Title: Introduction to CQM

Date: Aug 02, 2017 11:00 AM

URL: http://pirsa.org/17080006

Abstract: Categorical quantum mechanics is a research programme which aims to axiomatise (finite dimensional) quantum theory as an algebraic theory inside an abstract symmetric monoidal category. The central idea is that quantum observables can be axiomatised as certain Frobenius algebras, and that two observables are (strongly) complementary when their Frobenius algebras jointly form a Hopf algebra. The resulting theory is surprisingly powerful, especially when combined with its graphical notation. In this talk I'll introduce the main concepts and present some applications to quantum computation.

Introduction to Categorical Quantum Mechanics

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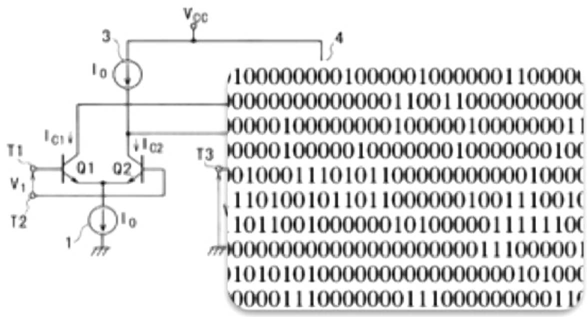
0.

Motivations

Quantum Computing

- **IDEA**: *exploit* quantum effects for computation
  - Fast algorithms (Shor, Grover)
  - Simulate physical systems (chemistry, materials)
  - Novel cryptographic protocols (QKD, blind computing)
  - …. and more.

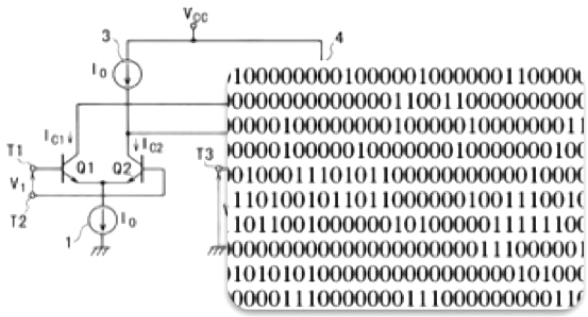
Motivations



$$\lambda x.\lambda y.\lambda z.xz(yz)$$

```
fun fact 0 = 1
  | fact x = x * fact (x-1)
```

Motivations



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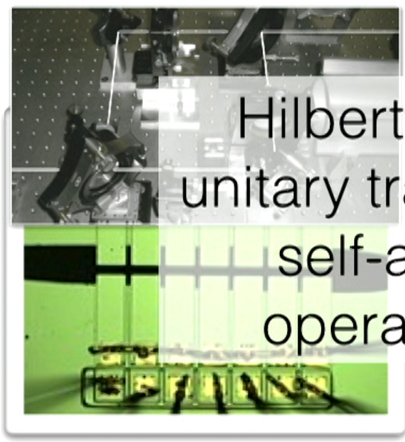


Hilbert space, unitary transforms, self-adjoint operators….

?

No-Cloning and No-Deleting

Theorem: There are no unitary operations D such that

$$D : |\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$$
$$D : |\phi\rangle \mapsto |\phi\rangle \otimes |\phi\rangle$$

unless $|\psi\rangle$and $|\phi\rangle$ are orthogonal [Wooters & Zurek 1982]

Theorem: There are no unitary operations E such that

$$E : |\psi\rangle \mapsto |0\rangle$$
$$E : |\phi\rangle \mapsto |0\rangle$$

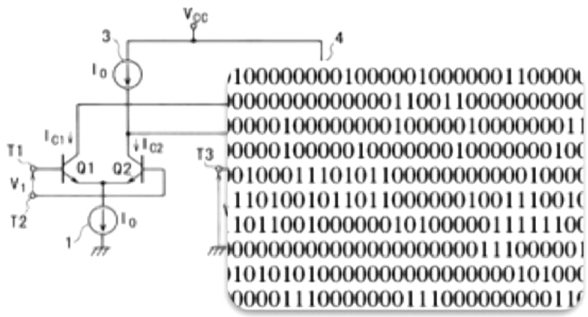
unless $|\psi\rangle$and $|\phi\rangle$ are orthogonal [Pati & Braunstein 2000]

Categorical Quantum Theory

Quantum theory as an *internal* theory in a monoidal category

- No assumption of linear structure
- Algebra : $A \otimes A \longrightarrow A$
- Coalgebra : $A \longrightarrow A \otimes A$
- And laws for their interaction.

Motivations



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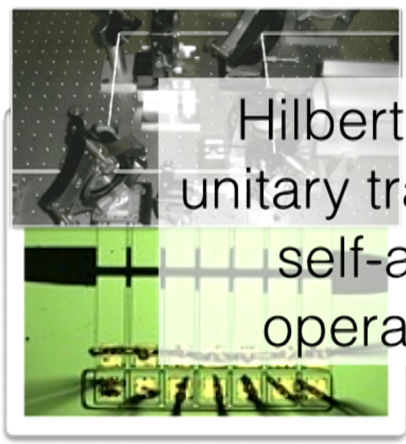

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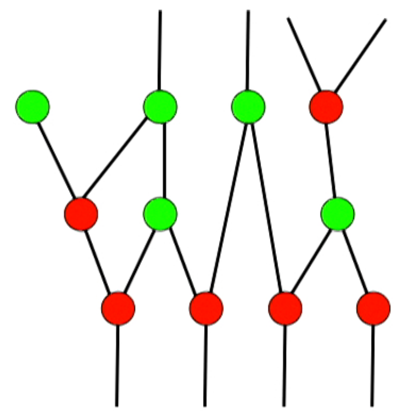

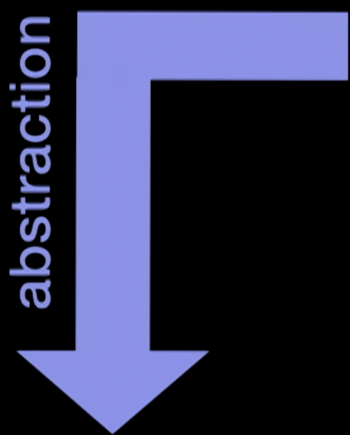

Quantum Theory

Hilbert spaces

Operators

Categorical Quantum Theory

Monoidal categories

(co)Algebras

Commutation rules

Categorical Quantum Theory

formalisation

Monoidal categories

(co)Algebras

Commutation rules

CQT as an "algebraic theory"

PROPs

Distributive laws

What I won't talk about:

- Mixed states, CP-maps, probabilities
- Causality
- Contextuality, Non-locality
- Infinite dimensional spaces

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doi : 10.1016/j.entcs.2006.12.018

- Mixed states, CP-maps, probabilities
- Causality arXiv:1107.6019
- Contextuality, Non-locality arXiv:1203.4988
- Infinite dimensional spaces

arXiv:1011.6123
arXiv:1605.04305

“Observables”

Non-degenerate, projective observables
on finite dimensional spaces
i.e.
Orthonormal bases

$$A = |a_1\rangle\,, |a_2\rangle\,, \ldots, |a_d\rangle$$

Unbiasedness and Phases

A state $|\psi\rangle$ is **unbiased** for a basis A if

$$|\langle a_i \mid \psi\rangle| = \frac{1}{\sqrt{d}}$$

Every unbiased state determines a unitary map via

$$U_\psi : |a_i\rangle \mapsto \sqrt{d}\langle a_i | \psi\rangle \, |a_i\rangle$$

This is called a **phase map** for A.

Unbiasedness and Phases

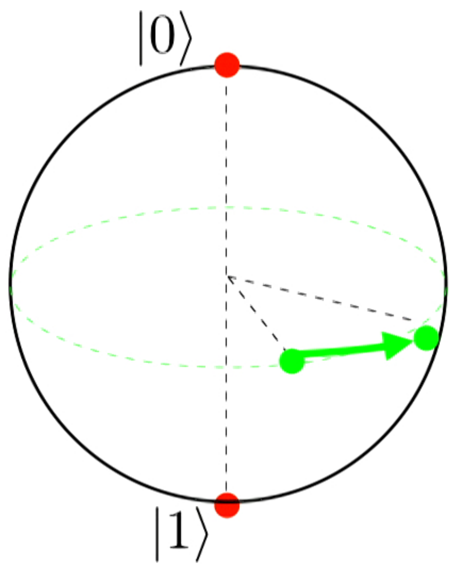
A state $|\psi\rangle$ is **unbiased** for a basis A if

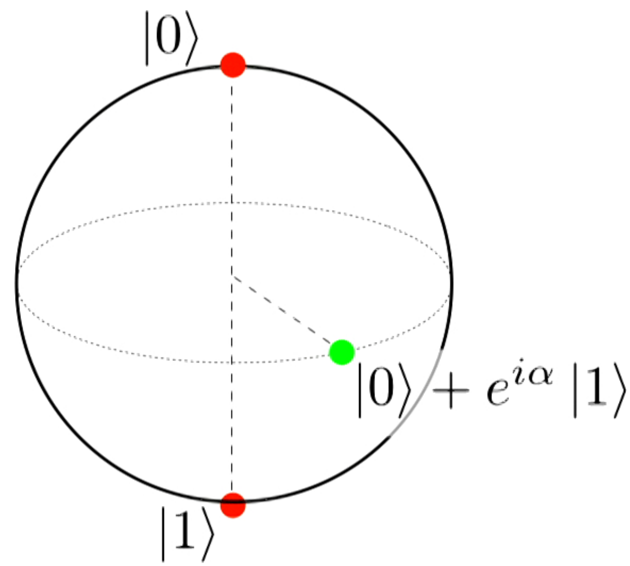
Proposition 1:
The phases of any observable form an abelian group

$$U_\psi : |a_i\rangle \mapsto \sqrt{d}\langle a_i \,|\, \psi\rangle \, |a_i\rangle$$

This is called a **phase map** for A.

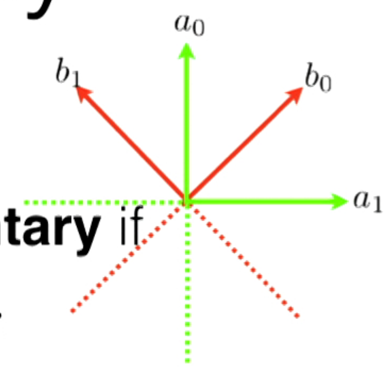
Phases & Unbiased Points





$$Z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

Complementarity



Two observables A and B are **complementary** if

$$|\langle a_i \mid b_j \rangle| = \frac{1}{\sqrt{d}} \qquad \forall i, j$$

(aka mutually unbiased)

They are **strongly complementary** if $\{U_{b_i}\}_i$ is a subgroup of the phase group Φ_A.

Observables are Frobenius Algebras

Theorem 2: Observables are in bijection with †-special commutative Frobenius algebras.

$$\delta : A \to A \otimes A \qquad \mu : A \otimes A \to A$$

$$\epsilon : A \to I \qquad \eta : I \to A$$

Via:

$$\delta :: |a_i\rangle \to |a_i\rangle \otimes |a_i\rangle \qquad \mu = \delta^\dagger$$

$$\epsilon :: |a_i\rangle \to 1 \qquad \eta = \epsilon^\dagger$$

Coecke, Pavlovic, and Vicary, "A new description of orthogonal bases", MSCS 23(3), 2013. arxiv:0810.0812

Strongly Complementary Observables are Hopf algebras

Theorem 3: Two observables are strongly complementary iff they form a Hopf algebra

$\delta_{\color{green}\bullet}$ $\epsilon_{\color{green}\bullet}$ $\mu_{\color{green}\bullet}$ $\eta_{\color{green}\bullet}$

$\mu_{\color{red}\bullet}$ $\eta_{\color{red}\bullet}$ $\delta_{\color{red}\bullet}$ $\epsilon_{\color{red}\bullet}$

Coecke and Duncan, "Interacting Quantum Observables: categorical algebra and diagrammatics", NJP 13(043016), 2011, arXiv:0906.4725.

Strongly Complementary Observables are Hopf algebras

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Frobenius δ ϵ μ η

Frobenius μ η δ ϵ

Coecke and Duncan, "Interacting Quantum Observables: categorical algebra and diagrammatics", NJP 13(043016), 2011, arXiv:0906.4725.

Standard Example

Given $|0\rangle, \ldots, |d-1\rangle$ an orthonormal basis.

$$\delta_{\color{green}\bullet} :: |n\rangle \mapsto |n\rangle \otimes |n\rangle$$

$$\mu_{\color{green}\bullet} :: |n\rangle \otimes |m\rangle \mapsto \begin{cases} |n\rangle & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon_{\color{green}\bullet} = \sum_{n \in \mathbb{Z}_D} \langle n|$$

$$\eta_{\color{green}\bullet} = \sum_{n \in \mathbb{Z}_D} |n\rangle$$

$$\mu_{\color{red}\bullet} :: |n\rangle \otimes |m\rangle \mapsto |n+m\rangle$$

$$\delta_{\color{red}\bullet} :: |n\rangle \mapsto \sum_{m+m'=n} |m\rangle \otimes |m'\rangle$$

$$\eta_{\color{red}\bullet} = |0\rangle$$

$$\epsilon_{\color{red}\bullet} = \langle 0|$$

Standard Example

Given $|0\rangle, \ldots, |d-1\rangle$ an orthonormal basis.

Proposition 4:
In complex Hilbert space all strongly complementary pairs are group algebras of abelian groups

Coecke, Duncan, Kissinger, Wang, "Strong Complementarity and Non-locality in Categorical Quantum Mechanics", *Proc. LiCS 2012*, arXiv:1203.4988.

$m+m'=n$

$\eta_\bullet = |0\rangle$ $\epsilon_\bullet = \langle 0|$

1.

Monoidal Categories

1. Symmetric Monoidal Categories

A monoidal category M is a category with a bifunctor, $\otimes$ or $\Box$,

$$\Box : M \times M \to M$$

written for objects a, b of M variously as a "product"

$$(a, b) \to a \Box b, a \otimes b, \text{or } a b$$

which is associative up to a natural isomorphism

$$\alpha : a(b c) \cong (a b)c \tag{1}$$

and is equipped with an element e, which is unit up to natural isomorphisms

$$\lambda : e a \cong a , \qquad \rho : a e \cong e . \tag{2}$$

These maps must satisfy certain commutativity requirements; for α, a pentagonal diagram

$$\begin{array}{ccccc} a(b(c d)) & \xrightarrow{\alpha} & (a b)(c d) & \xrightarrow{\alpha} & ((a b)c)d \\ \downarrow{\scriptstyle 1\alpha} & & & & \downarrow{\scriptstyle \alpha 1} \\ a((b c)d) & & \xrightarrow{\alpha} & & (a(b c))d\ , \end{array} \tag{3}$$

as in § VII.1.(5), and for λ and ρ the two commutativities

$$\begin{array}{ccc} a(e c) & \xrightarrow{\alpha} & (a e)c \\ \downarrow{\scriptstyle 1\lambda} & & \downarrow{\scriptstyle \rho 1} \\ a c & = & a c\ , \end{array} \qquad \lambda = \rho : e e \to e . \tag{4}$$

A *braiding* for a monoidal category M consists of a family of isomorphisms

$$\gamma_{a,b} : a \Box b \cong b \Box a \tag{5}$$

natural in a and $b \in M$, which satisfy for e the commutativity

$$\begin{array}{ccc} a \Box e & \xrightarrow{\gamma} & e \Box a \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \lambda} \\ a & = & a \end{array} \tag{6}$$

and which, with the associativity α, make both the following hexagonal diagrams commute (with the symbol $\Box$ omitted):

$$\begin{array}{ccc} (a b)c & \xrightarrow{\gamma} & c(a b) \\ \downarrow{\scriptstyle \alpha^{-1}} & & \downarrow{\scriptstyle \alpha} \\ a(b c) & & (c a)b \\ \downarrow{\scriptstyle 1\cdot\gamma} & & \downarrow{\scriptstyle \gamma\cdot 1} \\ a(c b) & \xrightarrow{\alpha} & (a c)b\ , \end{array} \qquad \begin{array}{ccc} a(b c) & \xrightarrow{\gamma} & (b c)a \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha^{-1}} \\ (a b)c & & b(c a) \\ \downarrow{\scriptstyle \gamma\cdot 1} & & \downarrow{\scriptstyle 1\cdot\gamma} \\ (b a)c & \xrightarrow{\alpha^{-1}} & b(a c)\ . \end{array} \tag{7}$$

Note that the first diagram replaces each $\gamma_{ab,c}$ which has a product $a b$ as first index by two γ's with single indices, while the second hexagonal diagram does the same for $\gamma_{a,bc}$ with a product as second index. Note also that the first hexagon of (7) for γ implies the second diagram for γ^{-1}, and conversely. Thus, when γ is a braiding for M, then γ^{-1} is also a braiding for M.

A *symmetric monoidal category*, as already defined in § VII. 7, is a category with a braiding γ such that every diagram

$$\begin{array}{ccc} a b & \xrightarrow{\gamma_{a,b}} & b a \\ & \searrow & \downarrow{\scriptstyle \gamma_{b,a}} \\ & & a b \end{array} \tag{8}$$

commutes. For this case, either one of the hexagons (7) implies the other.

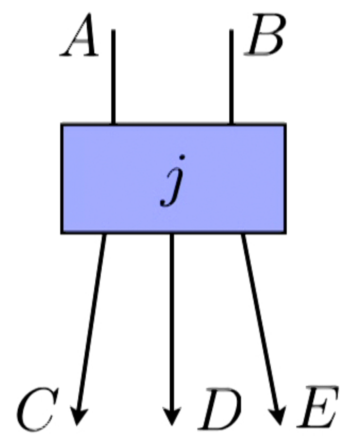
1bis.

Monoidal Categories (Graphically)


PAST / HEAVEN
FUTURE / HELL

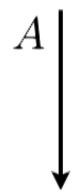
Diagrams

$$j : A \otimes B \to C \otimes D \otimes E$$



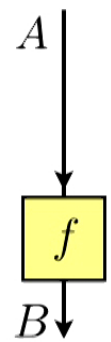
Categories

$$\mathrm{id}_A : A \to A$$

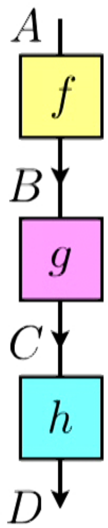


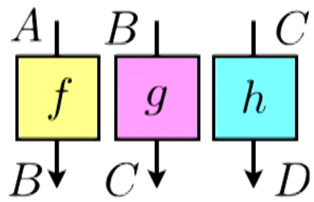
Categories

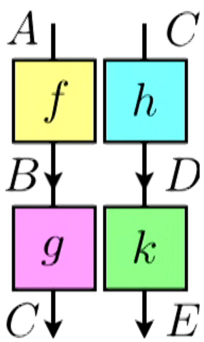
$$f \circ \mathrm{id}_A : A \to B$$



Monoidal Categories







Monoidal Categories

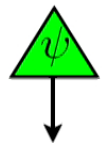
Monoidal categories have a special *unit* object called I which is a left and right identity for the tensor:

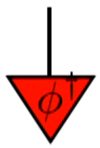
$$I \otimes A = A = A \otimes I$$

$$\mathrm{id}_I \otimes f = f = f \otimes \mathrm{id}_I$$

No lines are drawn for I in the graphical notation:

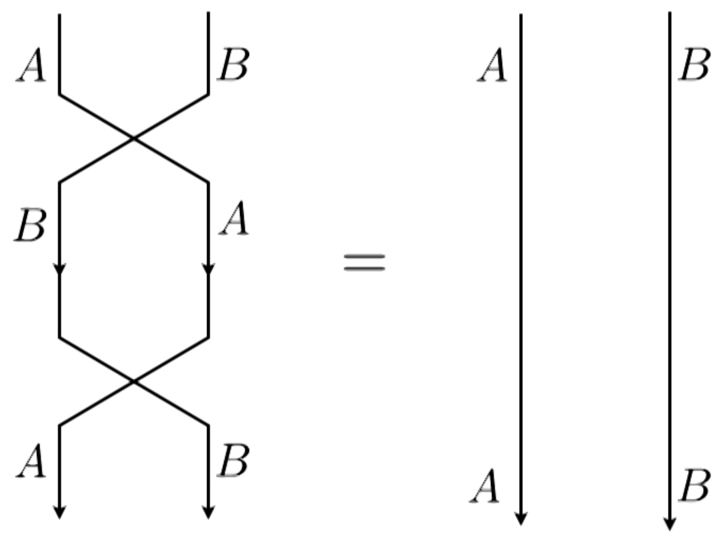
$\psi : I \to A$ $\quad$ $\phi^\dagger : A \to I$ $\quad$ $\phi^\dagger \circ \psi : I \to I$





Symmetric Monoidal Categories

$$\sigma_{A,B} : A \otimes B \to B \otimes A$$



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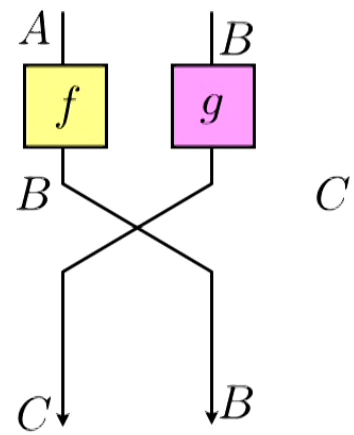

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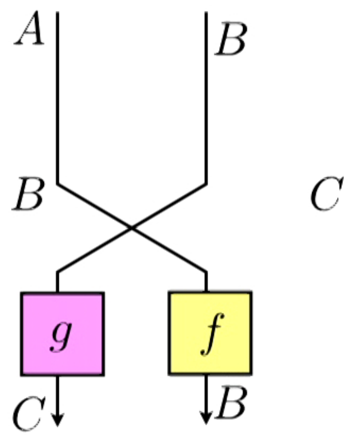

Compact Closure

A symmetric monoidal category is called *compact* if, for every object A, there exists a *dual* object A^* and arrows:

$$d : I \to A^* \otimes A \qquad e : A \otimes A^* \to I$$

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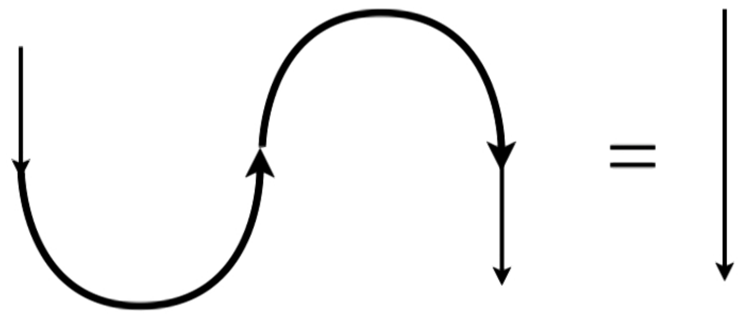


Defn 1: a †-*category* is a category equipped with an involutive, contravariant, identity-on-objects endofunctor.

"Unitary" : $f^\dagger = f^{-1}$ "Self-adjoint" : $f^\dagger = f$

Defn. 1a: a †-*functor* F is a functor satisfying

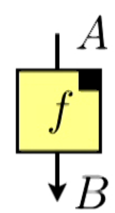
$$F(f^\dagger) = (Ff)^\dagger$$

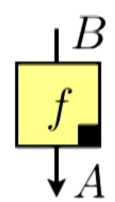
Defn. 1b: a †-*(symmetric) monoidal category* is an (S)MC where $\otimes$ is a †-functor and all the structure maps are unitary.

†-Categories

A category is a *†-category* if it is equipped with an involutive functor, $(\cdot)^\dagger$ which reverses the arrows while leaving the objects unchanged.

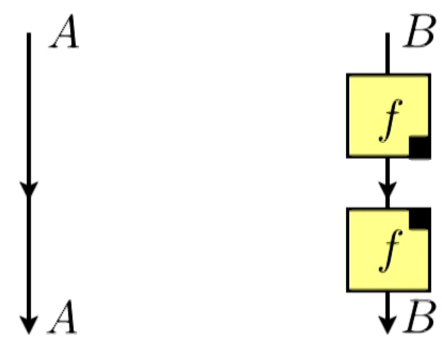
$f : A \to B$ $\qquad\qquad$ $f^\dagger : B \to A$





†-Categories

An arrow $f : A \to B$ is called *unitary* when:



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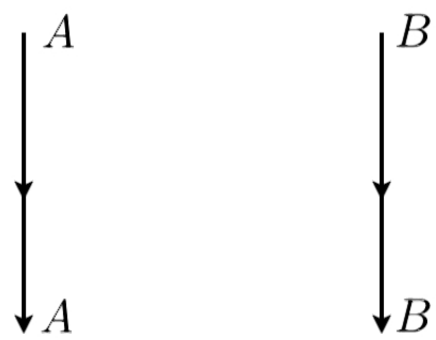


Examples

$\mathbf{fdHilb}$: $\dagger$ sends a linear map to its Hermitian adjoint

$\mathbf{Rel}$: $\dagger$ is relational converse

Any groupoid : $f^\dagger := f^{-1}$

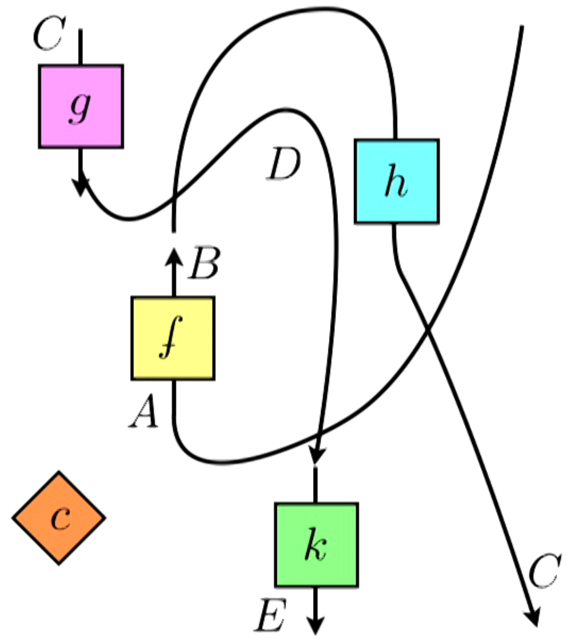
$\mathbf{Span(C)}$: $(A \xleftarrow{f} X \xrightarrow{g} B)^\dagger := B \xleftarrow{g} X \xrightarrow{f} A$

$\mathbf{Cospan(C)}$: same trick

$\mathbf{C} + \mathbf{C}^{\mathbf{op}}$: in the obvious way modulo one detail…

Graphical Calculus Theorem

Thm: one diagram can be deformed to another iff their denotations are equal by the structural equations of the category.



P. Selinger, "A survey of graphical languages for monoidal categories". In *New Structures for Physics*, volume 813 of Lecture Notes in Physics, pages 289–355. Springer, 2011. arXiv:0908.3347

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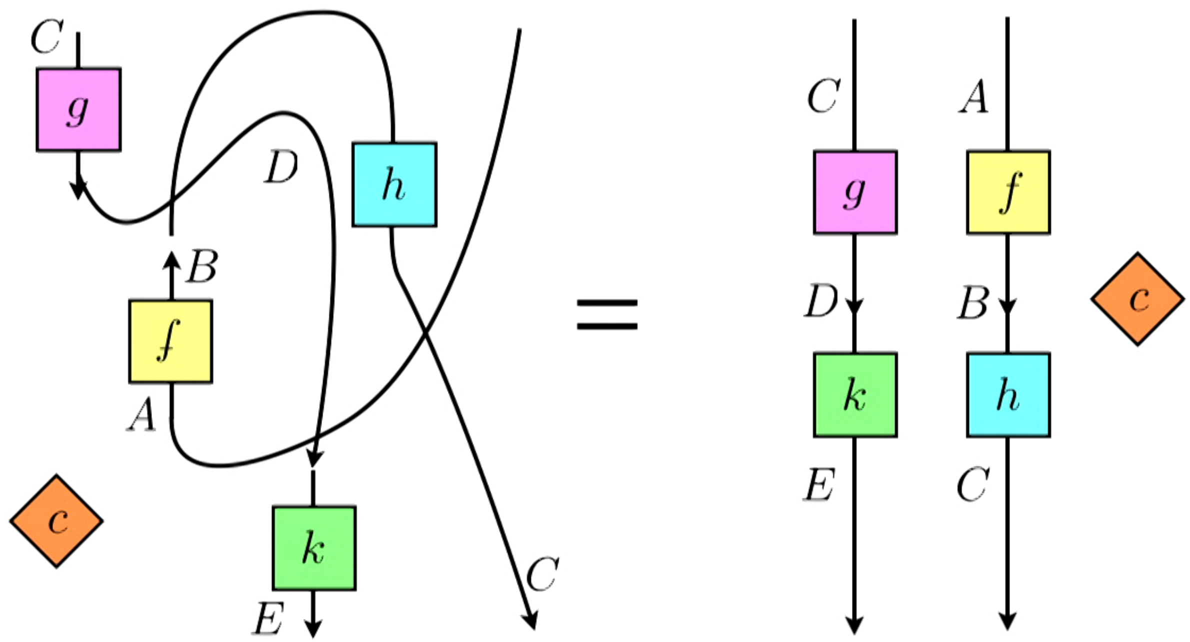

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The Category **FDHilb**

FDHilb is the category of finite dimensional complex Hilbert spaces. It is †-monoidal with the following structure.

- Objects: finite dimensional Hilbert spaces, $A,\ B,\ C$ etc
- Arrows: all linear maps
- Tensor: usual (Kronecker) tensor product; $I = \mathbb{C}$
- $f^\dagger$ is the usual adjoint (conjugate transpose)

A linear map $\psi : I \to A$ picks out exactly one vector. It is a ket and $\psi^\dagger : A \to I$ is the corresponding bra.

Hence $\psi^\dagger \circ \phi : I \to I$ is the inner product $\langle\psi \mid \phi\rangle$.

2.

Composing PROPs

COMPOSING PROPS

Dedicated to Aurelio Carboni on the occasion of his sixtieth birthday

STEPHEN LACK

Abstract. A PROP is a way of encoding structure borne by an object of a symmetric monoidal category. We describe a notion of *distributive law* for PROPs, based on Beck's distributive laws for monads. A distributive law between PROPs allows them to be composed, and an algebra for the composite PROP consists of a single object with an algebra structure for each of the original PROPs, subject to compatibility conditions encoded by the distributive law. An example is the PROP for bialgebras, which is a composite of the PROP for coalgebras and that for algebras.

Lack, "Composing PROPs", Theory and Applications of Categories 13(9), 2004.

PROs and PROPs

Defn 4. A *PRO* is a strict monoidal category whose objects are the natural numbers.

Let $\mathbb{T}$ be a PRO and let $\mathbf{C}$ be strict monoidal category.

Defn 4a: a $\mathbb{T}$-algebra in $\mathbf{C}$ is a strict monoidal functor from $\mathbb{T}$ to $\mathbf{C}$.

Defn 4b: a morphism of PROs $\mathbb{T} \to \mathbb{S}$ is a T-algebra in $\mathbb{S}$ which is the identity on objects.

Let $\mathbf{PRO}$ be the category of PROs and their morphisms.

PROs and PROPs

Defn 5. A *PROP* is a symmetric PRO; similarly for algebras.

Let $\mathbf{PROP}$ be the (sub)category of PROPs and their morphisms.

Example: the full subcategory of $\mathbf{fdHilb}$ with objects

$$\mathbb{C}^{2^n} \cong (\mathbb{C}^2)^{\otimes n}$$

Example: Let $\mathbb{P}$ be the PRO of permutations:

$$\mathbb{P}(n,n) = S_n \qquad \mathbb{P}(n,m) = \emptyset \ (n \neq m)$$

PROs and PROPs

All PROPs "contain the permutations" hence we can view $\mathbf{PROP}$ as the coslice category:

$$\begin{array}{c}\mathbb{P}\\ \downarrow\\ \mathbf{PRO}\end{array}$$

The disjoint union in $\mathbf{PROP}$ is defined by the push out in $\mathbf{PRO}$:

$$\begin{array}{ccc} & \mathbb{P} & \\ \swarrow & & \searrow \\ \mathbb{T} & & \mathbb{S} \\ \searrow & & \swarrow \\ & \mathbb{T}+\mathbb{S} & \end{array}$$

PROs and PROPS

Syntactic presentation of a PROP:

$$(\Sigma, E)$$

Generators
symbols with arity and coarity

Relations
equations between terms of same type

The disjoint union is very simple:

$$(\Sigma_1, E_1) + (\Sigma_2, E_2) = (\Sigma_1 + \Sigma_2, E_1 + E_2)$$

Composing PROPs

PROPs are monads in a certain (complicated) category.
Distributive laws of monads produce composite monads — can do this for PROPs!

$$\lambda : \mathbb{T} ; \mathbb{S} \Rightarrow \mathbb{S} ; \mathbb{T}$$

This boils down to an equation

$$n \xrightarrow{t} k \xrightarrow{s} m$$
$$\Downarrow$$
$$n \xrightarrow[s']{} k' \xrightarrow[t']{} m$$

for every composable pair.

Lack, "Composing PROPs", Theory and Applications of Categories 13(9), 2004.

Example 1

Let G be any group; define a PRO $G^{\times}$ by

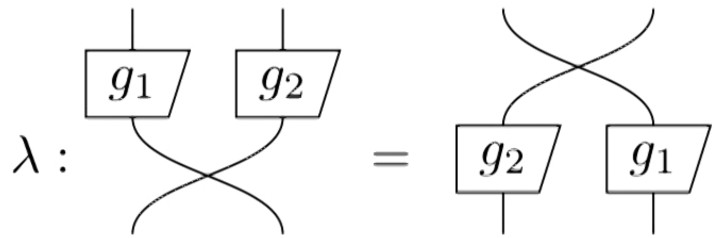
$$G^{\times}(n,n) = \prod_{n} G \qquad G^{\times}(n,m) = \emptyset \ \ (n \neq m)$$

The initial PROP $\mathbb{P}$ can be viewed as PRO with one generator $c : 2 \longrightarrow 2$ subject to $c^2 = \mathrm{id}$ and the Yang-Baxter equation

Example 1

We have a distributive law:

$$\lambda : \; G^{\times};\mathbb{P} \to \mathbb{P};G^{\times}$$

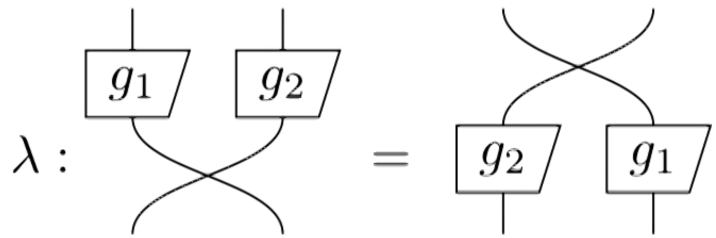


Every morphism $f : n \longrightarrow n$ in $\mathbb{P};G^{\times}$ consists of an element of S_n followed by an n-vector of elements of G.

Example 1

We have a distributive law:

$$\lambda : \; G^{\times};\mathbb{P} \to \mathbb{P};G^{\times}$$



Every morphism $f : n \longrightarrow n$ in $\mathbb{P};G^{\times}$ consists of an element of S_n followed by an n-vector of elements of G.

Example 1

NB:

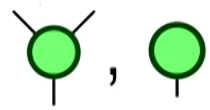
1. Both $G^{\times}$ and $\mathbb{P}$ are groupoids, hence $\dagger$-categories, and so is $\mathbb{P};G^{\times}$.

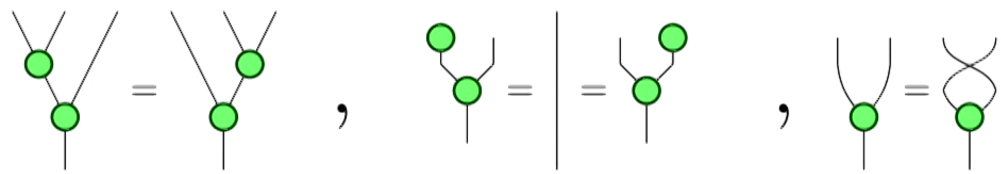
2. This construction yields a functor

$$\mathbb{P} : \mathbf{Grp} \to \mathbf{PROP}$$

Example 2

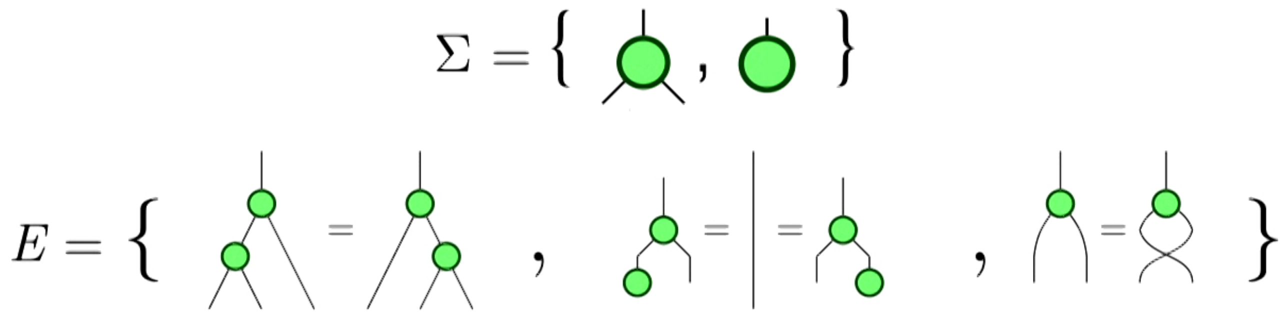
The PROP of commutative monoids $\mathbb{M}$

$\Sigma = \{$  $\}$

$E = \{$  $\}$

Example 2

The PROP of cocommutative comonoids $\mathbb{M}^{\mathrm{op}}$

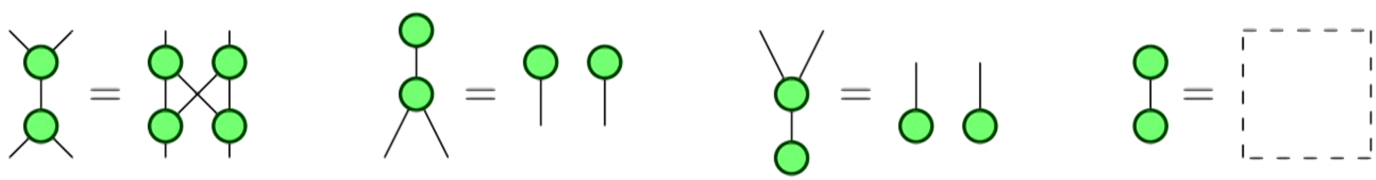


Example 2

The PROP $\mathbb{B}$ of **bialgebras** arises by a distributive law

$$\lambda_B : \mathbb{M};\mathbb{M}^{\mathrm{op}} \to \mathbb{M}^{\mathrm{op}};\mathbb{M}$$

generated by the equations



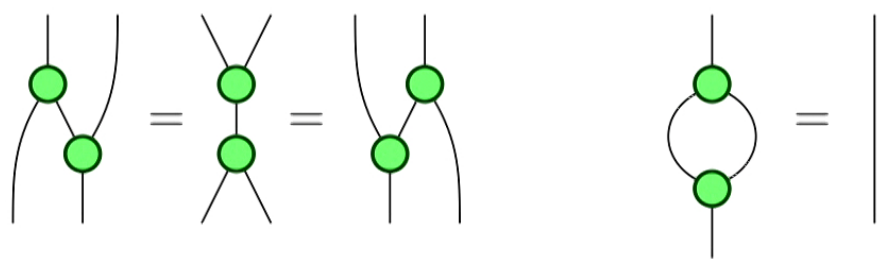
This is equivalent to $\mathbf{Span(FinSet)}$

Example 2

The PROP $\mathbb{F}$ of **special commutative Frobenius algebras*** arises by a distributive law

$$\lambda_F : \mathbb{M}^{\mathrm{op}};\mathbb{M} \to \mathbb{M};\mathbb{M}^{\mathrm{op}}$$

generated by the equations



This is equivalent to $\mathbf{Cospan(FinSet)}$

*aka commutative separable algebras

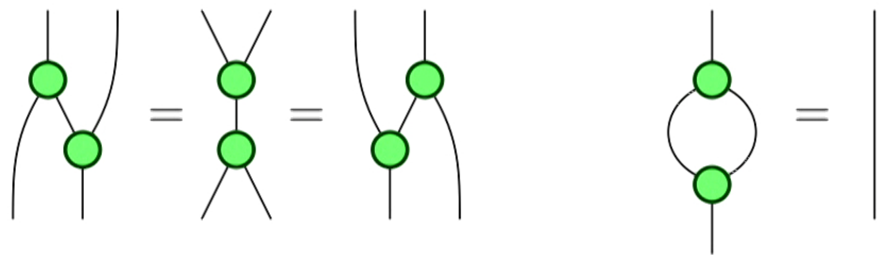
Rosebrugh, Sabadini, Walters, "Generic commutative separable algebras and cospans of graphs", *Theory and Applications of Categories* 15(6), 2005.

Example 2

The PROP $\mathbb{F}$ of **special commutative Frobenius algebras*** arises by a distributive law

$$\lambda_F : \mathbb{M}^{\mathrm{op}};\mathbb{M} \to \mathbb{M};\mathbb{M}^{\mathrm{op}}$$

generated by the equations



This is equivalent to $\mathbf{Cospan(FinSet)}$

†-PROPs

Defn: a †-PRO(P) is a PRO(P) which is also a †-monoidal category. Their algebras are required to preserve †.

NB : all of $\mathbb{P}G$, $\mathbb{B}$, and $\mathbb{F}$ have a natural †-structure

3.

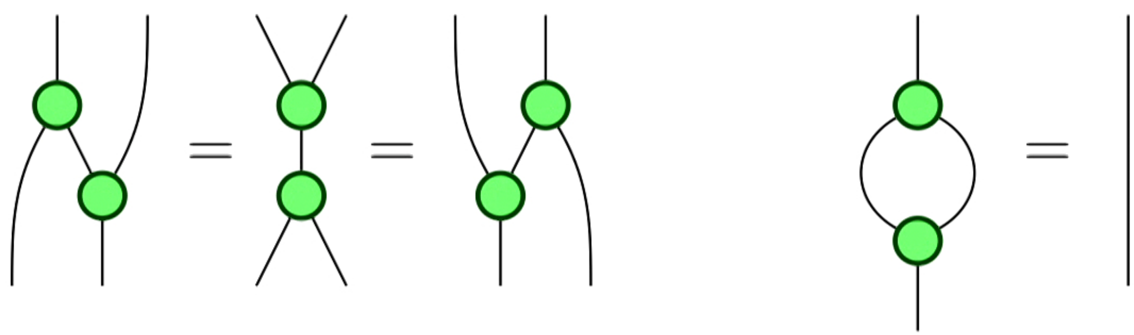
Frobenius Algebras

Frobenius Algebras

The PROP $\mathbb{F}$ of **special commutative Frobenius algebras** arises by a distributive law

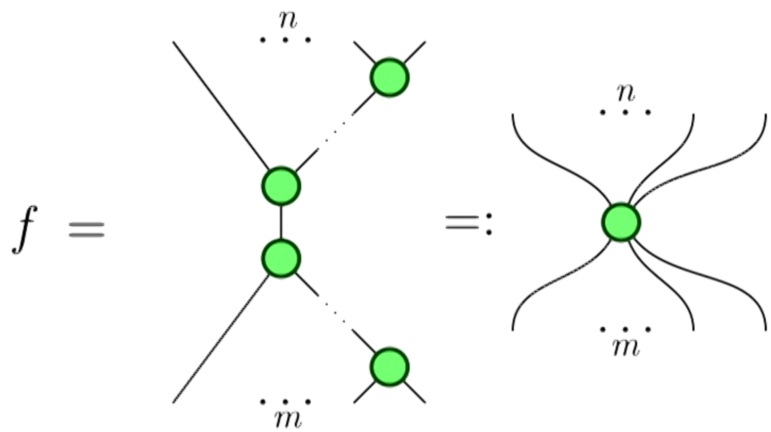
$$\lambda_F : \mathbb{M}^{\mathrm{op}};\mathbb{M} \to \mathbb{M};\mathbb{M}^{\mathrm{op}}$$

generated by the equations



Frobenius Algebras

Theorem: let $f : n \longrightarrow m$ be connected in $\mathbb{F}$. Then



Corollary: $\mathbb{F}$ is self-dual †-compact.

Coecke and Paquette, "POVMs and Naimark's theorem without sums", Proc QPL 2006. arXiv:quant-ph/0608072

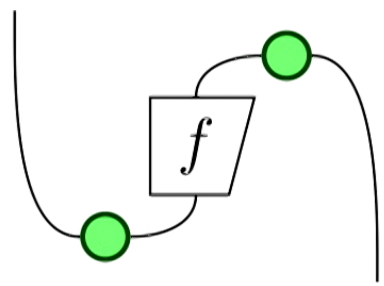
Let A be an $\mathbb{F}$-algebra in some $\dagger$-category $\mathbf{C}$, and let

$$f : A^n \longrightarrow A^m$$

be a morphism.

Define the $\circ$-**transpose** by

$$f^{\circ} =$$



Define the $\circ$-**conjugate** by

$$f_{\circ} = (f^{\dagger})^{\circ} = (f^{\circ})^{\dagger}$$

The map f is $\circ$-real if $f = f_{\circ}$ (or equiv. $f^{\dagger} = f^{\circ}$)

Let A be an $\mathbb{F}$-algebra in some †-category $\mathbf{C}$, and let

$$f : A^n \longrightarrow A^m$$

be a morphism.

Define the ○-**transpose** by

Proposition 5:
In $\mathbb{F}$ itself every thing is ○-real

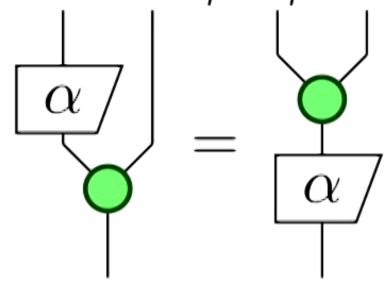
Define the ○-**conjugate** by

$$f_{\circ} = (f^{\dagger})^{\circ} = (f^{\circ})^{\dagger}$$

The map f is ○-real if $f = f_{\circ}$ (or equiv. $f^{\dagger} = f^{\circ}$)

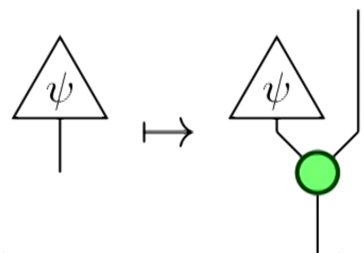
Phases

Defn. a map $\alpha{:}A{\longrightarrow}A$ is called a *pre-phase* if:



A pre-phase is a *phase* if it is unitary.

Defn: a point $\psi{:}I{\longrightarrow}A$ is called *unbiased* if the map



is a phase.

$$\Lambda(\psi) : \psi \mapsto \mu \circ (\psi \otimes \mathrm{id})$$

Phases

Lemma. let $\alpha: A \longrightarrow A$ be a phase; then there exists an unbiased point $\psi: I \longrightarrow A$ such that:

1. $\alpha = \Lambda(\psi)$;
2. $\alpha^{\circ} = \alpha$;
3. $\alpha^{\dagger} = \Lambda(\psi_{\circ})$;
4. $\mu(\psi \otimes \psi_{\circ}) = \eta$.

Corollary: α is a phase iff $\alpha^{\dagger}$ is a phase.

Phases

Lemma. let $\alpha: A \longrightarrow A$ be a phase; then there exists an unbi

Proposition 6:
1. The phases are an abelian group
2. The unbiased points are an abelian group
3. They're isomorphic

$$3.\ \ \alpha^{\dagger} = \Lambda(\psi_{\circ});$$
$$4.\ \ \mu(\psi \otimes \psi_{\circ}) = \eta.$$

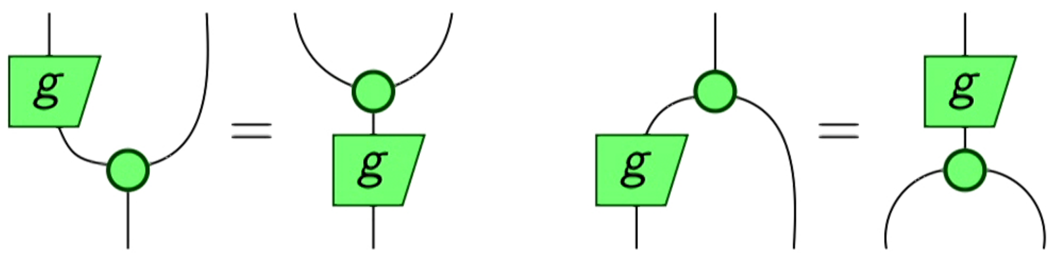
Corollary: α is a phase iff $\alpha^{\dagger}$ is a phase.

Phases

Let G be an abelian group; define the PROP $G^{\times}$ by

$$\Sigma = \{g : 1 \to 1 \mid g \in G\} \qquad E = \{g \circ h = gh\}$$

Quotient $\mathbb{F} + G^{\times}$ by the equations



(P)

Frob. algebras with phases

Recall $\mathbb{F}$ is itself a composite $\mathbb{M};\mathbb{M}^{\mathrm{op}}$ so we can view $\mathbb{F}G$ as an *iterated* distributive law for $\mathbb{M};G^{\times};\mathbb{M}^{\mathrm{op}}$.

This yields a factorisation:

$$f = n \xrightarrow[\mathbb{M}]{\nabla} m \xrightarrow[G^{\times}]{g} m \xrightarrow[\mathbb{M}^{\mathrm{op}}]{\Delta} n'$$

So $\mathbb{F}G$ is the PROP of Frob.algs. with *phases.*

Frob. algebras with phases

Recall $\mathbb{F}$ is itself a composite $\mathbb{M};\mathbb{M}^{\mathrm{op}}$ so we can view $\mathbb{F}G$ as an *iterated* distributive law for $\mathbb{M};G^{\times};\mathbb{M}^{\mathrm{op}}$.

This

Proposition 7:
This gives a functor $\mathbb{F} : \mathbf{AbGrp} \to \dagger\mathbf{PROP}$

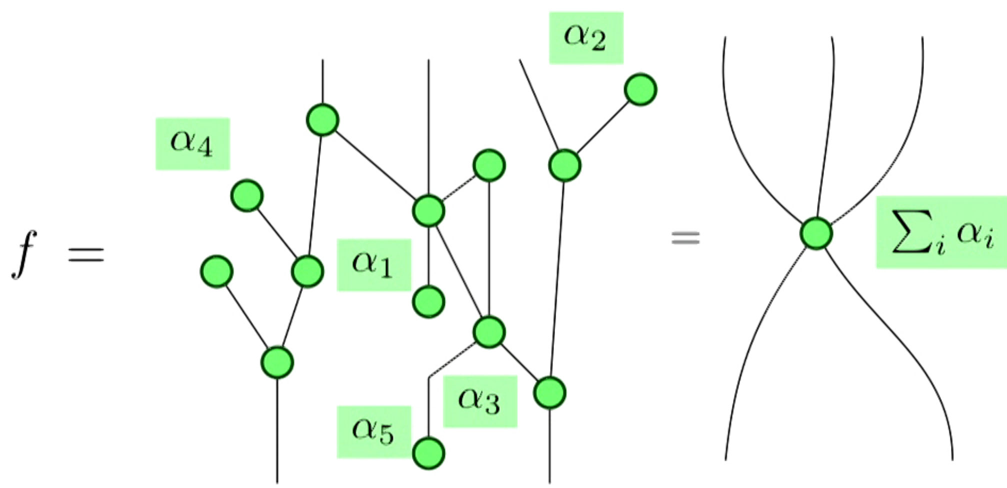
$$f = n \xrightarrow{\nabla} m \xrightarrow{g} m \xrightarrow{\Delta} n'$$

$$\mathbb{M} \qquad G^{\times} \qquad \mathbb{M}^{\mathrm{op}}$$

So $\mathbb{F}G$ is the PROP of Frob.algs. with *phases.*

Frob. algebras with phases

Theorem: let $f : n \longrightarrow m$ be connected in $\mathbb{F}G$. Then

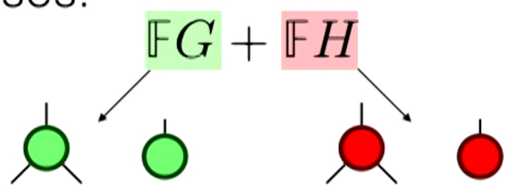


4.

Frobenius-Hopf Algebras

Two Frobenius Algebras

We can form the coproduct i.e. *non*-interacting Frobenius algebras with phases.



Factorisation:

$$f = n \xrightarrow{g_1} d_1 \xrightarrow{h_1} d_2 \xrightarrow{g_2} d_3 \xrightarrow{h_2} \cdots \xrightarrow{g_k} m$$

Two Frobenius Algebras

No interactions:

f unitary implies $f \in \mathbb{P}G + \mathbb{P}H \quad \cong \quad \mathbb{P}(G * H)$.

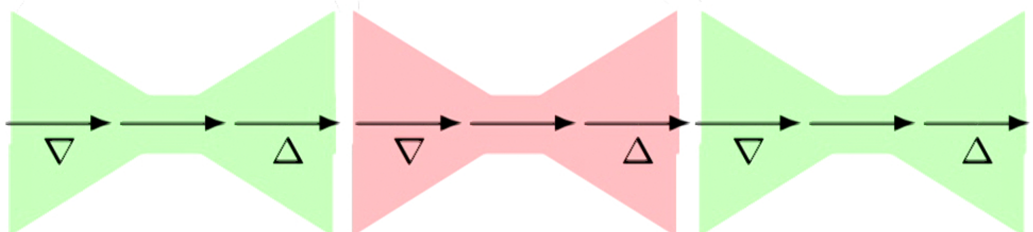
$$f^{\color{green}\bullet} \neq f^{\color{red}\bullet}$$

$$f_{\color{green}\bullet} = f_{\color{red}\bullet} \ \textit{implies}\ f \in \boldsymbol{P}1$$

Two Frobenius Algebras

Factorisation:

$$f = n \xrightarrow{g_1} d_1 \xrightarrow{h_1} d_2 \xrightarrow{g_2} d_3 \xrightarrow{h_2} \cdots \xrightarrow{g_k} m$$



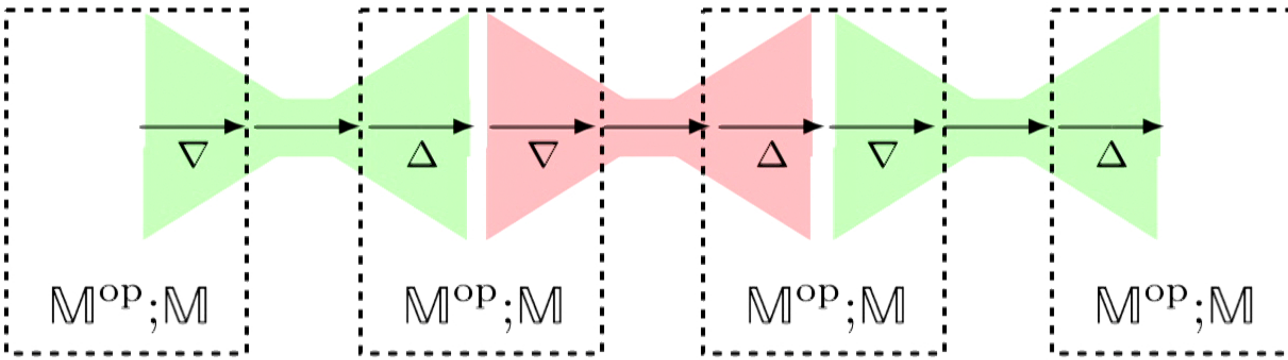
Two Frobenius Algebras

No interactions:

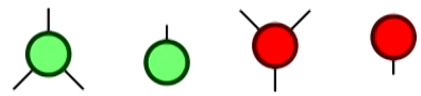
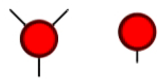
f unitary implies $f \in \mathbb{P}G + \mathbb{P}H \quad \cong \quad \mathbb{P}(G * H)$.

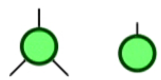
$$f^{\circ} \neq f^{\bullet}$$

$$f_{\circ} = f_{\bullet} \;\; implies \;\; f \in \boldsymbol{P}1$$

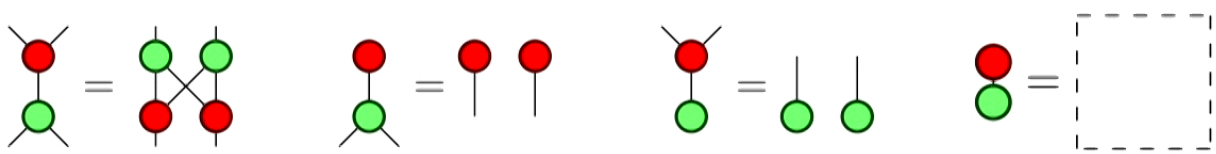


Bialgebras

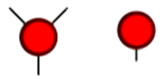
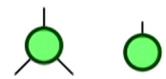
Defn: A bialgebra is 4-tuple: 
where:  is a monoid

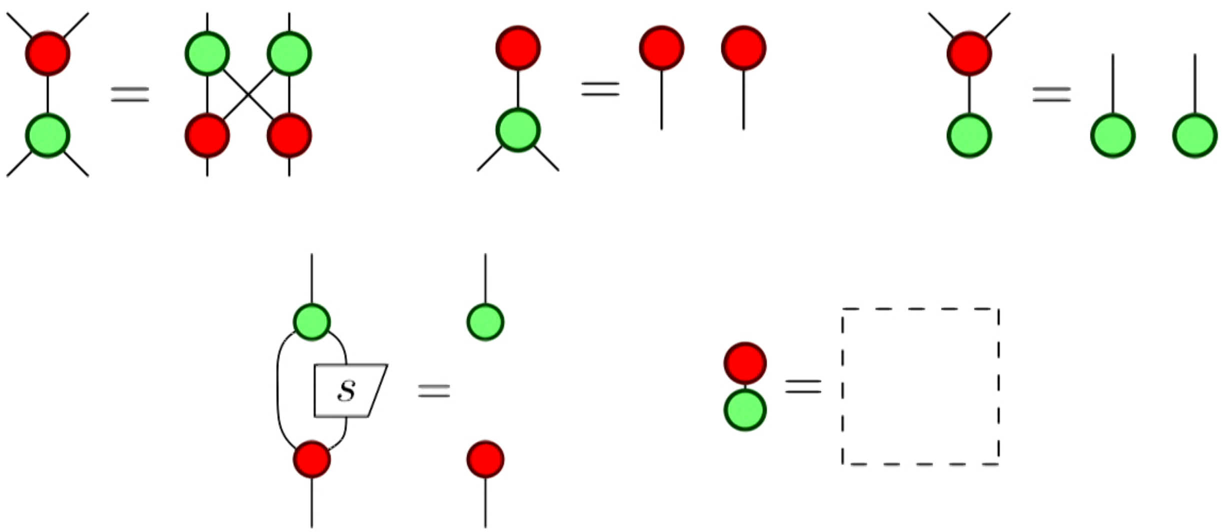
 is a comonoid

jointly satisfying the distribution laws:



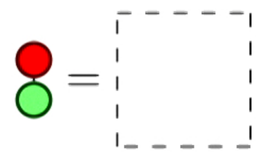
Hopf Algebra Axioms

 is a monoid  is a comonoid



Normalisation

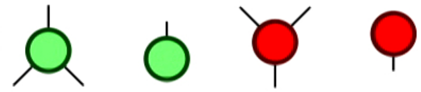
Oh uh:

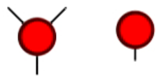


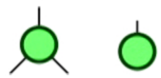
This is false in the usual model. We want:

$$|\langle a_i \mid b_j \rangle| = \frac{1}{\sqrt{d}} \qquad \forall i, j$$

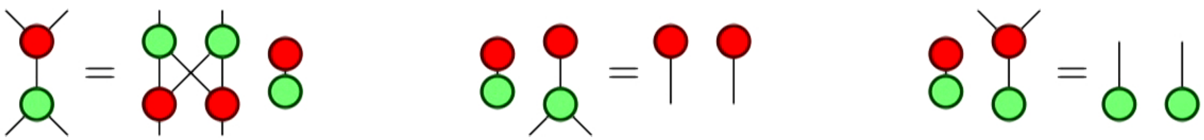
Scaled Bialgebras

Defn: A *scaled* bialgebra is 4-tuple: 

where:  is a †SCFA

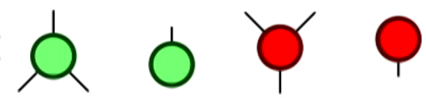
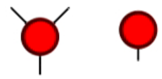
 is a †SCFA

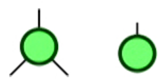
jointly satisfying the commutation laws:



(B)

Scaled Bialgebras

Defn: A *scaled* bialgebra is 4-tuple: 
where:  is a †SCFA

 is a †SCFA

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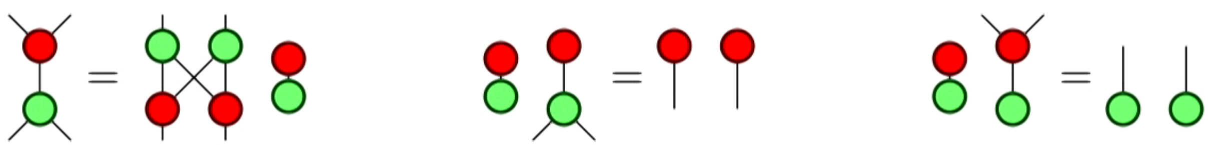
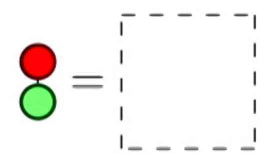
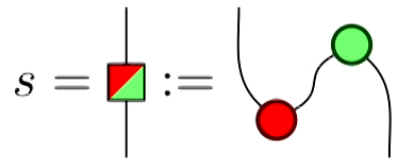


(B)

Not this one: 

The antipode

The antipode can be defined as:

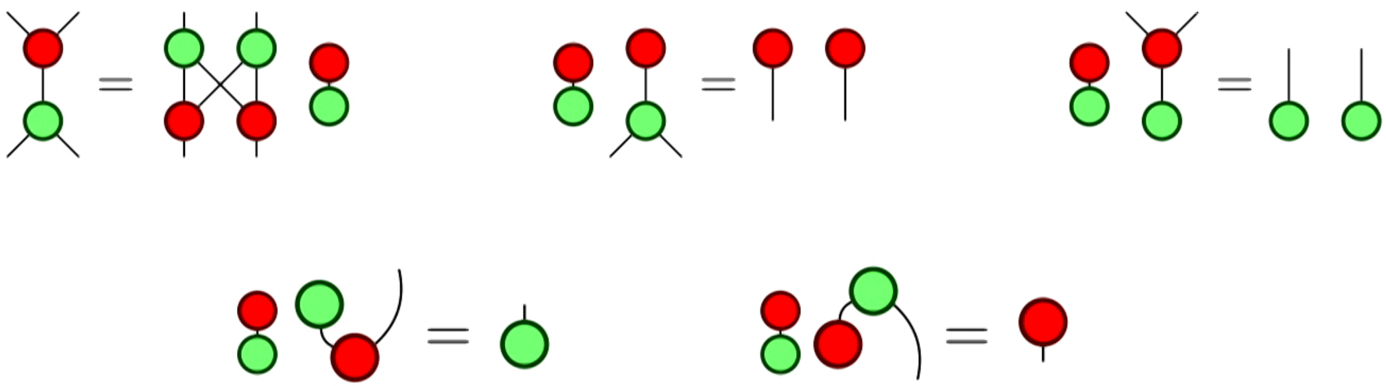


Then we have:

Theorem: the scaled bialgebra $(\delta_{\circ}, \epsilon_{\circ}, \mu_{\bullet}, \eta_{\bullet})$ is Hopf iff

$(+)$

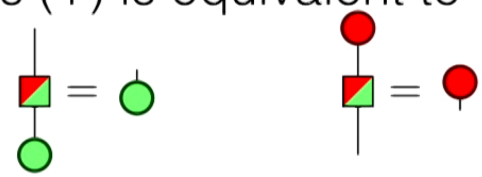
Frobenius-Hopf Algebra Axioms



The antipode

(+)

NB: the equations (+) is equivalent to



Emphasis:

1) it is an interaction between the (co)monoid structures of the opposing colours

2) it is **not** a full distributive law between them

Interacting Frobenius Algebras

Definition: Let $\mathbb{IF}(G,H)$ denote the PROP obtained from imposing (B+) on $\mathbb{F}G + \mathbb{F}H$.

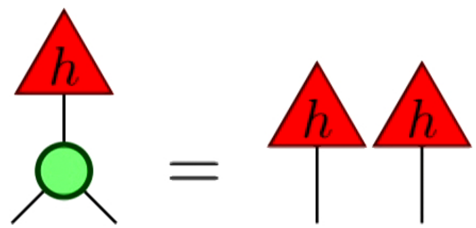
$$\boldsymbol{IF} : \mathbf{Ab} \times \mathbf{Ab} \to \dagger\text{-}\mathbf{PROP}$$

5.

Classical Structure

Set-like elements

Defn. A point $h : I \longrightarrow A$ is -*set-like* if it satisfies



Prop. The set-like elements of a commutative Hopf algebra form an abelian group, with $h^{-1} = sh$

Corollary: if the -set-like elements are -unbiased, they are a subgroup of the phase group.

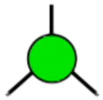
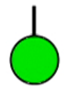
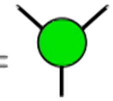
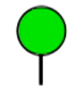
Set-like elements

Lemma. If $h : I \longrightarrow A$ is ●-*set-like* then

(1) it is not ●-unbiased

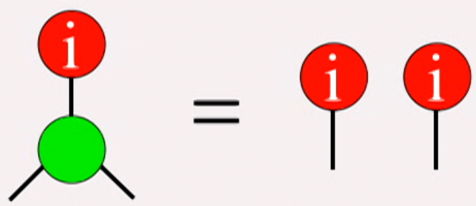
(2) it is ●-unbiased iff it is ●- real.

Two kinds of points

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

Two kinds of points

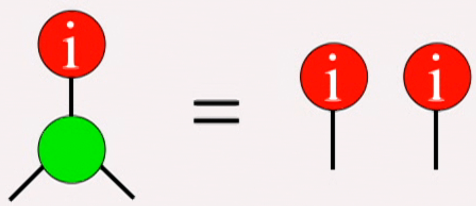
Set-like Points



Those points which can be copied by δ

Two kinds of points

Set-like Points

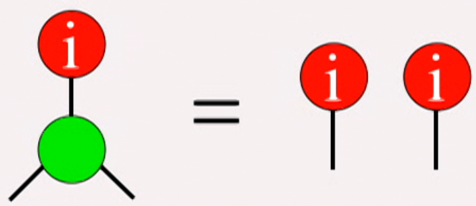


Those points which can be copied by δ

Unbiased Points

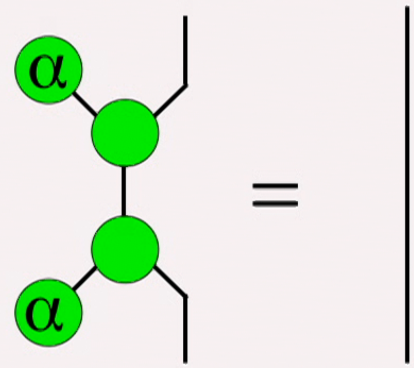
Two kinds of points

Set-like Points

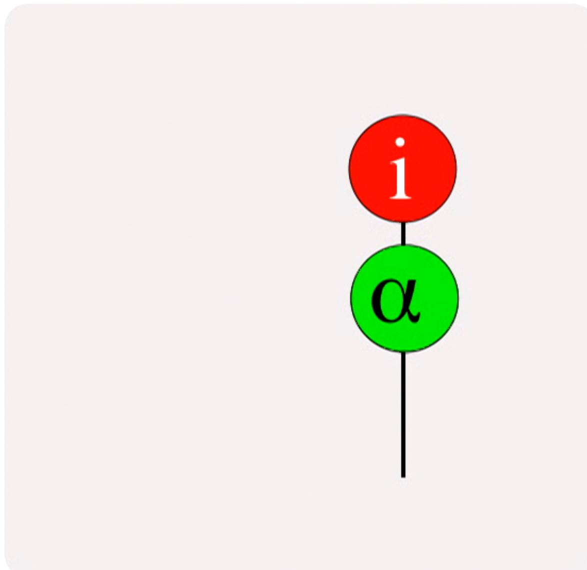


Those points which can be copied by δ

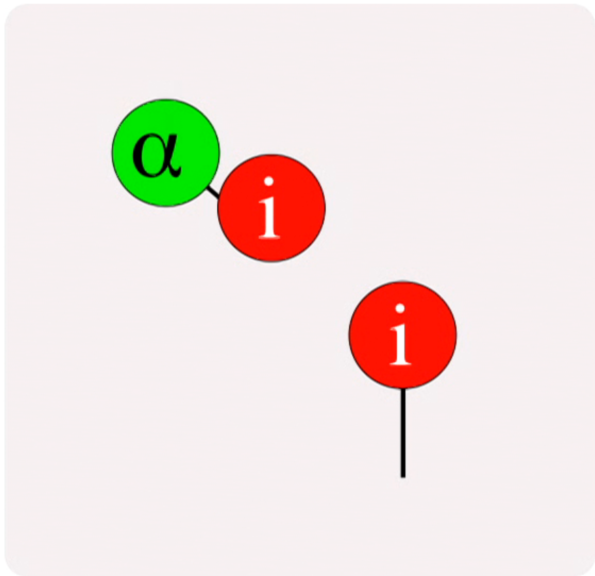
Unbiased Points



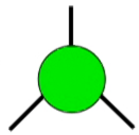
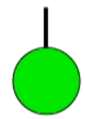
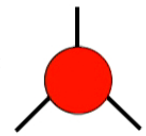
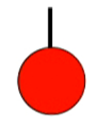
Set-like points are eigenvectors



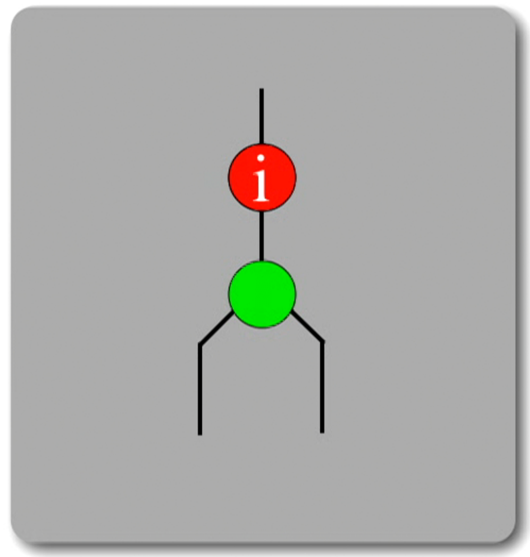
Set-like points are eigenvectors



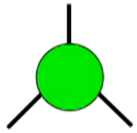
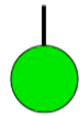
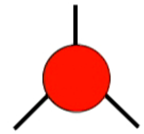
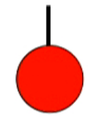
Comonoid Homomorphism

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

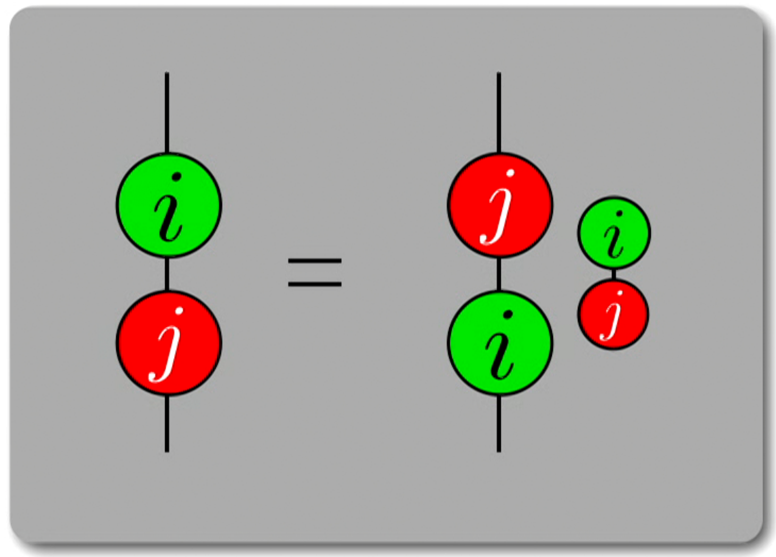
The classical maps are comonoid homomorphisms



Classical Phases Commute

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

The classical maps satisfy canonical commutation relations



More interactions

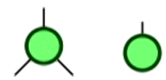
Let G be an abelian group with subgroup G_k, and likewise H and H_k. Define a PROP $\mathbb{IFK}(G \geq G_k, H \geq H_k)$ by imposing the equalities

upon $\mathbb{IF}(G, H)$ for every g in G_k and every h in H_k

Classical Maps

By definition every ●-set-like element in $\mathbb{IFK}(G \geq G_k, H \geq H_k)$ determines a ●-phase map: we call these ●-*classical* maps.

Lemma: let $k : 1 \longrightarrow 1$ be ●-classical; then

(1) k is a coalgebra morphism for 

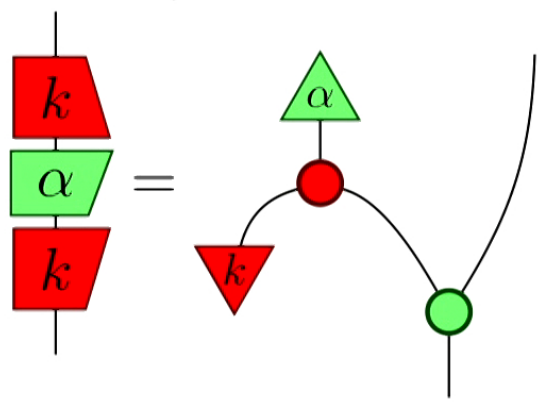
(2) $k^\dagger = sks$

Classical Maps Generate New Phases

So far the only -phase maps have been elements of G. This is no longer true.

Theorem: Let α be a -phase and k a -classical map then $k^{\dagger}\alpha k$ is a -phase.

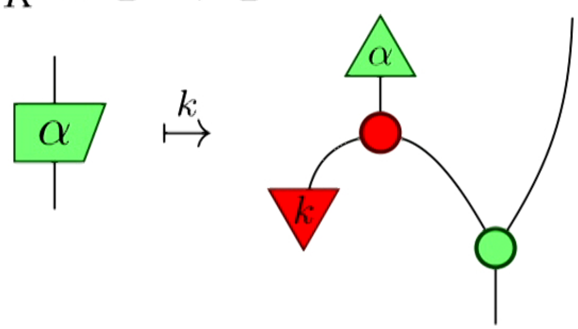
Proof:



Classical Maps Generate New Phases

Corollary 6.8. For H_{K} the group of -set-like elements and Φ the group of -phases there is a group action:

$$H_K \times \Phi \to \Phi$$



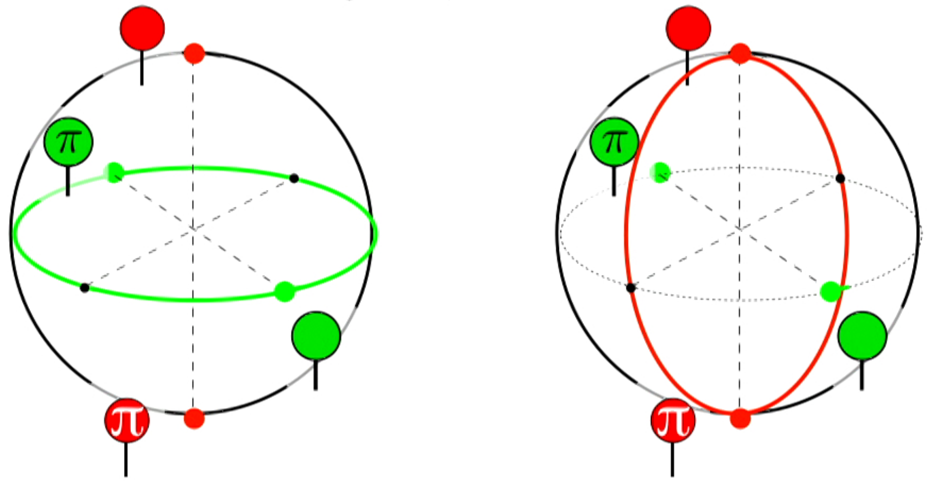
6.

The ZX-calculus

Z and X Observables

The Pauli **Z** and **X** observables are strongly complementary, with some additional features:

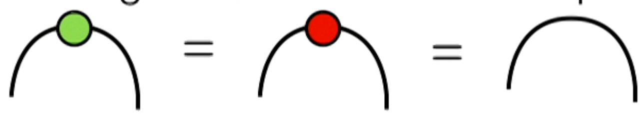
- The phase group is $[0,2\pi)$



- Dimension 2 implies two classical points
- The action of the classical group is $\alpha \mapsto -\alpha$

Z and X Observables

- Both observables generate the same compact structure



  - can just treat the diagram as an undirected graph

- the Z and X are related by a definable unitary
  - gives rise to colour change rule

ZX-calculus

Good Points:

+ Universal
+ Derived from the basic algebra of complementarity
+ Powerful algebraic theory
+ Can represent almost anything

Bad point:

- Need to impose operational meaning post-hoc

Meh Point:

- Not complete

ZX-calculus

Good Points:

+ Universal
+ Derived from the basic algebra of complementarity
+ Powerful algebraic theory
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+ **Now complete for the Clifford + T fragment!**

Bad point:

- Need to impose operational meaning post-hoc

Meh Point:

~~- Not complete~~

Jeandel, Perdrix, Vilmart
A Complete Axiomatisation of the ZX-Calculus
for Clifford+T Quantum Mechanics
1705.11151

ZX-calculus

Good Points:

+ Universal
+ Derived from the basic algebra of complementarity
+ Powerful algebraic theory
+ Can represent almost anything
+ Now complete for the Clifford + T fragment!
+ **Now universally complete!**

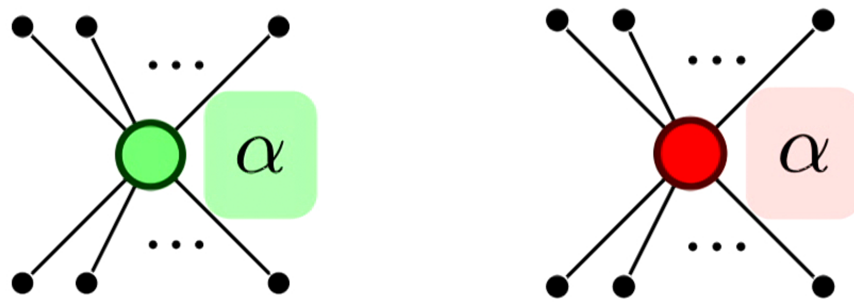
Bad point:

- Need to impos

Ng and Wang
A universal completion of the ZX-calculus
1706.09877

-Calculus
anics

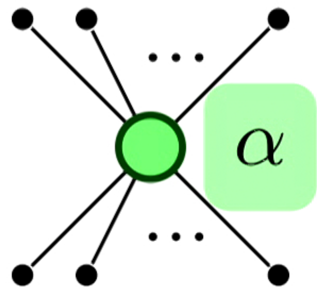
ZX-calculus syntax



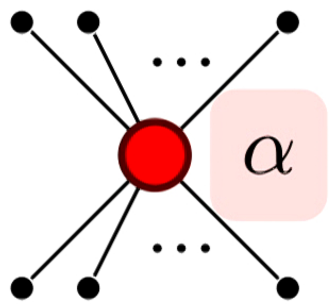
$$\alpha \in [0, 2\pi)$$

Defn: A *diagram* is an undirected open graph generated by the above vertices.

ZX-calculus semantics



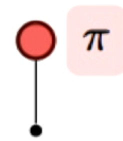
$$|0\rangle^{\otimes n} \mapsto |0\rangle^{\otimes m}$$
$$|1\rangle^{\otimes n} \mapsto e^{i\alpha} |1\rangle^{\otimes m}$$



$$|+\rangle^{\otimes n} \mapsto |+\rangle^{\otimes m}$$
$$|-\rangle^{\otimes n} \mapsto e^{i\alpha} |-\rangle^{\otimes m}$$

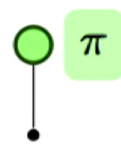
Representing Qubits

$[\![$  $]\!] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

$[\![$ 

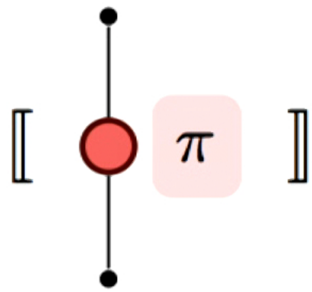
$]\!] = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

$[\![$  $]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$

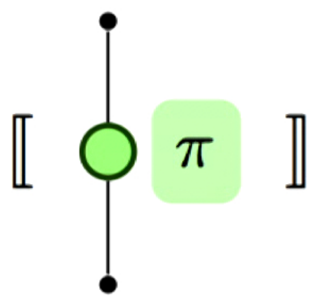
$[\![$ 

$]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$

Representing Paulis

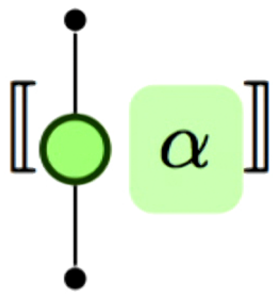


$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

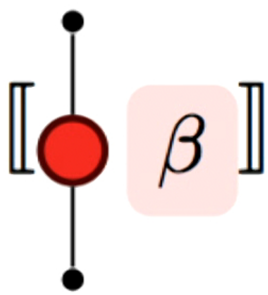


$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Representing Phase shifts

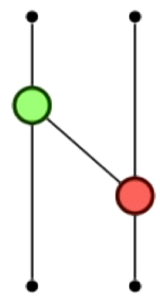
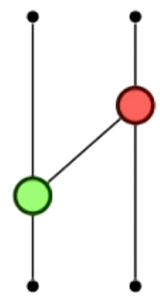


$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$



$$= \begin{pmatrix} \cos\frac{\beta}{2} & -i\sin\frac{\beta}{2} \\ -i\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}$$

Representing CNot

$\wedge X = [\![$  $]\!] = [\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

The ZX-calculus is universal

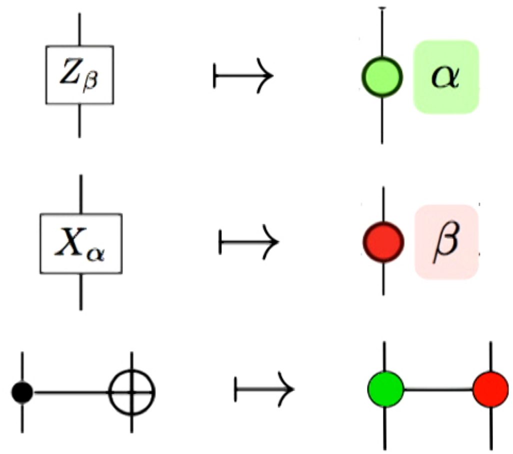
Theorem: Let U be a unitary map on n qubits; then there exists a ZX-calculus term D such that:

$$[\![D]\!] = U$$

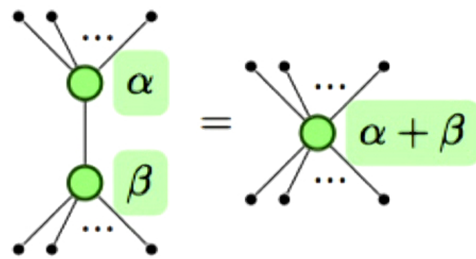
The ZX-calculus is universal

Theorem: Let U be a unitary map on n qubits; then there exists a ZX-calculus term D such that:

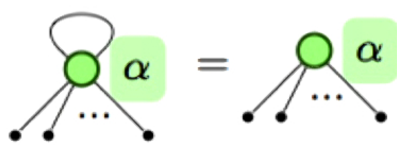
$$[\![D]\!] = U$$



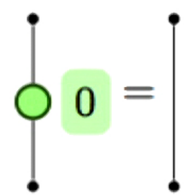
Equations



(spider)

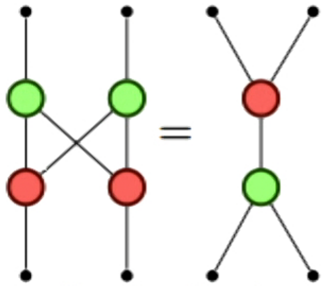


(anti-loop)

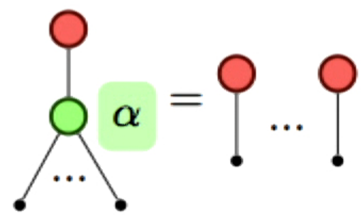


(identity)

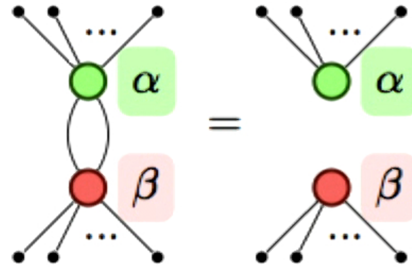
Equations



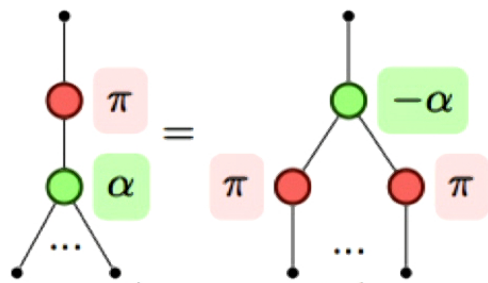
(bialgebra)



(copying)

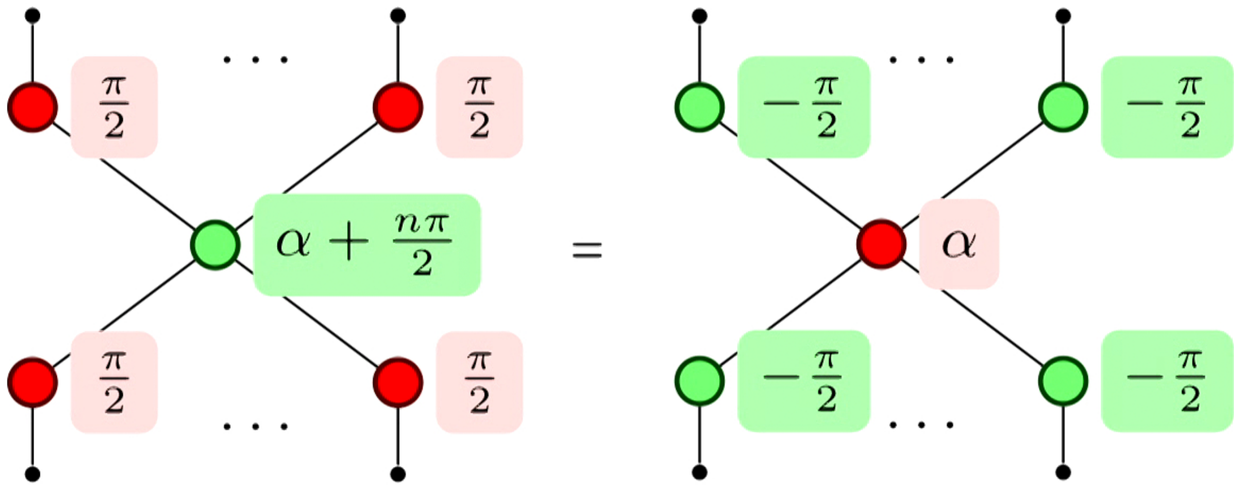


(hopf)



(π-commute)

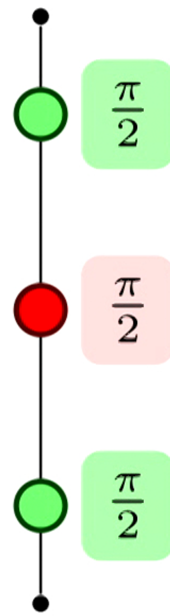
Equations



(colour change)

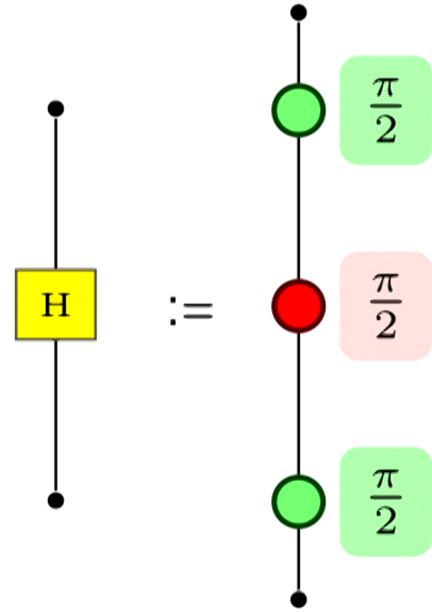
Representing Hadamard

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = [\![$$

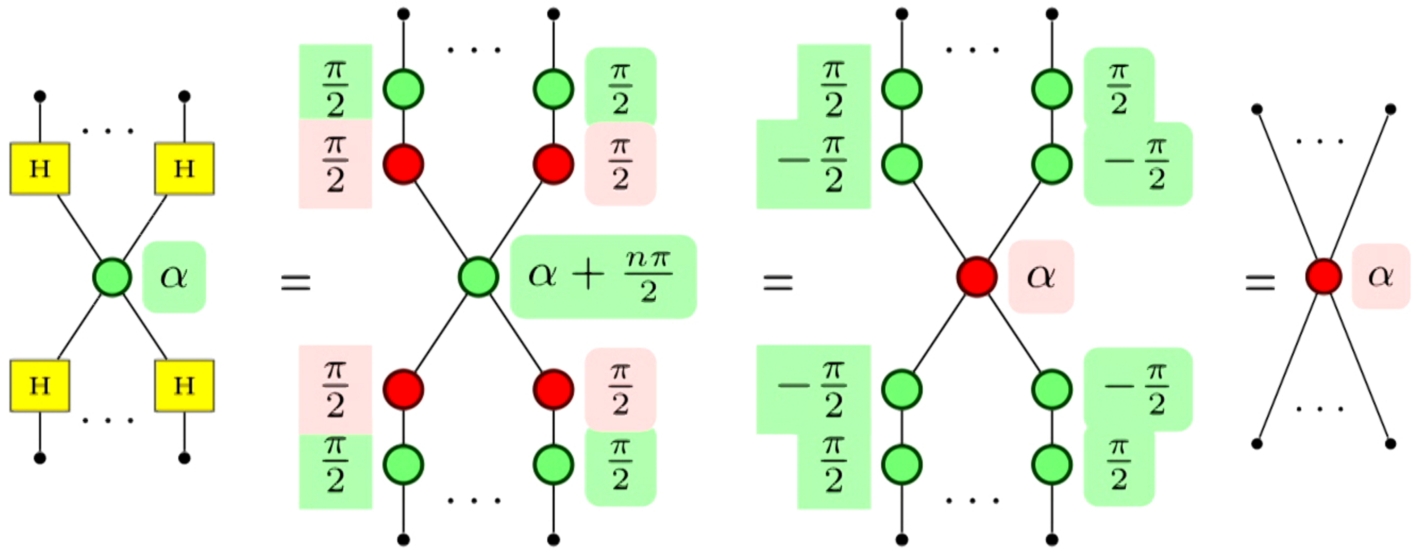


$$]\!]$$

More on the Hadamard

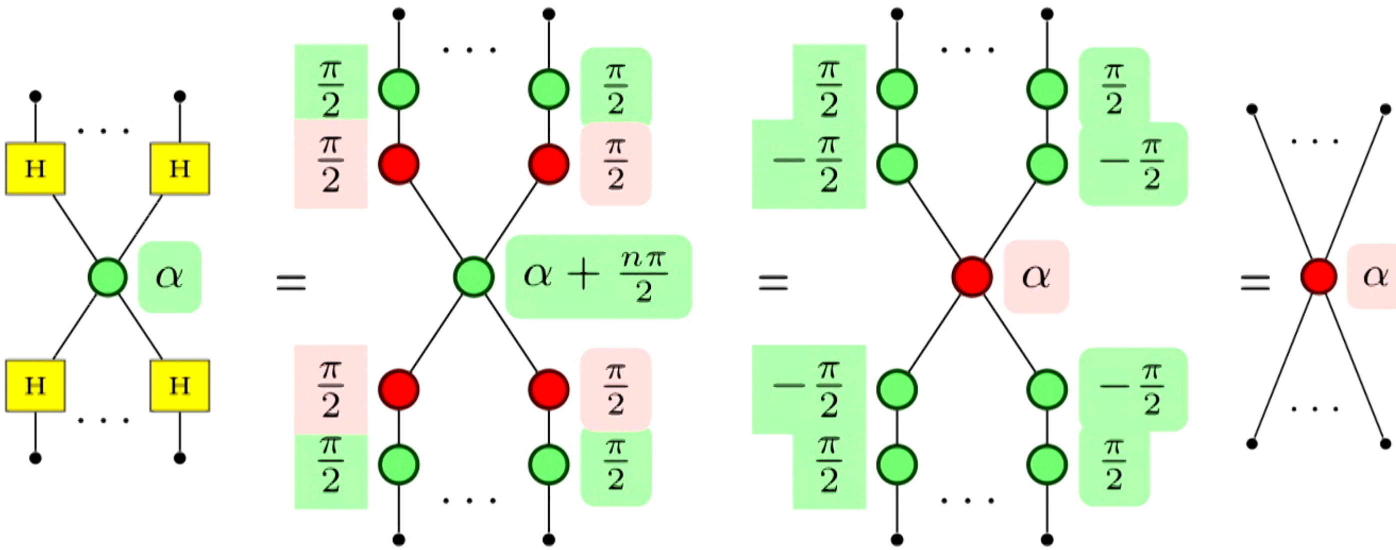


More on the Hadamard



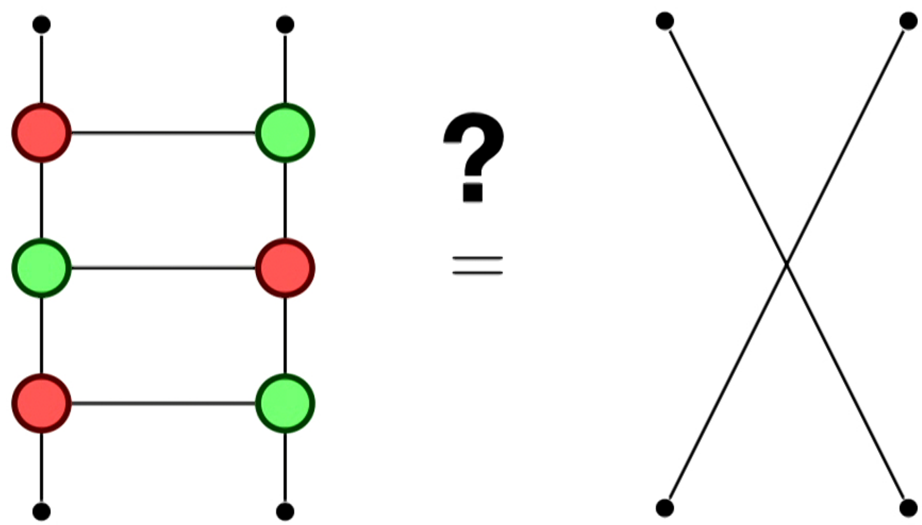
Corollary: total symmetry between red and green

More on the Hadamard

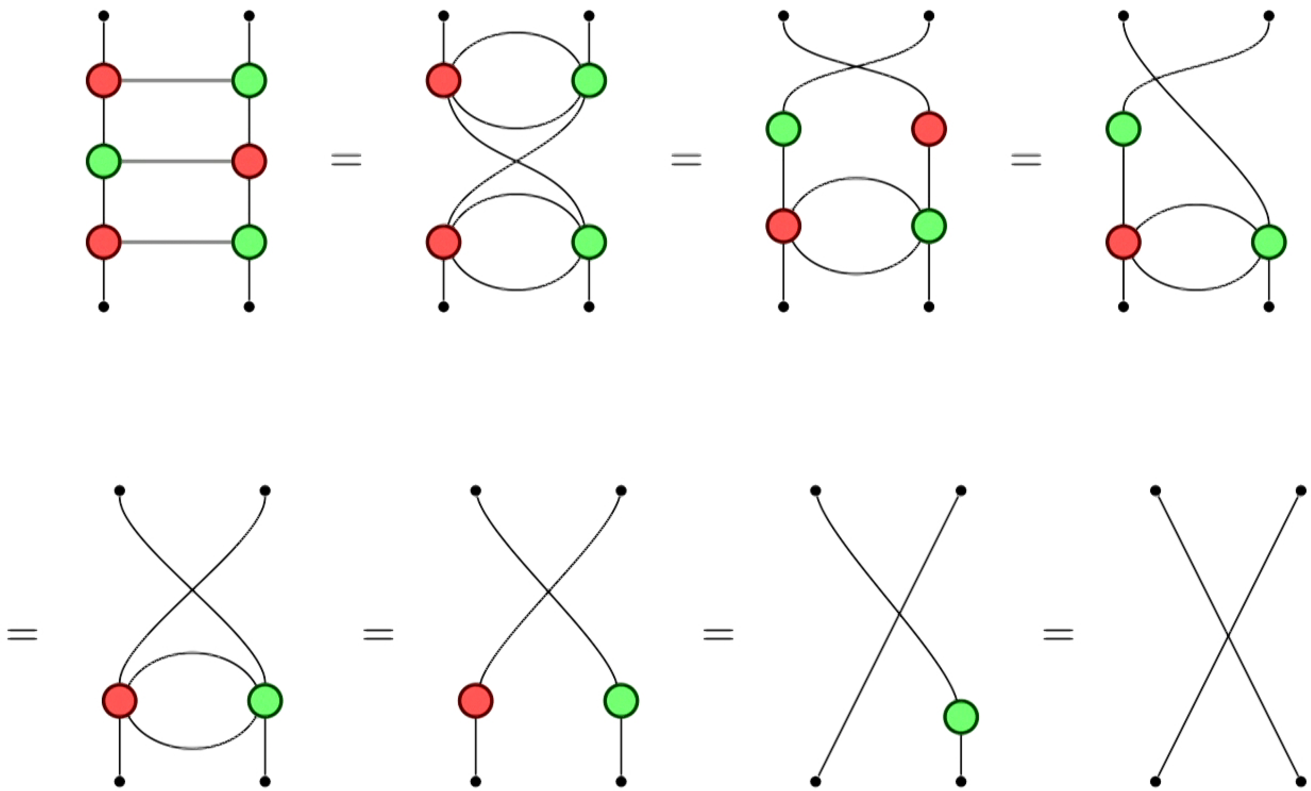


Corollary: total symmetry between red and green

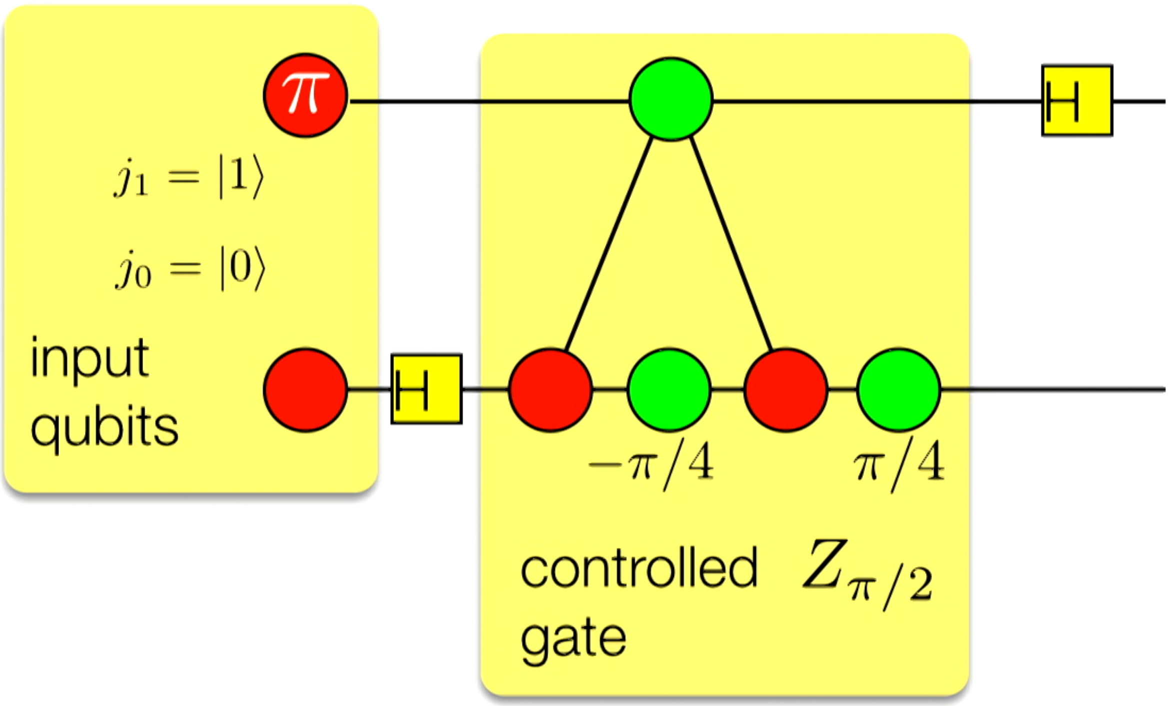
Example: CNOTS



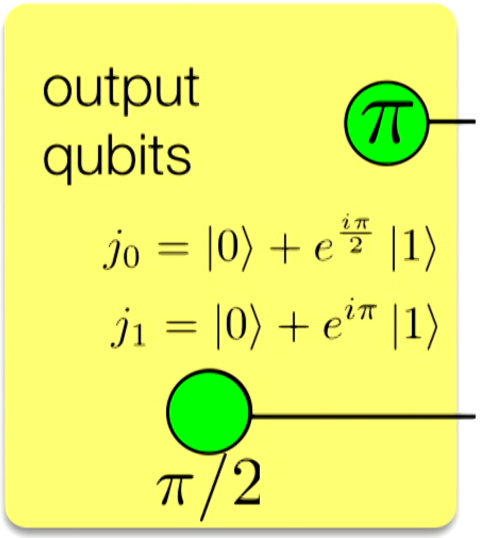
Example: CNOTS



Example: 2-Qubit Quantum Fourier Transform



Example: 2-Qubit Quantum Fourier Transform



Application 1: MBQC

1WQC is a quantum computer design based on single qubit projective measurements on a graph state.

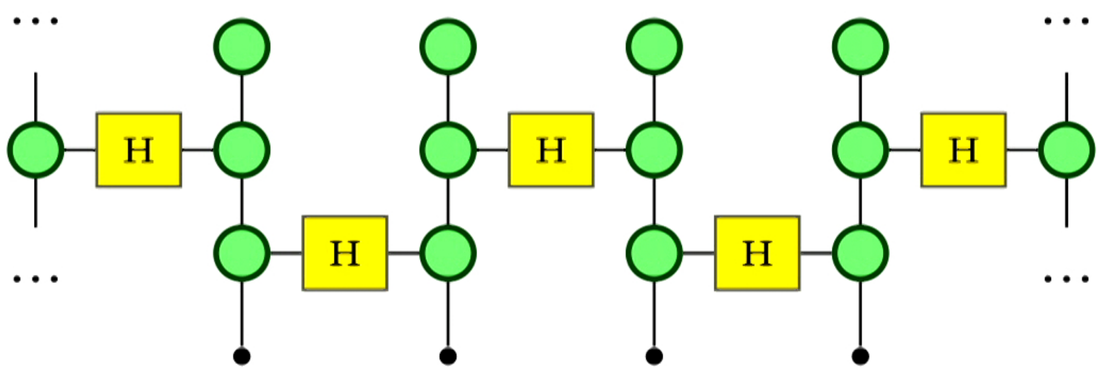
- We can use the ZX-calculus to translate from the 1WQC to circuit model
- Relies on the Hopf algebra normal form
- Produces circuits with minimal space complexity

Graph States

Let G = (V,E) be a simple, undirected graph. Then define:

$$|G\rangle = \bigotimes_{(v,u)\in E} CZ_{vu} \bigotimes_{v\in V} |+\rangle$$

Viewed as circuit we get this:

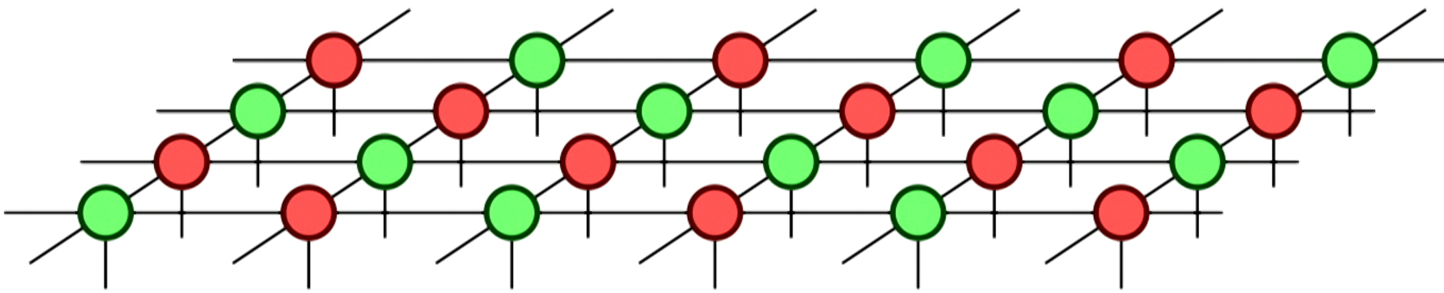


Graph States

Let G = (V,E) be a simple, undirected graph. Then define:

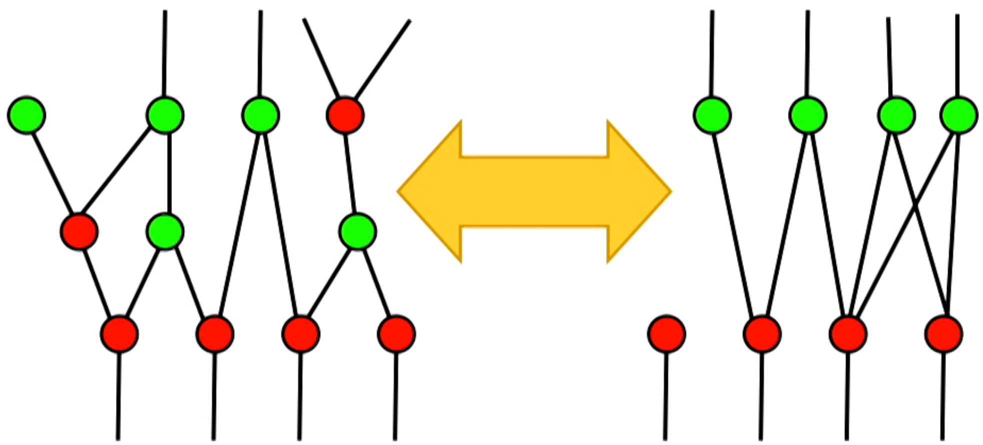
$$|G\rangle = \bigotimes_{(v,u)\in E} CZ_{vu} \bigotimes_{v\in V} |+\rangle$$

Or in 2D:

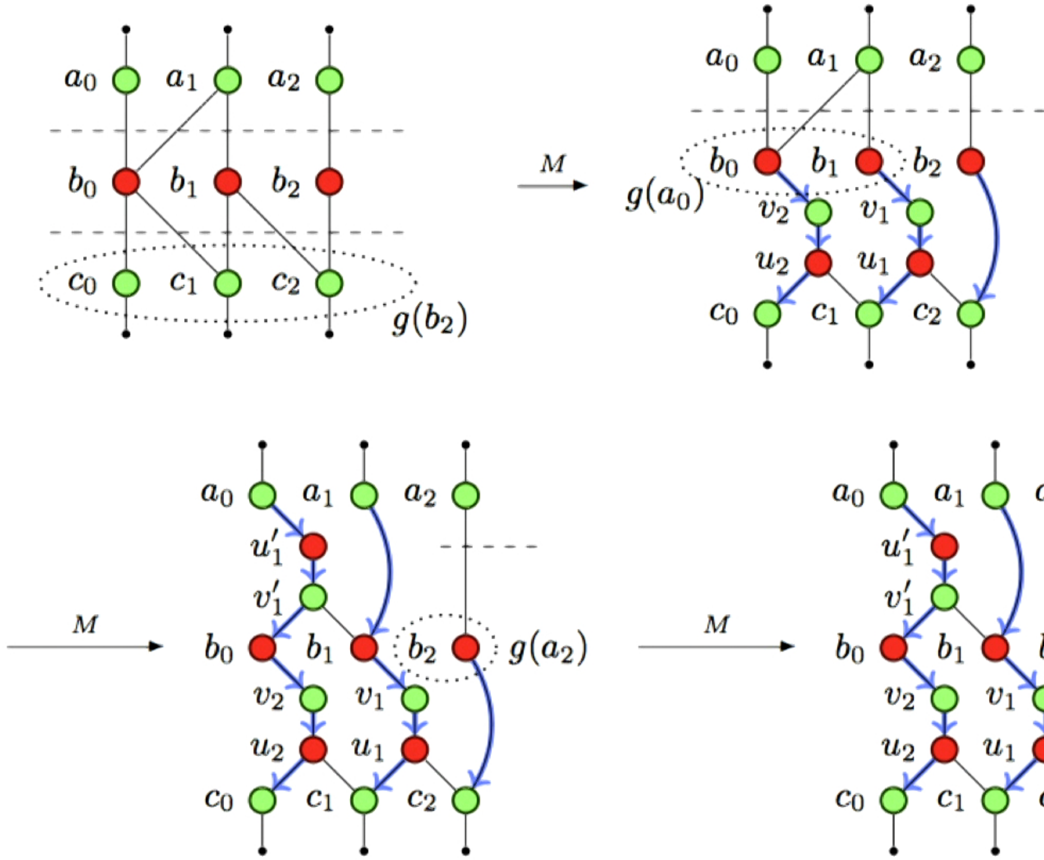


Hopf Algebra Equivalence

Thm: any Hopf algebra expression can be put into normal form:



GFlow Strategy

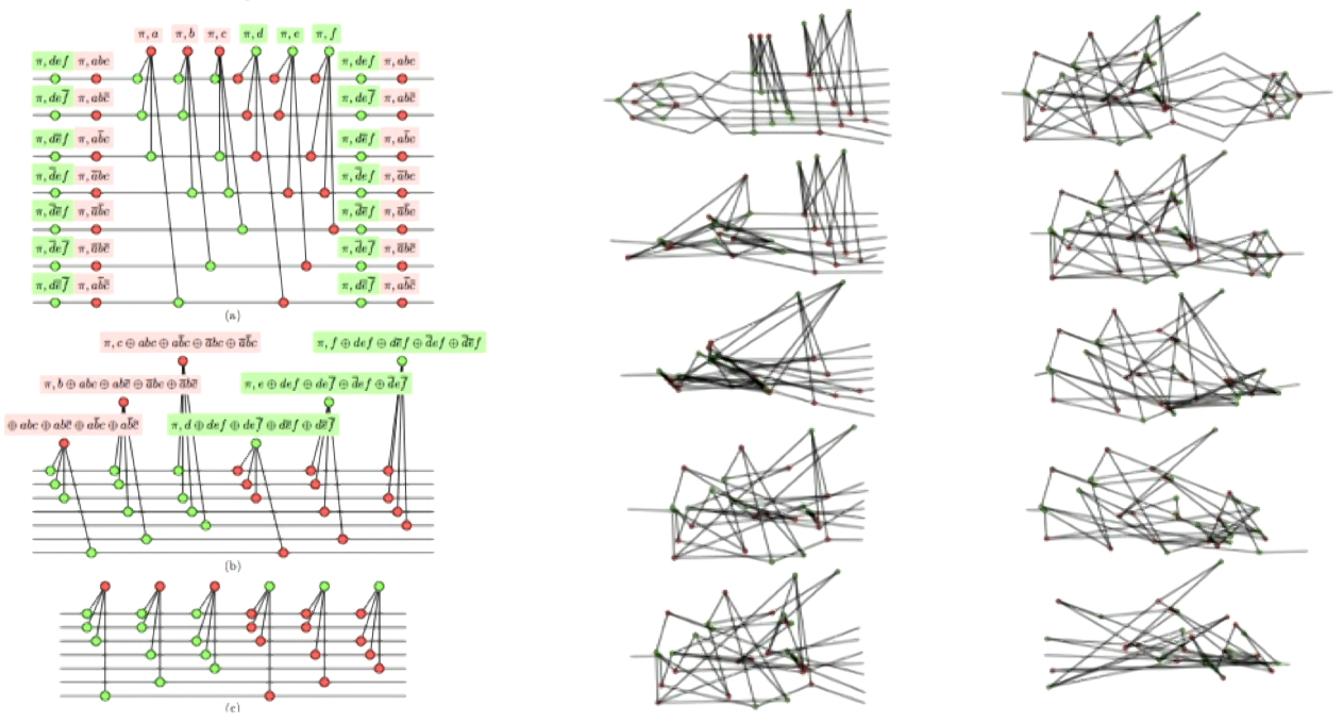


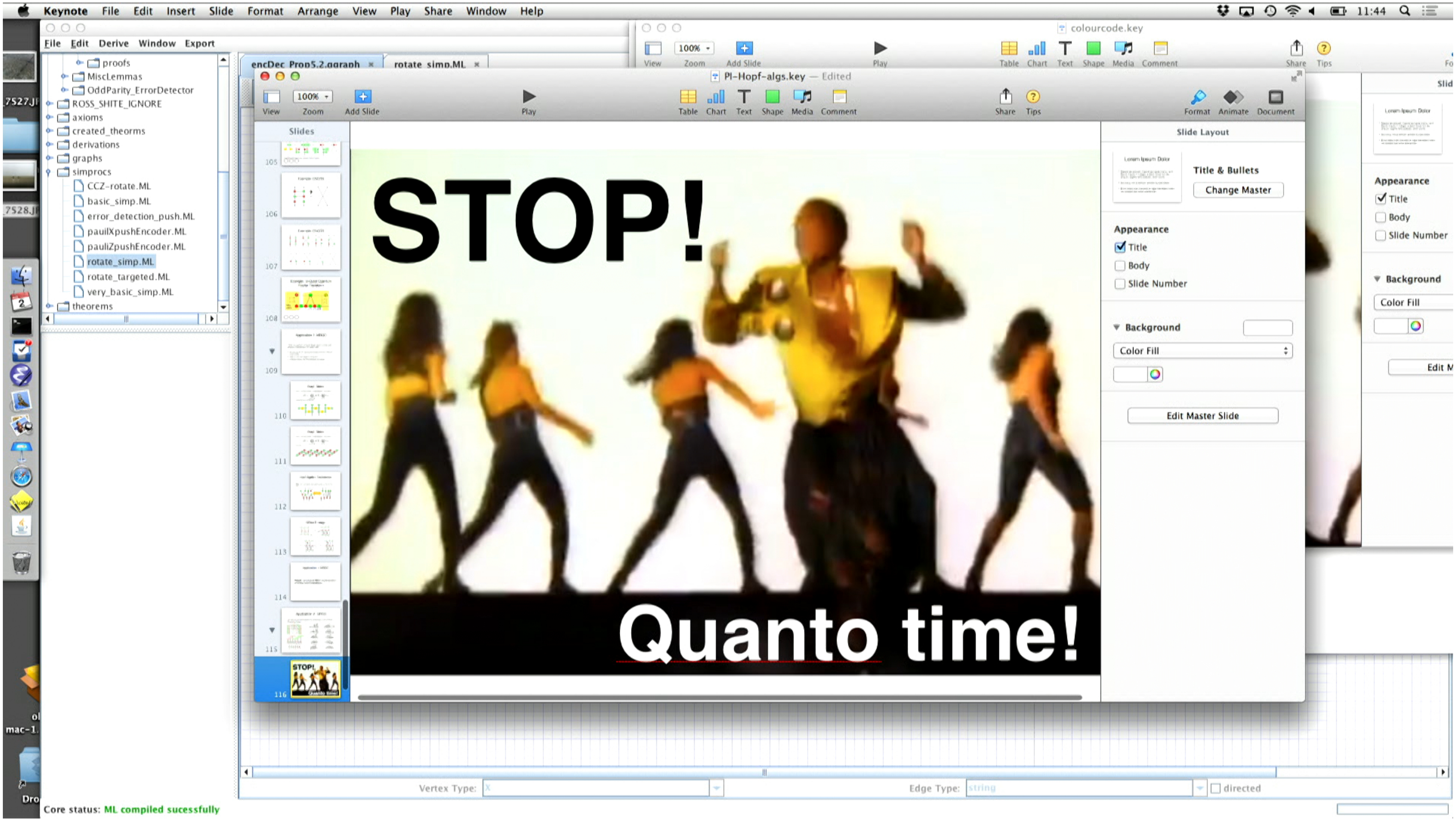
Application 1: MBQC

Result: complicated MBQC *implementation* to simpler circuit *specification*.

Application 2: QECC

ZX-calculus can demonstrate the correctness Quantum Error Correcting Codes:

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Slides
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Slide Layout
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Vertex Type:
Edge Type:
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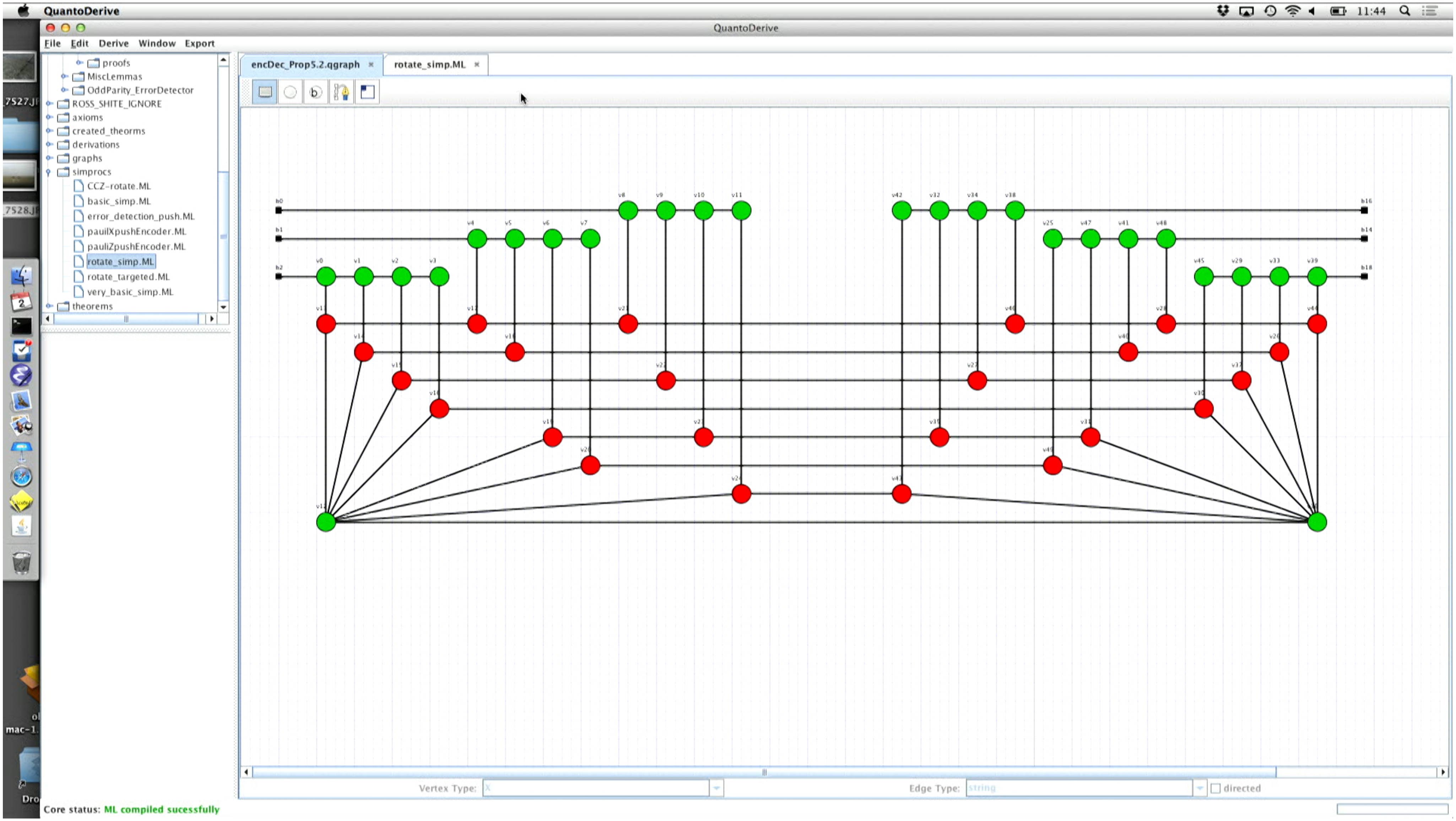
Core status: ML compiled sucessfully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pauliXpushEncoder.ML
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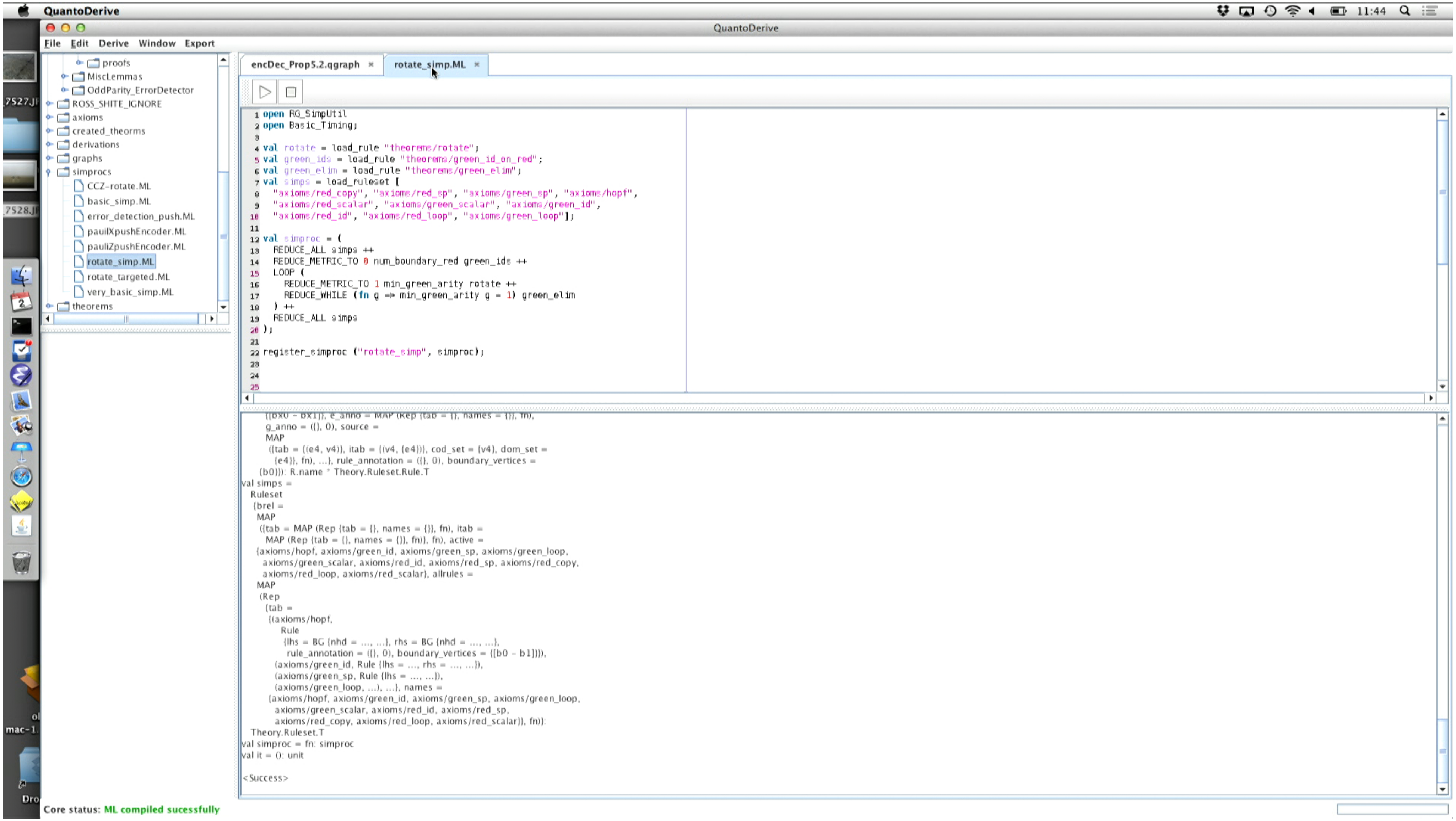
rotate_simp.ML
Vertex Type:
Edge Type:
directed
Core status: ML compiled sucessfully



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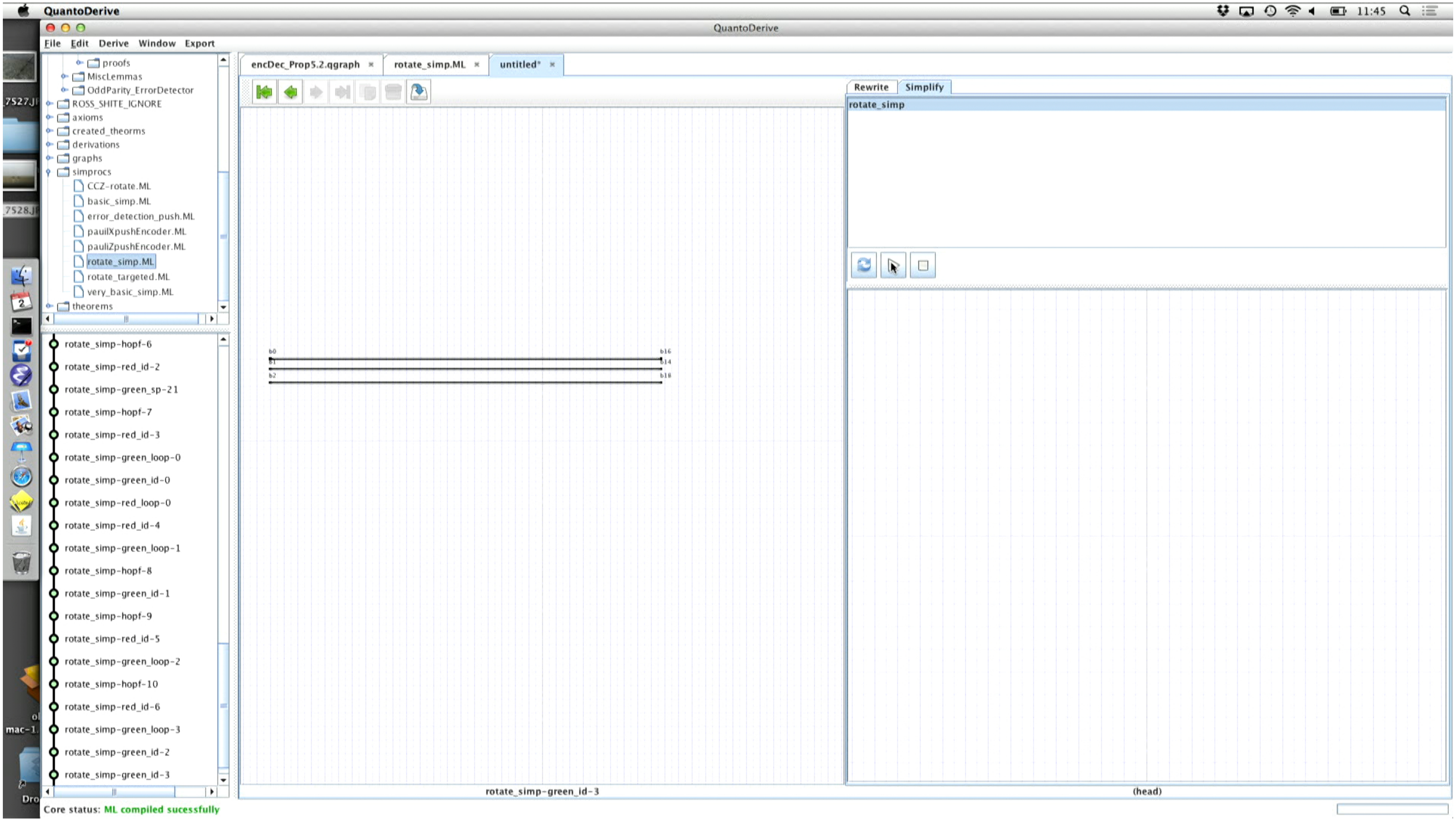
CCZ-rotate.ML
basic_simp.ML
error_detection_push.ML
pauliXpushEncoder.ML
pauliZpushEncoder.ML
rotate_simp.ML
rotate_targeted.ML
very_basic_simp.ML
theorems
encDec_Prop5.2.qgraph
rotate_simp.ML
open RG_SimpUtil
open Basic_Timing;
val rotate = load_rule "theorems/rotate";
val green_ids = load_rule "theorems/green_id_on_red";
val green_elim = load_rule "theorems/green_elim";
val simps = load_ruleset [
"axioms/red_copy", "axioms/red_sp", "axioms/green_sp", "axioms/hopf",
"axioms/red_scalar", "axioms/green_scalar", "axioms/green_id",
"axioms/red_id", "axioms/red_loop", "axioms/green_loop"];
val simproc = (
REDUCE_ALL simps ++
REDUCE_METRIC_TO 0 num_boundary_red green_ids ++
LOOP (
REDUCE_METRIC_TO 1 min_green_arity rotate ++
REDUCE_WHILE (fn g => min_green_arity g = 1) green_elim
) ++
REDUCE_ALL simps
);
register_simproc ("rotate_simp", simproc);
g_anno = ({}, 0), source =
MAP
({tab = {(e4, v4)}, itab = {(v4, {e4})}, cod_set = {v4}, dom_set =
{e4}}, fn), ...}, rule_annotation = ({}, 0), boundary_vertices =
{b0})): R.name * Theory.Ruleset.Rule.T
val simps =
Ruleset
{brel =
MAP
({tab = MAP (Rep {tab = {}, names = {}}, fn), itab =
MAP (Rep {tab = {}, names = {}}, fn)}, fn), active =
{axioms/hopf, axioms/green_id, axioms/green_sp, axioms/green_loop,
axioms/green_scalar, axioms/red_id, axioms/red_sp, axioms/red_copy,
axioms/red_loop, axioms/red_scalar}, allrules =
MAP
(Rep
{tab =
{(axioms/hopf,
Rule
{lhs = BG {nhd = ..., ...}, rhs = BG {nhd = ..., ...},
rule_annotation = ({}, 0), boundary_vertices = {[b0 - b1]}}),
(axioms/green_id, Rule {lhs = ..., rhs = ..., ...}),
(axioms/green_sp, Rule {lhs = ..., ...}),
(axioms/green_loop, ...), ...}, names =
{axioms/hopf, axioms/green_id, axioms/green_sp, axioms/green_loop,
axioms/green_scalar, axioms/red_id, axioms/red_sp,
axioms/red_copy, axioms/red_loop, axioms/red_scalar}}, fn)}:
Theory.Ruleset.T
val simproc = fn: simproc
val it = (): unit
<Success>
Core status: ML compiled sucessfully



QuantoDerive
File Edit Derive Window Export
proofs
MiscLemmas
OddParity_ErrorDetector
ROSS_SHITE_IGNORE
axioms
created_theorms
derivations
graphs
simprocs
CCZ-rotate.ML
basic_simp.ML
error_detection_push.ML
pauliXpushEncoder.ML
pauliZpushEncoder.ML
rotate_simp.ML
rotate_targeted.ML
very_basic_simp.ML
theorems
rotate_simp-hopf-6
rotate_simp-red_id-2
rotate_simp-green_sp-21
rotate_simp-hopf-7
rotate_simp-red_id-3
rotate_simp-green_loop-0
rotate_simp-green_id-0
rotate_simp-red_loop-0
rotate_simp-red_id-4
rotate_simp-green_loop-1
rotate_simp-hopf-8
rotate_simp-green_id-1
rotate_simp-hopf-9
rotate_simp-red_id-5
rotate_simp-green_loop-2
rotate_simp-hopf-10
rotate_simp-red_id-6
rotate_simp-green_loop-3
rotate_simp-green_id-2
rotate_simp-green_id-3
encDec_Prop5.2.qgraph
rotate_simp.ML
untitled*
Rewrite
Simplify
rotate_simp
rotate_simp-green_id-3
(head)
Core status: ML compiled sucessfully

References

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