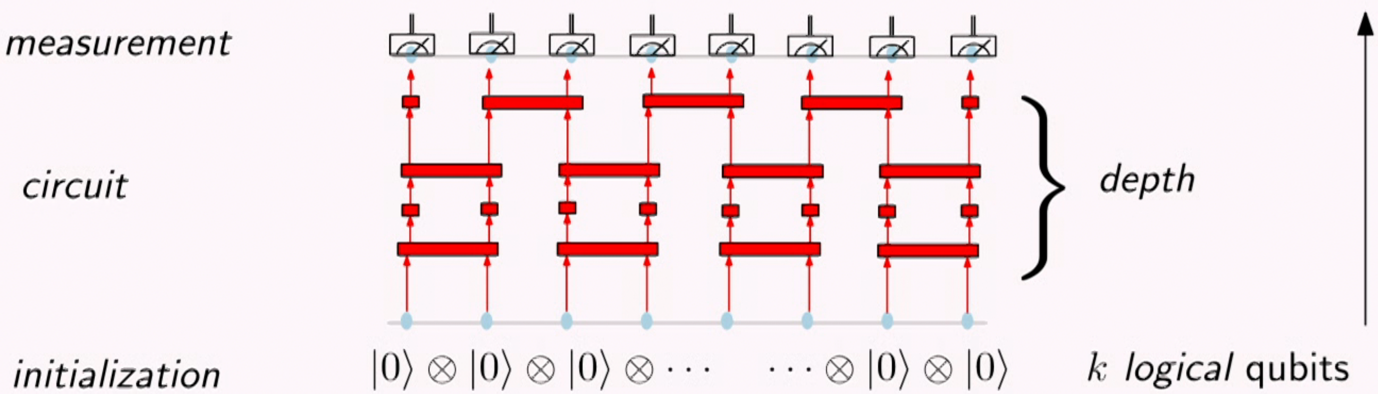


Quantum noise on n qubits

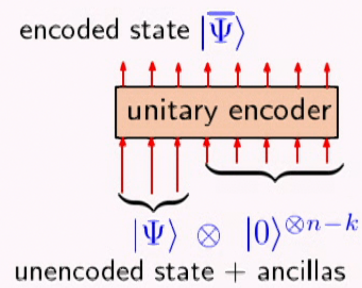
Quantum noise on n qubits is represented by a completely positive trace-preserving map (CPTPM)

$$\mathcal{N} : \mathcal{B}((\mathbb{C}^2)^{\otimes n}) \to \mathcal{B}((\mathbb{C}^2)^{\otimes n})$$

Operational problem: can we recover information subjected to such noise?

Procedure: (isometrically) embed/ "encode"

$$\begin{array}{ccc} (\mathbb{C}^2)^{\otimes k} & \to & \mathcal{L} \subset (\mathbb{C}^2)^{\otimes n} \\ \Psi & \mapsto & \overline{\Psi} \end{array}$$



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Using the Kraus decomposition $\mathcal{N}(\rho) = \sum_{E \in \mathcal{E}} E \rho E^\dagger$
it can be shown that it suffices to protect against against a certain set of errors $\mathcal{E}$
where an error is a linear map $E : (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$

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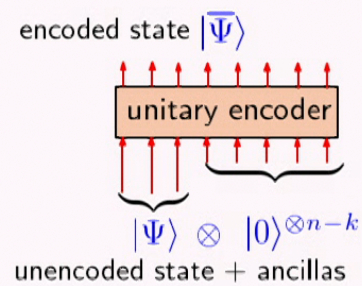

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Mathematical problem: Is there a recovery CPTPM $\mathcal{R} : \mathcal{B}((\mathbb{C}^2)^{\otimes n}) \to \mathcal{B}((\mathbb{C}^2)^{\otimes n})$

such that for ``suitable'' ρ $\quad \mathcal{R}(E\rho E^\dagger) \propto \rho \qquad \text{for all } E \in \mathcal{E}$

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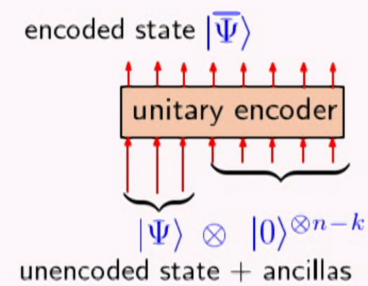

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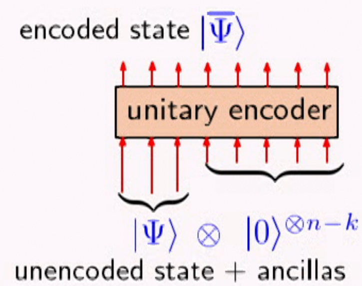

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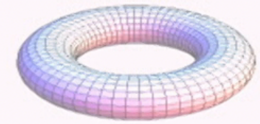
QEC condition:
[Knill, Laflamme]

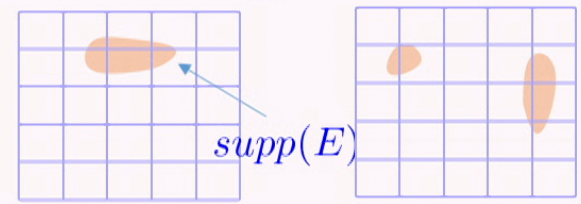
$\mathcal{L}$ protects against errors $\mathcal{E}$ $\quad\Leftrightarrow\quad$ $\langle\overline{\Psi}|E^\dagger F|\overline{\varphi}\rangle = c(E,F)\langle\overline{\Psi}|\overline{\varphi}\rangle$
for all $E, F \in \mathcal{E}$, $\overline{\Psi}, \overline{\varphi} \in \mathcal{L}$

``Topological'' error-correcting codes

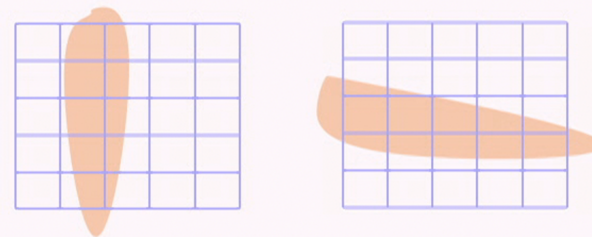


Def: A **"topological" code**:

protects against all local errors, e.g., and more generally errors with "topologically trivial" support



does not protect against errors with topologically non-trivial support, e.g.,



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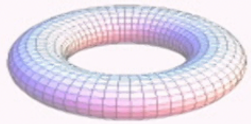

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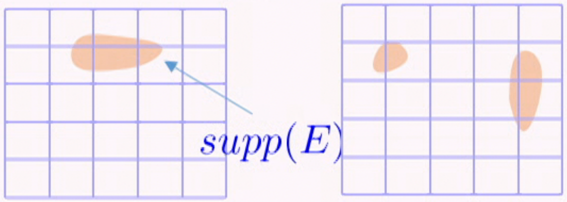

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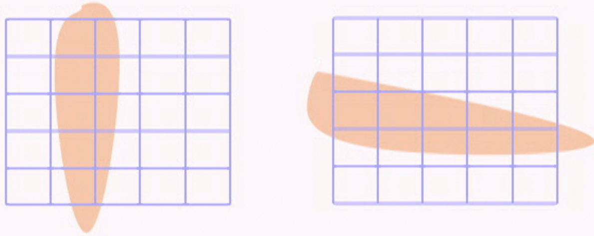
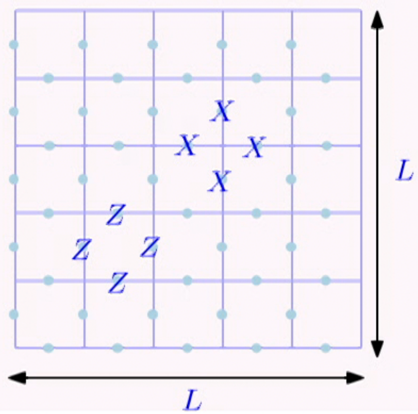


Example: Kitaev's toric code



$n = 2L^2$ qubits on the edges of a edges of a $L \times L$ periodic lattices

$$\mathcal{L} = \{\Psi \in (\mathbb{C}^2)^{\otimes n} \,|A_v\Psi = B_p\Psi = \Psi \quad \text{for all } v, p\}$$

$A_v = X^{\otimes 4}$ for each vertex v

$B_p = Z^{\otimes 4}$ for each plaquette p

$k = \log_2 \dim \mathcal{L} = 2$ encoded qubits

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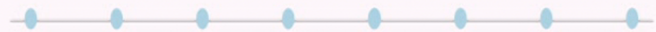

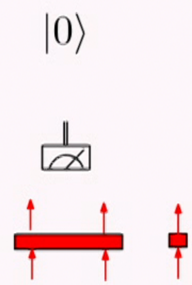

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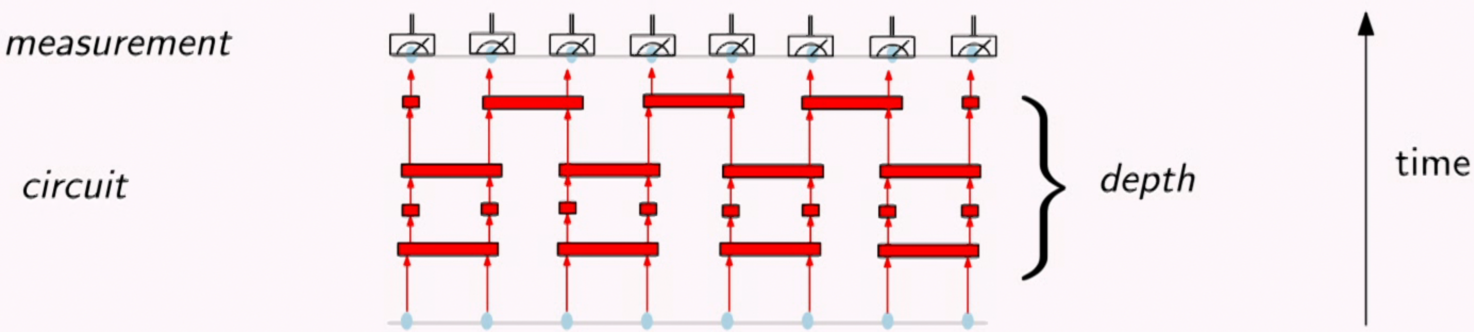

initialization $|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \cdots \quad \cdots \otimes |0\rangle \otimes |0\rangle$ k *logical* qubits

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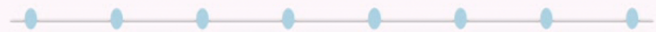
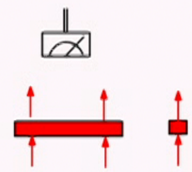


$|0\rangle$



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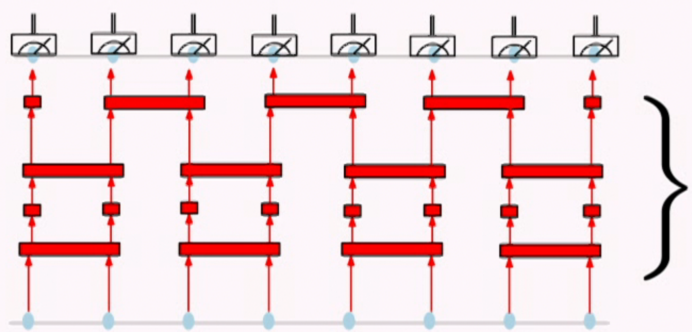
measurement

circuit



depth

time

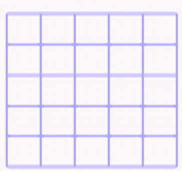
initialization $|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \cdots \quad \cdots \otimes |0\rangle \otimes |0\rangle$ k *logical* qubits

Logical operators and gates

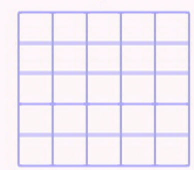
Given: error-correcting code $\mathcal{L} \cong (\mathbb{C}^2)^{\otimes k} \subset (\mathbb{C}^2)^{\otimes n}$

A operator $F : (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ is **logical** if $F\mathcal{L} \subset \mathcal{L}$.

A logical unitary $U : (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ is an **implementation** of a unitary $U_L : (\mathbb{C}^2)^{\otimes k} \to (\mathbb{C}^2)^{\otimes k}$ if



$$\begin{tikzcd}
\Psi \in (\mathbb{C}^2)^{\otimes k} \arrow[r, "U_L"] \arrow[d, "\text{encode}"] & U_L\Psi \in (\mathbb{C}^2)^{\otimes k} \arrow[d, "\text{encode}"] \\
\overline{\Psi} \in \mathcal{L} \subset (\mathbb{C}^2)^{\otimes n} \arrow[r, "U"] & U\overline{\Psi} \in \mathcal{L} \subset (\mathbb{C}^2)^{\otimes n}
\end{tikzcd}$$



Goal: characterize unitaries $U_L : (\mathbb{C}^2)^{\otimes k} \to (\mathbb{C}^2)^{\otimes k}$ which have "fault-tolerant" implementations

i.e., unitary automorphisms of $\mathcal{L}$ with certain properties

The code space of Kitaev's toric code

Logical operators in Kitaev's toric code

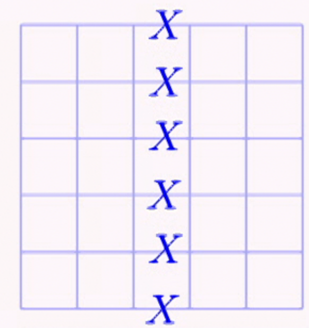
The operators $\overline{X}_1, \overline{Z}_1, \overline{X}_2, \overline{Z}_2$

- preserve the code space $\mathcal{L}$, i.e., are *logical*
- satisfy Pauli commutation relations

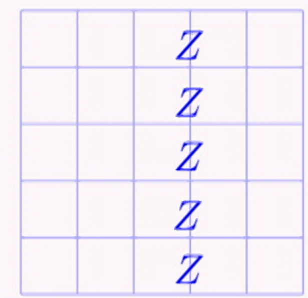
$\Rightarrow$ They define a factorization of the code space $\mathcal{L} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ such that

$\overline{X}_1 \cong X \otimes I$
$\overline{Z}_1 \cong Z \otimes I$
$\overline{X}_1 \cong X \otimes I$
$\overline{X}_2 \cong I \otimes X$

$\overline{X}_1 =$



$\overline{Z}_2 =$



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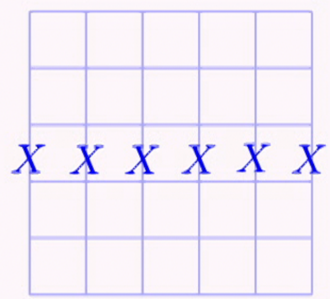
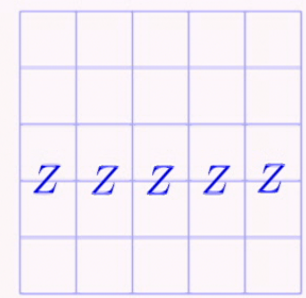
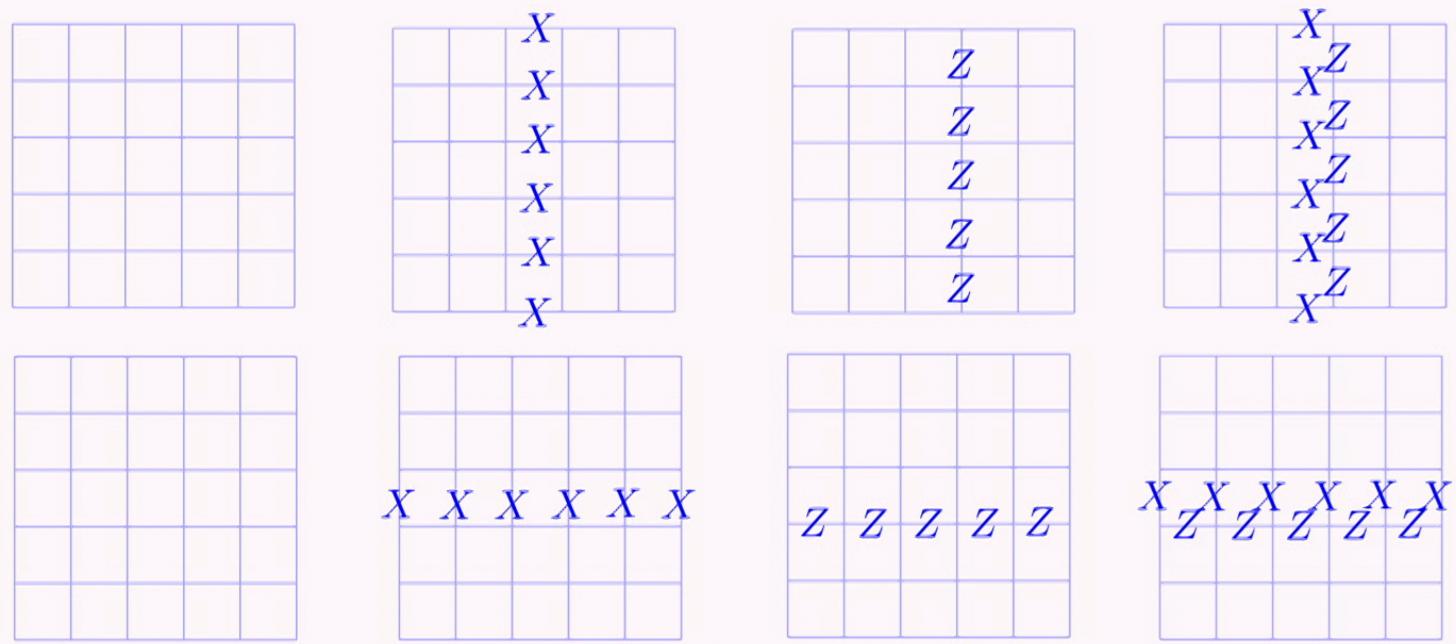


$\overline{Z}_1 =$



Logical operators in Kitaev's toric code: commuting subalgebras



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these 4 commute:

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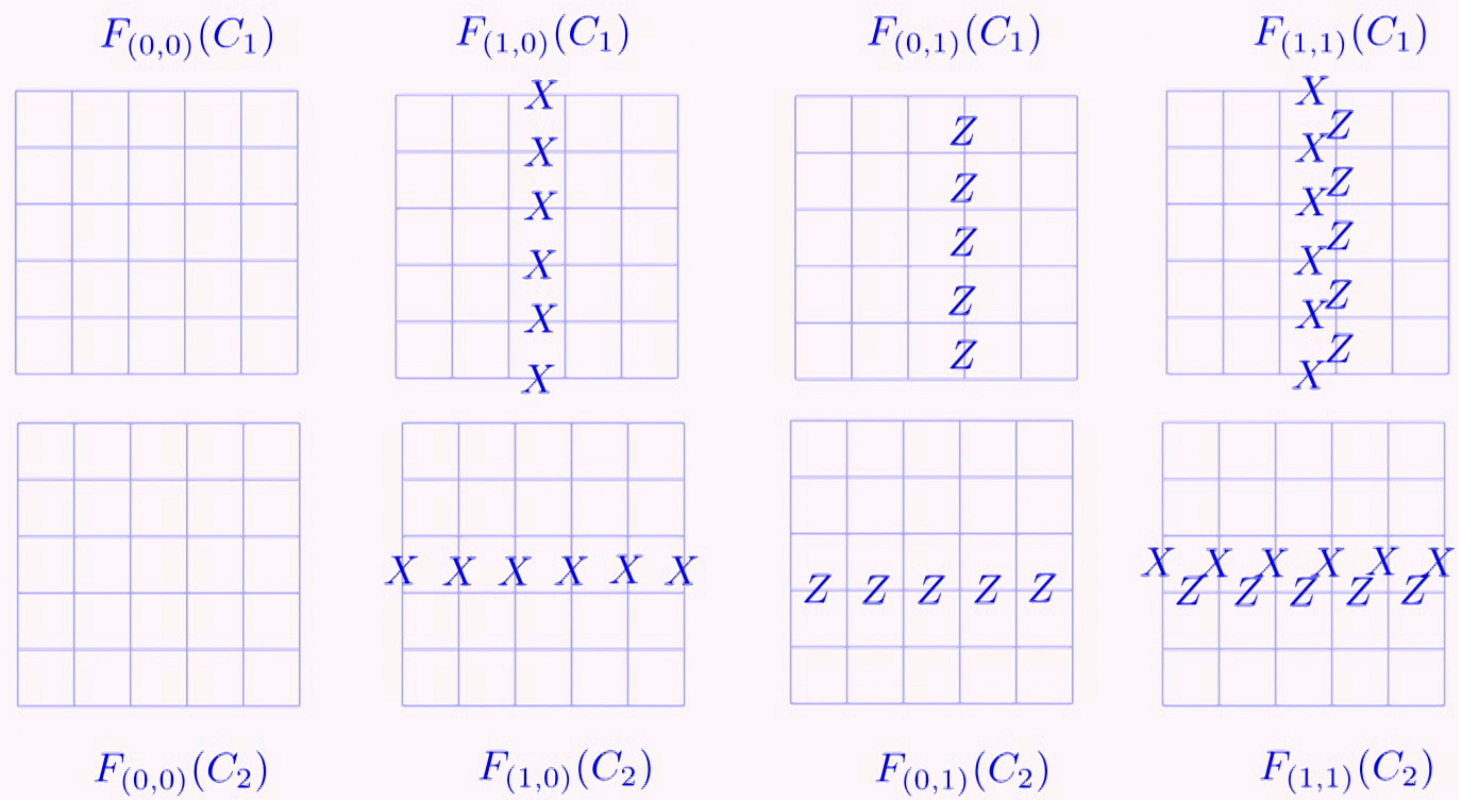
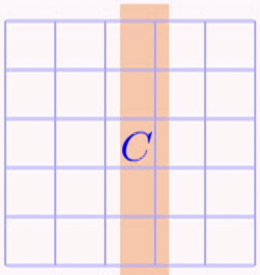


``Flux''-basis states associated with loops on a torus



$F_{(\alpha,\beta)}(C) =$ 

$\Rightarrow$ For every closed, non-contractible loop C, there is a family of logical operators $\{F_{(\alpha,\beta)}(C)\}_{(\alpha,\beta)\in\mathbb{Z}_2\times\mathbb{Z}_2}$ satisfying

$$F_{(\alpha,\beta)}(C)F_{(\alpha',\beta')}(C) = F_{(\alpha+\alpha',\beta+\beta')}(C)$$

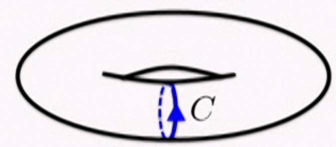
i.e., these form a representation of the **Verlinde algebra** $\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$

$$(\alpha,\beta) * (\alpha',\beta') = (\alpha+\alpha',\beta+\beta')$$

we can use the following 4 orthogonal projections to label basis states of the code space:

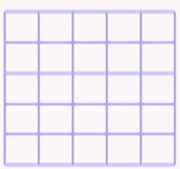
idempotents

$$\begin{aligned}
P_{(0,0)}(C) &= \tfrac{1}{2}(\text{id} + X^{\otimes L})\cdot\tfrac{1}{2}(\text{id} + Z^{\otimes L}) && |1\rangle_C \\
P_{(1,0)}(C) &= \tfrac{1}{2}(\text{id} - X^{\otimes L})\cdot\tfrac{1}{2}(\text{id} + Z^{\otimes L}) && |e\rangle_C \\
P_{(0,1)}(C) &= \tfrac{1}{2}(\text{id} + X^{\otimes L})\cdot\tfrac{1}{2}(\text{id} - Z^{\otimes L}) && |m\rangle_C \\
P_{(1,1)}(C) &= \tfrac{1}{2}(\text{id} - X^{\otimes L})\cdot\tfrac{1}{2}(\text{id} - Z^{\otimes L}) && |\epsilon\rangle_C
\end{aligned}$$



Every non-contractible closed loop C gives rise to a basis $\mathcal{B}_C$ of the code space

Fault-tolerant gates (on Kitaev's toric code)



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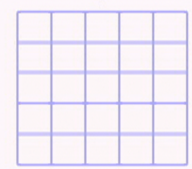

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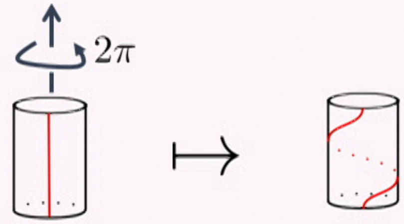
Fault-tolerant execution logical gates: two ways

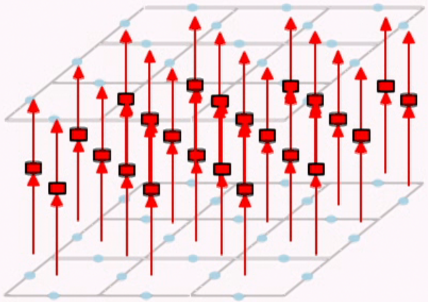
1) Apply code deformation (sequence of codes)

- generalizes to other models: mapping class group representation
- gives universal gate sets (in certain models)!



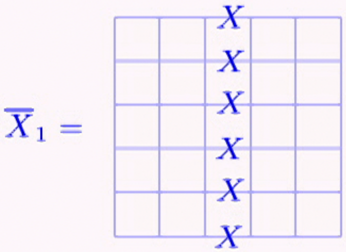
2) Apply a short (transversal) quantum circuit

- gives certain Clifford operations
- generalization?

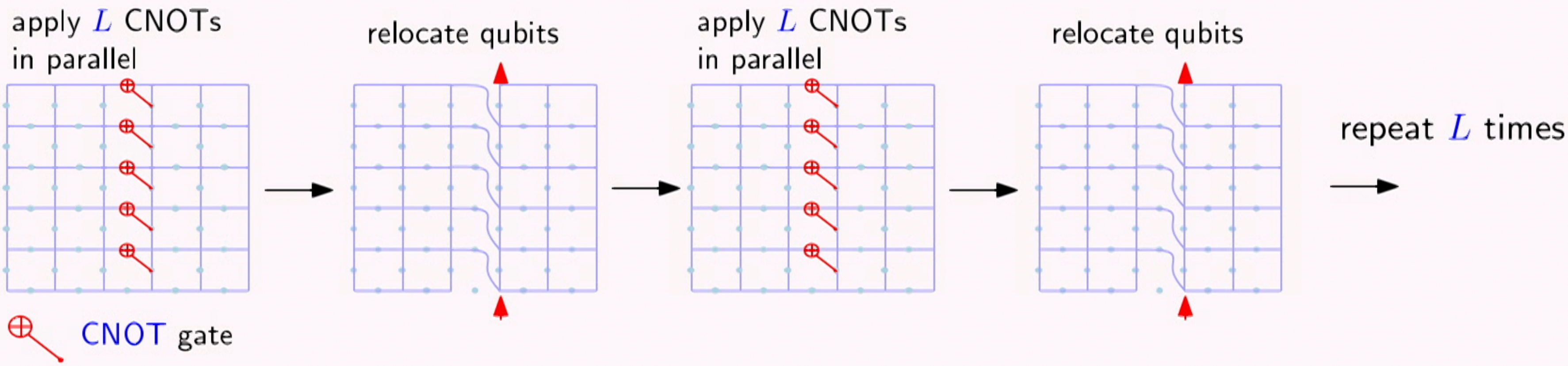


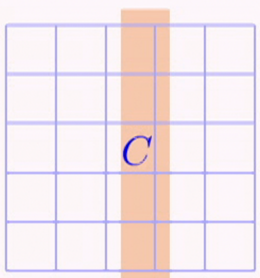
Special case: apply a string-operator

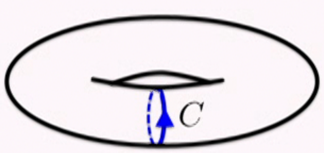
- only gives logical Pauli operators
- does not generalize



Mapping class group representation and toric code

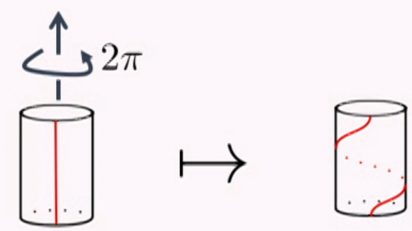




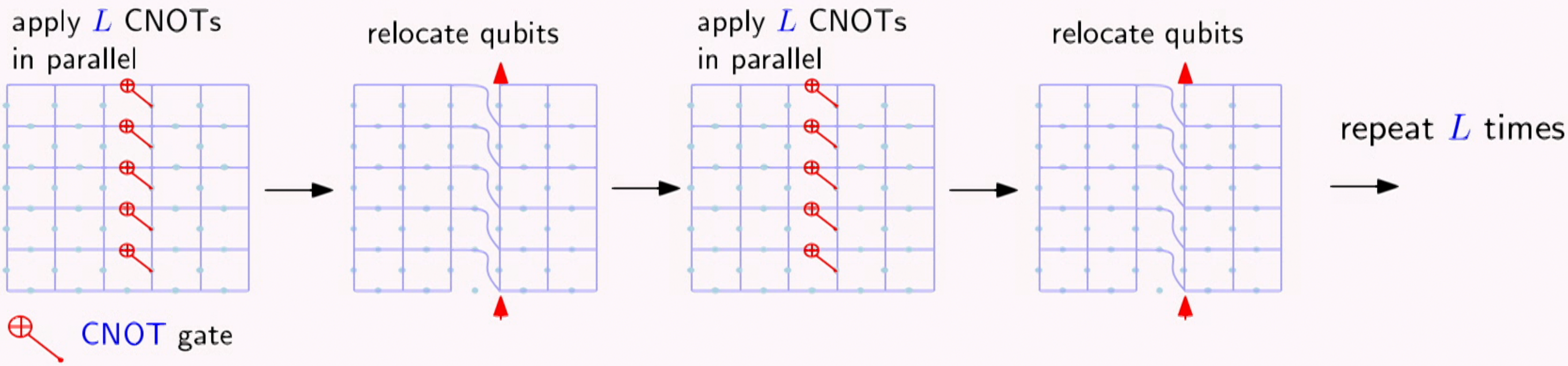


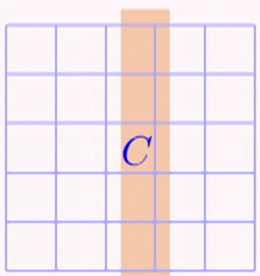
basis states of the code space:

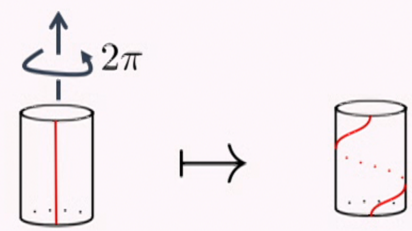
$\|1\rangle_C$	are **eigenvectors**	1
$\|e\rangle_C$	**of this operation**	-1
$\|m\rangle_C$	with eigenvalues	-1
$\|\epsilon\rangle_C$		1



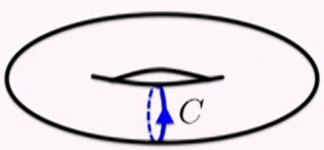
Mapping class group representation and toric code







$\Rightarrow$ For every closed, no-contractible loop C, there is a logical gate $U(C)$ implementable in depth L



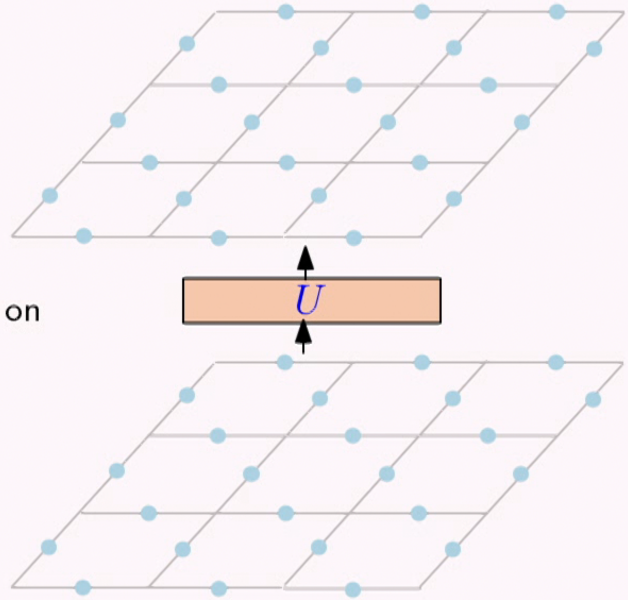
Each C defines an element $\vartheta_C \in$ MCG of the mapping class group of the torus (twisting along C).

$\vartheta_C \mapsto U(C)$ gives a (projective) representation of MCG

Transversal gates are protected

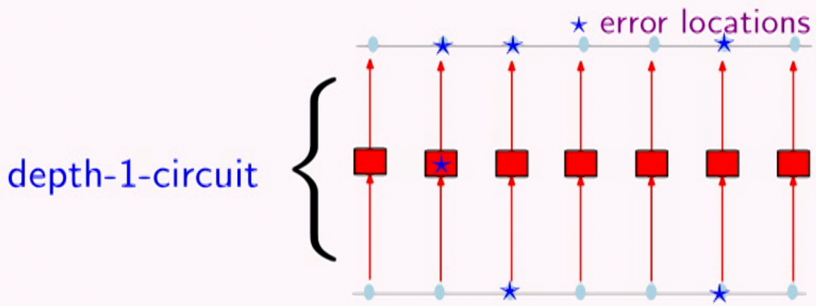
unitary U
preserving codespace $\mathcal{L}$:

fault-tolerance
properties depend on
structure of U



Transversal gates are protected

transversal gate ≡ implementable by a depth-1-circuit



unitary U
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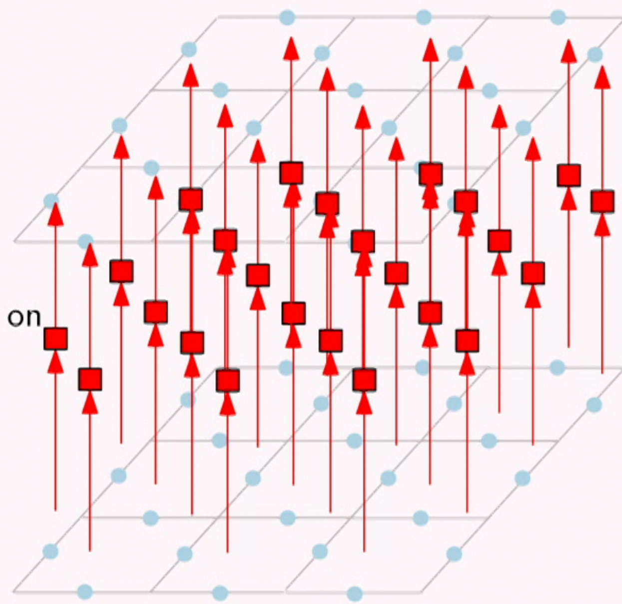
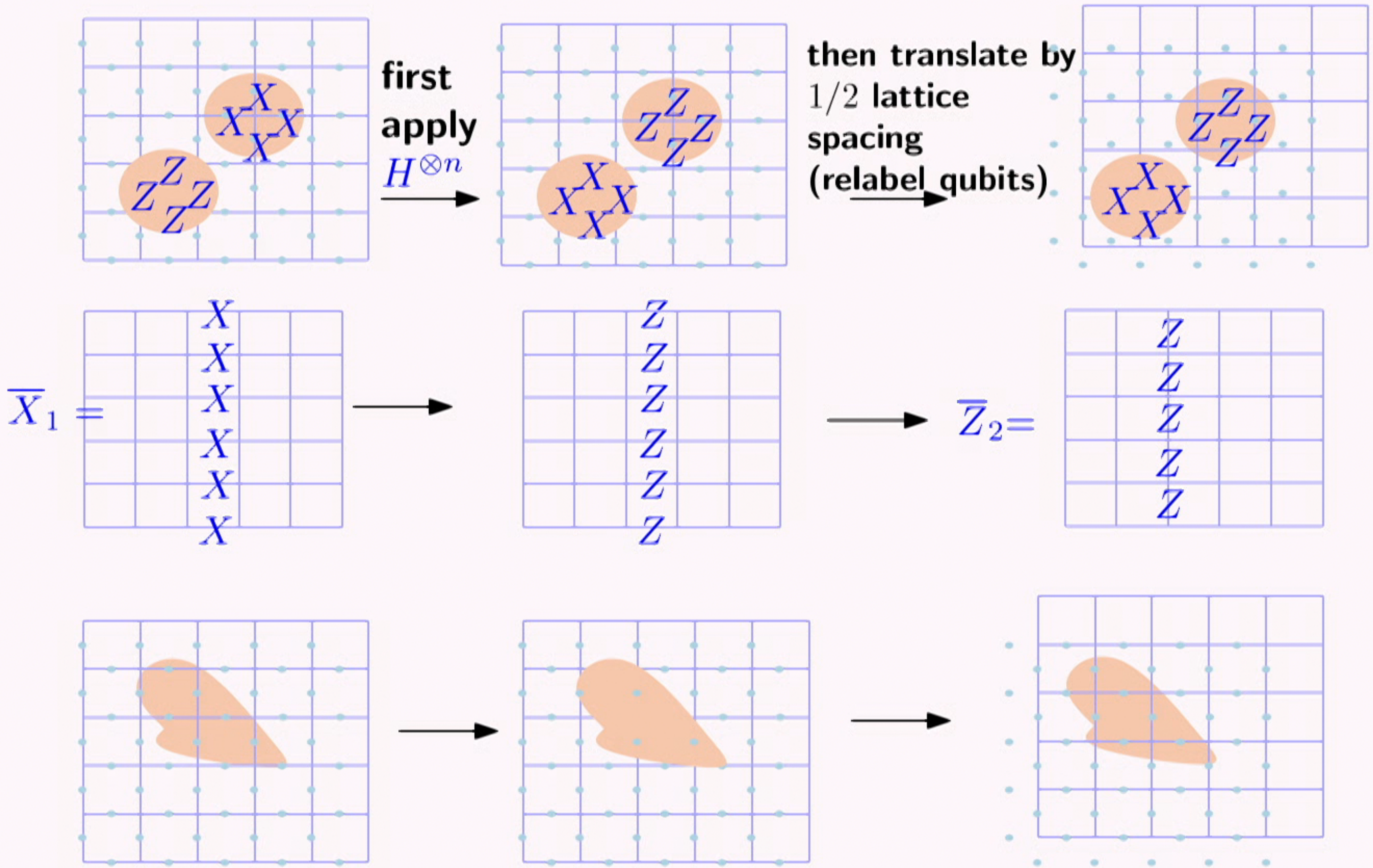


fault-tolerance properties depend on structure of U

when applying a transversal gate:

- preexisting errors do not spread
- faulty unitaries only introduce local errors

Example: Robust implementation of a gate in Kitaev's code



$\Rightarrow$ operation is logical

overall effect on logical operators:

$$\begin{array}{ccc} \bar{X}_1 & \mapsto & \bar{Z}_2 \\ \bar{Z}_1 & \mapsto & \bar{X}_2 \\ \bar{X}_2 & \mapsto & \bar{Z}_1 \\ \bar{Z}_2 & \mapsto & \bar{X}_1 \end{array}$$

implements the gate SWAP $\circ\, (H \otimes H)$

in a **locality-preserving way**: support of errors only minimally changed

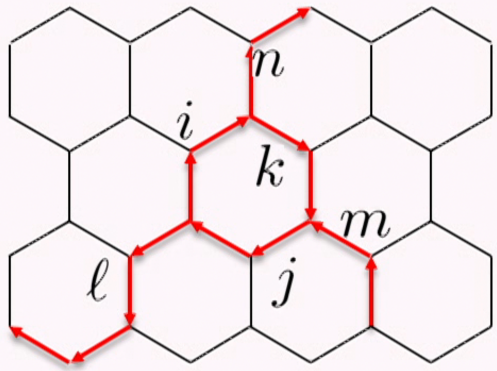
Table of contents

- Motivation: **quantum fault-tolerance**
- Case study: **Kitaev's toric code**
  - ground state (labeling)
  - mapping class group representation
  - protected gates
- Our work: The **Turaev-Viro code**
  - relationship to 3-manifold invariants
  - ground states
  - mapping class group representations
  - protected gates

The Levin-Wen/Turaev-Viro code

ingredients:

- finite set of “particle labels”
- involution operation on particle labels
- set of allowed triples
- scalars and a tensor



local Hilbert space $\mathbb{C}^d$
associated to every edge

Code space $\mathcal{L} \subset (\mathbb{C}^d)^{\otimes N}$

$$\mathcal{L} = \{|\Psi\rangle \mid B_p|\Psi\rangle = |\Psi\rangle \ \forall p, A_v|\Psi\rangle = |\Psi\rangle \ \forall v\}$$

Levin & Wen, Phys.Rev. B71 (2005) 045110

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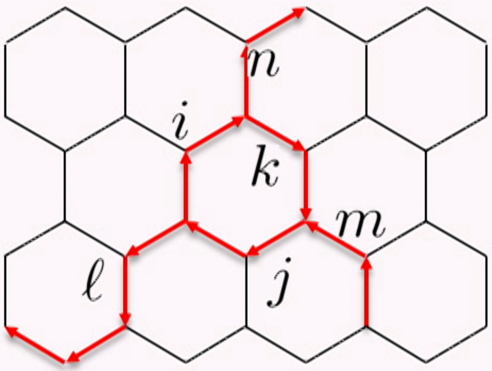
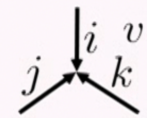


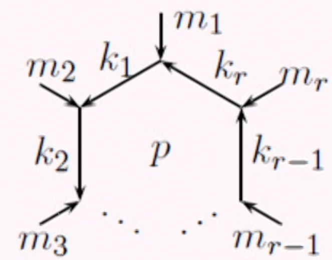
local Hilbert space $\mathbb{C}^d$
associated to every edge

vertex operator: $A_v = \sum_{(i,j,k) \text{ allowed}} |ijk\rangle\langle ijk|$



plaquette operator: $$B_p = \frac{1}{\mathcal{D}^2} \sum_{\vec{k},\vec{k}',\vec{m}} \sum_i d_i \left(\prod_{t=1}^{r} F^{m_t k_t^* k_{t-1}}_{i k'_{t-1} (k'_t)^*} \right) |\vec{k}', \vec{m}\rangle\langle \vec{k}, \vec{m}|$$

$|\vec{k}, \vec{m}\rangle =$



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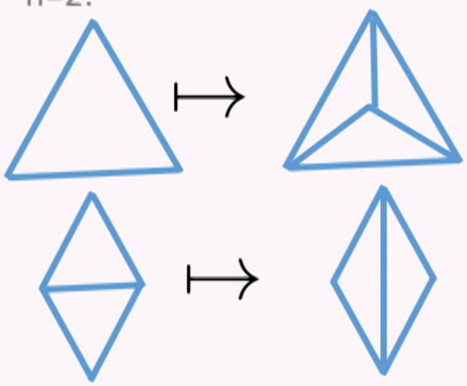
Manifold-invariants from triangulations

Consider **closed n-manifolds modulo homeomorphism**

FACT: For n=2,3, every equivalence class has a triangulated representative.

FACT (Pachner): n-manifolds homeomorphic $\Longleftrightarrow$
triangulations related sequence of Pachner moves.

Pachner moves: finite list of local changes of triangulation, e.g., in n=2:



Manifold-invariants from triangulations

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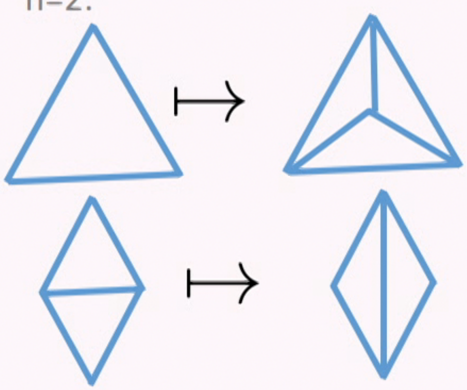
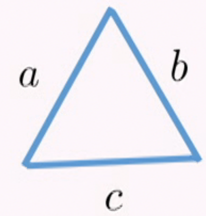


Recipe for constructing invariants:

- associate scalar to every triangulation
- show invariance under Pachner moves

Example: State-sum invariants



$\mapsto \quad F_{abc}$

associate scalar with (colored) triangle

define invariant by summing over edge colorings:

$$I(M) = \mathcal{D}^{-\#\text{triangles}} \sum_{\phi} \prod_{\text{triangles } t} g_t^{\phi}$$

triangulated 2-manifold

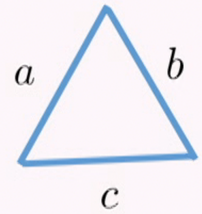
sum over all colorings

Compatibility with Pachner moves

$I($  $) = I($  $)$

$I($  $) = I($  $)$

Example: State-sum invariants



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Compatibility with Pachner moves

$I($  $) = I($  $)$

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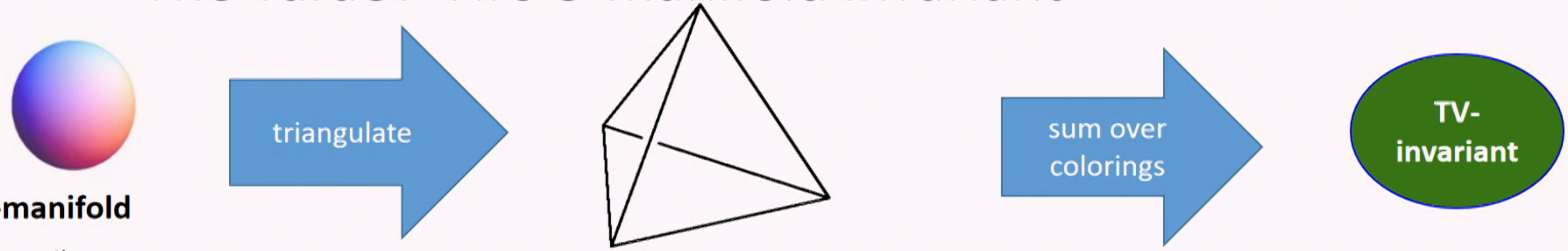
is equivalent to algebraic conditions

$$\mathcal{D}^{-1} F_{abc} = \mathcal{D}^{-3} \sum_{x,y,z} F_{axz} F_{xby} F_{zyc}$$

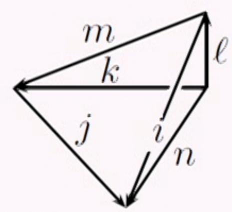
$$\sum_x F_{abx} F_{cxd} = \sum_y F_{ayc} F_{dyb}$$

The Turaev-Viro 3-manifold invariant



3-manifold

(closed)



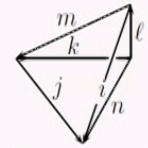
$$\mapsto \frac{F^{i^* j m}_{k \ell^* n}}{\sqrt{d_m d_n}}$$

scalar associated with (colored) tetrahedron

$$\mathsf{TV}(M) = \mathcal{D}^{-2|V_M|} \sum_{\text{colorings } \phi} \prod_{\text{edges } e} d_{\phi(e)} \prod_{\text{tetrahedra} t} g_t^\phi$$

sum over all ``allowed'' colorings

Algebraic conditions for invariance
(via Pachner moves)



$\mapsto \dfrac{F^{i^*jm}_{k\ell^*n}}{\sqrt{d_m d_n}}$

$$\mathrm{TV}_{\mathcal{C}}(M) = \mathcal{D}^{-2|V_M|} \sum_{\text{colorings } \phi} \prod_{\text{edges } e} d_{\phi(e)} \prod_{\text{tetrahedra} t} g_t^{\phi}$$

If

$$\left.\begin{aligned}
1 &= 1^* \\
d_1 &= 1 \\
d_i &= d_{i^*} \\
\mathcal{D} &= \sqrt{\textstyle\sum_i d_i^2} \\
d_i d_j &= \sum_k \delta_{ijk} d_k \\
\sum_m \delta_{ijm^*}\delta_{mkl^*} &= \sum_m \delta_{jkm^*}\delta_{iml^*} \\
F^{ijm}_{k\ell n}\delta_{ijm}\delta_{k\ell m^*} &= F^{ijm}_{k\ell n}\delta_{i\ell n}\delta_{jkn^*} \\
\textstyle\sum_n F^{m\ell q}_{kpn} F^{jip^*}_{mns} F^{jsn}_{\ell kr} &= F^{jip^*}_{q^*kr} F^{r^*iq^*}_{m\ell s} \\
(F^{ijm}_{k\ell n})^* &= F^{i^*j^*m^*}_{k^*\ell^*n^*} \\
F^{ijm}_{k\ell n} = F^{jim}_{\ell kn^*} = F^{\ell km^*}_{jin} &= F^{imj}_{k^*n\ell}\sqrt{\tfrac{d_m d_n}{d_j d_\ell}} \\
F^{ii^*1}_{j^*jk} &= \sqrt{\tfrac{d_k}{d_i d_j}}\,\delta_{ijk}
\end{aligned}\right\}$$

*: involution on set of colors

1: special color

$\delta_{ijk} \in \mathbb{N} \cup \{0\}$

$F^{ijm}_{k\ell n} \in \mathbb{R}$

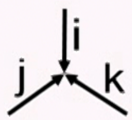
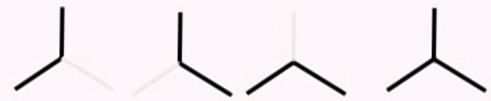
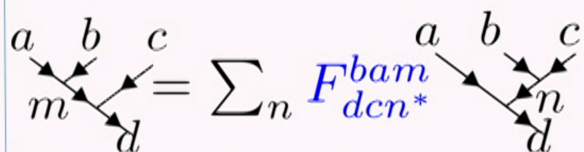
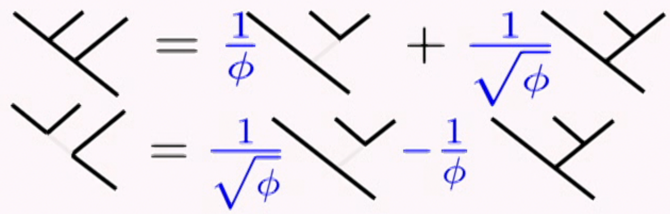
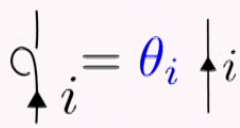
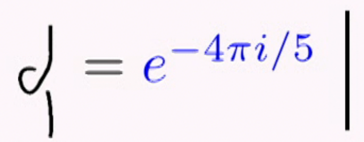
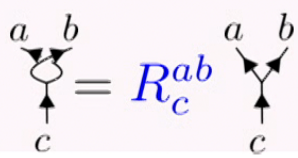
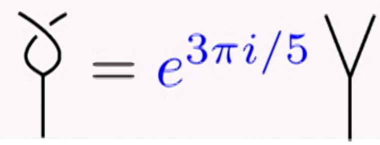
$d_i \in \mathbb{R}_{>0}$

then $\mathrm{TV}_{\mathcal{C}}$ is a 3-manifold invariant

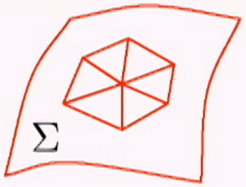
A *spherical category* $\mathcal{C}$ is/provides a solution to these equations.

(Barrett and Westbury, hep-th/9311155)

Data/example of a modular category

		modular category $\mathcal{C}$	example: category Fib
Unitary, braided, semisimple, *	Particles (colors)	$\{1, i, j, \ldots\}, *$ $\quad \uparrow i = \downarrow i^*$	$\{1, \tau\} \quad \tau^* = \tau$
	Fusion rules	 (set of) allowed triples $\delta_{ijk} \in \{0,1\}$	
	q-dim	$\bigcirc_i = d_i$	$\bigcirc = \phi = \frac{\sqrt{5}+1}{2}$
	F-matrix		
	top. phase		
	R-matrix		

The Turaev-Viro code $\subset (\mathbb{C}^d)^{\otimes |E|} \cong$ edge colorings of surface triangulation



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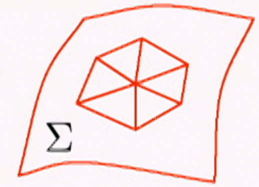
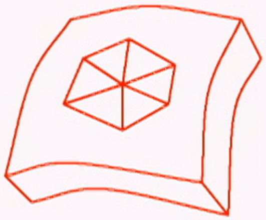
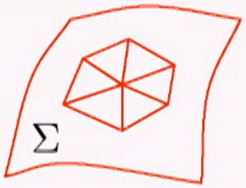


$\Sigma \times [-1, 1]$



(extend triangulation from $\Sigma \times \{\pm 1\}$)

The Turaev-Viro code $\subset (\mathbb{C}^d)^{\otimes |E|} \cong$ edge colorings of surface triangulation



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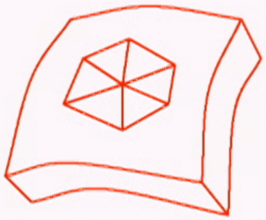

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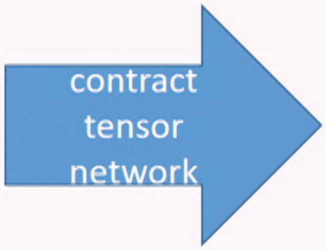

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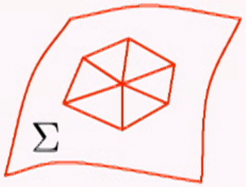

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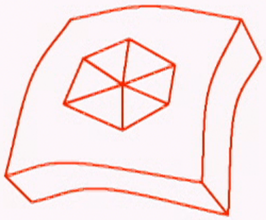


(extend triangulation from $\Sigma \times \{\pm 1\}$)

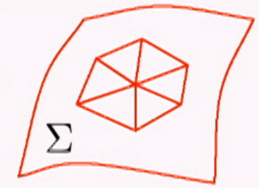
contract tensor network

$(\mathbb{C}^d)^{\otimes |E|}$

$\uparrow$

$(\mathbb{C}^d)^{\otimes |E|}$

The Turaev-Viro code $\subset (\mathbb{C}^d)^{\otimes |E|} \cong$ edge colorings of surface triangulation



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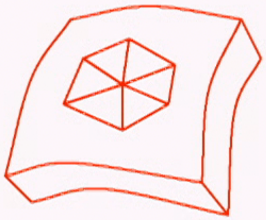

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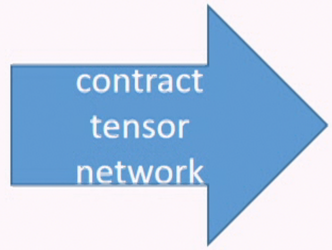
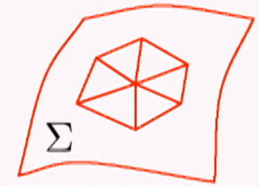


$(\mathbb{C}^d)^{\otimes |E|}$

$(\mathbb{C}^d)^{\otimes |E|}$

Turaev-Viro code: support of this projection in the Hilbert space $(\mathbb{C}^d)^{\otimes |E|}$

The Turaev-Viro code $\subset (\mathbb{C}^d)^{\otimes |E|} \cong$ edge colorings of surface triangulation



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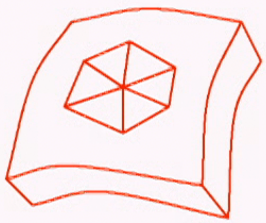

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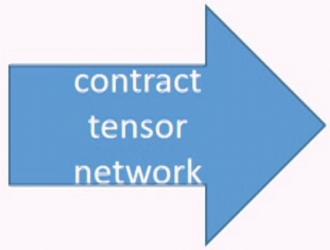


Turaev-Viro code: support of this projection in the Hilbert space $(\mathbb{C}^d)^{\otimes |E|}$

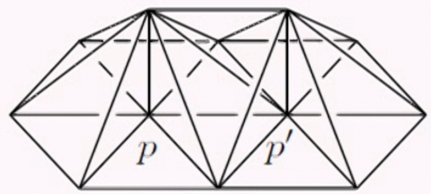
Local stabilizers: attaching blisters - set of local operators which are

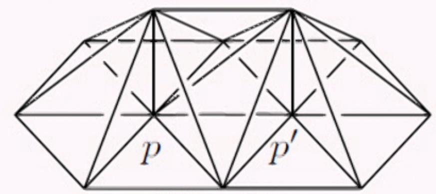
- projections
- mutually commuting
- stabilize code space

$\mapsto$

Blisters: properties from (manifold)invariance

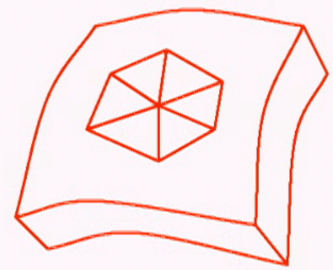
commuting:





$$B_p B_{p'} = B_{p'} B_p$$

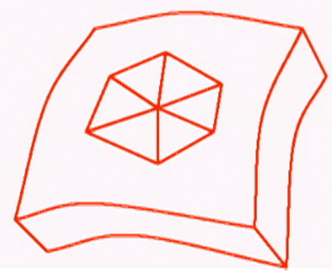
stabilize code space:



$$B_p P_{\mathrm{TV}} = P_{\mathrm{TV}}$$

project onto code space

$$\prod_p B_p =$$



The code space of the Turaev-Viro code

Three mathematical theorems underlie this beautiful model

(1) given a UFC C, we can construct a Turaev-Viro unitary (2 + 1)-TQFT [BW],

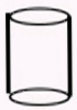
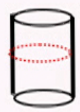
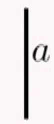
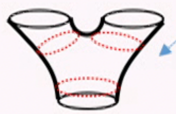
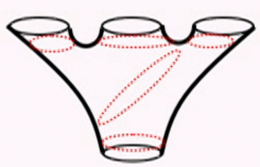
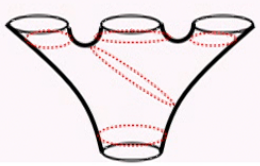
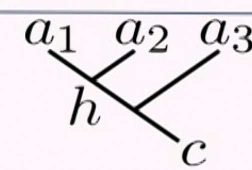
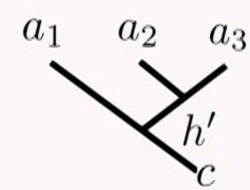
(2) the Drinfeld center Z(C) or quantum double D(C) of a UFC C is always modular [Mue], and

(3) the Turaev-Viro (2+ 1)-TQFT based on C is equivalent to the Reshetikhin-Turaev (2 + 1)-TQFT based on the center Z(C) [BK, TV].

Chang et al.: ON ENRICHING THE LEVIN-WEN MODEL WITH SYMMETRY, arXiv:1412.6589

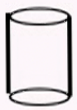
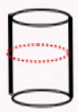
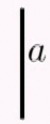
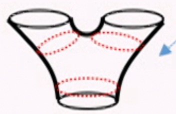
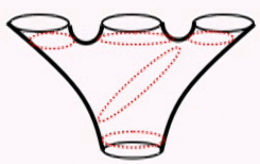
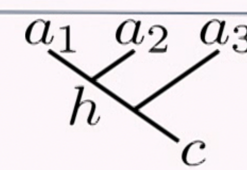
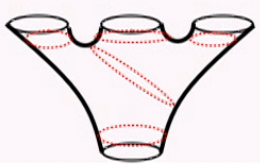
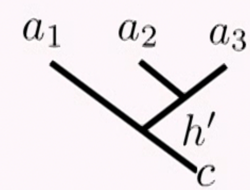
``Standard bases'' from maximal sets of commuting observables

Any DAP-decomposition correspond to a "complete set of observables" and defines a basis of the code space.

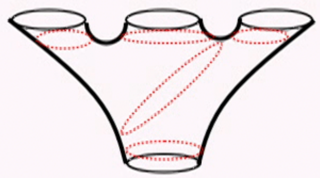
surface	DAP-decomposition(s)		elements of standard basis/bases
		use idempotents of the Verlinde algebra for each loop	
			
	 		 

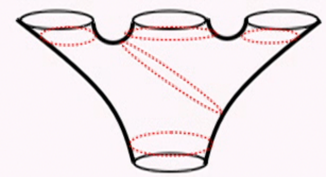
``Standard bases'' from maximal sets of commuting observables

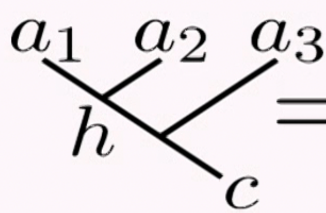
Any DAP-decomposition correspond to a "complete set of observables" and defines a basis of the code space.

surface	DAP-decomposition(s)		elements of standard basis/bases
		use idempotents of the Verlinde algebra for each loop	
			
		<- analogy to three spin-1/2s: $(\vec{S}_1+\vec{S}_2)^2$ $(\vec{S}_1+\vec{S}_2+\vec{S}_3)^2$ S^Z_{total}	
		$(\vec{S}_2+\vec{S}_3)^2$ $(\vec{S}_1+\vec{S}_2+\vec{S}_3)^2$ S^Z_{total}	

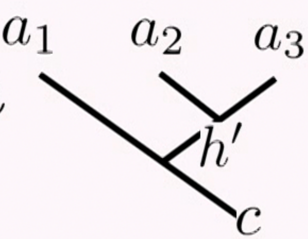
F-move: basis change between bases associated with different DAP-decompositions







$$= \sum_{h'} F^{a_2 a_1 h}_{c a_3 h'}$$

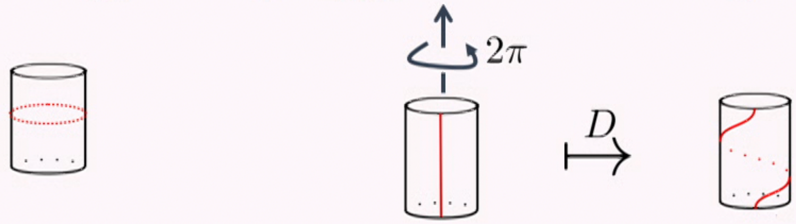


some (controlled) unitary $U(a_2, a_1, a_3, c)_{h,h'}$

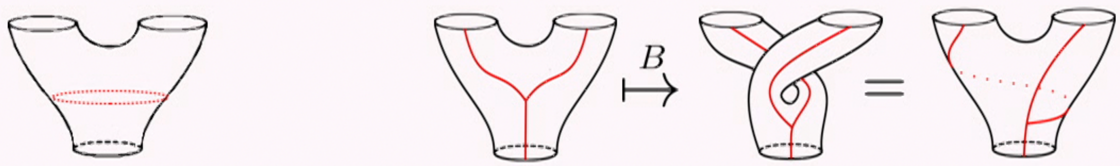
....analogous to spin-1/2- 6j symbols

Mapping class group (generators) and basis elements

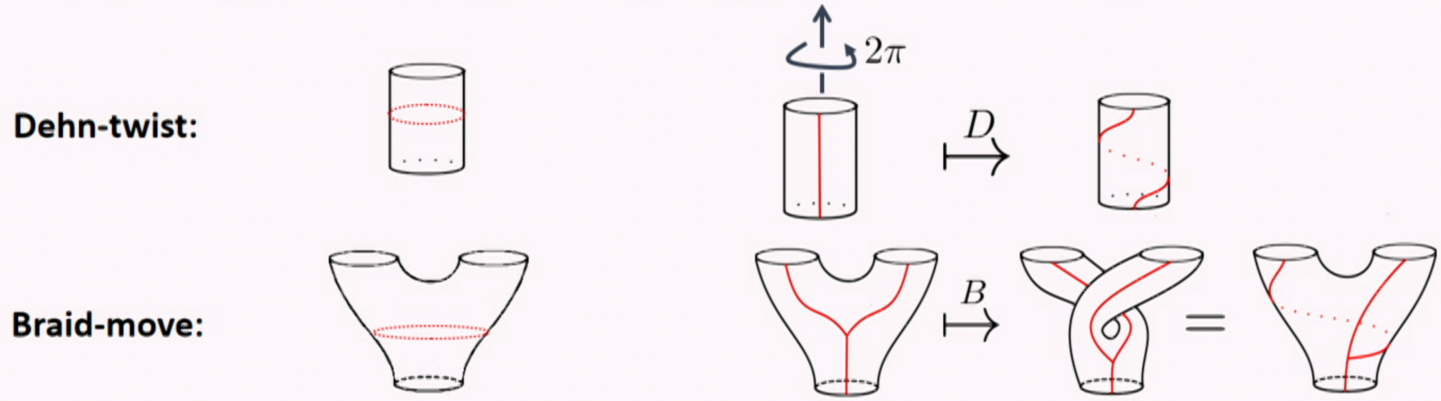
Dehn-twist:

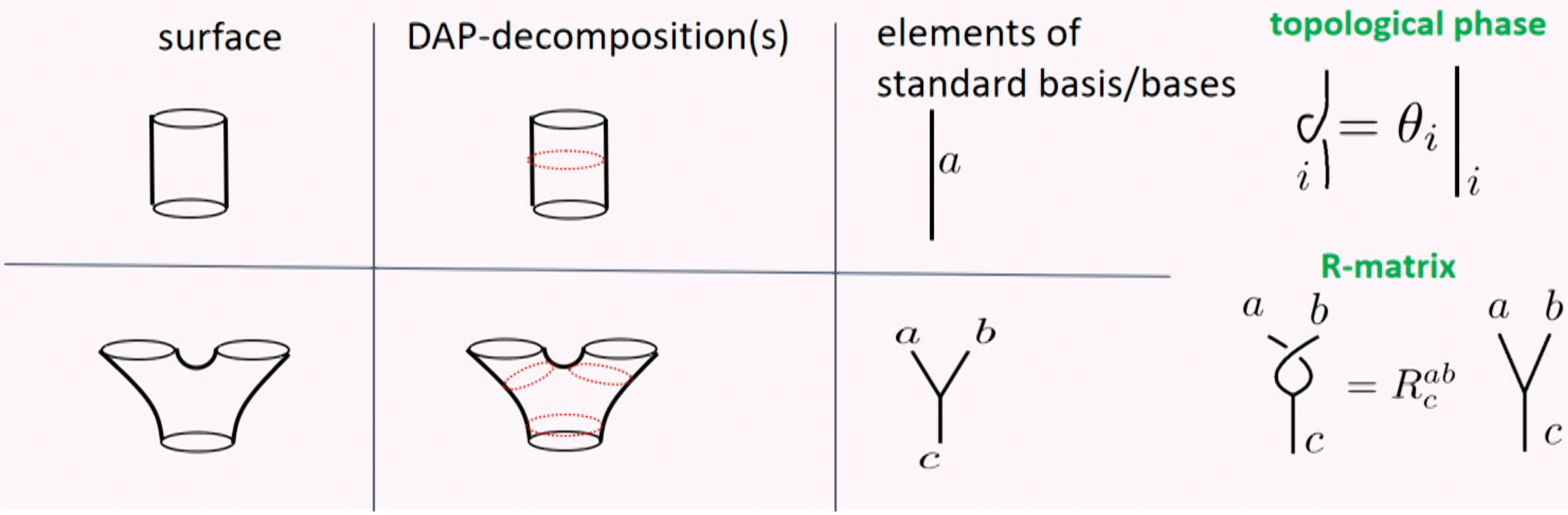


Braid-move:



Mapping class group (generators) and basis elements



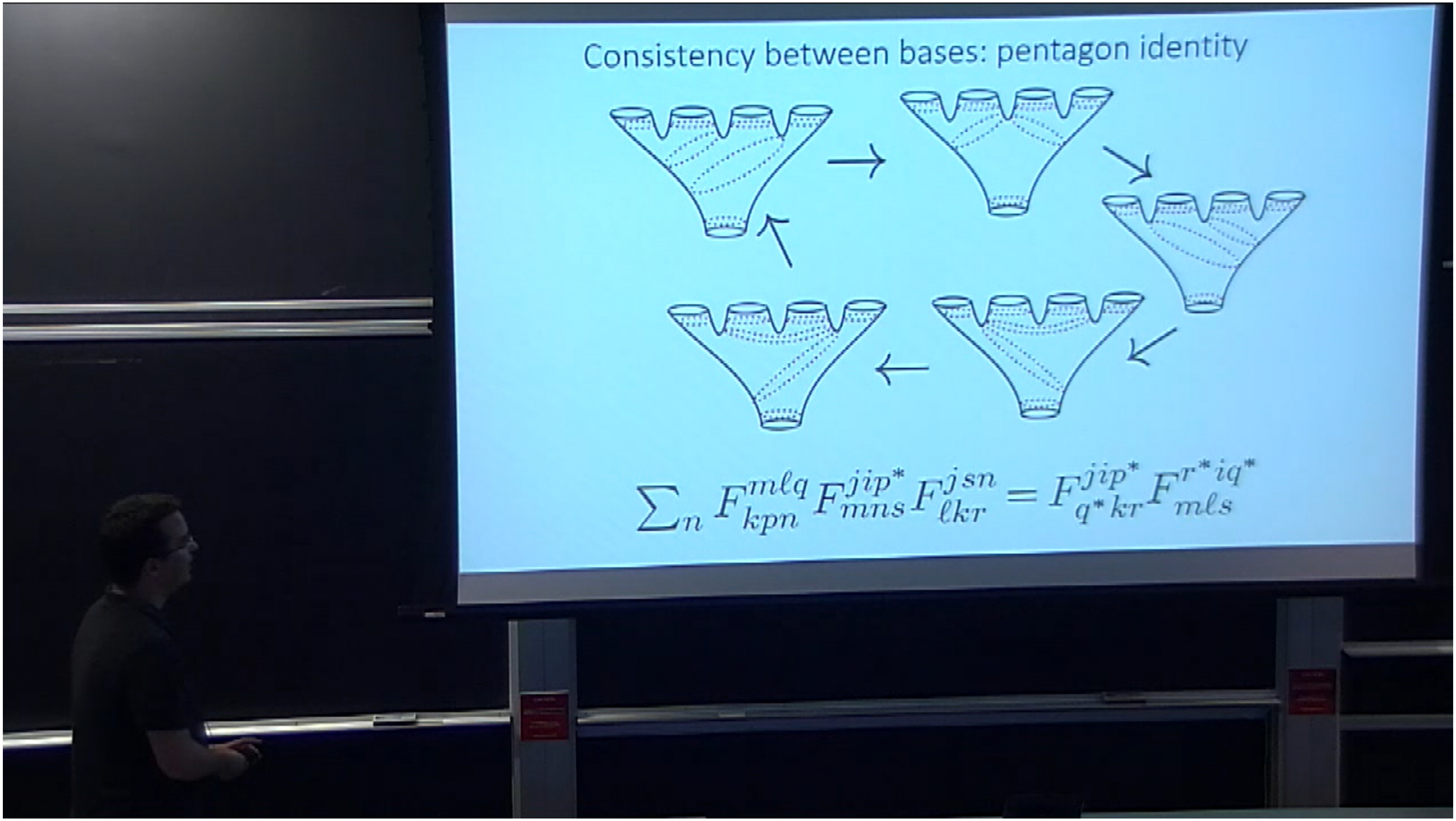


$D|i\rangle = \theta_i|i\rangle$

D = twist

$B|b, a; c\rangle = R_c^{ab}|a, b; c\rangle$

B = braid


Consistency between bases: pentagon identity
$\sum_n F^{m\ell q}_{kpn} F^{jip^*}_{mns} F^{jsn}_{\ell kr} = F^{jip^*}_{q^* kr} F^{r^* iq^*}_{m\ell s}$

Conditions for MCG-representations:

(Moore and Seiberg)

•Consistency of basis changes:

$$\sum_n F^{m\ell q}_{kpn} F^{jip^*}_{mns} F^{jsn}_{\ell kr} = F^{jip^*}_{q^*kr} F^{r^*iq^*}_{m\ell s}$$

(pentagon-identity)

spherical

•Compatibility of basis changes with action of braiding generators:

$$R^{ki}_{m} F^{k^*i^*m}_{\ell j^* g} R^{kj}_{g} = \sum_n F^{i^*k^*m}_{\ell j^* n} R^{kn}_{\ell} F^{j^*i^*n}_{\ell k^* g}$$

$$\theta_i = (R^{i^*i}_{1})^*$$

(hexagon-identity)

braided

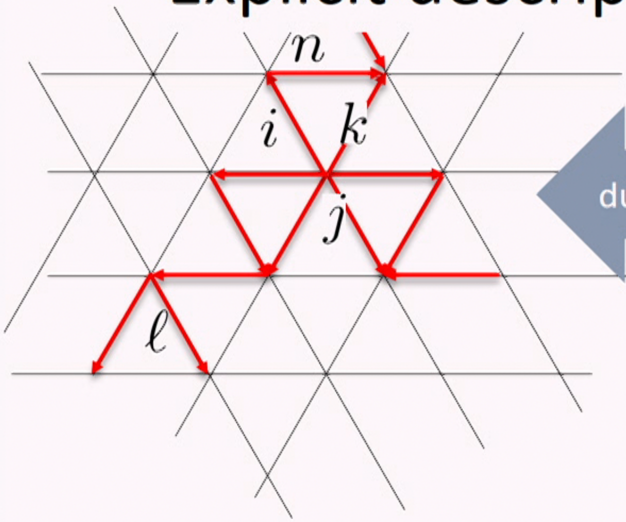
• unitarity of representation:

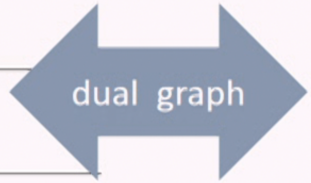
....

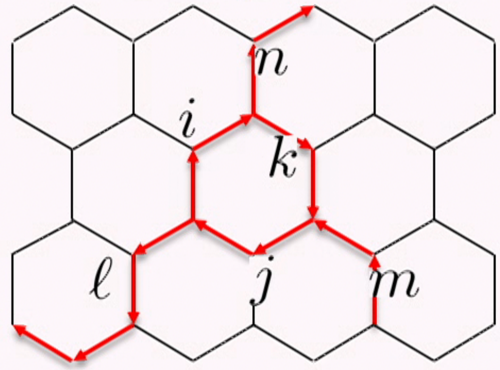
modular category

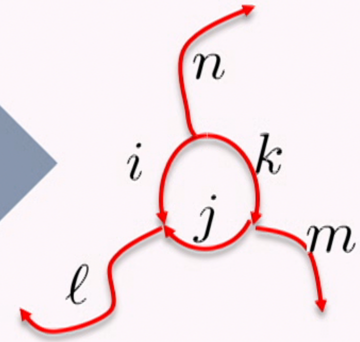
Basis states for the Turaev-Viro code

Explicit descriptions of code spaces: three descriptions







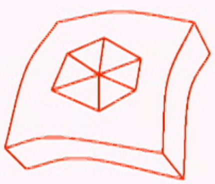


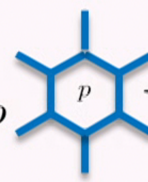
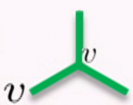
local Hilbert space $\mathbb{C}^d$
associated to every edge

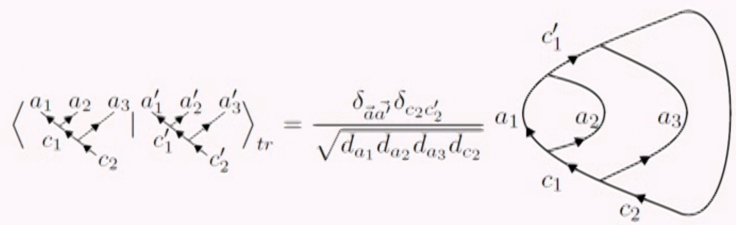
- **Turaev-Viro subspace**
defined using $\Sigma \times [-1, 1]$

- **ground space of Levin-Wen**
qudit lattice Hamiltonian

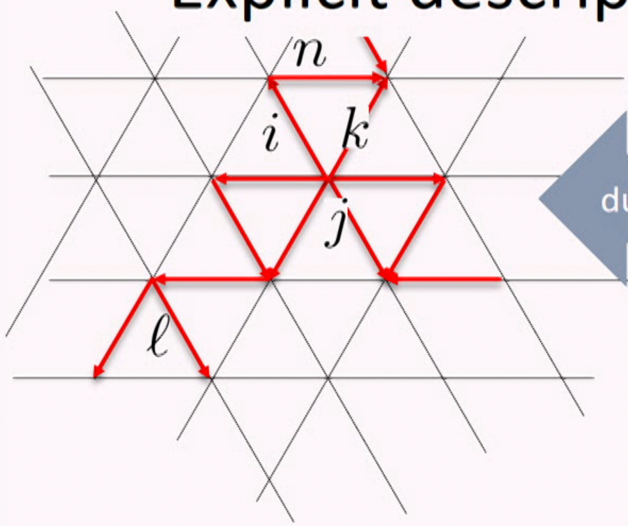
- **ribbon graph space $\mathcal{H}_\Sigma$**

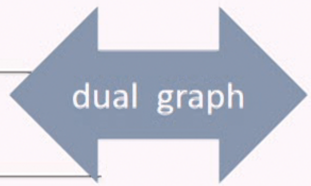


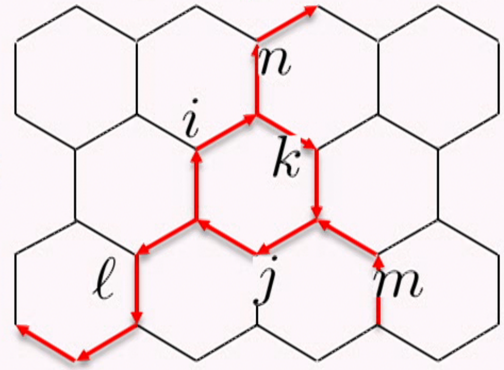
$$H = -\sum_p$$  $$-\sum_v$$ 

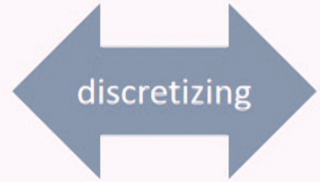


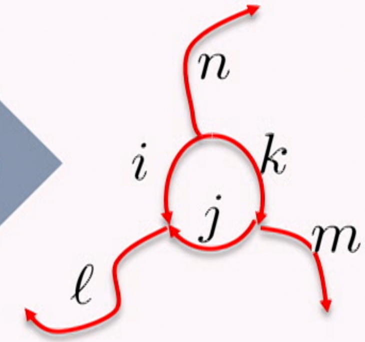
Explicit descriptions of code spaces: three descriptions











local Hilbert space $\mathbb{C}^d$
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- **Turaev-Viro subspace**
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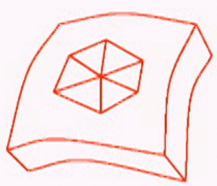
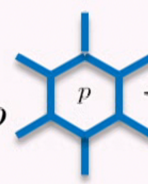
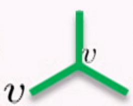
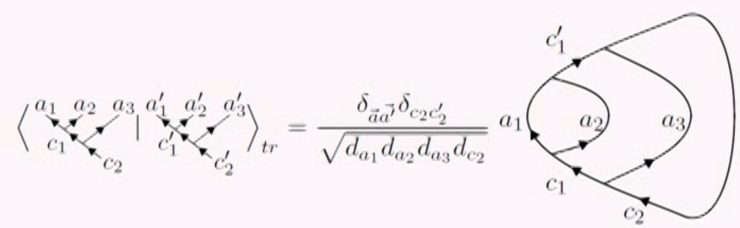
  qudit lattice Hamiltonian

- **ribbon graph space** $\mathcal{H}_\Sigma$

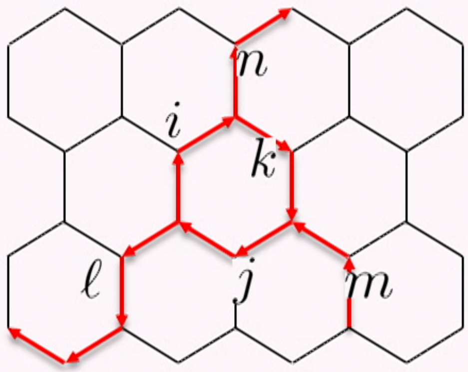


$H = -\sum_p$  $-\sum_v$ 

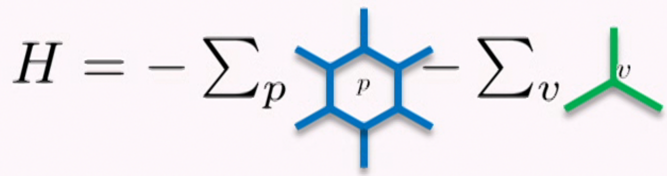
Fact: These Hilbert spaces are **isomorphic**.
(statement is independent of triangulation used)



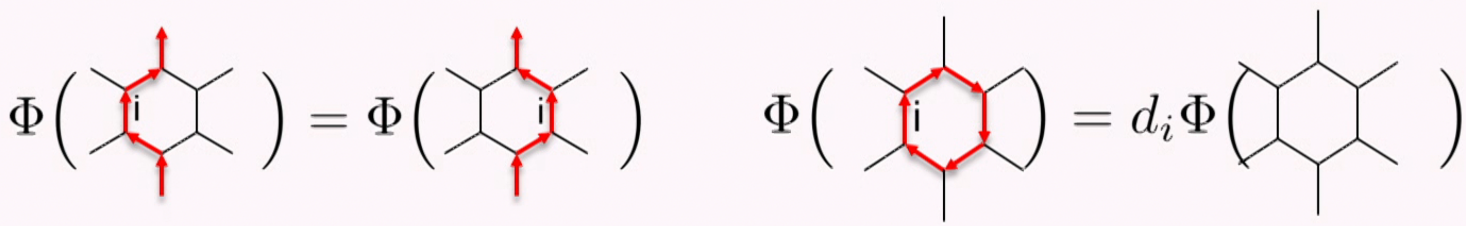
Levin-Wen ground space and local relations



qudit lattice Hamiltonian



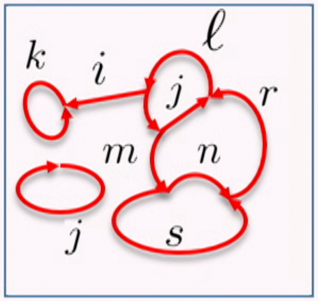
ground state coefficients in computational basis satisfy discrete local “skein” relations, e.g.,



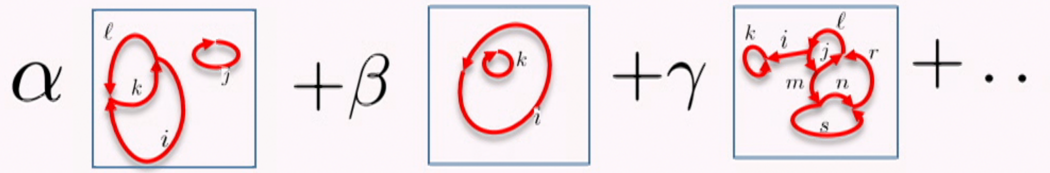
Consequence: Ground space is isomorphic to Hilbert space of ribbon graphs (“pictures”) modulo local equivalence relations

Ribbon graphs Hilbert space $\mathcal{H}_\Sigma$ for general category

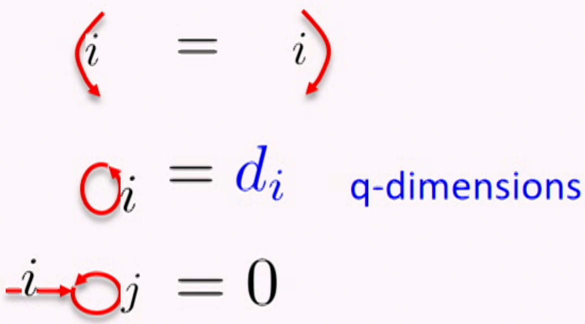
trivalent *labeled directed* graphs (with loops) embedded in Σ



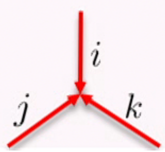
State: formal linear combination of ribbon graphs



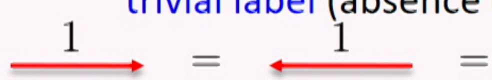
modulo **local relations**



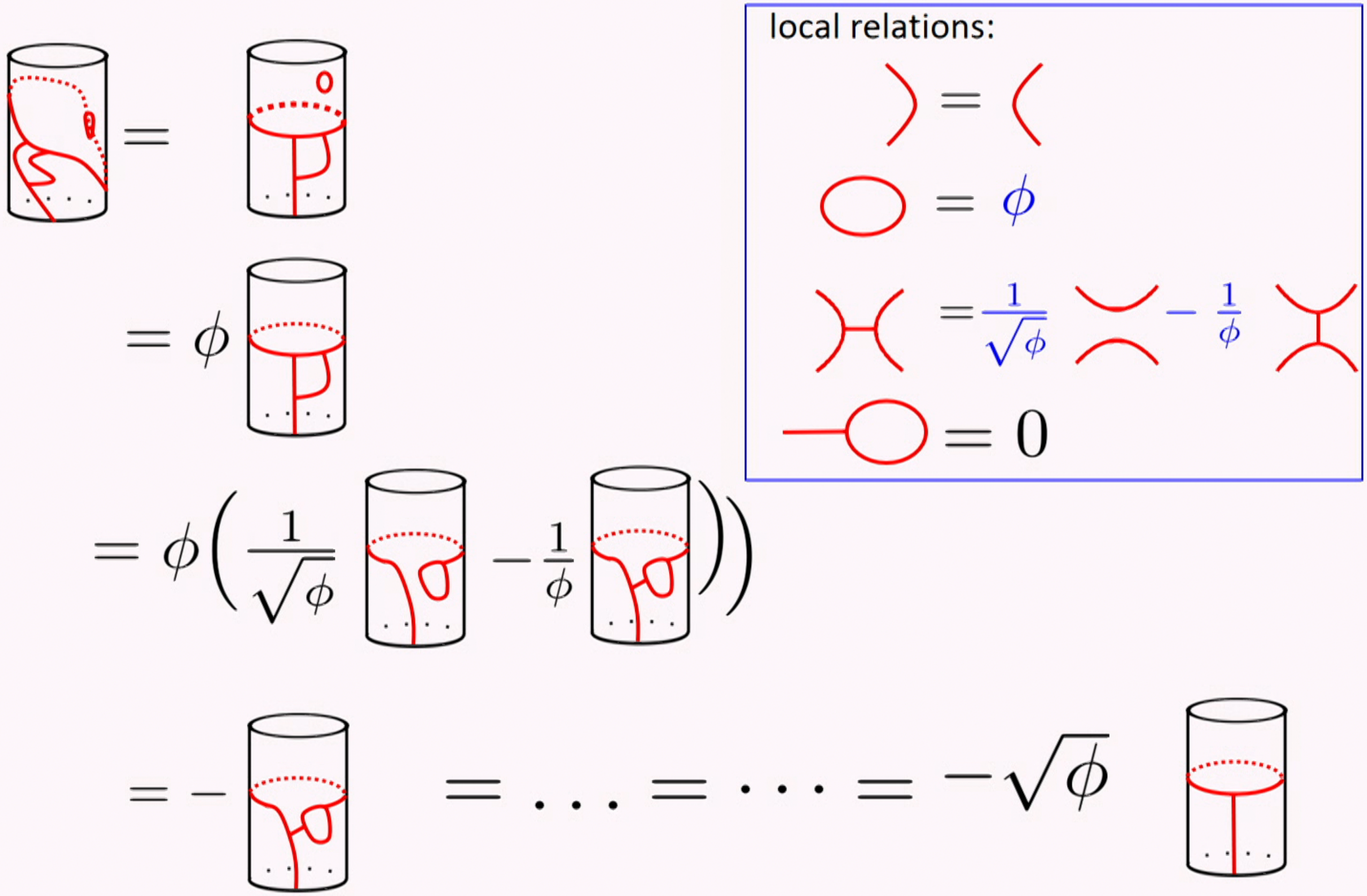
fusion rules
(set of allowed triples):



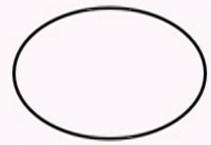
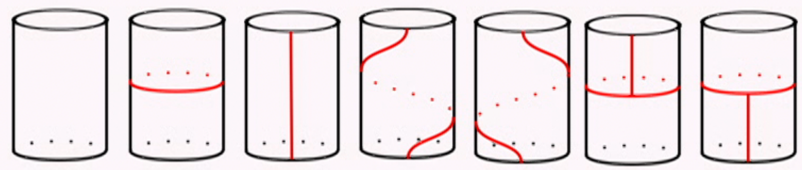
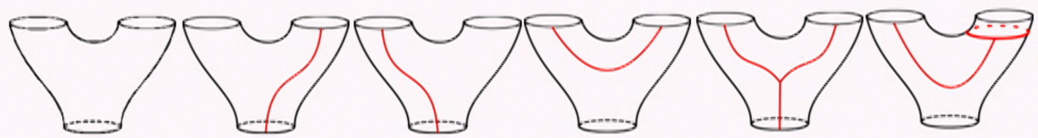
trivial label (absence of string):



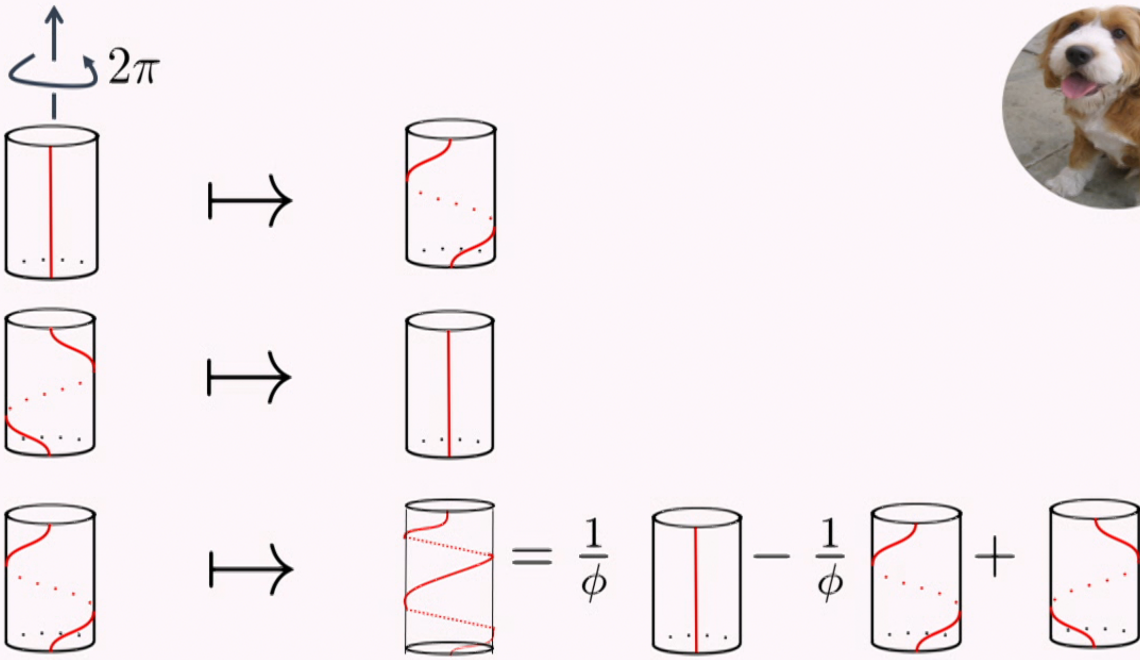
Example of relations in $\mathcal{H}_\Sigma$ for category Fib

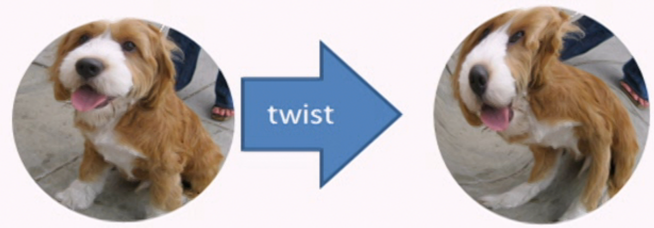


Ribbon graph bases of $\mathcal{H}_\Sigma$ for Fib

Surface Σ	$dim \mathcal{H}_\Sigma$	Example basis
Disc (1-punctured sphere)	**1**	
Annulus (2-punctured sphere)	**7**	
Pair of pants (3-punctured sphere)	**65**	 ...
n-punctured sphere	$2^{\Omega(n)}$	

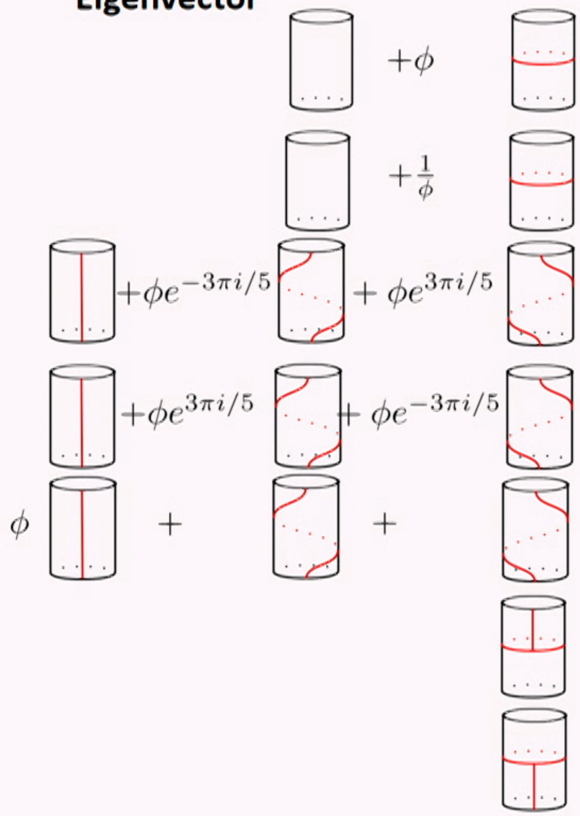
Action of Dehn twist on $\mathcal{H}_{\Sigma_2}$ for Fib





Freedman, Nayak, Walker, Wang,
On Picture (2+1)-TQFTs,
arXiv:0806.1926

Goal: identify "**fusion tree basis**" (eigenvectors of twist)



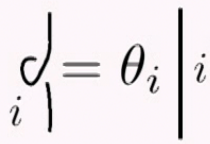
Eigenvector	eigenvalue (twist)	name	boundary labels
row 1	1	$1 \otimes 1$	$1, 1$
row 2	1	$\tau \otimes \tau$	
row 3	$e^{4\pi i/5}$	$1 \otimes \tau$	τ, τ
row 4	$e^{-4\pi i/5}$	$\tau \otimes 1$	
row 5	1	$\tau \otimes \tau$	
row 6	1	$\tau \otimes \tau$	$\tau, 1$
row 7	1	$\tau \otimes \tau$	$1, \tau$

Anyonic fusion basis obtained by diagonalization

fusion space basis element

$$V_i^i = \mathbb{C}\,\Big|\,i$$

topological phase



anyon type

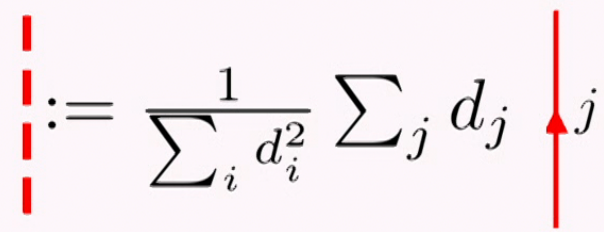
i

of “doubled” theory

$Fib \otimes Fib'$

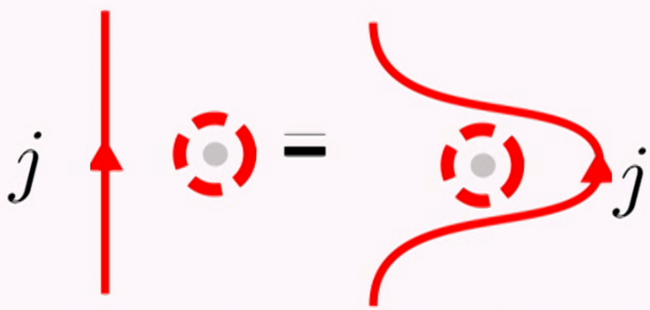
multiplicity index for different realizations as subspaces of $\mathcal{H}_{\Sigma_2}$

Tool for describing anyonic fusion basis states: "vacuum" ribbons

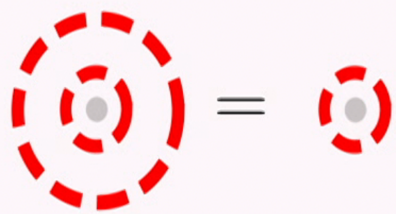


Properties:

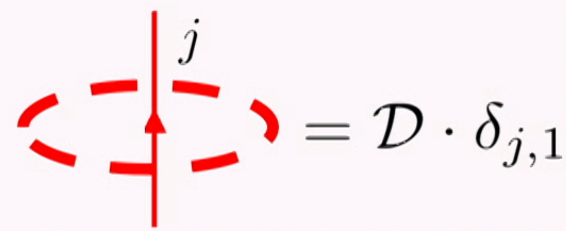
"removal of holes"



"doubling"



"removal of enclosed strings"

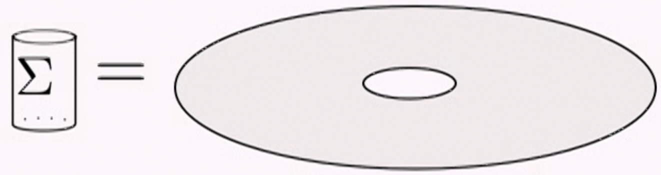


Anyonic fusion basis from "doubled" manifold $\Sigma \times [-1,1]$

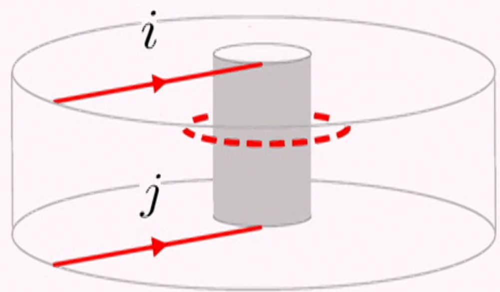
Goal: find anyonic fusion basis states on Σ

Intermediate step: identify relevant ribbon graphs on $\Sigma \times [-1,1]$

Example: find element for annulus $i \otimes j$



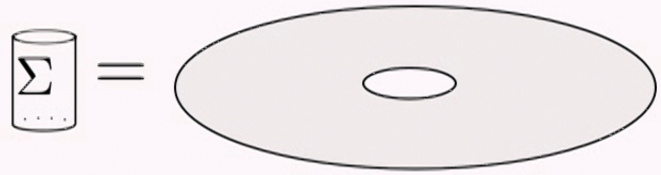
some ribbon graph on



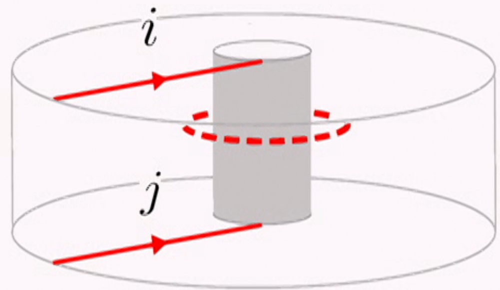
Anyonic fusion basis from“doubled” manifold $\Sigma \times [-1,1]$

Goal: find anyonic fusion basis states on Σ

Example: find element for annulus $\uparrow i \otimes j$

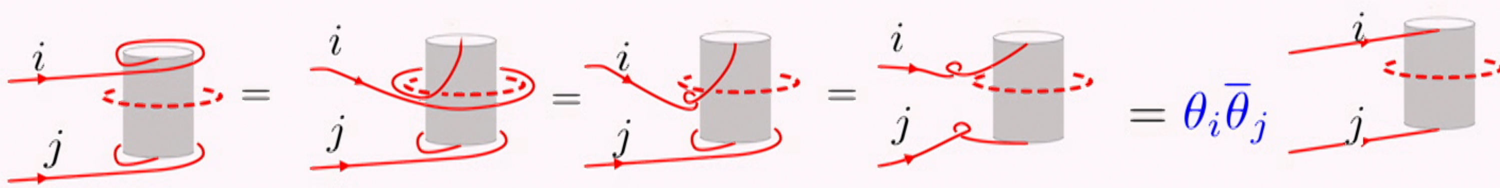


some ribbon graph on



Intermediate step: identify relevant ribbon graphs on $\Sigma \times [-1,1]$

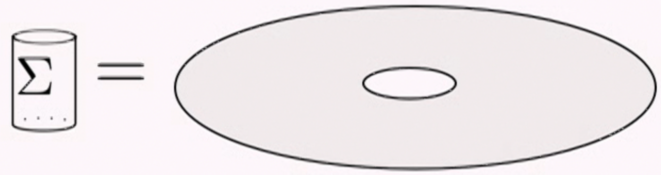
simple derivation of topological phase:



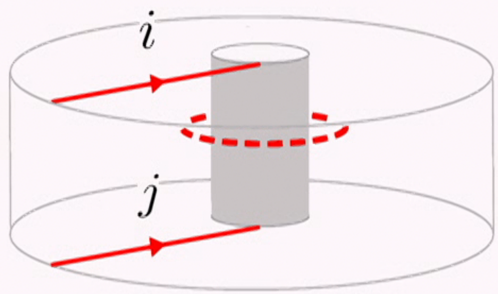
Anyonic fusion basis from "doubled" manifold $\Sigma \times [-1,1]$

Goal: find anyonic fusion basis states on Σ

Example: find element $i \otimes j$ for annulus

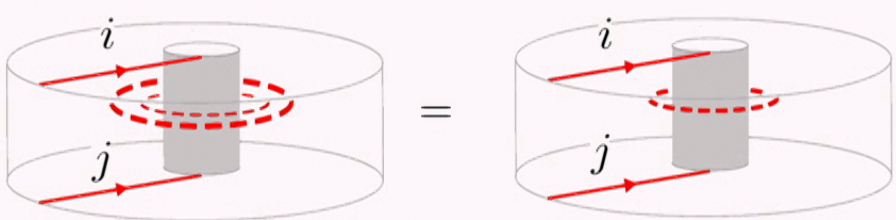


some ribbon graph on



Intermediate step: identify relevant ribbon graphs on $\Sigma \times [-1,1]$

simple derivation of idempotency property:



Anyonic fusion basis from“doubled” manifold $\Sigma \times [-1,1]$

Goal: find anyonic fusion basis states on Σ

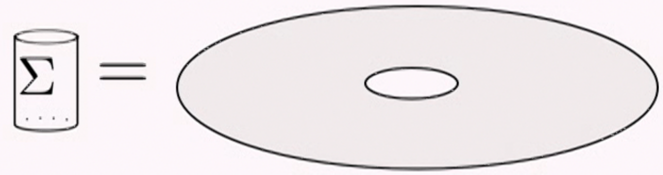
Intermediate step: identify relevant ribbon graphs on $\Sigma \times [-1,1]$

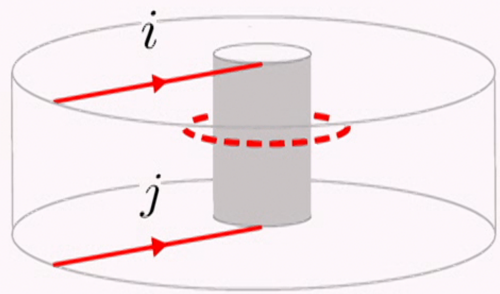
Map ribbon graphs

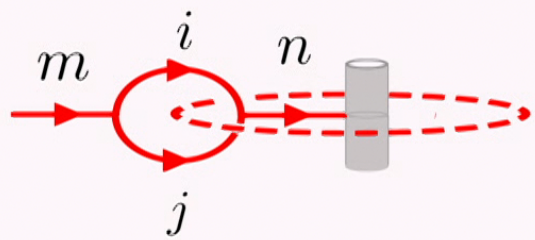
$$\Sigma \times [-1,1] \to \Sigma$$

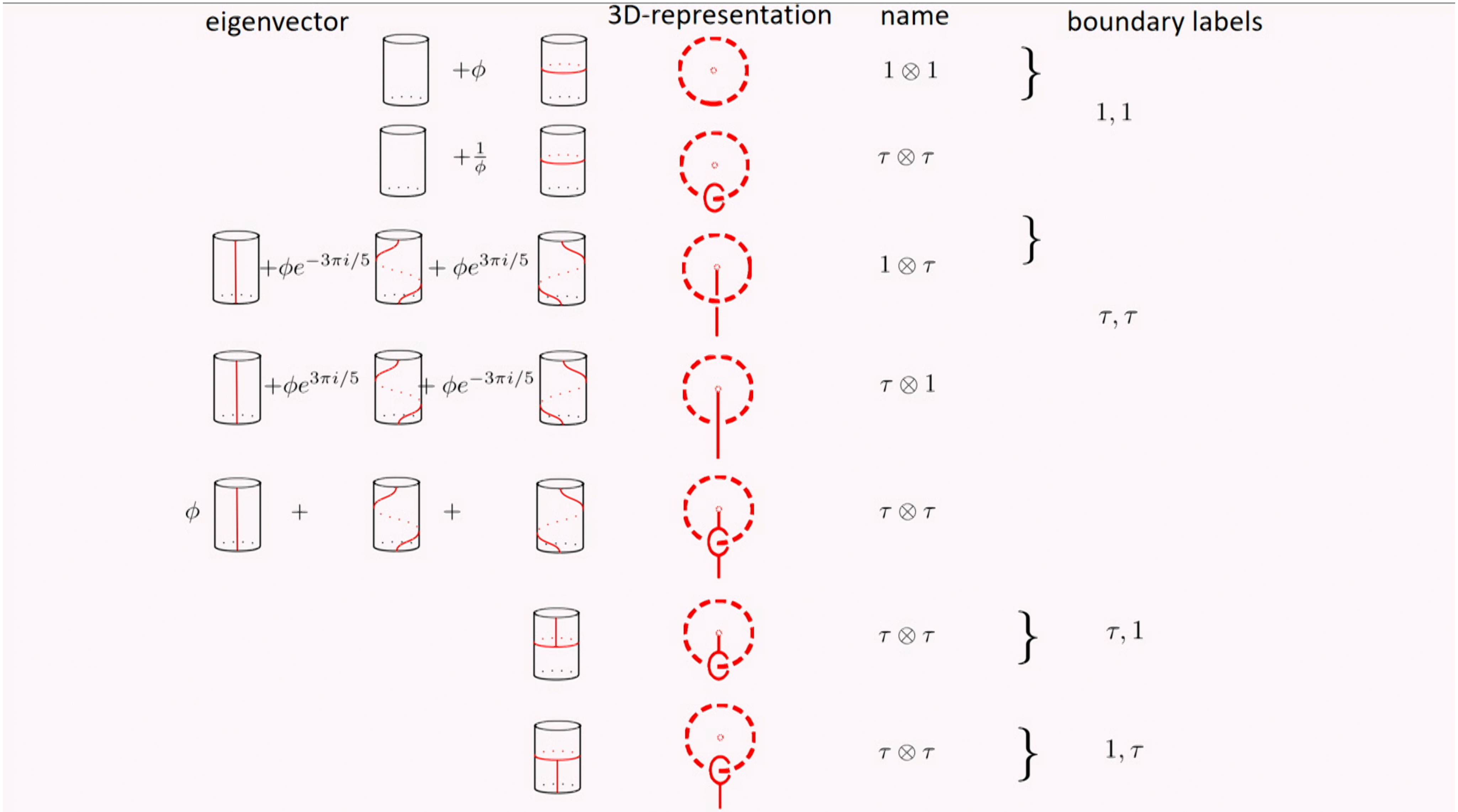
by connecting up boundary ribbons, and projecting

Example: find element for annulus $\uparrow i \otimes j$

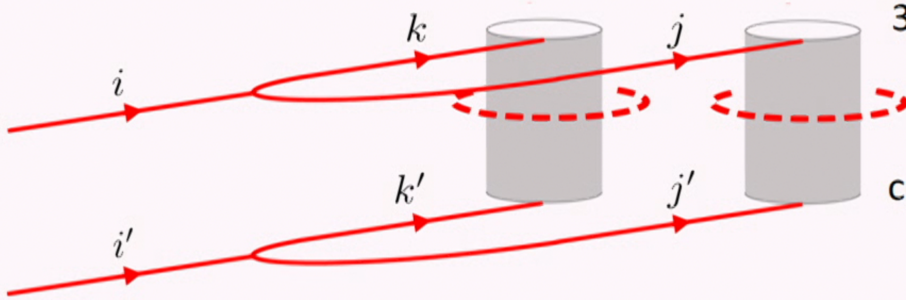






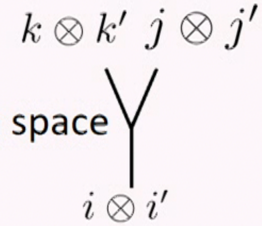

eigenvector
3D-representation
name
boundary labels
$+\phi$
$+\frac{1}{\phi}$
$+\phi e^{-3\pi i/5}$
$+\phi e^{3\pi i/5}$
$+\phi e^{3\pi i/5}$
$+\phi e^{-3\pi i/5}$
ϕ
$+$
$+$
$1\otimes 1$
$\tau\otimes\tau$
$1\otimes\tau$
$\tau\otimes 1$
$\tau\otimes\tau$
$\tau\otimes\tau$
$\tau\otimes\tau$
$1, 1$
τ, τ
$\tau, 1$
$1, \tau$

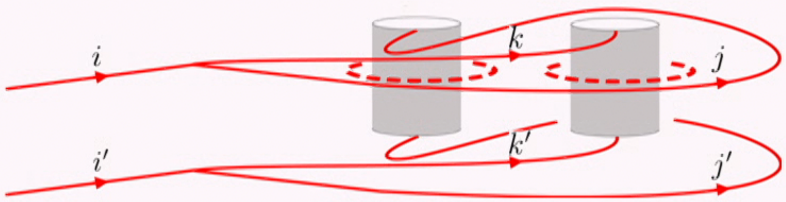
3D-ribbon graphs for 2-anyon fusion spaces



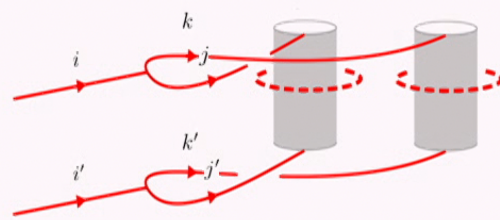
3-punctured sphere x$[-1,1]$

corresponds to fusion space

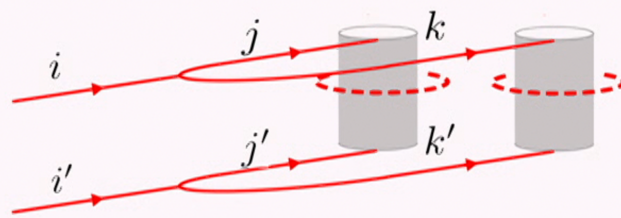




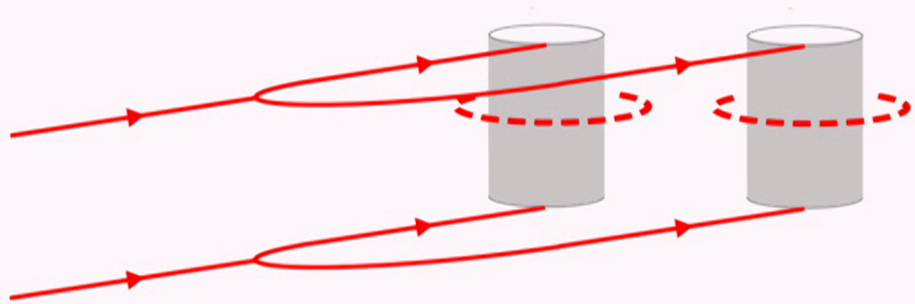
$=$

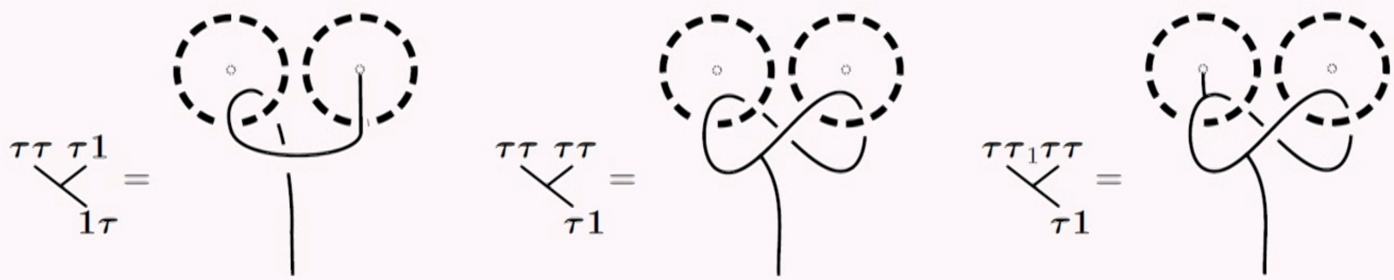


$$= R_i^{jk}\,\overline{R_{i'}^{j'k'}}$$

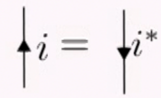
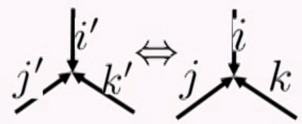
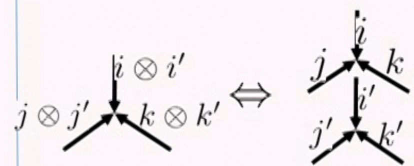
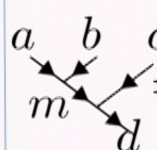
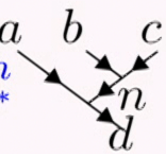
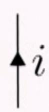


2-anyon fusion basis for Fib





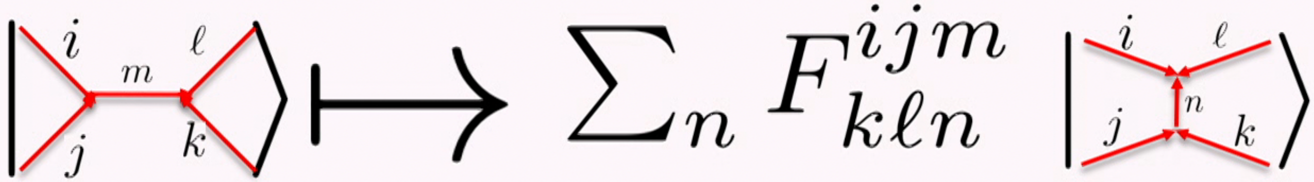
Derived categories: basic data

		modular tensor category $\mathcal{C}$	dual category $\mathcal{C}'$	doubled category $\mathcal{C}\otimes\mathcal{C}'$
Unitary, braided, semisimple, *	Particles	$\{1, i, j, \ldots\}, *$ 	$\{i' \mid i \in \mathcal{C}\}$	$\left\{ i \otimes j' \mid \begin{smallmatrix} i\in\mathcal{C}, \\ j'\in\mathcal{C}' \end{smallmatrix} \right\}$
	Fusion rules	 (set of) allowed triples		
	q-dim	 $= d_i$	$d_{i'} = d_i$	$d_{i\otimes j'} = d_i d_{j'}$
	F-matrix	 $= \sum_n F^{bam}_{dcn^*}$ 	$F^{b'a'm'}_{d'c'n'^*} = F^{bam}_{dcn^*}$	$F \otimes F'$
	top. phase	 $= \theta_i$ 	$\theta_{i'} = \overline{\theta}_i$	$\theta_{i\otimes j'} = \theta_i \theta_{j'}$
	R-matrix	 $= R^{ab}_c$ 	$R^{a'b'}_{c'} = \overline{R^{ab}_c}$	$R \otimes R'$

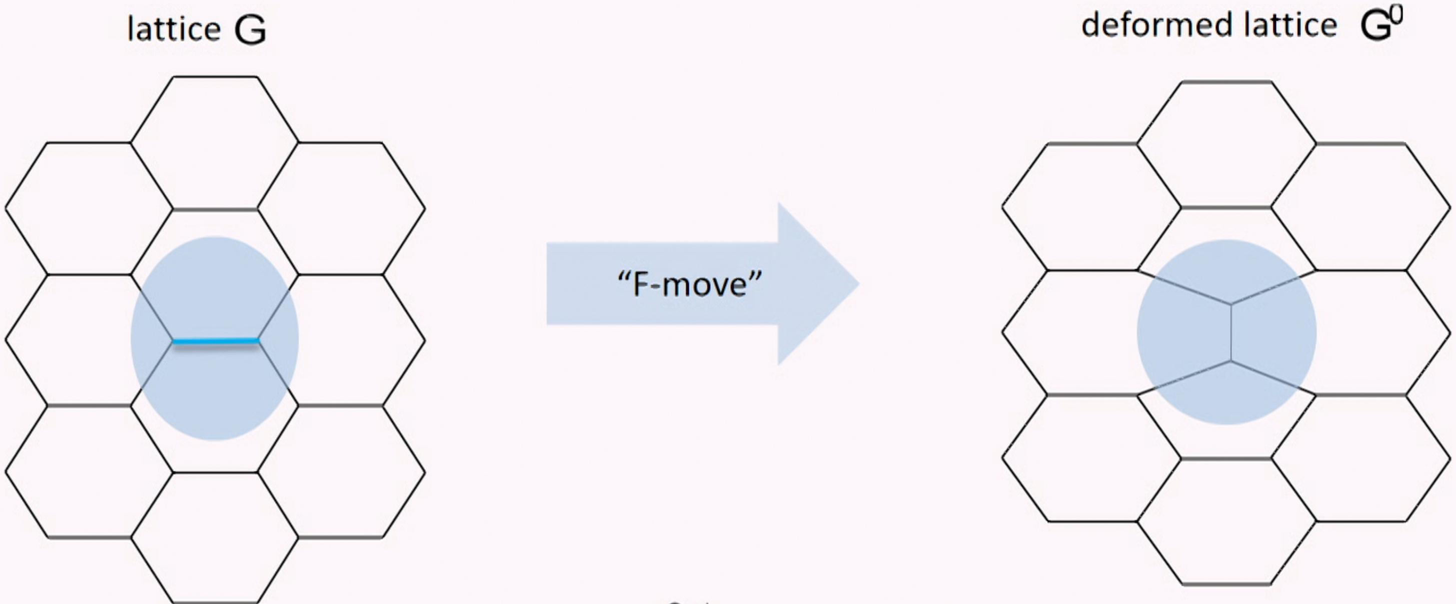
Computation with Turaev-Viro codes

Computation with Turaev-Viro codes

Different lattices and F-move isomorphism

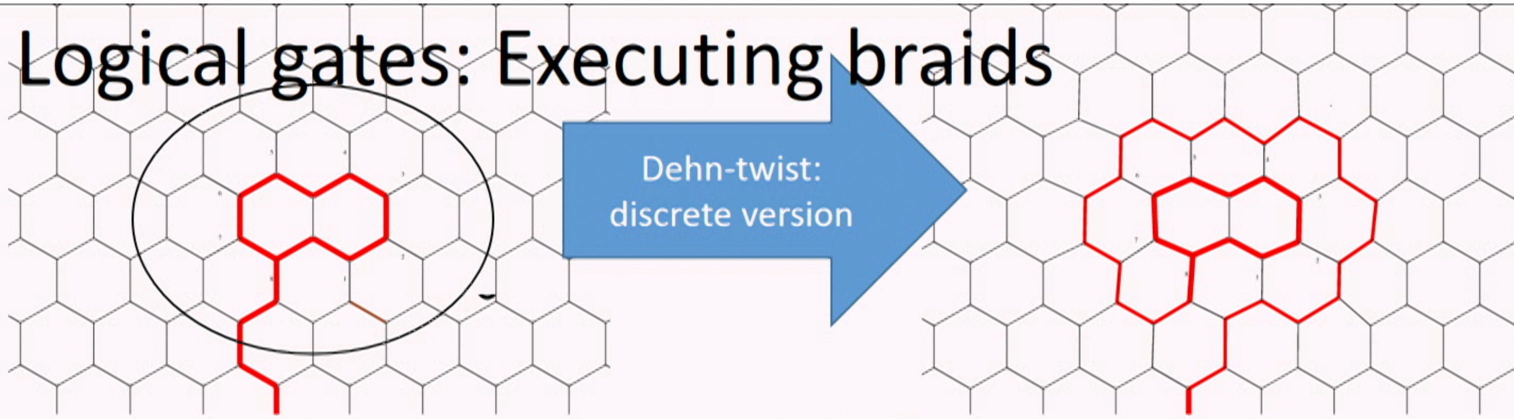


For unitary tensor categories, this is a **unitary 5-qudit gate.**



$\mathcal{H}_{\mathcal{G}}$ ground space of spins on original lattice $\xrightarrow{\cong}$ $\mathcal{H}_{\mathcal{G}'}$ ground space of spins on deformed lattice

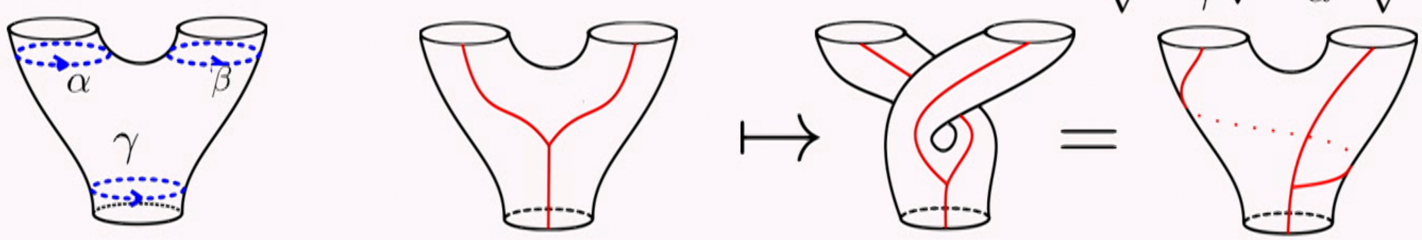
Logical gates: Executing braids



Can be implemented by sequence of $O(|\gamma|^2)$ F-moves (5-qudit gates)

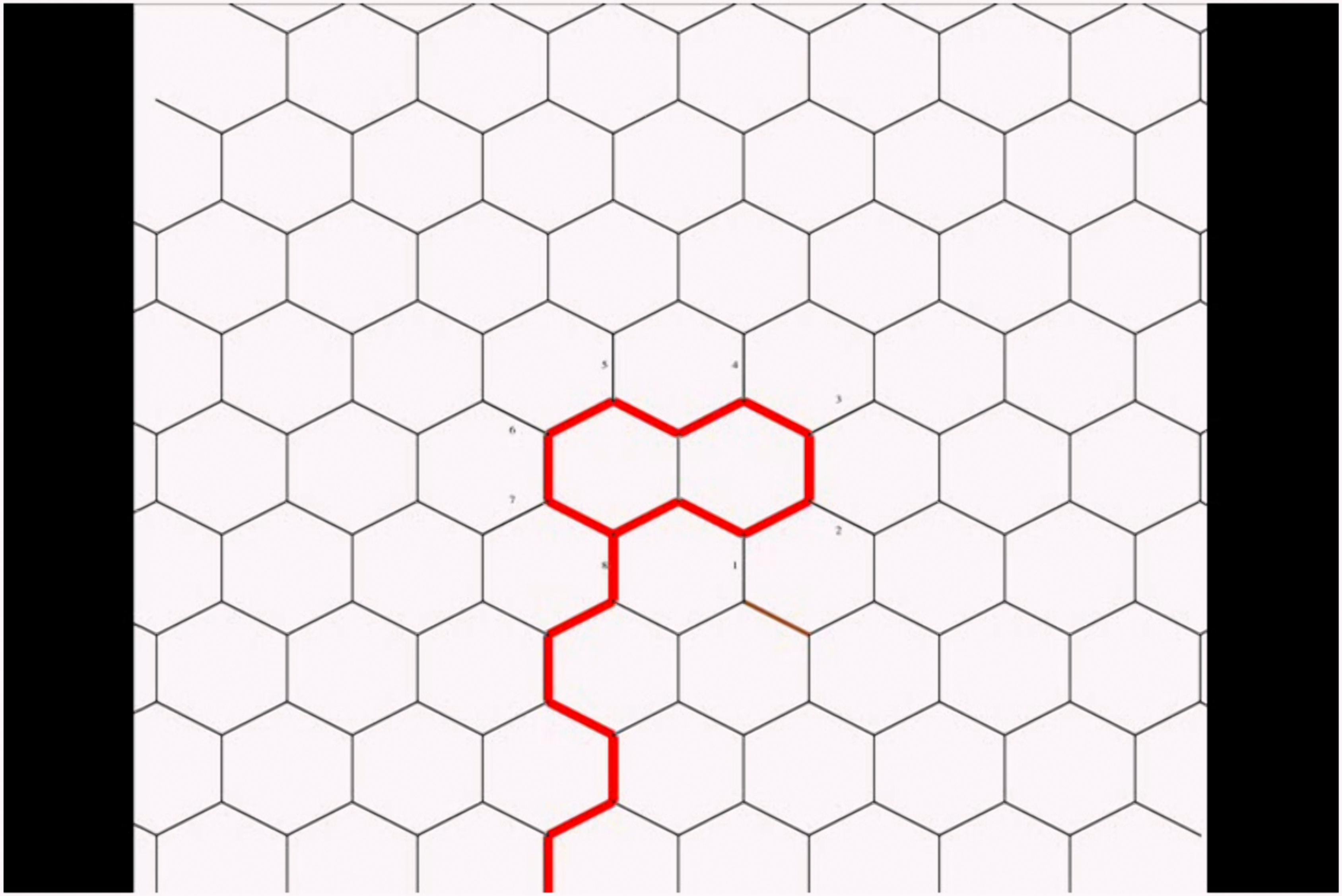
π -twists can be implemented similarly, therefore braids:

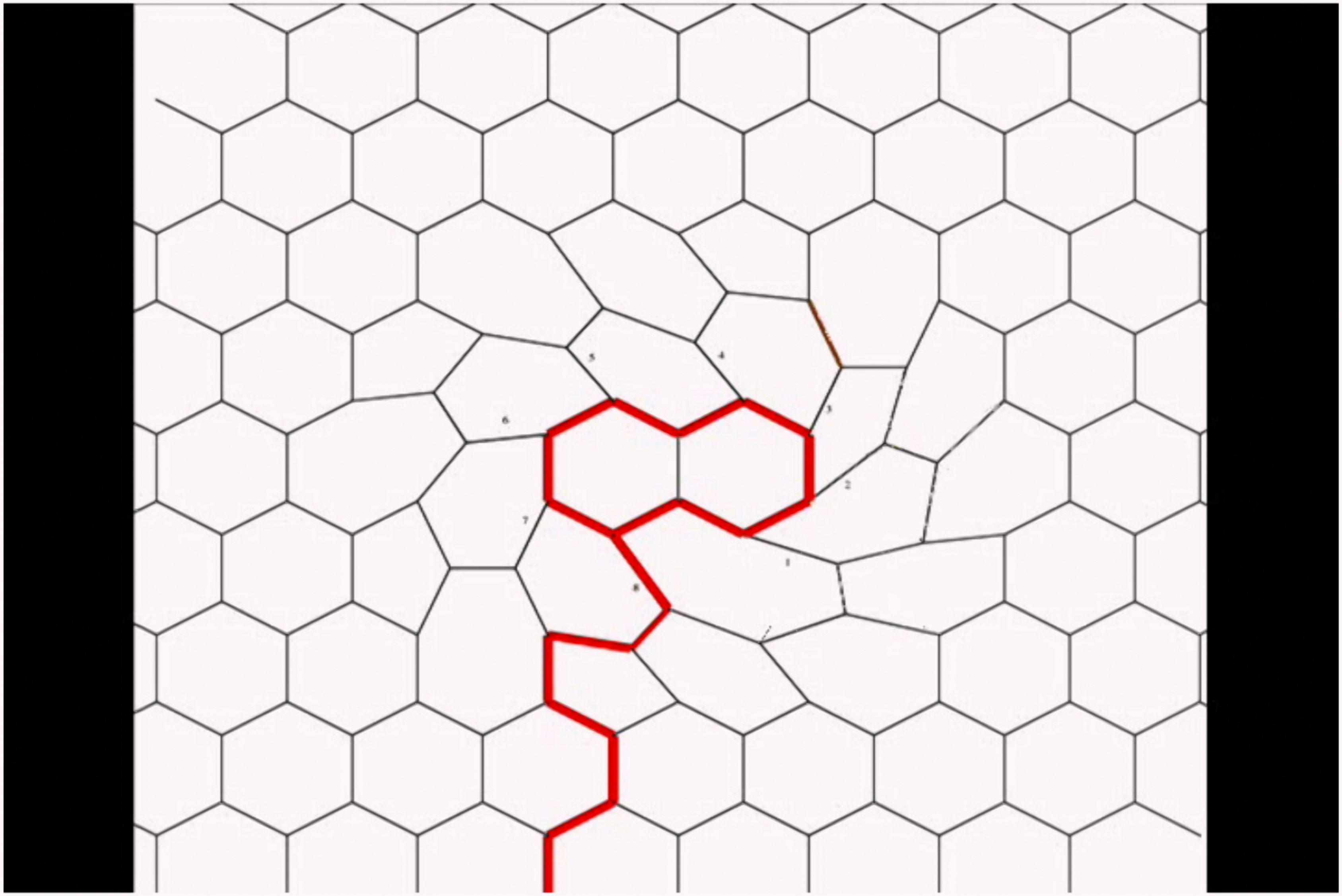
$$B = \sqrt{D_\gamma}\sqrt{D_\alpha^{-1}}\sqrt{D_\beta}$$



universal gate set:

- braids generate dense subgroup of unitaries on subspace of $\mathcal{H}_\Sigma$ for (doubled) Fib
- for approriate encoding, approximation of universal gate set by Solovay-Kitaev (Freedman, Larsen, Wang'02)

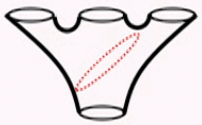

5
4
3
6
7
2
8
1

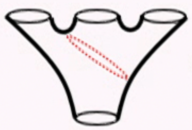
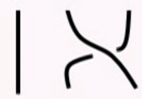

5
4
6
3
2
7
1
8

Example "topological" qubit in Fib

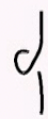
Qubit encoding: $|0\rangle \mapsto$  $|1\rangle \mapsto$ 

Braids:

 $= \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{pmatrix} =$ 

 $= \begin{pmatrix} \frac{1}{\sqrt{\phi}} & \frac{1}{\phi} \\ -\frac{1}{\phi} & \frac{1}{\sqrt{\phi}} \end{pmatrix}^{-1} \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\phi}} & \frac{1}{\phi} \\ -\frac{1}{\phi} & \frac{1}{\sqrt{\phi}} \end{pmatrix} =$ 

 $= \frac{1}{\sqrt{\phi}} \quad -\frac{1}{\phi}$ 

 $= e^{-4\pi i/5}$ 

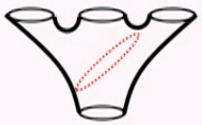
 $= \frac{1}{\phi} \quad + \frac{1}{\sqrt{\phi}}$ 

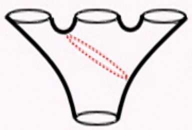
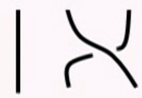
 $= e^{3\pi i/5}$ 

Example "topological" qubit in Fib

Qubit encoding: $|0\rangle \mapsto$  $|1\rangle \mapsto$ 

Braids:

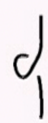
 $= \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{pmatrix} =$ 

 $= \begin{pmatrix} \frac{1}{\sqrt{\phi}} & \frac{1}{\phi} \\ -\frac{1}{\phi} & \frac{1}{\sqrt{\phi}} \end{pmatrix}^{-1} \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\phi}} & \frac{1}{\phi} \\ -\frac{1}{\phi} & \frac{1}{\sqrt{\phi}} \end{pmatrix} =$ 

NOT-gate approximation accuracy 10^(-4)
compiled with Solovay-Kitaev

 $= \frac{1}{\sqrt{\phi}} \quad -\frac{1}{\phi}$ 

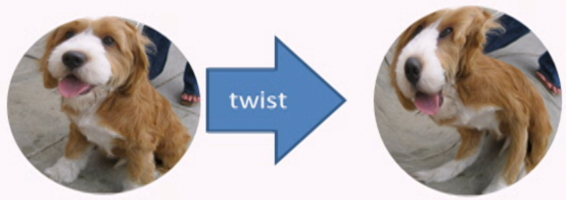
 $= \frac{1}{\phi} \quad + \frac{1}{\sqrt{\phi}}$ 

 $= e^{-4\pi i/5}$ 

 $= e^{3\pi i/5}$ 

Bonesteel et al.,
Phys. Rev. Lett. 95,
140503 (2005)

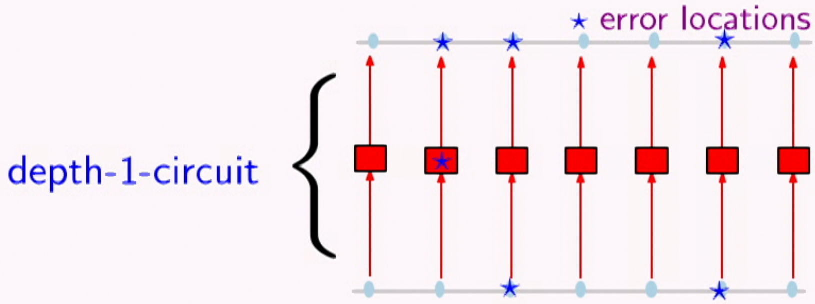
Gate sets obtained from the mapping class group



TQFT	*mapping class group (braiding)* *contained in*
$D(\mathbb{Z}_2)$	Pauli group
abelian anyon model	generalized Pauli group
Fibonacci model	universal
Ising model	Clifford group

Limitations on transversal gates are protected

transversal gate ≡ implementable by a depth-1-circuit



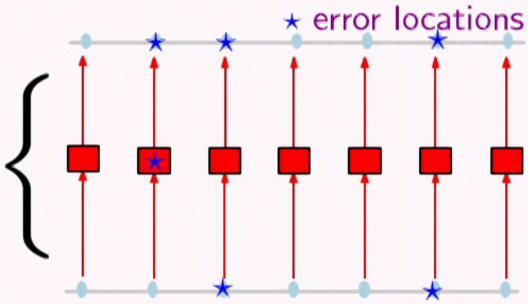
when applying a transversal gate:

- preexisting errors do not spread
- faulty unitaries only introduce local errors

Limitations on transversal gates are protected

transversal gate ≡ implementable by a depth-1-circuit

depth-1-circuit



when applying a transversal gate:

- preexisting errors do not spread
- faulty unitaries only introduce local errors

.....but limited

General (non-stabilizer) codes:

Theorem: Transversal encoded gates generate a **finite group**.

[Eastin, Knill '09]

Proof uses theory of Lie groups.

2D surface codes:

Theorem: Suppose the stabilizer group has no generators of weight 2. Then all transversal gates are in the **Clifford group**.

[Sarvepalli, Raussendorf '09]

Proof uses theory of matroids.

Limitations for protected gates for local stabilizer codes

Clifford hierarchy

$\mathcal{C}_1$ =Pauli group

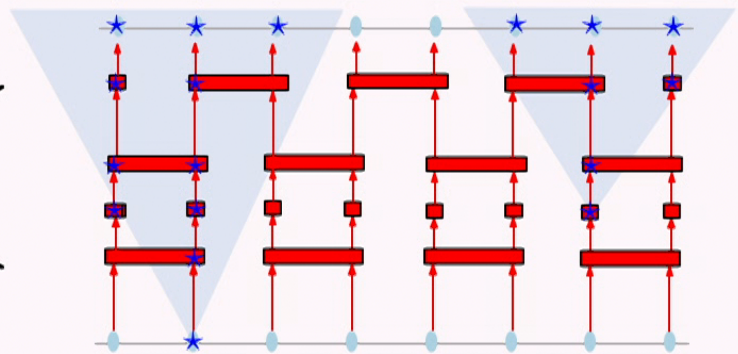
$\mathcal{C}_2$ =Clifford group

$$\mathcal{C}_{j+1} = \{U \in \mathsf{U}(2^k) \mid U\mathcal{C}_1 U^\dagger \subseteq \mathcal{C}_j\}$$

Theorem: [Bravyi,K '13] For a D-dimensional local stabilizer code: protected gates belong to $\mathcal{C}_D$

protected gate $\equiv$ implementable by constant-depth quantum circuit

constant-depth quantum circuit



Limitations for protected gates for local stabilizer codes

Clifford hierarchy

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$\mathcal{C}_2 =$ Clifford group

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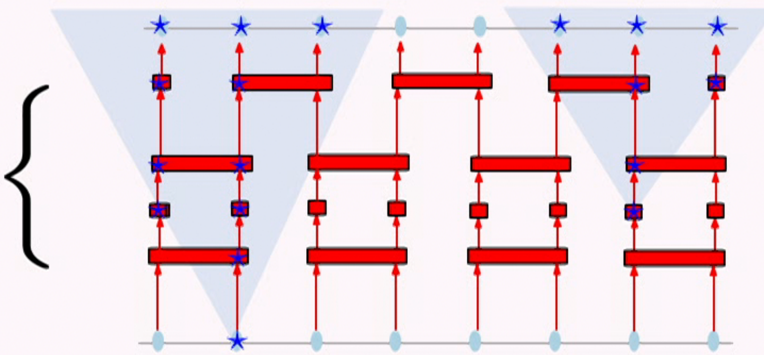

Corollary: For any

- 2-dimensional local stabilizer code
- family $\{\mathcal{L}_L\}_L$ of D-dimensional local stabilizer codes such that $k = k(L)$ independent of L

the set of protected gates is **not computationally universal**

Limitations for protected gates for local stabilizer codes

Clifford hierarchy

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the set of protected gates is **not computationally universal**

protected gate $\equiv$ implementable by constant-depth quantum circuit

Bombin'13: There are codes saturating this bound.

[Pastawski, Yoshida '14] tradeoffs and generalization to *subsystem codes*

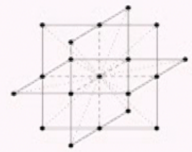
$D = 2$

$\mathcal{C}_2$ (Cliffords)

if gates in $\mathcal{C}_2$ then $d \leq O(L)$

$p_{\text{loss}} < 1/2$

$D = 3$



$\mathcal{C}_3$

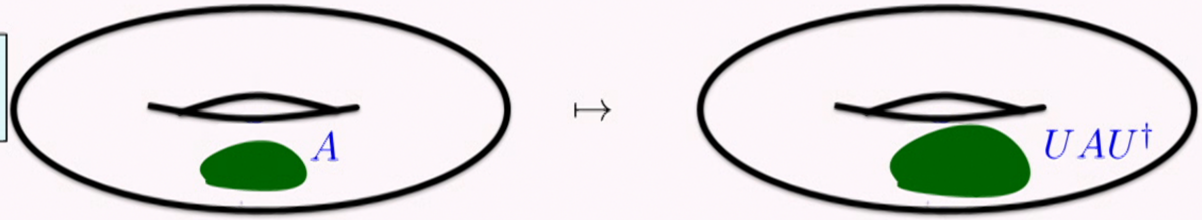
if gates in $\mathcal{C}_3$ then $d \leq O(L)$

$p_{\text{loss}} < 1/3$

only gates in $\mathcal{C}_2$ if energy barrier is macroscopic

Limitations on protected gates in TQFTs: results

Definition: A gate U is **protected** if it preserves locality



Limitations on protected gates in TQFTs: results

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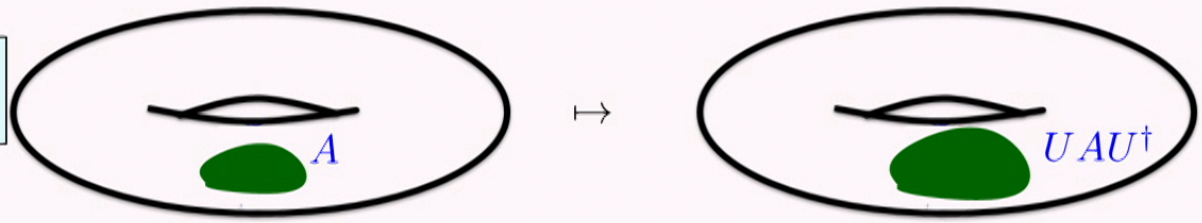

TQFT	*mapping class group (braiding) contained in*	*locality-preserving unitaries contained in*
$D(\mathbb{Z}_2)$	Pauli group	Clifford group
abelian anyon model	generalized Pauli group	generalized Clifford group
Fibonacci model	universal	global phase (trivial)
Ising model	Clifford group	Pauli group
generic anyon model	model-dependent	finite group
generic anyon model	universal	global phase (trivial)

Results

Matrix representation of protected gates in DAP-basis

Suppose $U : \mathcal{H}_{phys} \to \mathcal{H}_{phys}$ is a protected gate

Lemma: Let $\mathcal{C} = \{C_j\}$ be a DAP-decomposition, $\mathcal{B}_\mathcal{C}$ be the associated basis of $\mathcal{H}_\Sigma$.
The matrix $\mathbf{U}_\mathcal{C}$ representing U in this basis is **unitary monomial**:

$$\underbrace{\mathbf{U}_\mathcal{C} = \mathbf{\Pi}_\mathcal{C}}_{\text{permutation matrix}}\underbrace{\mathbf{D}_\mathcal{C}}_{\text{diagonal unitary}}$$

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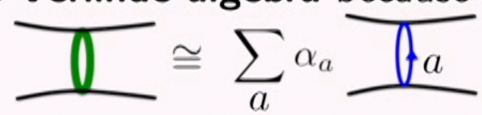
$$\mathbf{U}_{\mathcal{C}} = \mathbf{\Pi}_{\mathcal{C}} \mathbf{D}_{\mathcal{C}}$$

permutation matrix — diagonal unitary

Proof sketch:

for any loop C_j consider $A \mapsto UAU^\dagger$ for logical operators supported around C_j

This realizes an **isomorphism of the Verlinde algebra** because



hence $UP_a(C_j)U^\dagger = P_{\pi_j(a)}(C_j)$ for a permutation π_j of particle labels

Then extend to whole DAP-decomposition

Matrix representation of protected gates in DAP-basis

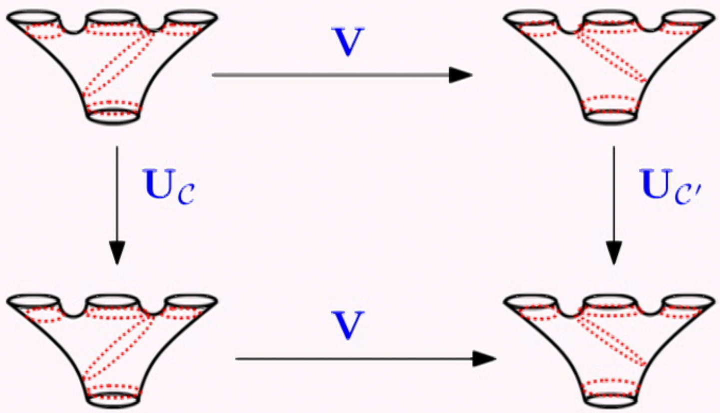
Suppose $U : \mathcal{H}_{phys} \to \mathcal{H}_{phys}$ is a protected gate

Lemma: Let $\mathcal{C} = \{C_j\}$ be a DAP-decomposition, $\mathcal{B}_\mathcal{C}$ be the associated basis of $\mathcal{H}_\Sigma$.
The matrix $\mathbf{U}_\mathcal{C}$ representing U in this basis is **unitary monomial**:

$$\mathbf{U}_\mathcal{C} = \mathbf{\Pi}_\mathcal{C}\mathbf{D}_\mathcal{C}$$

permutation matrix

diagonal unitary



Consequence: For two bases $\mathcal{B}_\mathcal{C}$ and $\mathcal{B}_{\mathcal{C}'}$ related by a unitary $\mathbf{V}$ we must have

$$\mathbf{V}\mathbf{\Pi}_\mathcal{C}\mathbf{D}_\mathcal{C} = \mathbf{\Pi}_{\mathcal{C}'}\mathbf{D}_{\mathcal{C}'}\mathbf{V}$$

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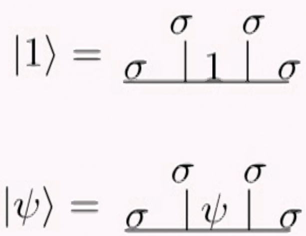
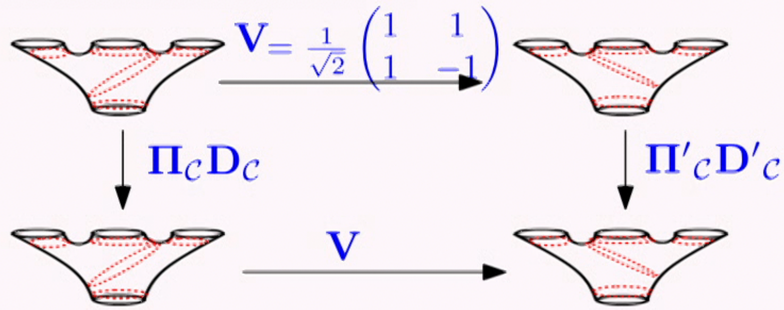
Example: 4 Ising-σ anyons

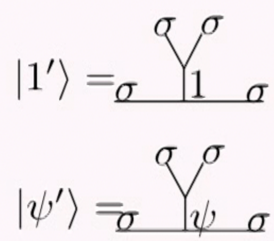
Protected gates belong to the Pauli group.

basis $\mathcal{B}_\mathcal{C}$





basis $\mathcal{B}_{\mathcal{C}'}$



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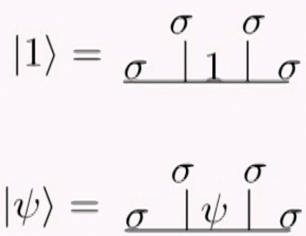

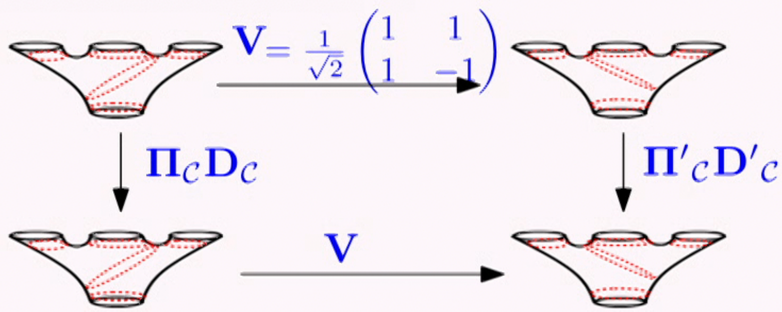

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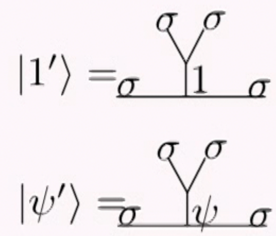

Matrix representation of protected gates in DAP-basis

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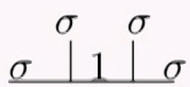
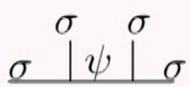
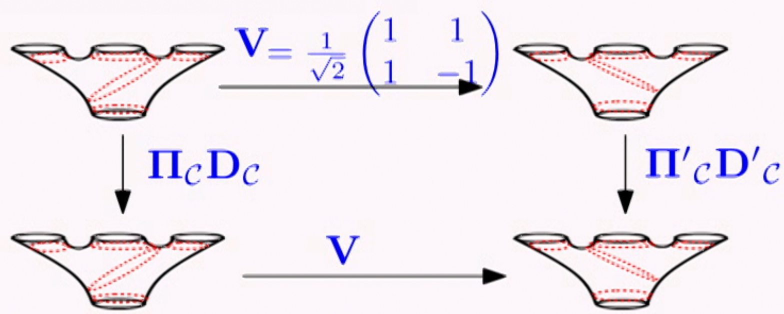
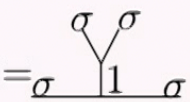
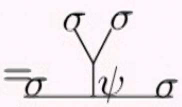
$|1\rangle =$ 

$|\psi\rangle =$ 



basis $\mathcal{B}_{\mathcal{C}'}$

$|1'\rangle =$ 

$|\psi'\rangle =$ 

		$\mathbf{\Pi} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
$\mathbf{\Pi}' =$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$(\mathbf{D}_{\mathcal{C}}, \mathbf{D}_{\mathcal{C}'}) = e^{i\varphi}(\mathsf{diag}(1,1), \mathsf{diag}(1,1))$	$e^{i\varphi}(\mathsf{diag}(1,1), \mathsf{diag}(1,-1))$
	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$e^{i\varphi}(\mathsf{diag}(1,1), \mathsf{diag}(1,-1))$	$e^{i\varphi}(\mathsf{diag}(1,-1), \mathsf{diag}(1,1))$

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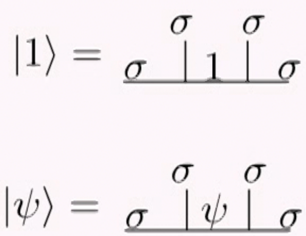

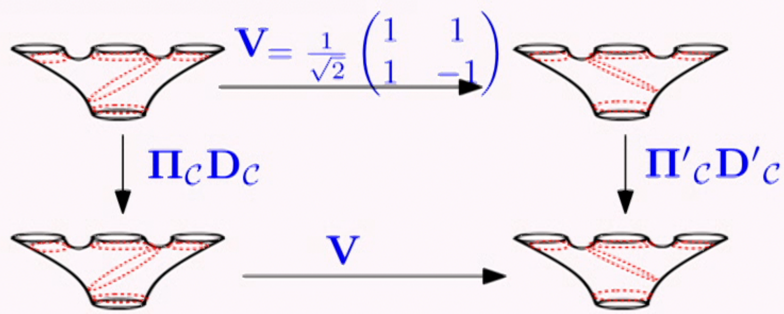

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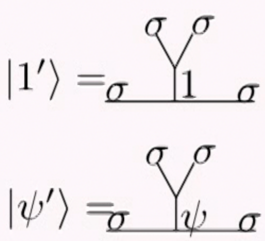

$\mathbf{\Pi}' =$	$\mathbf{\Pi} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\mathbf{U} = \mathbf{\Pi}\mathbf{D}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	
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Suppose $U : \mathcal{H}_{phys} \to \mathcal{H}_{phys}$ is a protected gate

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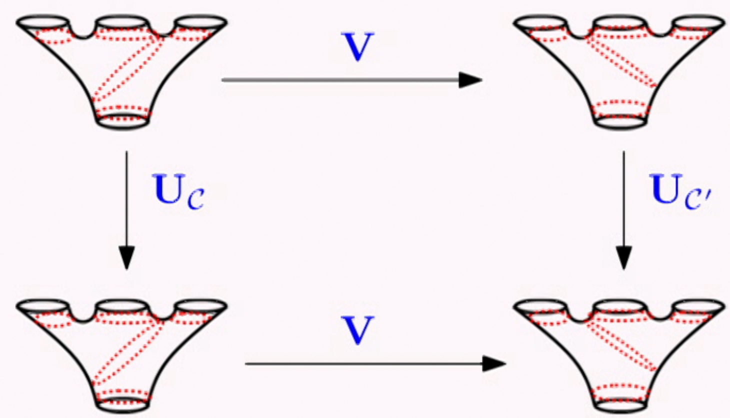

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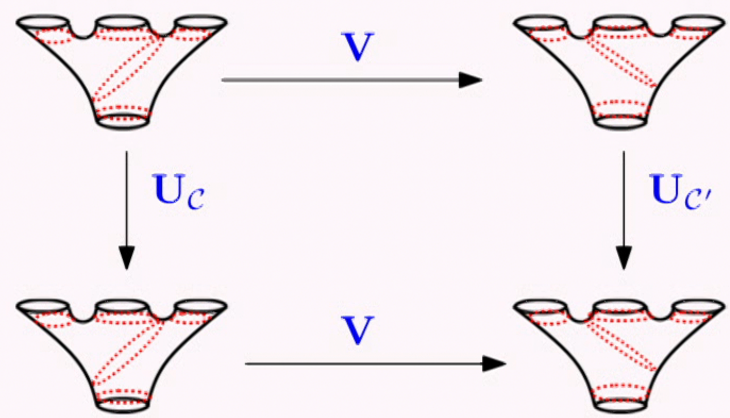

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Consequence: $\mathbf{V}(\vartheta)\mathbf{\Pi}_\mathcal{C}\mathbf{D}_\mathcal{C}\mathbf{V}(\vartheta)^\dagger$ is unitary monomial matrix for any $\vartheta \in \mathsf{MCG}_\Sigma$

Universality and absence of protected gates

Theorem: If $V : \mathsf{MCG}_\Sigma \to PU(\mathcal{H}_\Sigma)$ has a dense image, then there is no non-trivial protected gate.

Consequence: $\mathbf{V}(\vartheta)\mathbf{\Pi}_\mathcal{C}\mathbf{D}_\mathcal{C}\mathbf{V}(\vartheta)^\dagger$ is unitary monomial matrix for any $\vartheta \in \mathsf{MCG}_\Sigma$

Conclusions and open problems

- Turaev-Viro codes offer a rich class of examples for potential platforms for topological quantum computation.

- The mapping class group representation can be “decomposed” using the string-net formalism

- Explicit constructions of protected/transversal gates for TQFTs?

(cf. ``braided autoequivalence’’: Barkeshli et al., Symmetry, Defects, and Gauging of Topological Phases, arXiv:1410.4540)

- Performing syndrome-measurement & error correction, thresholds for fault-tolerance?

- Higher-dimensional generalizations?

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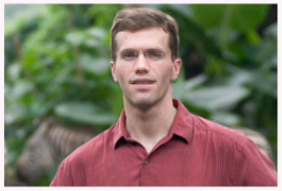
Thank you!



Ben Reichardt

Greg Kuperberg

Quantum Computation with Turaev-Viro codes,
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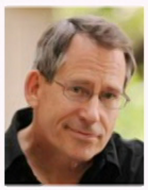
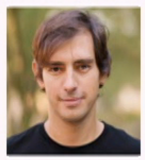
Sumit Sijher



John Preskill



Fernando Pastawski

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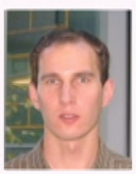

Sergey Bravyi

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