Title: Kitaev models based on unitary quantum groupoids

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Abstract: Kitaev originally constructed his quantum double model based on finite groups and anticipated the extension based on Hopf algebras, which was achieved later by Buerschaper, etc. In this talk, we will present the work on the generalization of Kitaev model for quantum groupoids and discuss its ground states.

Kitaev models based on quantum groupoids

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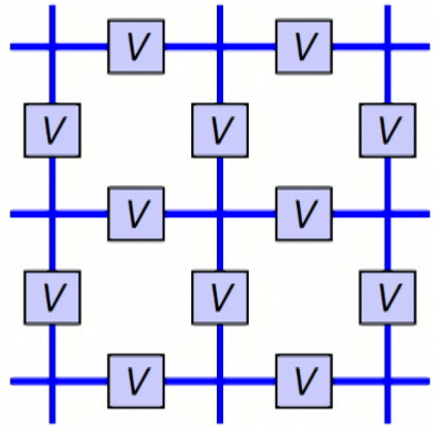
Motivation

- Kitaev built exactly solvable lattice model based on finite groups whose ground state degeneracy is protected by topology from local perturbation. He also predicted the extension to Hopf algebra, which was achieved by Buerschaper et al.
- Every fusion category is the representation category of some weak Hopf algebra. There should be lattice model based on certain algebra dual to Levin-Wen model based on fusion category.

Lattice models

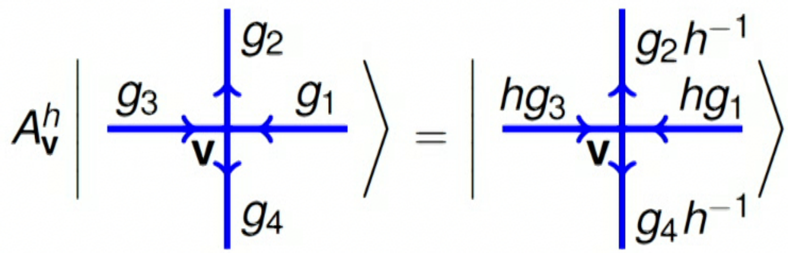


- Place spins/qubits at each edge of a oriented lattice on a closed surface.
- Hilbert space $\mathcal{L} = \bigotimes_{edges} V$.

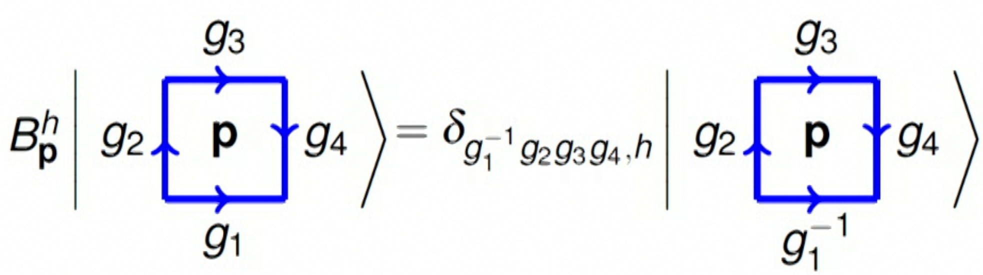
Quantum double model based on finite group G

Hilbert space $\mathcal{L} = \bigotimes_{edges} \mathbb{C}[G]$

- Vertex operator $A_{\mathbf{v}}^h$:



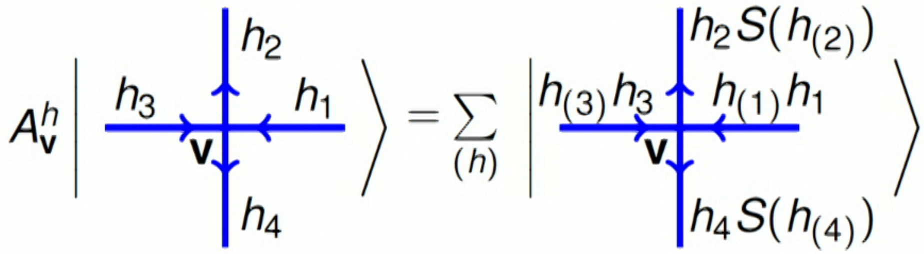
- Plaquette operator $B_{\mathbf{p}}^h$:



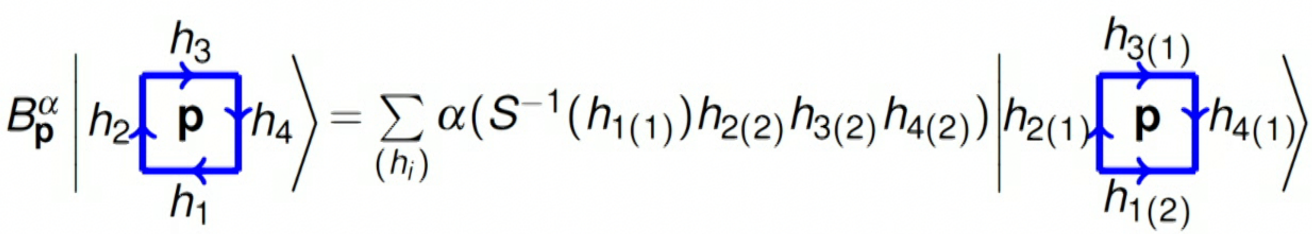
Quantum double model based on C^* Hopf algebra H

Hilbert space $\mathcal{L} = \bigotimes_{edges} H$

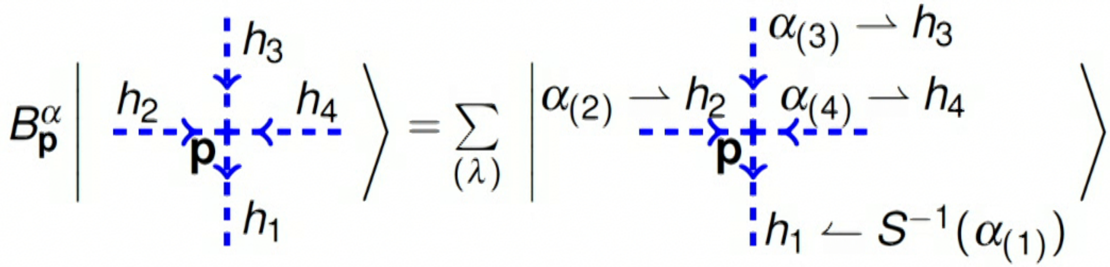
- Vertex operator $A_{\mathbf{v}}^h$:



- Plaquette operator $B_{\mathbf{p}}^\alpha$:



Dual picture of plaquette operator



where H is considered as a left H^*-module with the action

$$\alpha \rightharpoonup x := \langle \alpha, x_{(2)} \rangle x_{(1)}$$

and a right H^*-module with the action

$$x \leftharpoonup \alpha := \langle \alpha, x_{(1)} \rangle x_{(2)}.$$

- At a site $(\mathbf{v}, \mathbf{p})$, $A_{\mathbf{v}}^h$ and $B_{\mathbf{p}}^\alpha$ give rise to a representation of quantum double $D(H)$

$$A_{\mathbf{v}}^h B_{\mathbf{p}}^\alpha = \sum_{(h),(\alpha)} B_{\mathbf{p}}^{\alpha_{(2)}} A_{\mathbf{v}}^{h_{(2)}} \langle \alpha_{(3)}, S^{-1}(h_{(3)}) \rangle \langle \alpha_{(1)}, h_{(1)} \rangle$$

- When $\Lambda \in Cocom(H) = \{x \mid \Delta(x) = \Delta^{op}(x)\}$ and $\lambda \in Cocom(H^*)$, we have $[A_{\mathbf{v}}^\Lambda, A_{\mathbf{v}'}^\Lambda] = [B_{\mathbf{p}}^\lambda, B_{\mathbf{p}'}^\lambda] = [A_{\mathbf{v}}^\Lambda, B_{\mathbf{p}}^\lambda] = 0$.
- Further, when Λ and λ are $*$-invariant idempotents i.e. $\Lambda^* = \Lambda$, $\lambda^* = \lambda$ and $\Lambda^2 = \Lambda$, $\lambda^2 = \lambda$ (e.g. Haar integral), we have exactly solvable Hamiltonian

$$H = -\sum_{\mathbf{v}} A_{\mathbf{v}}^\Lambda - \sum_{\mathbf{p}} B_{\mathbf{p}}^\lambda$$

whose ground state $|\Psi>$ is a common eigenspace of all $A_{\mathbf{v}}^\Lambda$'s and $B_{\mathbf{p}}^\lambda$'s for eigenvalue 1, i.e., $A_{\mathbf{v}}^\Lambda|\Psi> = |\Psi>$ and $B_{\mathbf{p}}^\lambda|\Psi> = |\Psi>$ for all **v** and **p**.

Quantum Groupoids

A finite quantum groupoid (weak Hopf algebra) H is a finite dimensional vector space with structures of associative algebra by multiplication $m : H \otimes H \to H$ and unit $\eta \in H$ and coassociative coalgebra by comultiplication $\Delta : H \to H \otimes H$ and counit $\varepsilon : H \to \mathbb{C}$ s.t.

(1) $\Delta(ab) = \Delta(a)\Delta(b)$

$$(\Delta \otimes id)\Delta(\eta) = (\Delta(\eta) \otimes \eta)(\eta \otimes \Delta(\eta)) = (\eta \otimes \Delta(\eta))(\Delta(\eta) \otimes \eta)$$

(2) $\varepsilon(abc) = \varepsilon(ab_{(1)})\varepsilon(b_{(2)}c) = \varepsilon(ab_{(2)})\varepsilon(b_{(1)}c)$

(3) $h_{(1)}S(h_{(2)}) = \varepsilon(\eta_{(1)}h)\eta_{(2)}$

$$S(h_{(1)})h_{(2)} = \varepsilon(h\eta_{(2)})\eta_{(1)}$$

$$S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h)$$

- Integral
- C*, quasitriangular, ribbon, ...
- *Rep*(*H*) is tensor category with $U \otimes V := \Delta(\eta) \cdot (U \otimes_{\mathbb{C}} V)$ and tensor unit $H_t = \{\varepsilon(\eta_{(1)}h)\eta_{(2)} \mid h \in H\}$
  duality, braided, ribbon, modular, ...

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- The quantum double $D(H)$ for a quantum groupoid H is defined over the vector space $H^{*cop} \otimes H$ with multiplication

$$(\alpha \otimes x)(\beta \otimes y) = \alpha\beta_{(2)} \otimes x_{(2)} y \langle \beta_{(3)}, S^{-1}(x_{(3)})\rangle \langle \beta_{(1)}, x_{(1)}\rangle$$

Let J be the two-sided ideal spanned by

$$\alpha(z \rightharpoonup \varepsilon) \otimes x - \alpha \otimes zx, \; z \in H_t, \; x \in H, \; \alpha \in H^*$$
$$\alpha(\varepsilon \leftharpoonup w) \otimes x - \alpha \otimes wx, \; w \in H_s, \; x \in H, \; \alpha \in H^*$$

$D(H) := H^{*cop} \otimes H/J$ is a quantum groupoid with unit $[\varepsilon \otimes \eta]$ and

$$\Delta_D([\alpha \otimes x]) = [\alpha_{(1)} \otimes x_{(1)}] \otimes [\alpha_{(2)} \otimes x_{(2)}]$$
$$\varepsilon_D([\alpha \otimes x]) = \varepsilon(x)\alpha(\eta)$$
$$S_D([\alpha \otimes x]) = [S^{-1}(\alpha_{(2)}) \otimes S(x_{(2)})]\langle \alpha_{(3)}, S^{-1}(x_{(3)})\rangle \langle \alpha_{(1)}, x_{(1)}\rangle$$

Quantum Double Model from Quantum Groupoid

- At any site $(\mathbf{v},\mathbf{p})$, $A_{\mathbf{v}}^h$ and $B_{\mathbf{p}}^\alpha$ satisfy:

$$A_{\mathbf{v}}^h B_{\mathbf{p}}^\alpha = \sum_{(h),(\alpha)} B_{\mathbf{p}}^{\alpha_{(2)}} A_{\mathbf{v}}^{h_{(2)}} \langle \alpha_{(3)}, S^{-1}(h_{(3)})\rangle \langle \alpha_{(1)}, h_{(1)}\rangle$$

$$B_{\mathbf{p}}^\alpha A_{\mathbf{v}}^{zh} = B_{\mathbf{p}}^{\alpha(z\rightharpoonup\varepsilon)} A_{\mathbf{v}}^h$$

$$B_{\mathbf{p}}^\alpha A_{\mathbf{v}}^{wh} = B_{\alpha(\varepsilon\leftharpoonup w)} A_{\mathbf{v}}^h$$

- For $*$-invariant idempotents $\Lambda \in Cocom(H)$ and $\lambda \in Cocom(H^*)$

$$H = -\sum_{\mathbf{v}} A_{\mathbf{v}}^\Lambda - \sum_{\mathbf{p}} B_{\mathbf{p}}^\lambda$$

is an exactly solvable Hamiltonian.
Remark: such Λ and λ exist for C^*-quantum groupoid.

Finite Groupoid

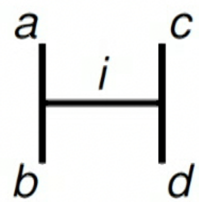
For a finite groupoid G (a category with finitely many morphisms such that each morphism is invertible), its groupoid algebra $\mathbb{C}(G)$ is a finite quantum groupoid.

$$gh = \text{composition whenever well defined or 0 otherwise}$$
$$\eta = \sum \mathrm{id}_x, \quad id_x \text{ is the identity morphism of object } x$$
$$\Delta(g) = g \otimes g$$
$$\varepsilon(g) = 1$$
$$S(g) = g^{-1}$$

Kitaev-Kong Quantum Groupoid $H_{\mathcal{C}}$

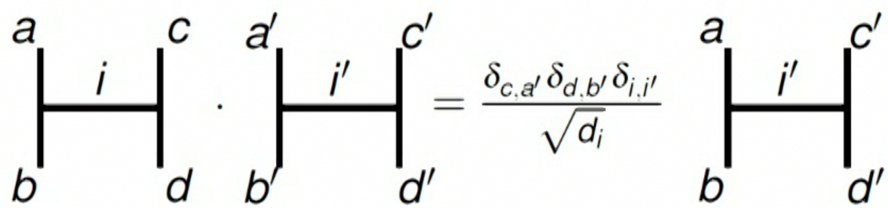
Kitaev and Kong constructed a C^*-quantum groupoid $H_{\mathcal{C}}$ from a unitary fusion category $\mathcal{C}$. For simplicity, we assume $\mathcal{C}$ is mulitiplicity free and self-dual.

- As vector space, $H_{\mathcal{C}}$ is spanned by

$e^{ab}_{i;cd} =$ 

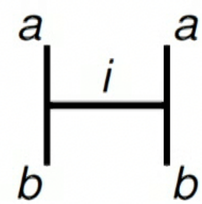
where $a, b, c, d, i \in I_{\mathcal{C}}$.

- Multiplication



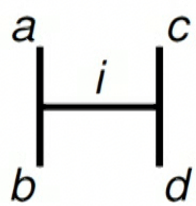
- Unit

$$\eta = \sum_{a,b,i} \sqrt{d_i}$$

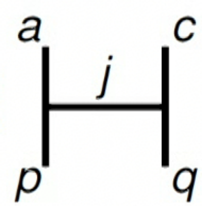


- Comultiplication

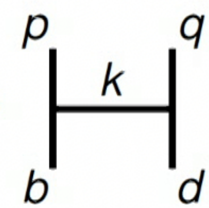
$$\Delta \left(\right.$$



$$\left. \right) = \sum_{\substack{j,k \\ p,q}} \frac{\sqrt{d_i d_p d_q}}{\sqrt{d_b d_c}} F^{iap}_{k,jk} F^{idq}_{j,kc}$$

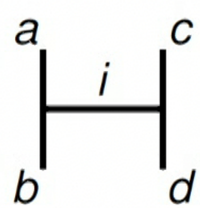


$$\otimes$$



- Counit

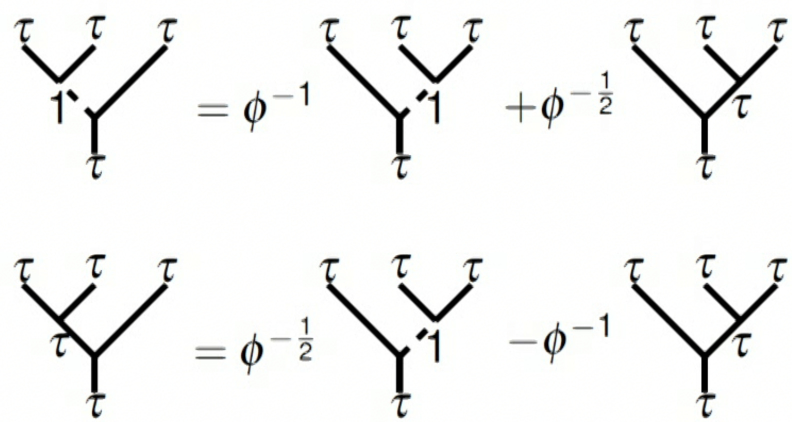
$$\varepsilon \left(\right.$$



$$\left. \right) = \delta_{a,b} \delta_{c,d} \delta_{i,1}$$

Fibonacci Category $\mathcal{F}$

- Simple objects: 1 and τ;
- Self dual: $1^* = 1$ and $\tau^* = \tau$;
- Quantum dimension: $d_1 = 1$ and $d_\tau = \phi = \frac{1+\sqrt{5}}{2}$;
- Fusion rule: $\tau^2 = 1 + \tau$.
- F-moves:

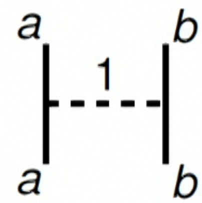


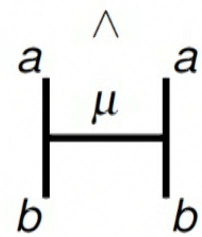
$H_{\mathcal{F}} = M_{2\times 2} \oplus M_{3\times 3}$ is 13 dimensional quantum groupoid with basis $\{e^{\mathbf{11}}_{\mathbf{1};\mathbf{11}}, e^{\mathbf{11}}_{\mathbf{1};\tau\tau}, e^{\tau\tau}_{\mathbf{1};\mathbf{11}}, e^{\tau\tau}_{\mathbf{1};\tau\tau}, e^{\tau\tau}_{\tau;\tau\tau}, e^{\tau\tau}_{\tau;\mathbf{1}\tau}, e^{\tau\tau}_{\tau;\tau\mathbf{1}}, e^{\mathbf{1}\tau}_{\tau;\tau\tau}, e^{\tau\mathbf{1}}_{\tau;\tau\tau}, e^{\tau\mathbf{1}}_{\tau;\tau\mathbf{1}}, e^{\mathbf{1}\tau}_{\tau;\mathbf{1}\tau}, e^{\tau\mathbf{1}}_{\tau;\mathbf{1}\tau}, e^{\mathbf{1}\tau}_{\tau;\tau\mathbf{1}}\}$. $H_{\mathcal{F}}$ is actually the Temperley-Lieb-Jones algebra TLJ_4 generated by

$$\begin{aligned} u_1 =& \phi e^{\mathbf{11}}_{\mathbf{1};\mathbf{11}} + \phi^{\frac{3}{2}} e^{\mathbf{1}\tau}_{\tau;\mathbf{1}\tau}, \\ u_2 =& \phi^{-1} e^{\mathbf{11}}_{\mathbf{1};\mathbf{11}} + \phi^{-\frac{1}{2}} e^{\mathbf{11}}_{\mathbf{1};\tau\tau} + \phi^{-\frac{1}{2}} e^{\tau\tau}_{\mathbf{1};\mathbf{11}} + e^{\tau\tau}_{\mathbf{1};\tau\tau} + \phi^{-\frac{1}{2}} e^{\mathbf{1}\tau}_{\tau;\mathbf{1}\tau} + e^{\mathbf{1}\tau}_{\tau;\tau\mathbf{1}} \\ &+ e^{\tau\mathbf{1}}_{\tau;\mathbf{1}\tau} + \phi^{\frac{1}{2}} e^{\tau\mathbf{1}}_{\tau;\tau\mathbf{1}}, \\ u_3 =& \phi e^{\mathbf{11}}_{\mathbf{1};\mathbf{11}} + \phi^{\frac{1}{2}} e^{\tau\mathbf{1}}_{\tau;\tau\mathbf{1}} + e^{\tau\mathbf{1}}_{\tau;\tau\tau} + e^{\tau\tau}_{\tau;\tau\mathbf{1}} + \phi^{-\frac{1}{2}} e^{\tau\tau}_{\tau;\tau\tau}. \end{aligned}$$

subject to the relations $u_k^2 = \phi u_k$, $u_k u_{k\pm 1} u_k = u_k$, $u_i u_j = u_j u_i$ if $|i-j| > 1$ and $1 - \phi^2 u_2 - \phi(u_1 + u_3) + \phi(u_1 u_2 + u_2 u_1 + u_2 u_3 + u_3 u_2) + \phi^2 u_1 u_3 - u_1 u_2 u_3 - u_3 u_2 u_1 - \phi(u_2 u_1 u_3 + u_3 u_1 u_2) + u_2 u_1 u_3 u_2 = 0$ (the 4th Jones-Wenzl idempontent).

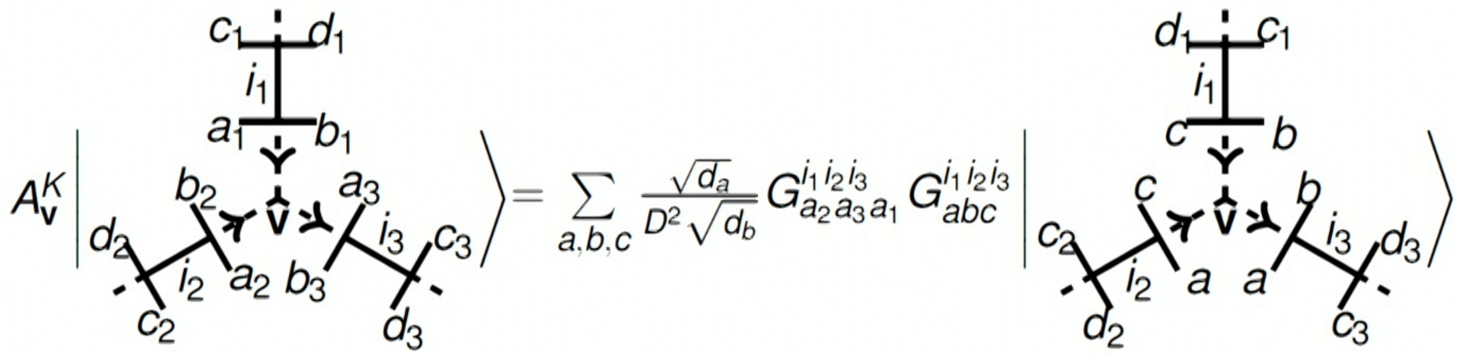
Solvable Hamiltonian for $H_{\mathcal{C}}$

- $$\Lambda = \frac{1}{D^2} \sum_{a,b} d_a d_b$$ 

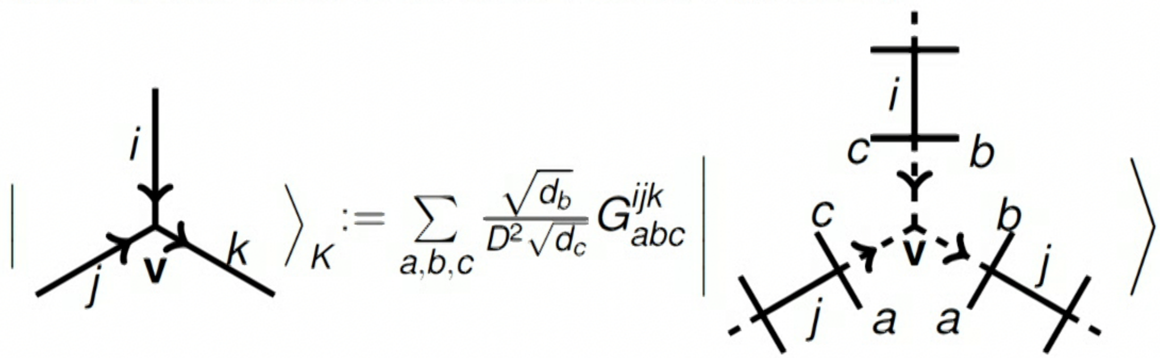
- $$\lambda = \frac{1}{D^2} \sum_{a,b,\mu} \sqrt{d_\mu}$$ 

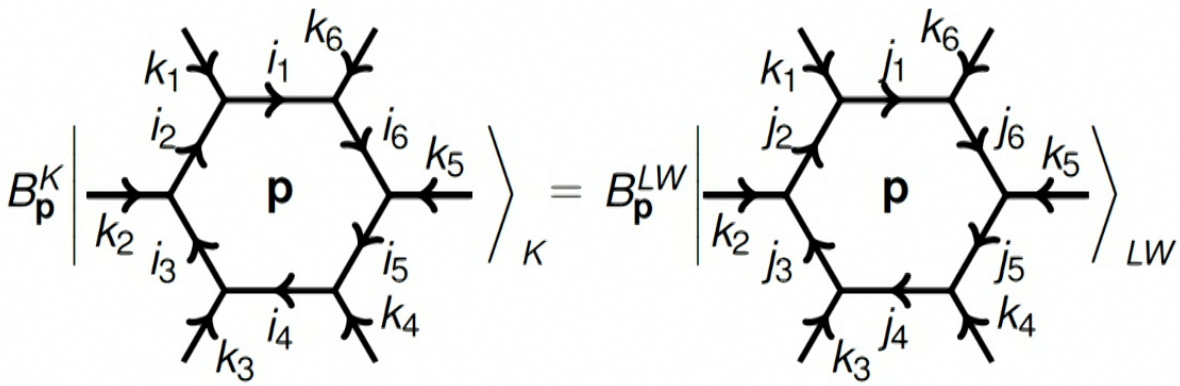
Exactly solvable Hamiltonian $H^K = -\sum_{\mathbf{v}} A^K_{\mathbf{v}} - \sum_{\mathbf{p}} B^K_{\mathbf{p}}$.

Ground States



$A_{\mathbf{v}}^K(\mathcal{L}^K)$ has a basis around vertex **v** given by





Theorem

Given a trivalent lattice Γ on a closed oriented surface Σ, the ground space $L^K(\Sigma, \Gamma)$ of the Kitaev model based on $H_{\mathcal{C}}$ is canonically isomorphic to the ground space $L^{LW}(\Sigma, \Gamma)$ of Levin-Wen Models based on $\mathcal{C}$. As a consequence, $L^K(\Sigma, \Gamma)$ is canonically isomorphic to the target space $Z_{TV}(\Sigma)$ of the Turaev-Viro TQFT based on $\mathcal{C}$.

Interest

- Non-semisimple Hopf algebra $\rightarrow$?
  Lattice model realization of Kuperberg invariant of 3-manifolds
- 3D lattice model $\leftarrow$? trialgebra

Thank You!