Title: Interacting Hopf monoids and Graphical Linear Algebra

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Abstract: The interaction of Hopf monoids and Frobenius monoids is the productive nucleus of the ZX calculus, where famously each Frobenius monoid-comonoid pair corresponds to a complementary basis and the Hopf structure describes the interaction between the bases. The theory of Interacting Hopf monoids (IH), introduced by Bonchi, Sobocinski and Zanasi, features essentially the same Hopf-Frobenius interaction pattern. The free symmetric monoidal category generated by IH is isomorphic to the category of linear relations over the field of rationals: thus the string diagrams of IH are an alternative graphical language for elementary concepts of linear algebra. IH has a modular construction via distibutive laws of props, and has been applied as a compositional language of signal flow graphs. In this talk I will outline the equational theory, its construction and applications, as well as report on ongoing and future work.

Interacting Hopf monoids and Graphical Linear Algebra

Pawel Sobocinski, Southampton

Hopf algebras in … Categorical Quantum Mechanics
Perimeter Institute
2 August 2017

pedagogy

fields $\longrightarrow$ linear algebra $\longrightarrow$ Hopf algebra

copedagogy

Hopf algebra $\longrightarrow$ linear algebra $\longrightarrow$ fields

plan

- **Bialgebras and Hopf algebras - a “computer science” perspective**
- Interacting Hopf Monoids
- Graphical Linear Algebra

“computer science” perspective

- **Syntax → Semantics**
  - regular expressions → regular languages
  - context free grammars → context free languages
  - programming languages → domain theory (complete partial order, Scott topology, etc)
  - syntax of predicate logic → models of predicate logic
  - algebraic theory → models (universal algebra)

(commutative) monoids and groups a la 1930s universal algebra - syntax

- (presentation of) algebraic theory: pair T = (Σ, E) of finite sets
- for commutative monoids:
  - signature Σ, arity: $\Sigma \to \mathbf{N}$
    - $\cdot$: 2
    - e : 0
  - equations E (pairs of **typed terms**)
    - $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
    - $x \cdot y = y \cdot x$
    - $x \cdot e = x$
- For abelian groups, additionally
  - signature: $(-)^{-1}$: 1
  - equations: $x \cdot x^{-1} = e$

(commutative) monoids and groups a la universal algebra - semantics

- To give a **model**
  - Pick carrier set X
    - $\cdot : 2$ ➡ $\cdot : X^2 \to X$
    - $e : 0$ ➡ $e : X^0 \to X$
    - $(-)^{-1} : 1$ ➡ $(-)^{-1} : X^1 \to X$
  - For every evaluation of variables σ: Var→X, each equation must hold
- So, e.g. a **model of the algebraic theory of monoids** is the same thing as a **monoid**, in the classical sense

functorial semantics, 1960s

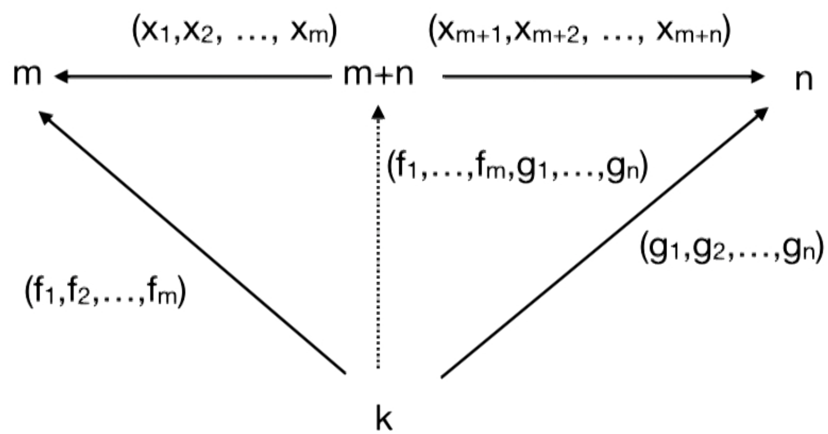
- Lawvere was not happy with universal algebra
  - too set theory specific
    - (e.g. topological groups morally should be a model)
  - too much ad hoc extraneous machinery
    - (e.g. countable set of variables, variable evaluation, etc.)
- Lawvere's 1963 doctoral thesis "Functorial semantics of algebraic theories" - universal algebra categorically

Lawvere theories

- Given algebraic theory (Σ,E), a category $\mathcal{L}_{(\Sigma,E)}$ with
  - objects: the natural numbers
  - arrows from m to n:
    - n-tuples of terms that (possibly) use variables $x_1, x_2, \ldots x_m$ modulo equations E
    - composition is substitution
- e.g. $2 \xrightarrow{(x_1 \cdot x_2)} 1 \qquad 2 \xrightarrow{(x_2 \cdot x_1)} 1 \qquad 1 \xrightarrow{(e,\, x_1)} 2 \xrightarrow{(x_2 \cdot x_1)} 1 \;=\; 1 \xrightarrow{(x_1 \cdot e)} 1$
- More concisely - “free category with products on the data of an algebraic theory”
- Fundamental fact:

classical model = cartesian functor $\mathcal{L} \to$ **Set**

products in a Lawvere theory



limitations of algebraic theories

- Copying and discarding built in

$$2 \xrightarrow{(x_1)} 1 \qquad 2 \xrightarrow{(x_2)} 1 \qquad 1 \xrightarrow{(x_1,\ x_1)} 2$$

- Consequently, there are also no bona fide operations with *coarities* other than one

$$1 \xrightarrow{c} 2 \quad = \quad 1 \xrightarrow{(c_1, c_2)} 2$$

- But in quantum mechanics, computer science, and elsewhere we often need to be more careful with resources

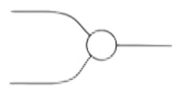
props

- A prop is a strict symmetric monoidal category with
  - strict means: ⊗ is associative on the nose
  - objects = natural numbers
  - $m \otimes n := m + n$ (I will usually write $m \oplus n$)
- Simple examples:
  - permutations of finite sets
  - functions between finite sets
  - **any Lawvere theory**
- prop homomorphism = identity on objects symmetric monoidal functor

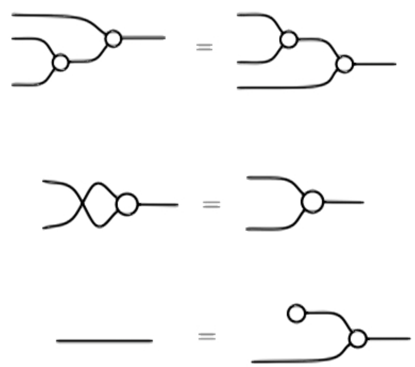
symmetric monoidal theories

- algebraic theory in the symmetric monoidal settings
- a symmetric monoidal theory is a pair of finite sets (Σ, E)
  - Σ signature, arity : $\Sigma \to N$, coarity : $\Sigma \to N$
  - E equations, pairs of **string diagrams** constructed from Σ, identity and symmetries

symmetric monoidal theory of commutative monoids

 $: (2,1)$

 $: (0,1)$

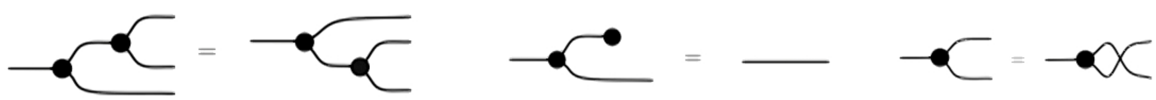


commutative monoid facts

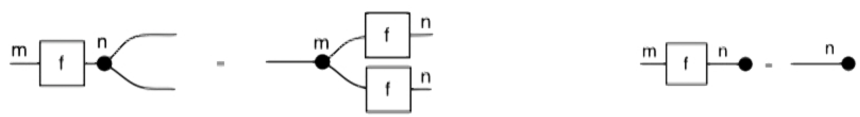
- the following are isomorphic as props
  - prop of commutative monoids
  - prop of functions between finite sets
- **not** isomorphic to the Lawvere theory of commutative monoids

folk theorem

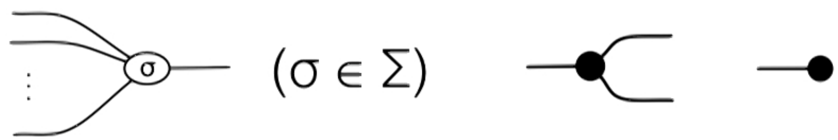
- A symmetric monoidal category **C** is cartesian iff
  - every object C∈**C** has a commutative comonoid structure Δ: C→C⊗C, c: C→I



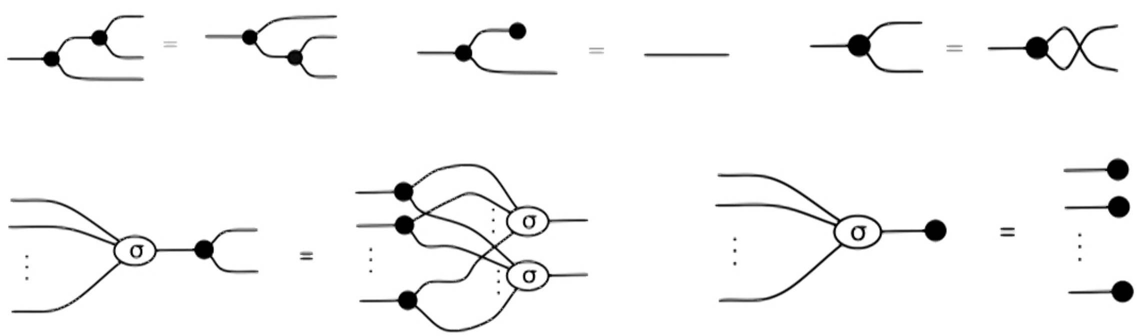
  - compatible with ⊗ in the obvious way
  - and every arrow f : m→n of **C** is a comonoid homomorphism, i.e.



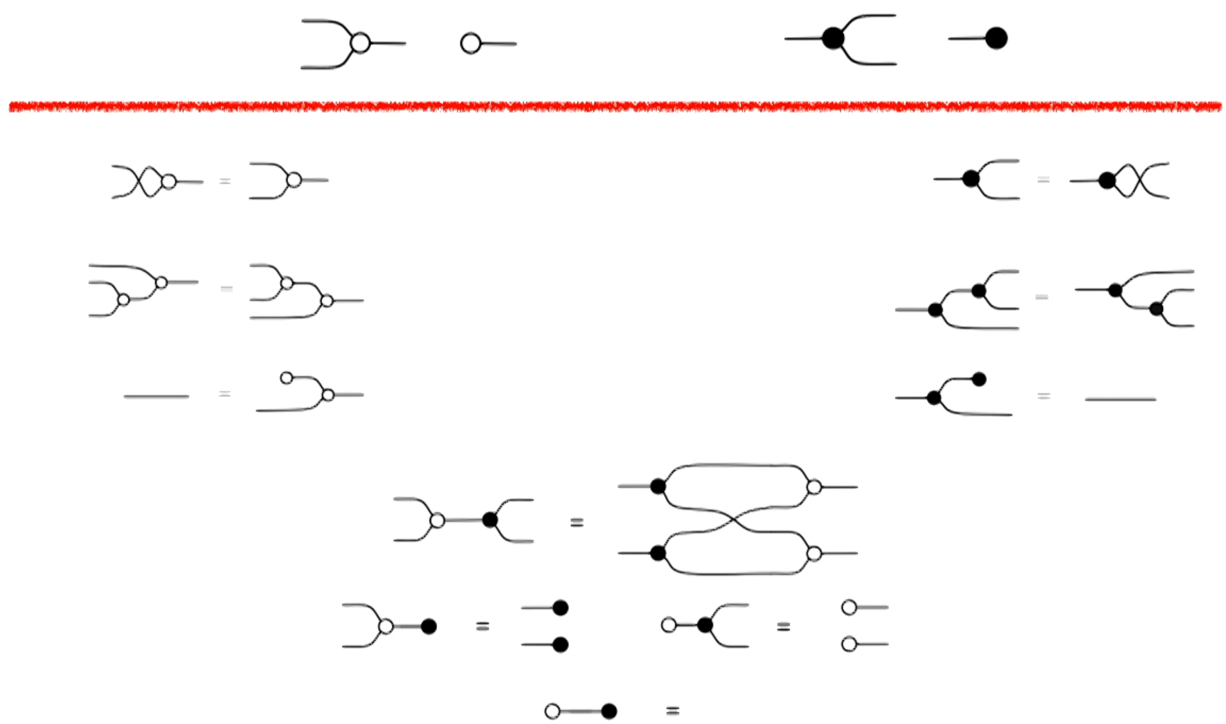
Lawvere theories as SMTs



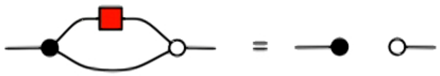
E +



Lawvere theory of commutative monoids as SMT



- e.g. the Hopf equation



is simply the SMT version of $x \cdot x^{-1} = e$

- Lawvere theory of commutative monoids = Symmetric monoidal theory of (co)commutative bialgebra

- Lawvere theory of abelian groups = Symmetric monoidal theory of (co)commutative Hopf algebras

So bialgebras and Hopf algebras are, respectively, monoids and groups in a **resource sensitive** universe.

plan

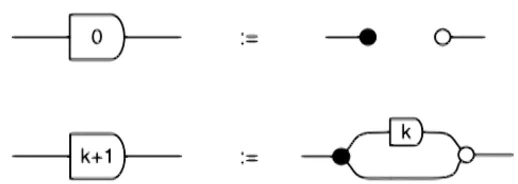
- Bialgebras and Hopf algebras - a “computer science” perspective
- **Interacting Hopf Monoids**
- Graphical Linear Algebra

bialgebra facts

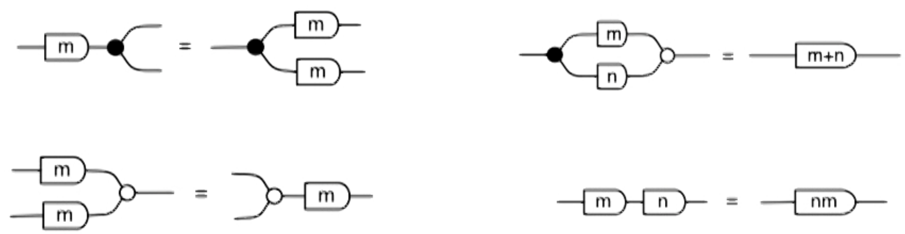
- All of the following are isomorphic as props
  - prop of (co)commutative bialgebras
  - Lawvere theory of commutative monoids
  - prop of spans of finite sets
  - prop of matrices of natural numbers
    - arrows $m \to n$ are $n \times m$ matrices with natural number entries
    - composition is matrix multiplication

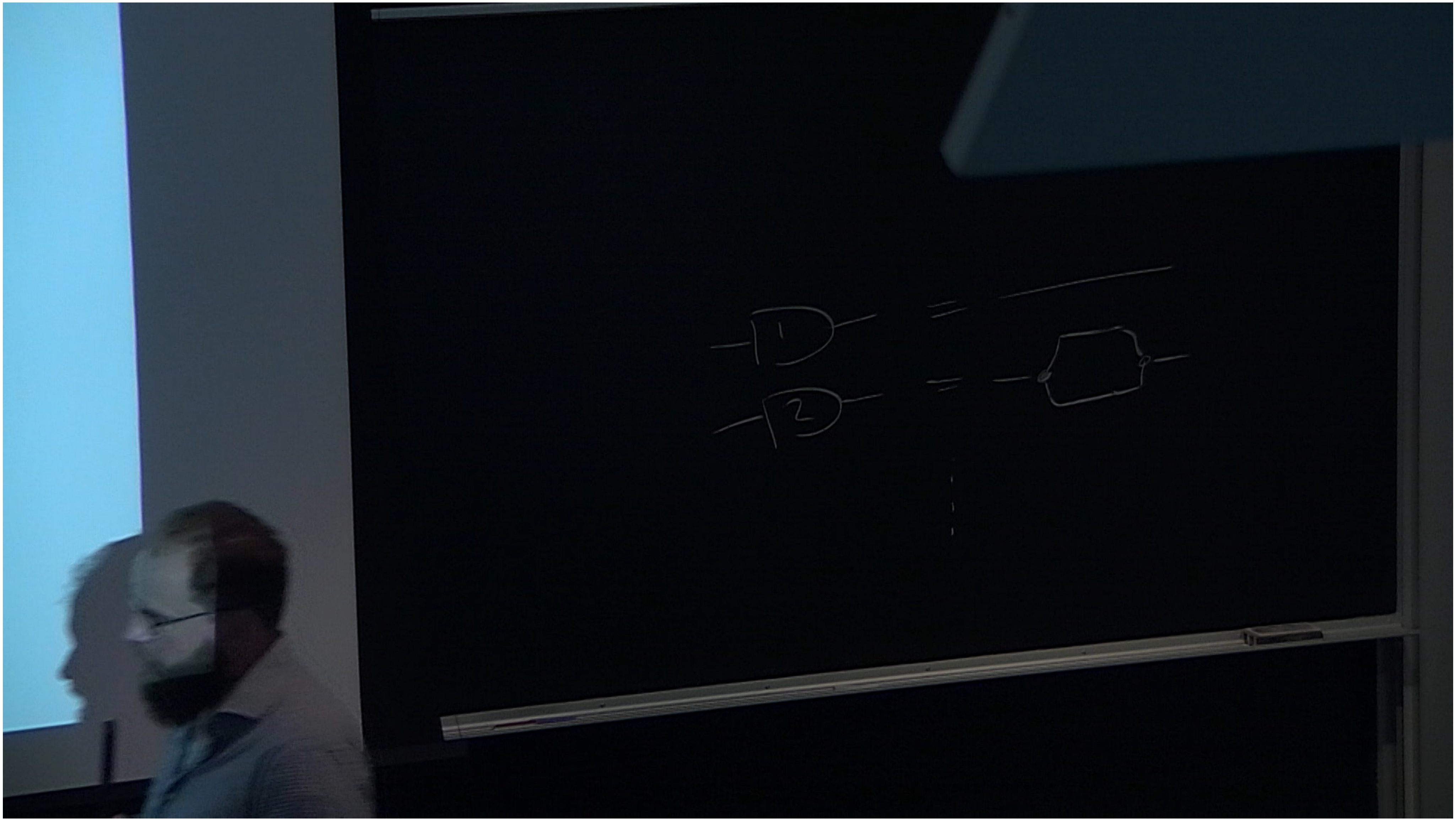
naturals as string diagrams

- naturals as *syntactic sugar*

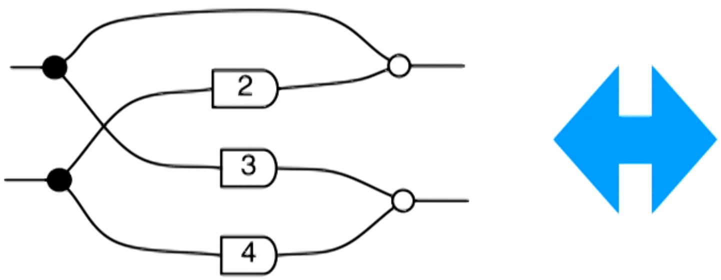


- some easy lemmas




1
2

correspondence with matrices

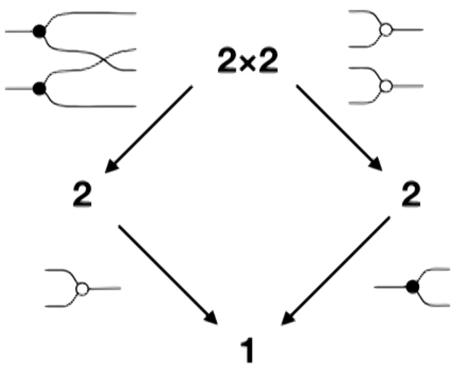


$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

- in general, the ijth entry is the *number of paths* from the jth port on the left to the ith port on the right

reading bialgebra equations off pullbacks

- S. Lack. Composing PROPs. TAC, Vol. 13, 2004, No. 9, pp 147-163.
- **distributive law** of props induced by pullbacks of finite sets & fns



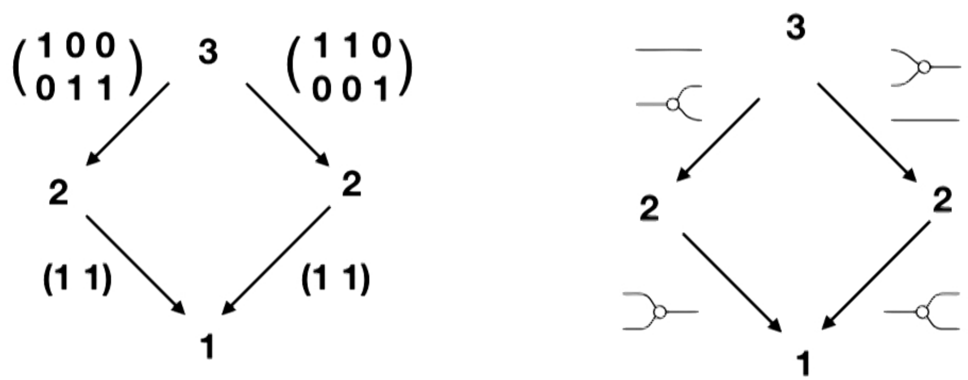
Hopf algebra facts

- The following are isomorphic as props
  - prop of (co)commutative Hopf algebras
  - Lawvere theory of abelian groups
  - prop of integer matrices
    - arrows $m \rightarrow n$ are $n \times m$ matrices with integer entries
    - composition is matrix multiplication

pullbacks of integer matrices

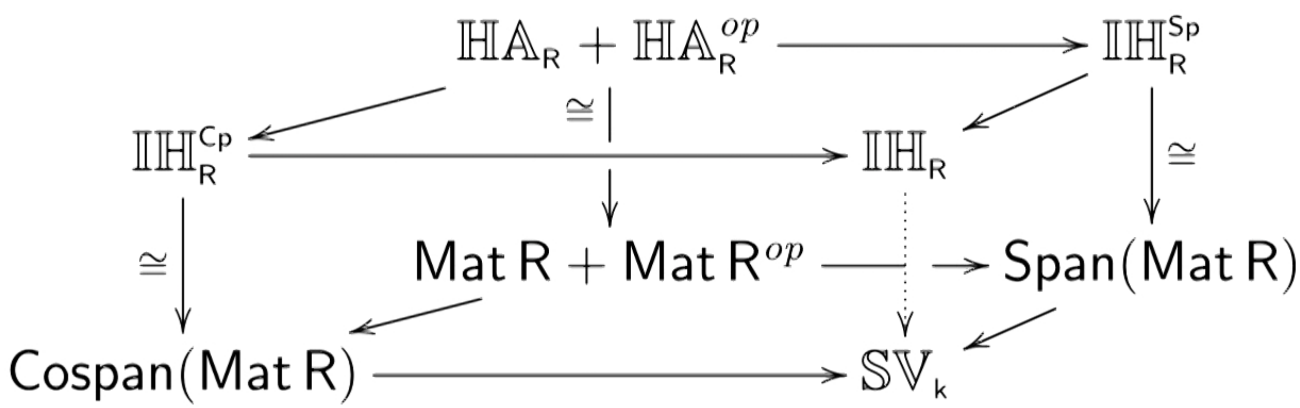
Can we use the “composing props” machinery to get a presentation for Span(Mat_Z)?

Yes! Mat_Z has pullbacks!

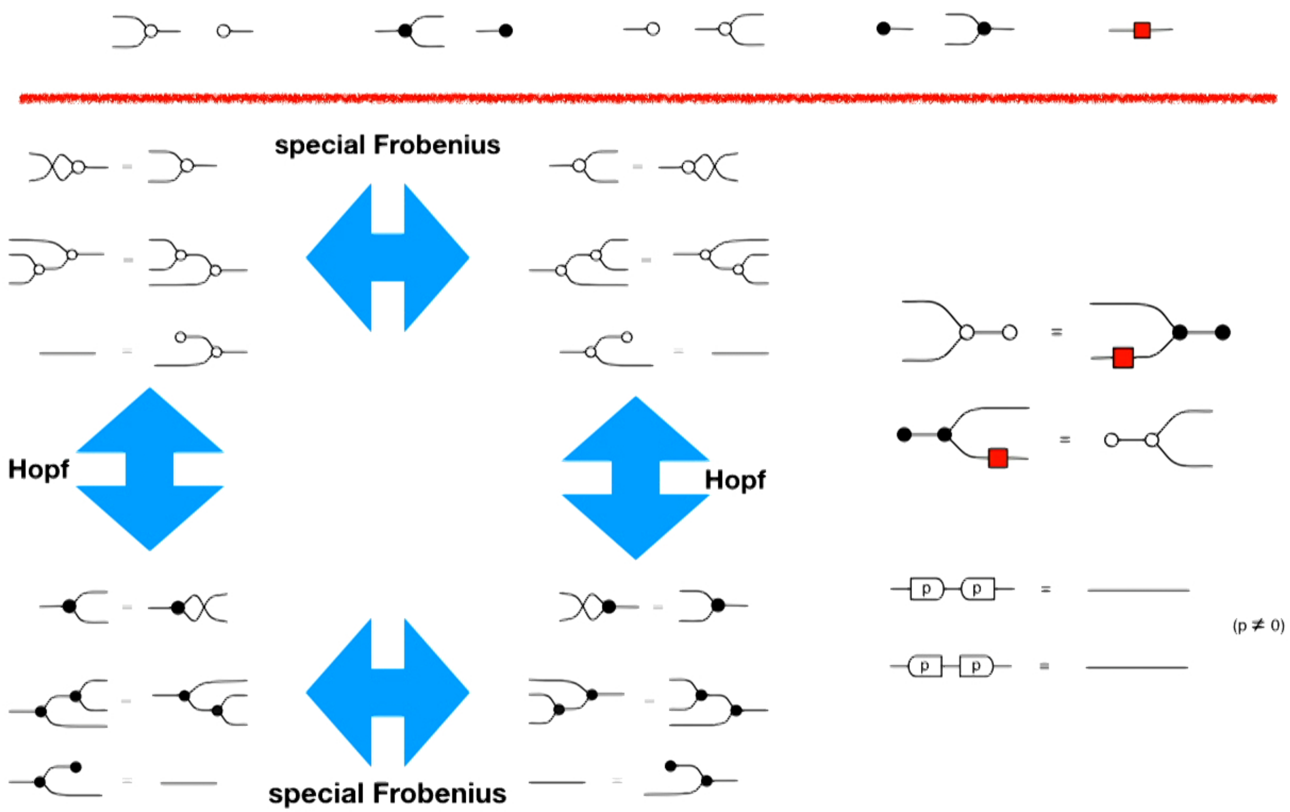


cube construction

Bonchi, S., Zanasi. Interacting Hopf Algebras. 1403.7048



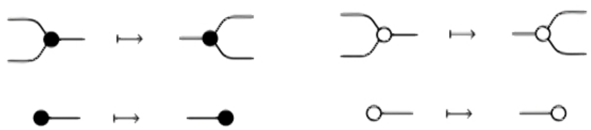
interacting Hopf algebras



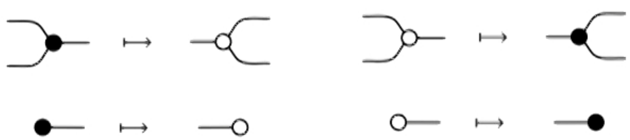
cf. Coecke, Duncan. Interacting quantum observables, NJP 2011

two symmetries, two daggers

- every equation, mirrored, is an equation
  - leads to the "mirror image" dagger

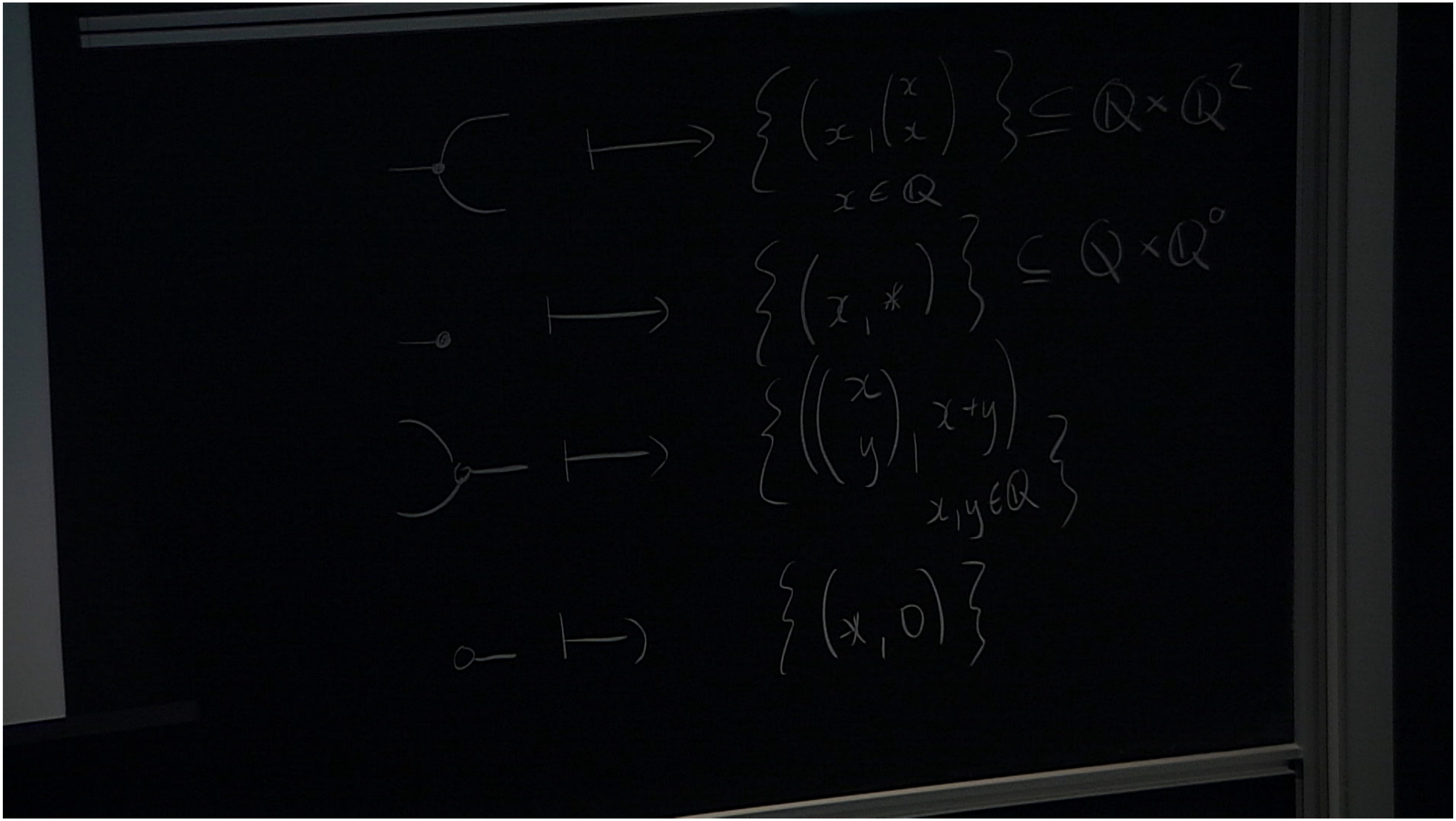


- every equation, colour swapped, is an equation
  - leads to the "bizarro" dagger



interacting Hopf algebra facts

- the following are isomorphic as props
  - the prop of interacting Hopf algebras
  - the prop of linear (additive) relations over **Q**
    - arrows m → n are linear subspaces of $\mathbf{Q}^m \times \mathbf{Q}^n$ as a **Q** vector space
    - composition is relational
    - monoidal product is **direct sum**


$\left\{ \left(x, \begin{pmatrix} x \\ x \end{pmatrix} \right) \right\}_{x \in \mathbb{Q}} \subseteq \mathbb{Q} \times \mathbb{Q}^2$
$\left\{ (x, *) \right\} \subseteq \mathbb{Q} \times \mathbb{Q}^0$
$\left\{ \left(\begin{pmatrix} x \\ y \end{pmatrix}, x+y \right) \right\}_{x,y \in \mathbb{Q}}$
$\left\{ (*, 0) \right\}$

generalisations

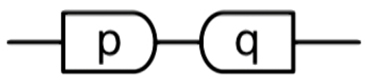
- Mat R has pullbacks if R is a PID
- following the same procedure, one gets a interacting Hopf algebras for R, presenting linear relations over the field of fractions of R
- taking R = k[x] is particularly interesting, with applications in computer science and systems and control theory
- *cf.*
  - Baez Erbele, Categories in Control
  - Jason Erbele PhD thesis, Categories in Control: applied PROPs, 2016

plan

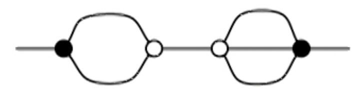
- Bialgebras and Hopf algebras - a “computer science” perspective
- Interacting Hopf Monoids
- **Graphical Linear Algebra**

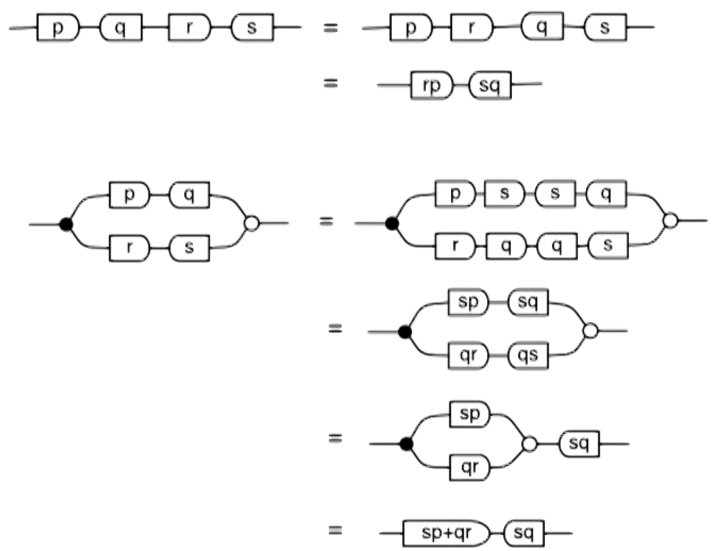
rational numbers

p/q :=



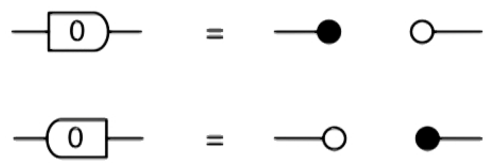
e.g. 2/3 is



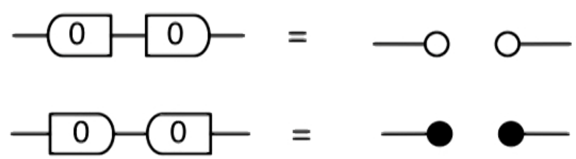


division by 0

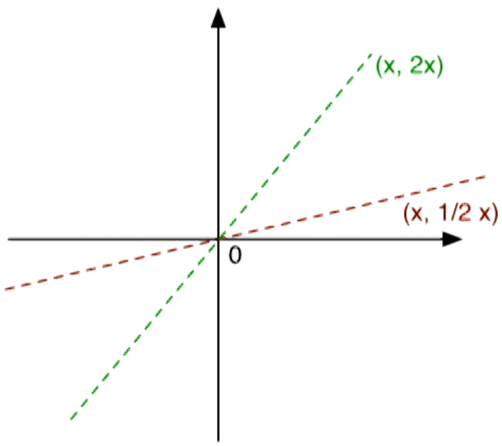
(fixing the deficiencies of the usual syntax)



Two ways of interpreting "0/0"



projective arithmetic ++



- projective arithmetic identifies rationals with 1-dim spaces (lines) of $\mathbf{Q}^2$
    - p -> { (x,px) | x ∈ **Q** }
    - ∞ : { (0, x) | x ∈ **Q** }
- The extended system includes all the subspaces of $\mathbf{Q}^2$, in particular:
    - the unique zero dimensional space { (0, 0) }
    - the unique two dimensional space { (x,y) | x,y ∈ **Q** }

division by 0

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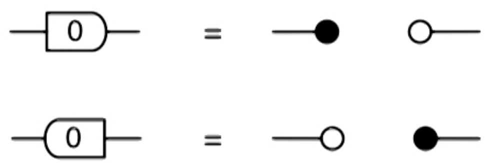

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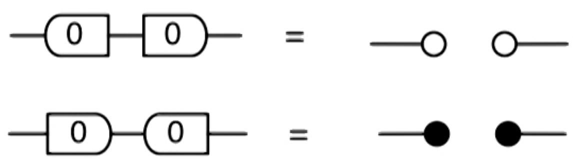
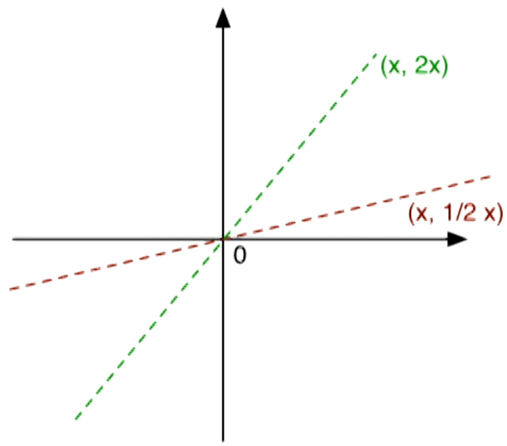


projective arithmetic ++



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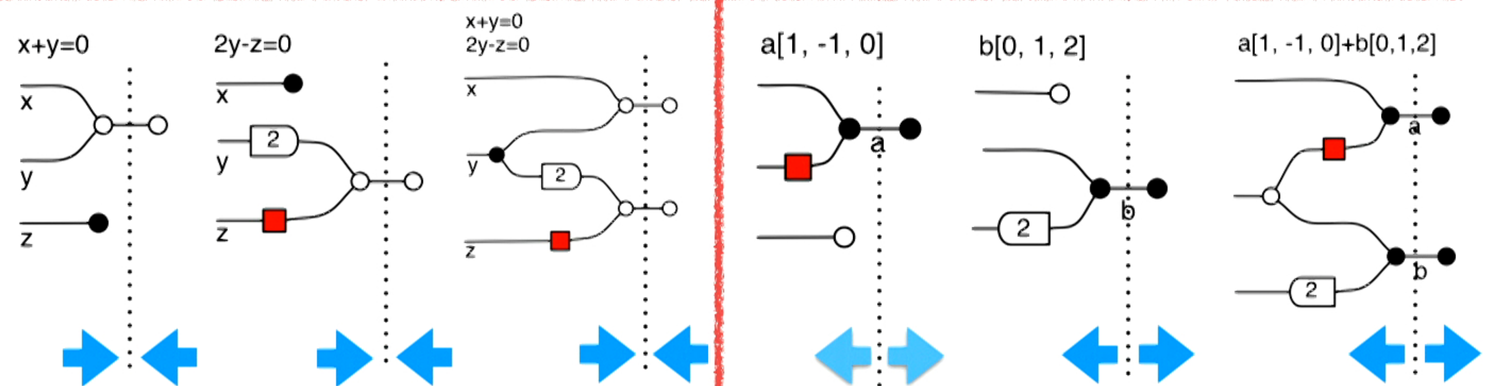
factorisations

- every diagram can be factorised as a span or a cospan of matrices
- gives two different canonical ways of representing spaces

solutions of a list of homogeneous equations

linear combinations of basis vectors



cospans

spans

elementary linear algebra...

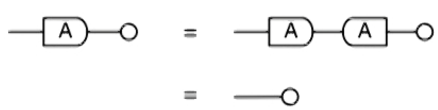
ker A = 

im A = 

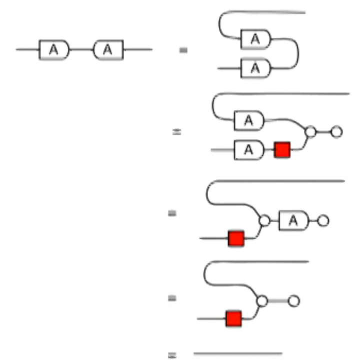
...

- **theorem**. A is injective iff ker A = 0

$\Rightarrow$



$\Leftarrow$



Bibliography

- Bonchi, S., Zanasi - Interacting bialgebras are Frobenius, FoSSaCS '14
- Bonchi, S., Zanasi - Interacting Hopf algebras, J Pure Applied Algebra 221:144–184, 2017
- Bonchi, S., Zanasi - The calculus of signal flow diagrams I: Linear Relations on Streams, Inf Comput 252:2–29, 2017
- Bonchi, S., Zanasi - A categorical semantics of signal flow graphs, CONCUR '14
- Bonchi, S., Zanasi - Full abstraction for signal flow graphs, PoPL '16
- Zanasi - Interacting Hopf Algebras: The theory of linear systems, PhD Thesis, ENS Lyon, 2015
- Bonchi, S., Zanasi - Lawvere Theories as composed PROPs, CMCS '16
- Fong, Rapisarda, S. - A categorical approach to open and interconnected dynamical systems, LiCS '16
- Bonchi, Gadducci, Kissinger, S. - Rewriting modulo symmetric monoidal structure, LiCS '16
- Bonchi, Gadducci, Kissinger, S. - Confluence of graph rewriting with interfaces, ESOP '17
- Bonchi, Holland, Pavlovic, S - Refinement for signal flow graphs, CONCUR '17
- Bonchi, Pavlovic, S. - Functorial semantics of Frobenius theories (in preparation)

GraphicalLinearAlgebra.net