Title: A tensor network framework for topological phases of quantum matter

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Abstract: We present a general scheme for constructing topological lattice models in any space dimension using tensor networks. Our approach relies on finding "simplex tensors" that satisfy a finite set of tensor equations. Given any such tensor, we construct a discrete topological quantum field theory (TQFT) and local commuting projector Hamiltonians on any lattice. The ground space degeneracy of these models is a topological invariant that can be computed via the TQFT, and the ground states are locally indistinguishable when the ground space is nondegenerate on the sphere. Any ground state can be realized by a tensor network obtained by contracting simplex tensors. Our models are exact renormalization fixed points, covering a broad range of models in the literature. We identify symmetries on the virtual level of the tensor networks of our models that generalize the topological invariance properties beyond fixed point models. This framework combined with recent tensor network techniques is convenient for studying excitations, their statistics, phase transitions, and ultimately for classification of gapped phases of many-body theories in 3+1 and higher dimensions.

A Tensor Network Framework for Topological Phases of Quantum Matter

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Institute for Quantum Information and Matter

Caltech

Joint work with Michael Walter and Dominic Williamson

- ~ D.Williamson, N.Bultinck, M. Marien, , J.Haegeman, N.Schuch, F.Verstraete - arXiv:1409.2150 [quant-ph]
- ~ S.Shukla, F. Pollmann, X.Chen - arXiv:1610.00608 [cond-mat.str-el]
- ~ K.Temme, M. Walter

Original: Let no one unversed in geometry enter here.

Let no one uninterested in geometry enter here.

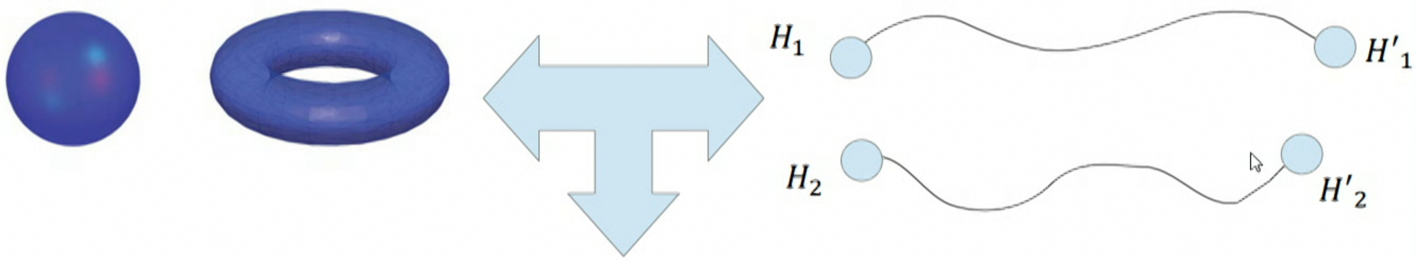
Motivations

- Topologist:

  Classify topologies

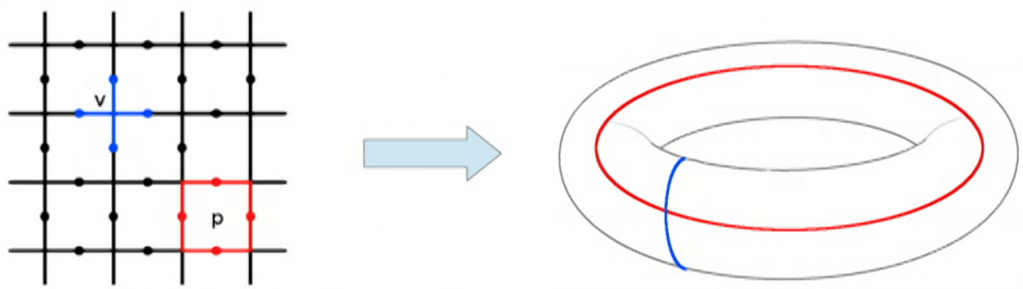
- Low/high energy physicist:

  Classify quantum matter



- Quantum information theorist & quantum memory engineer:

  Physical systems robust to local noise



Outline

- Motivations: Topology, Quantum matter, Quantum error correcting codes
- More Motivations:

  - Quantum matter

  - Quantum error correction
- Basics

  - Tensor Networks (notation & overview of achievements)

  - Topological Quantum Field Theory (state-sum TQFTs)
- Tensor Networks for TQFTs: numbers, states, operators, etc.
- A general tensor network framework: Tensor Networks $\supset$ TQFTs

  TQFTs are RG-fixed points. General TN framework: Away from RG-fixed points
- Examples: DW-models, 2+1D (Levin-Wen), 3+1D (Walker-Wang)
- Some problems:

  - excitations ~ extending TQFTs

  - Q. err. Correction performance

  - Phase transitions (perturbing tensors)

  - Higher form symmetries
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A brief history of classification of the phases of matter

- **Classical thermal physics:**

  Thermal phases:

  Solid, liquid, gas, etc.

- **Quantum phases at T=0 with local order parameter**

Landau's spontaneous sym. breaking

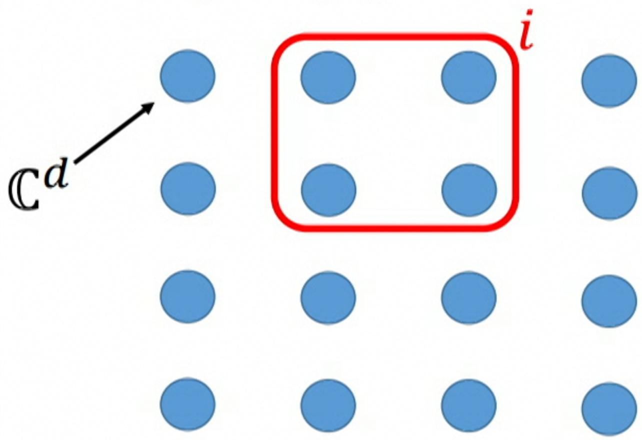
- **Quantum phases at T=0 with nonlocal order parameters:**

  Fractional quantum Hall systems, spin-liquids

Topological order

A computational approach for the classification of quantum matter

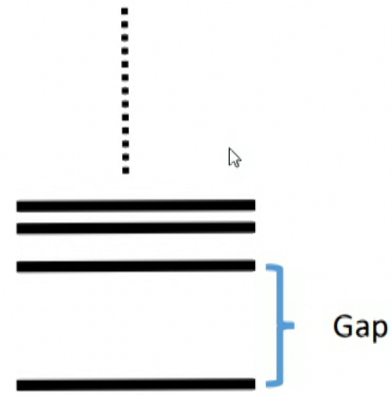
- Many-body system on a lattice, local Hamiltonian, gapped



$$|\Psi_N> \in \mathbb{C}^{d^{\otimes N}}$$

$$\updownarrow$$

$$H = \sum_i h_i$$



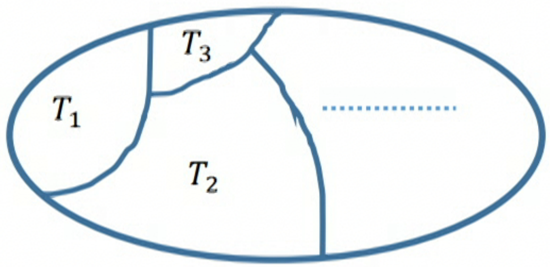
- Trivial phase: Product state ~ $|\varphi >^{\otimes N} \leftrightarrow h_i = (\mathbb{1} - |\varphi >< \varphi|)_i$

A computational approach for the classification of quantum matter

- Equivalence relation: gap preserving local unitaries

$$|\Psi\rangle \sim |\Psi'\rangle \text{ if } |\Psi'\rangle = U_{loc.}|\Psi\rangle$$

$$H \sim H' \quad \text{if} \quad H' = U_{loc.} H U_{loc.}^{\dagger}$$



- Phase diagram enriches when a symmetry is added

$$T_1 \rightarrow T_1^1, T_1^2, \dots$$

Quantum error correction

Let no one uninterested in error correction enter here.

- Error correction is fundamental

  Things that are persistent must be robust (correctable) against errors (noise).

  We are local, fortunately noise is also local.

  Examples: Languages

  Hard disks

  Quantum hard disks

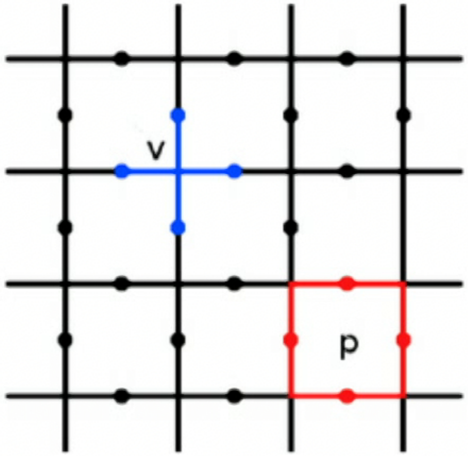
- Relation to topological order (gapped phases of quantum matter)

  Codespace: Ground state space of a local Hamiltonian

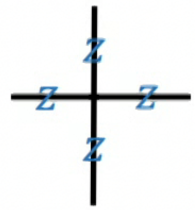
  Robustness: Ground state space is protected against local noise

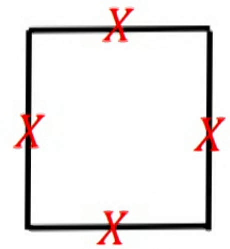
  (+ Hamiltonian perturbations)

Toric Code

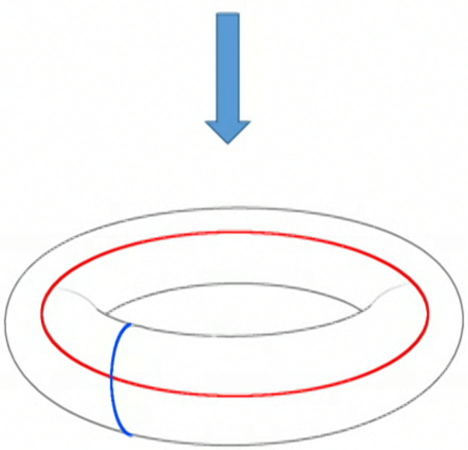


$$H = -\sum_v A_v + -\sum_p B_p$$

$A_v =$ 

$B_p =$ 

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$|\Psi_1 >= \sum_{loops}$ |even l_1 & even l_2 >

$|\Psi_2 >= \sum_{loops}$ |even l_1 & odd l_2 >

$|\Psi_3 >= \sum_{loops}$ |odd l_1 & even l_2 >

$|\Psi_4 >= \sum_{loops}$ |odd l_1 & odd l_2 >

Locally indistinguishable → Robust against Local noise

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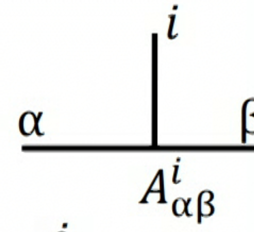
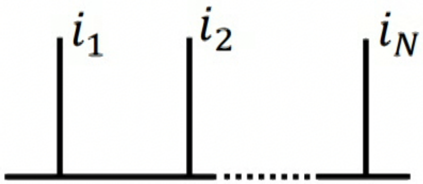
Basics-1: Tensor Networks

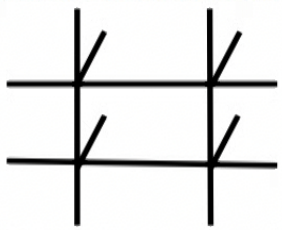


- Matrix product states

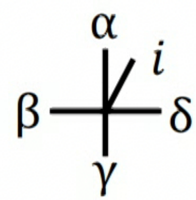


$$\equiv |\Psi> = \sum_{\alpha_1,\alpha_2,\ldots} tr(A^{i_1}A^{i_2}\ldots A^{i_N}) \; |i_1, i_2, \ldots, i_N>$$

- Projected entangled pair states (PEPS)

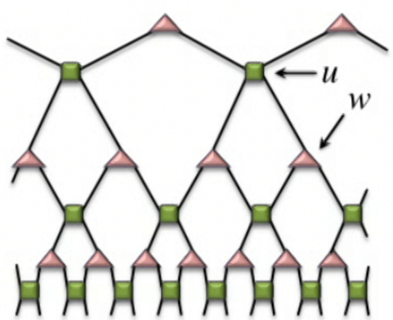


$$\equiv \; |\Psi>$$



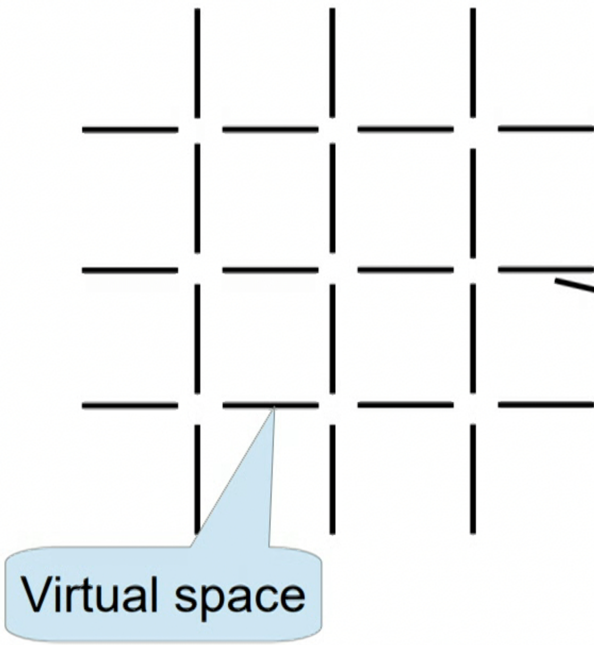
$\sim A^i_{\alpha\beta\gamma\delta}$

- Multiscale entanglement renormalization ansatz (MERA)



~ Renormalization (1D)

Basics-1: Tensor Networks



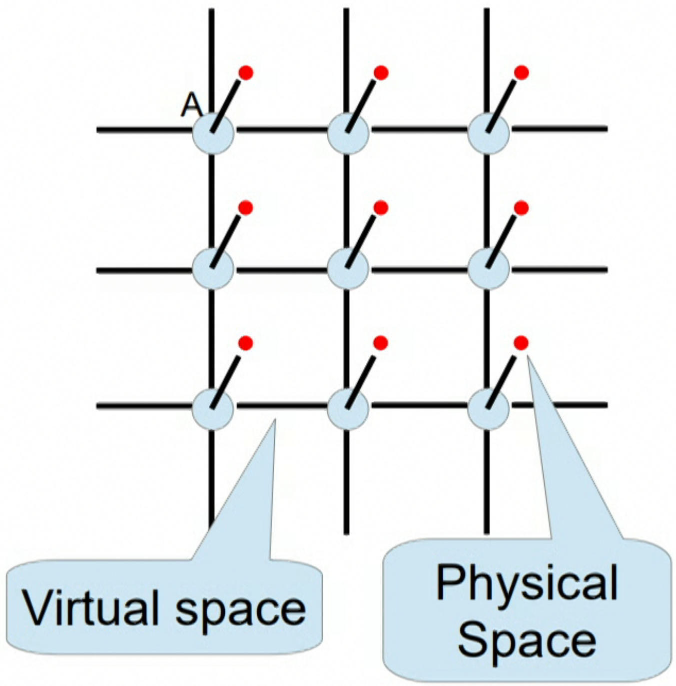
- Start with bipartite maximally entangled states between each nearest neighbour site:

$$\omega = \Sigma_{i=1}^{D} \mid i >\mid i >$$

- $\Psi' = \omega^{\otimes 2N}$

$$H' = \Sigma_i (I - \mid \omega >< \omega \mid)_i$$

Basics-1: Tensor Networks



- Insert a linear map at every site:

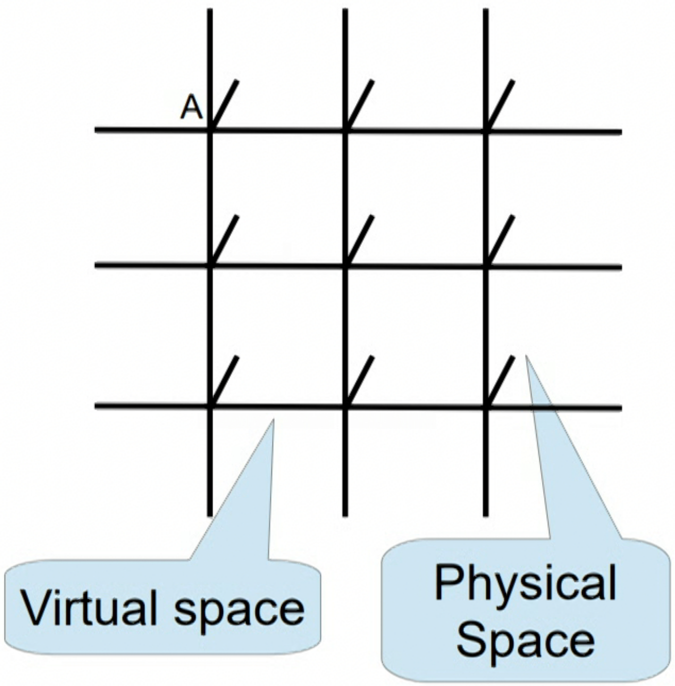
$$A{:}\,Virtual \to Physical$$

$$\Psi = A^{\otimes N}\omega^{\otimes 2N}$$

$$\Downarrow$$

$$HA^{\otimes N} = A^{\otimes N}H'$$

Basics-1: Pedagocigal Summary

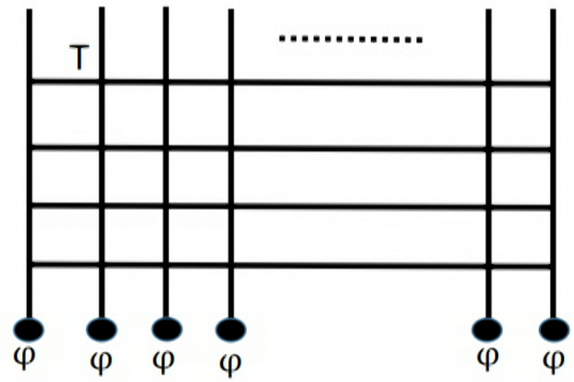


- There are virtual and physical Hilbert spaces
- The structure of the whole state is encoded in

  A (local tensor)
- Local tensor $\longrightarrow$ State
- State $\longrightarrow$ Local Hamiltonian
- Numerous other properties about entanglement entropy, efficient simulation of quantum systems, etc..

Euclidean Path Integrals ~ Tensor Networks

$$\sim |\Psi_{ground}>$$



- Given a local Hamiltonian H:

$$\sim \quad e^{-Ht}|\varphi^{\otimes N}>$$

- H is gapped $\longrightarrow$ Approximate ground state projector (constant depth circuit)
- Compress it to an MPS with finite bond dimension:

$$|\Psi_{ground}> \quad \sim$$

Tensor Network approach

- Take the ground states of many-body lattice systems as tensor network states (TNS).
- Find the conditions on the tensors ("iff" if possible), such that TNS

  satisfies the required physical properties.

  Topologically ordered TNS:
- Axiomatize the conditions on the TNS such that the ground state space is topologically ordered (= Physical properties (such as ground state degeneracy) depending on the topology).

TQFTs appear as fixed-points $\longrightarrow$ Tensor Network/Hamiltonian for TQFTs

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Basics-2: TQFT (state-sum)

- Goal: Assign numbers to n+1-dim manifolds:

$$M \nsim N \ iff \ \ Z(M) \neq Z(N)$$

- We can:

$$Z(M) = Z(N) \ if \ M \sim N$$

- How:

  - Triangulate the manifold M (glue n+1-simplices)
  - Put degrees of freedom on 0, 1, ..., n-simplices
  - Assign a tensor T to every n+1-simplex.
  - Obtain Z(M) by contracting (gluing) tensor (n+1-simplices) T.
  - T is such that Z(M) is invariant under retriangulations (=T satisfies Pachner equations).

Basics-2: TQFT (state-sum)

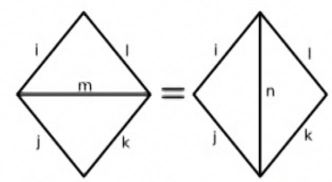
- 0-simplex:  (point)
- 1-simplex:  (line)
- 2-simplex:  (triangle)
- 3-simplex: 

  (tetrahedron)
- 4-simplex, ..., n-simplex, ...

- Gluing (a 1+1 D example):

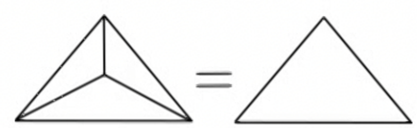
- Retriangulation invariance:

1-



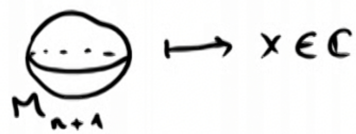
$$\sum_m T_{iml} T_{kmj} = \sum_n T_{ijn} T_{klr}$$

2-

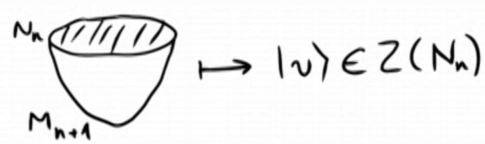


What do state-sum TQFTs do?

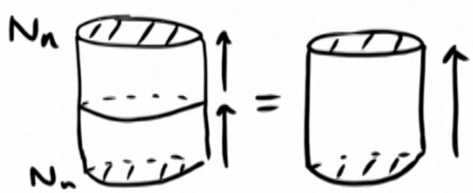
- A n+1 (state-sum) TQFT uses the local data T to assign $Z(M) \in \mathbb{C}$ to a closed n+1-manifold M.



- $Z(M) = |v> \in Z(N)$ to a n+1-manifold M with boundary closed n-manifold $N = \partial M$



- $Z(N \times I)$: $Z(N) \rightarrow Z(N)$ is projector on the vector space $Z(N)$



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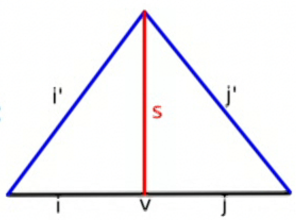
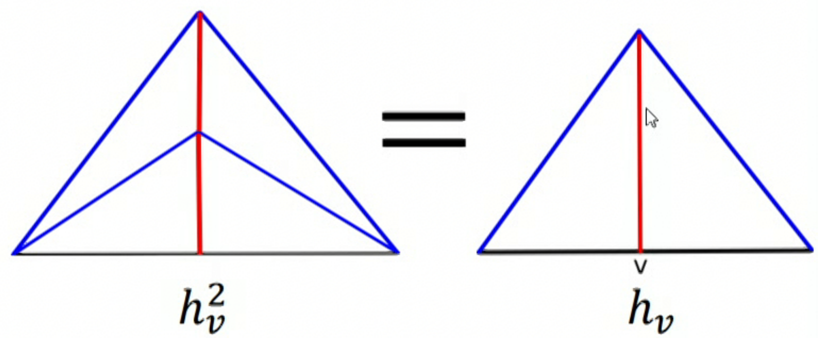
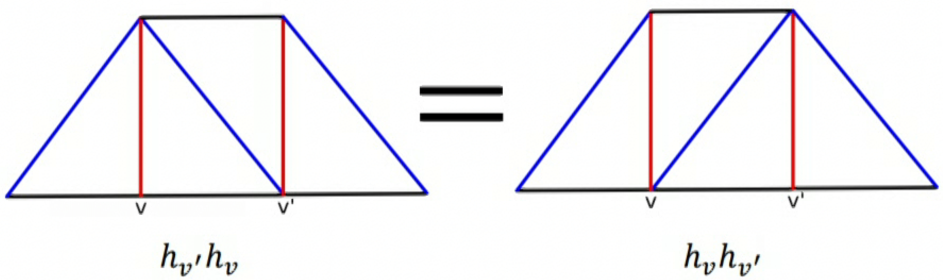
State-sum TQFTs ⇒ TN RG-fixed points

- Given a state sum TQFT (1+1D) (given a tensor T, satisfying Pachner equations) we can define a many-body Hamiltonian such that
- $H = -\Sigma_v h_v$ where $[h_v, h_{v'}] = 0 \; all \; v\,, v'$ and $h_v^2 = h_v$.

$h_v =$ 

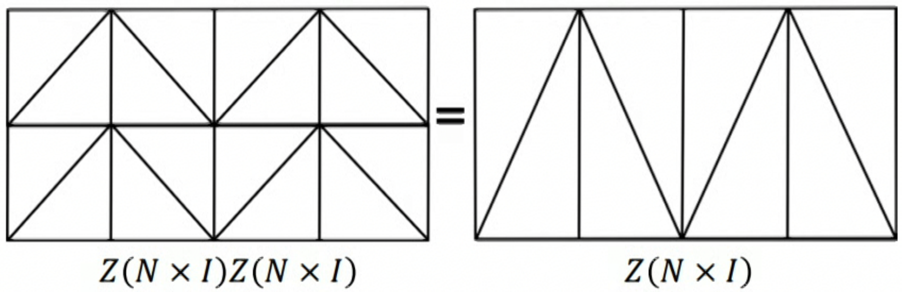




(Koenig, Reichardt, Kuperberg – 2010)

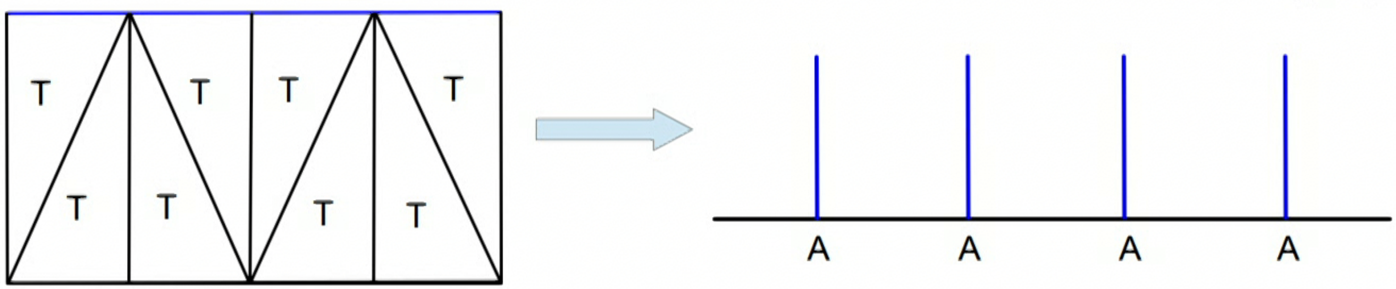
State-sum TQFTs ⟹ TN RG-fixed points

- “Cylinder map” $Z(N \times I)$ projects onto the ground state space.

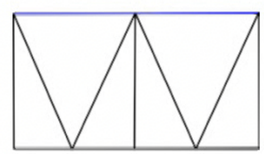


- Ground state is expressible in terms of a TNS, where

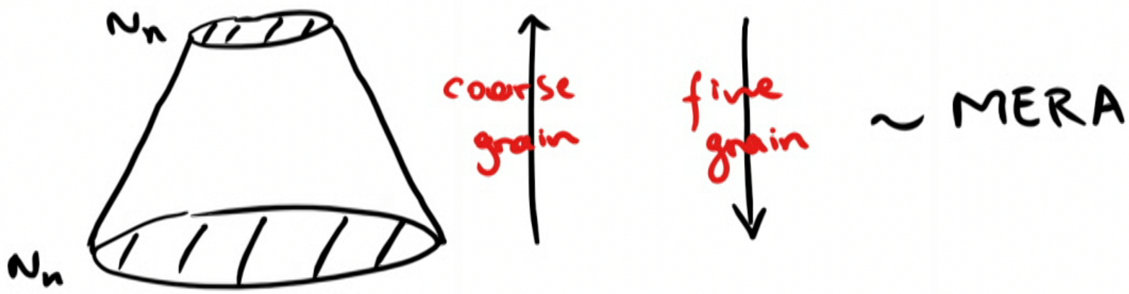
$A \sim T$

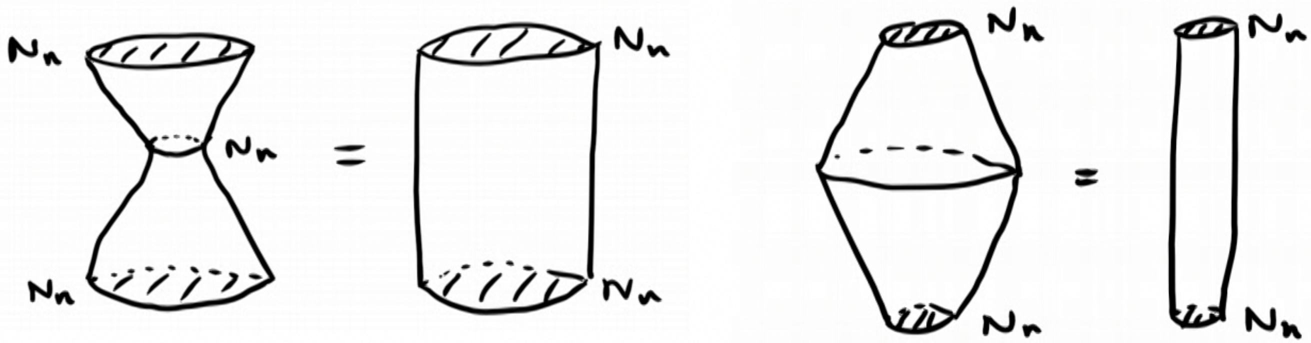


- Finer lattice ⟺ Coarser lattice



State-sum TQFTs ⟹ TN RG-fixed points





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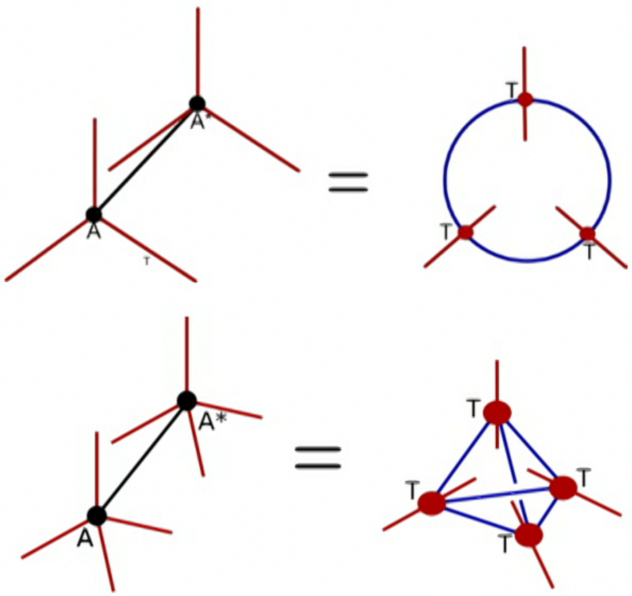
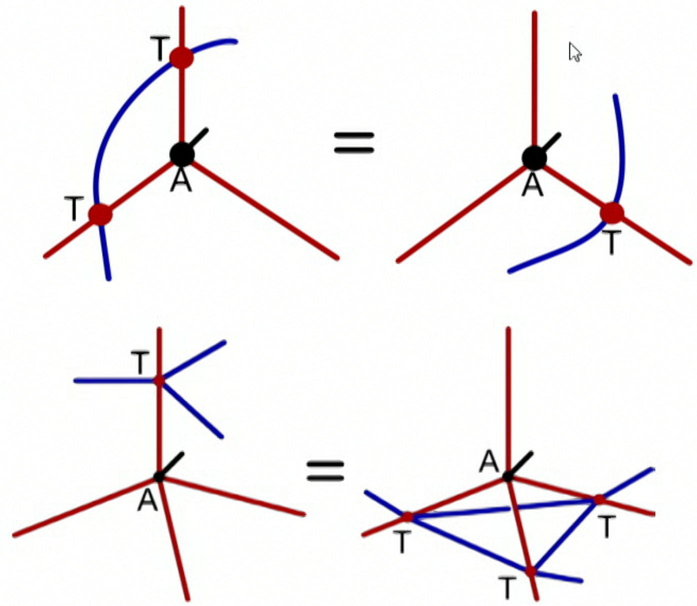
A general tensor network framework: Tensor network operators

- TNO-injectivity:
- The local tensor is injective within a subspace of the virtual degrees of freedom!

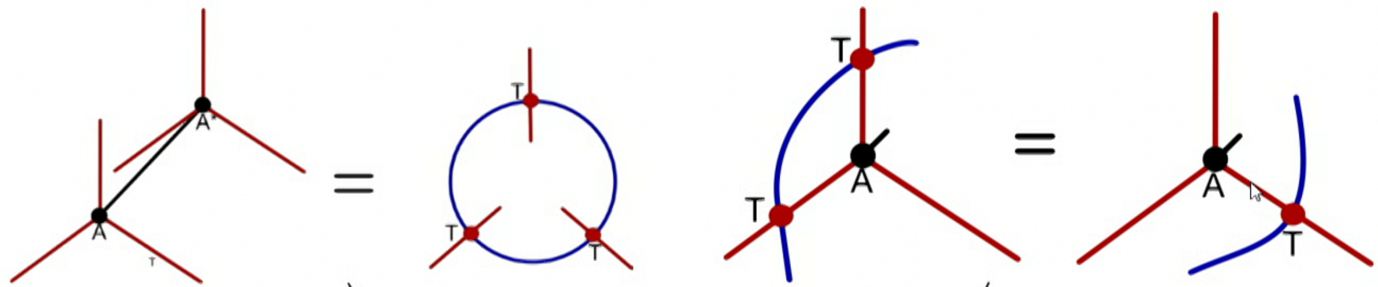


- Pulling through:
- TNOs can be deformed through the lattice freely!

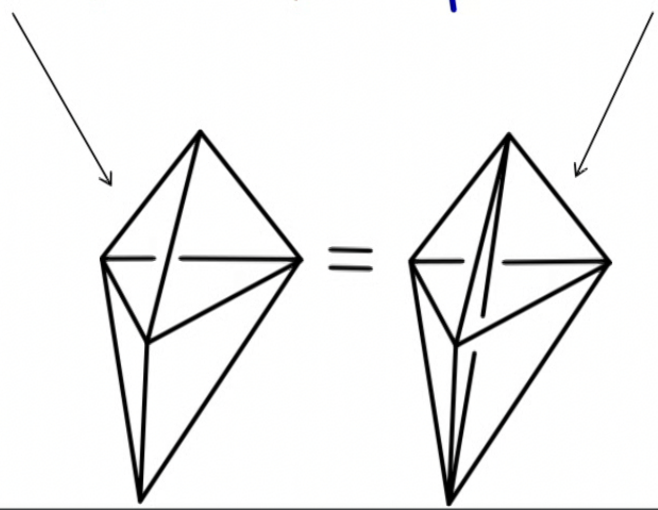


State-sum TQFTs $\subset$ TN framework

- Pachner eq. = TNO-injectivity
- Pachner eq. = Pulling through



- special case:



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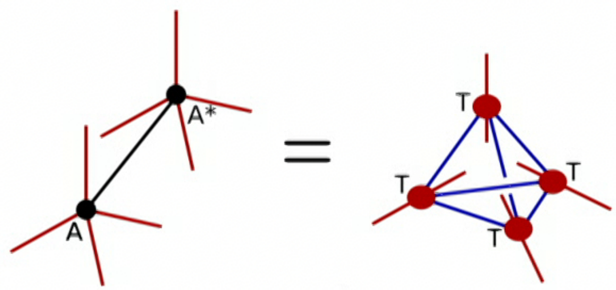
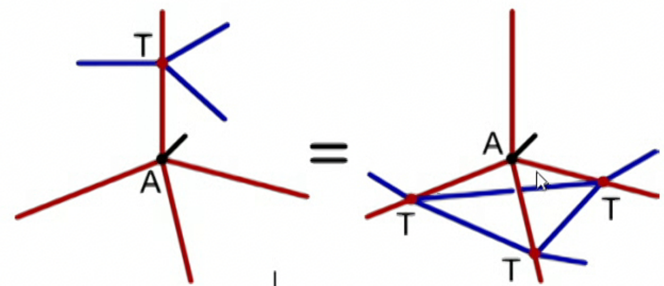
- Pachner eq. = TNO-injectivity



- special case: 2-4 Pachner equation

- Pachner eq. =Pulling through

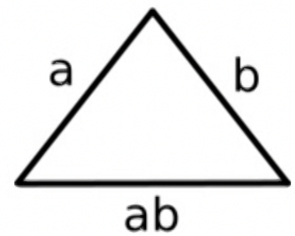


2-4 Pachner equation

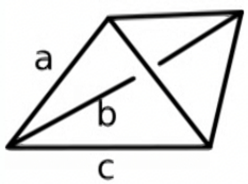
3-3 Pachner equation

Example-1: Dijkgraaf-Witten models in n+1 dimensions

- Finite group G
- Local degrees of freedom g in G
- live on 1-simplices

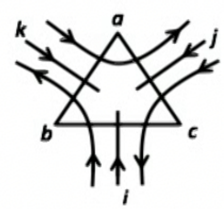


- n+1-simplex ~ T = $\omega_{n+1}(g_1, g_2, \ldots, g_{n+1})$ (satisfies cocycle equation)
- General construction of Hamiltonian and ground states
- n=2:



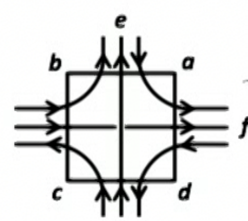
$\sim \omega_3(a,b,c)$

Local tensor



$\sim \omega_3(a,b,c)$

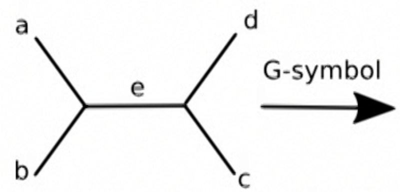
MPO

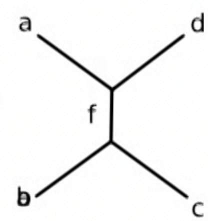


$\sim \omega_3(a,b,c)$

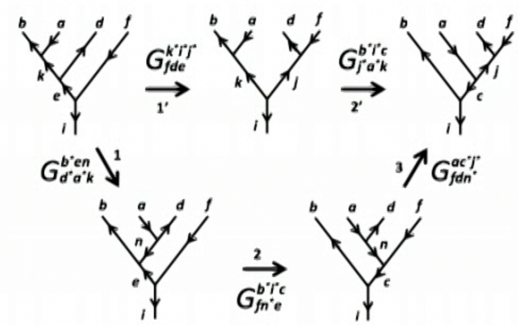
Example-2: Levin-Wen models in 2+1 dimensions

- A unitary fusion tensor category C:
- Objects: $i = 0,1,\dots,N$
- Tensor product: $a \otimes b = \bigoplus_c N_{abc} c$
- Associator:



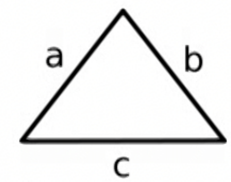


- Associator satisfies 'Pentagon equation'

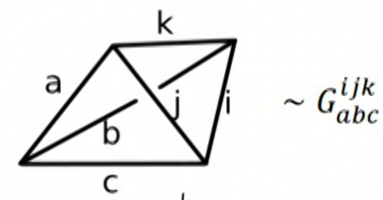


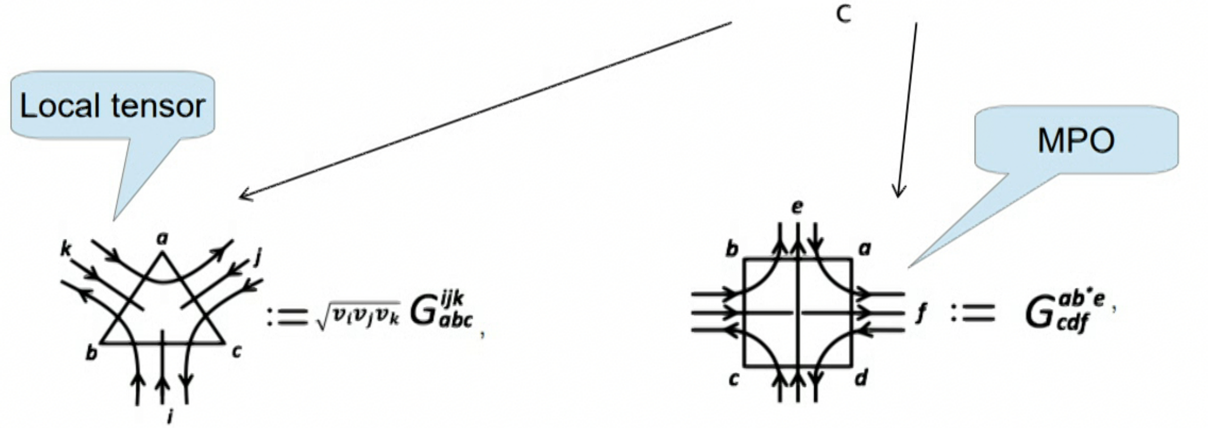
Example-2: Levin-Wen models in 2+1 dimensions

- Local degrees of freedom: objects in C assigned to the 1-simplices



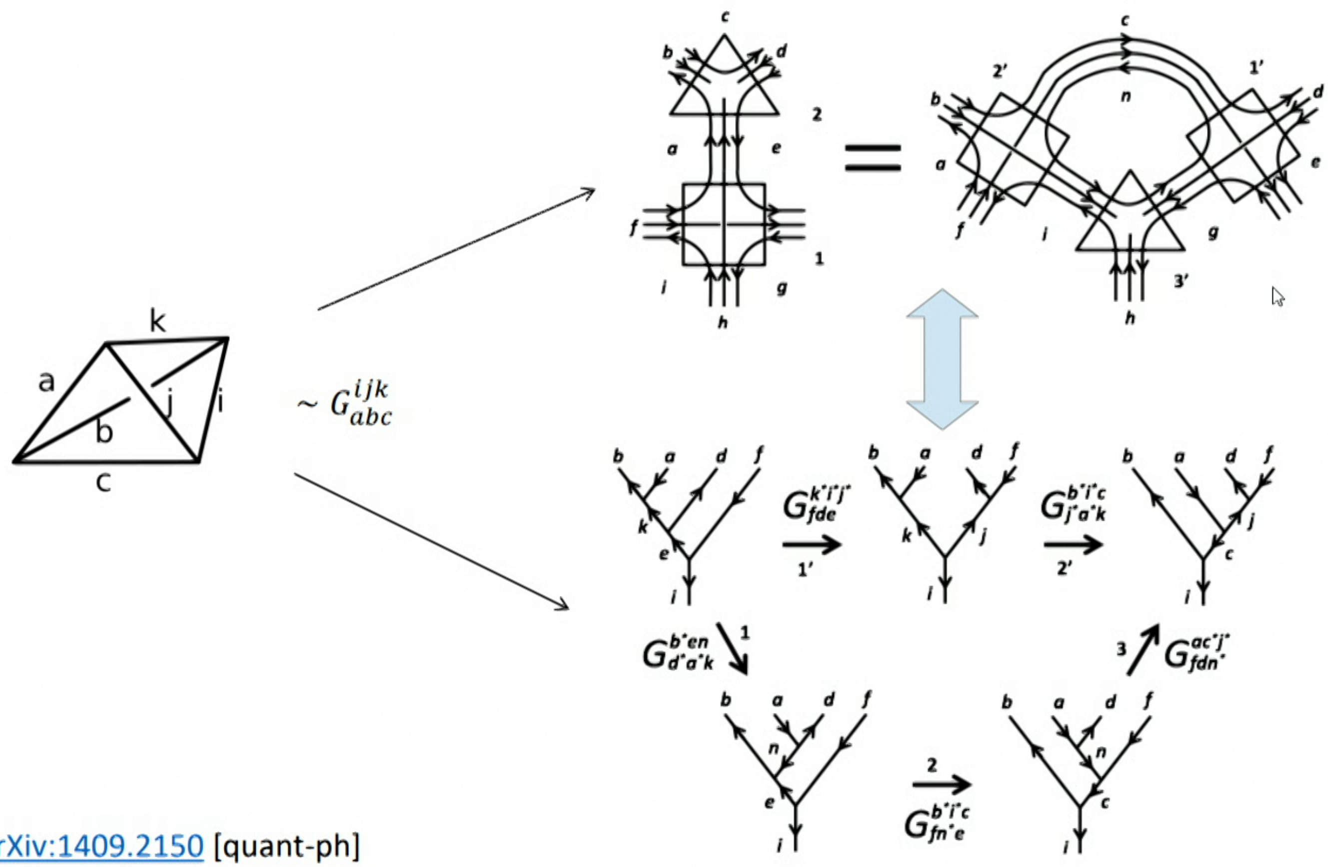
- 3-simplex: T ~ G-symbol





arXiv:1409.2150 [quant-ph]   Joint with Williamson, Bultinck, Marien, Haegeman, Schuch, Verstraete

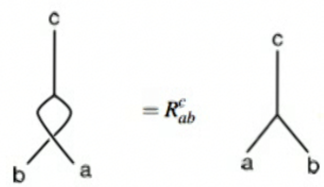
Example-2: Pulling through= Pentagon (Pachner) equation



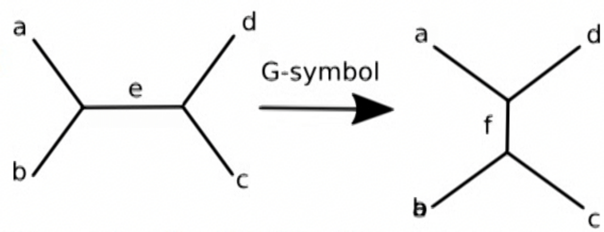
arXiv:1409.2150 [quant-ph]

Example-3: Walker-Wang models in 3+1 dimensions

- UBFC - C
- Objects: $i = 0,1,\ldots,N$
- Tensor product: $a \otimes b = \bigoplus_c N_{abc} c$
- Braiding:



- Associator:

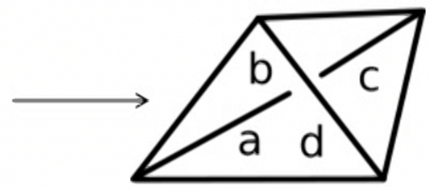


- Associator (G-symbol) satisfies 'Pentagon equation' and 'Hexagon equation' with the braiding morphism (R-symbol).

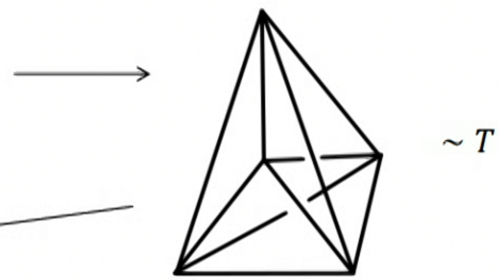
Joint with Walter & Temme

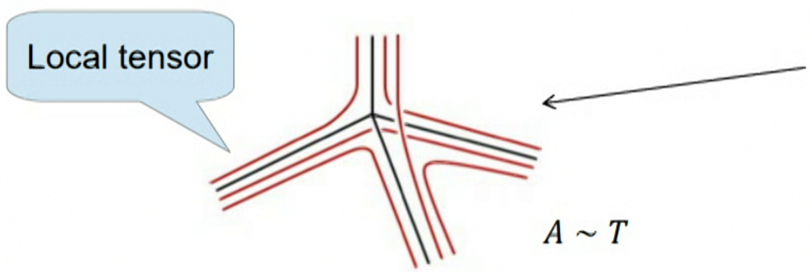
Example-3: Walker-Wang models in 3+1 dimensions

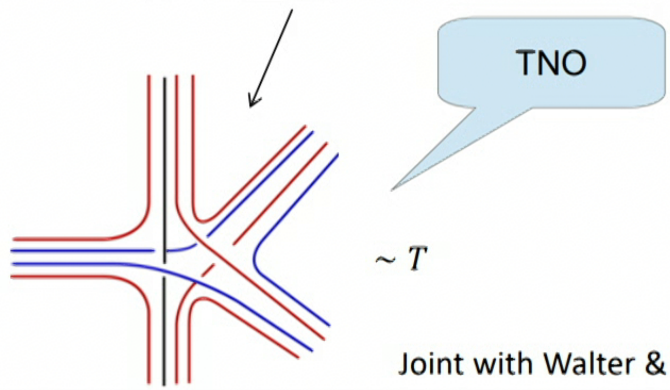
- Local degrees of freedom: objects in C are assigned to the 2-simplices



- 4-simplex: T ~ 15j-symbol

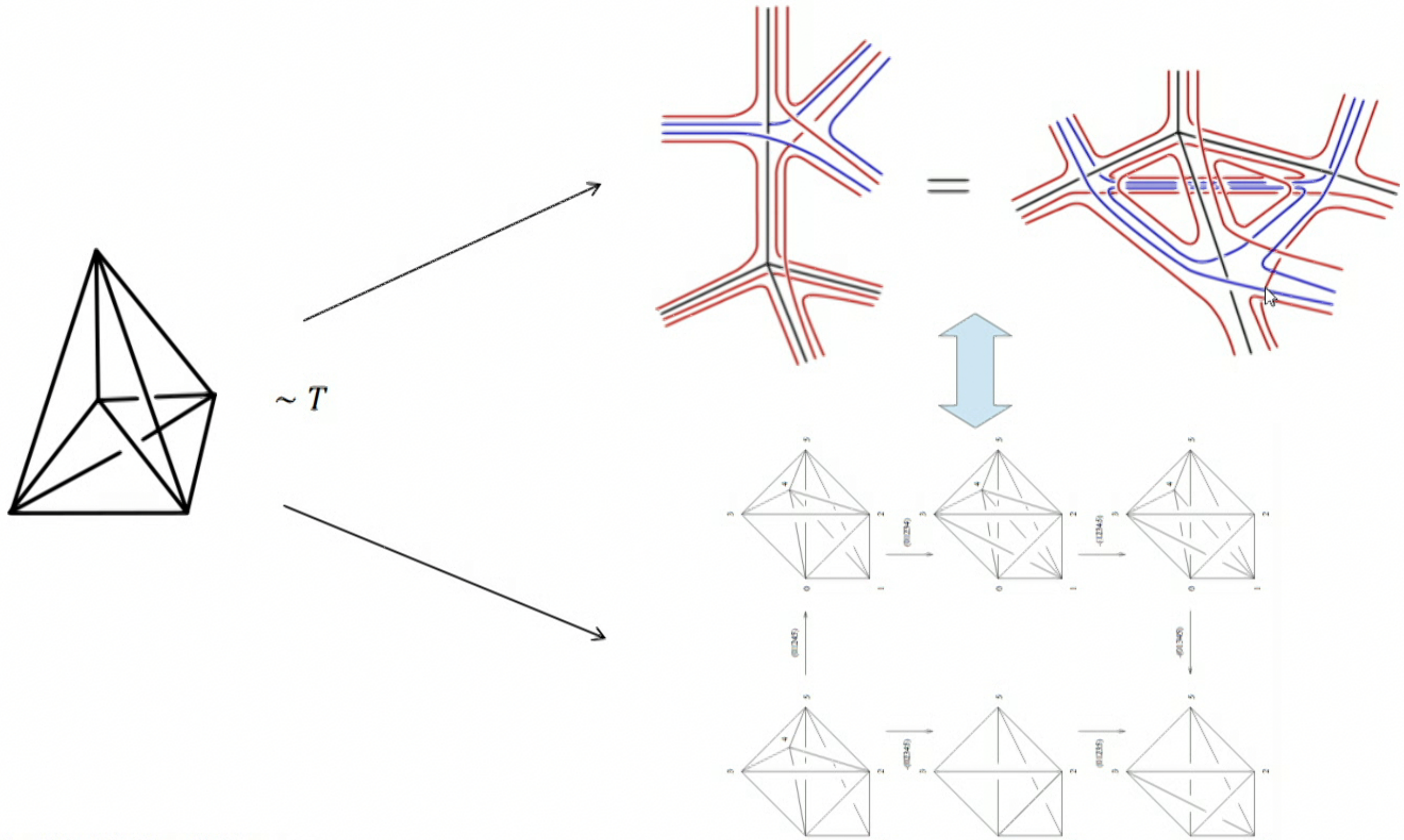






Joint with Walter & Temme

Example-3: Pulling through=Pachner equation



Joint with Walter & Temme

How to find a new model (n+1-dim)?

- A crude way: Solve the equations!

  1- Assign degrees of freedoms to 0, 1, ..., n-simplices

  2- Assign a tensor T to n+1 simplex

  3- Solve the Pachner equations for the tensor T (= retriangulation invariance in n+1 dimensions)

  4- Use what we saw to write down the model, tensor network ground states, tensor network operators, etc.

- A more systematic way: Come up with an n-category!

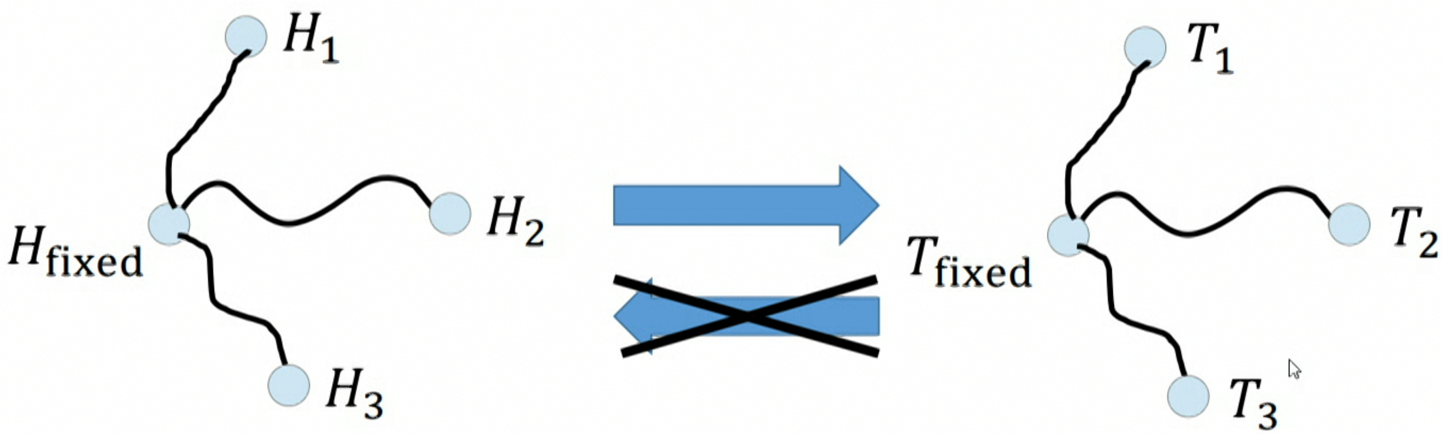
  1- Objects, 1-morphisms, 2-morphisms,... , n-morphisms, and an associator at level n+1.

  2- Assign objects to 0-simplices, 1-morphisms to 1-simplices, etc. And assign tensor T as the associator to the n+1 simplex

  3- T automatically satisfies Pachner equations.

  4- ...

Away from fixed points



- There are stable and unstable tensor variations

  Stable variations → Stay in the same phase

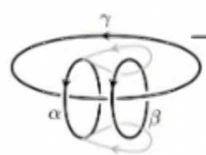
  Unstable variations → Going out of the phase (Boson condenstation)
- Stable variations are those that respect MPO-symmetries
- Numerical simulation of topological phase transition

Joint with Shukla, Pollmann, Chen arXiv:1610.00608 [cond-mat.str-el]

Outlook

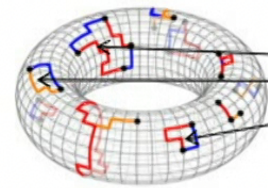
- Finding new models

  Understanding the excitations (loop-like excitations, braiding, etc.)



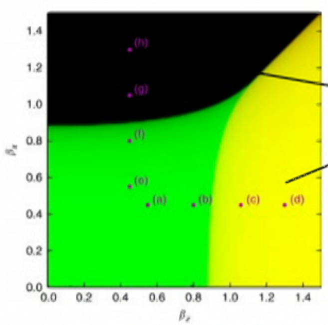
Unitary gates for quantum computation

- Performing/analyzing quantum error correction



Project onto local injectivity subspace

- Studying topological quantum phase transitions



Anyon condensation

- Extending Landau's theory to higher form symmetries

Symmetries on strings,surfaces, etc.

Summary

- Motivations, basics of tensor networks and state-sum TQFTs
- State-sum TQFTs ⇒ RG-fixed point models
- Characterization of topological order in tensor network states:
  - TNO injectivity
  - Pulling through (deformable TNOs)
- RG-fixed point models $\supset$ General TN framework
- RG-fixed point examples:
  - Dijkgraaf-Witten models (~group G & n+1-cocycle)
  - Levin-Wen (Turaev-Viro) models (~UFTC & F-symbol)
  - Walker-Wang (Crane-Yetter) models (~UBFC & 15j-symbol)
- Some problems:
  - excitations ~ extending TQFTs
  - Q. err. Correction performance
  - Phase transitions
  - Higher form symmetries