Title: From 3D TQFTs to 4D models with defects

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Abstract: I will explain a general strategy to lift (2+1)D topological phases, in particular string nets, to (3+1)D models with line defects. This allows a systematic construction of (3+1)D topological theories with defects, including an improved version of the Walker-Wang Model. It has also an interesting application to quantum gravity as it leads to quantum geometry realizations for which all geometric operators have discrete and bounded spectra. I will furthermore comment on some interesting (self-) duality relations that emerge in these constructions.

Lifting (2+1) TQFTs to (3+1) theories with line defects

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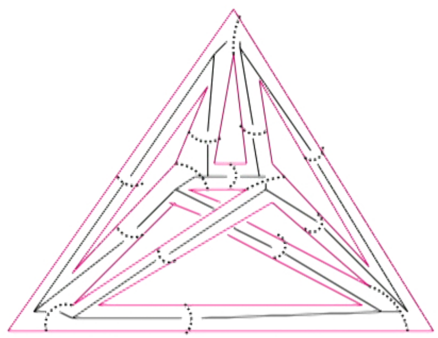

Delcamp-BD, 1606.02384 [hep-th] JMP 2017;

BD, 1701.02037 [hep-th], JHEP 2017

Kitaev models et al
PI, Aug 2017

Motivation

(2+1)D TQFT's — Kitaev model, string nets, Hopf algebra gauge theory, . . .

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(3+1)D theory — ?

Motivation

(2+1)D TQFT's	Kitaev model, string nets, Hopf algebra gauge theory, …	Hilbert space (code space), Operators
(3+1)D theory	(3+1) TQFTs with line defects eg. improved Walker-Wang, generalizations	Hilbert space (code space), Operators

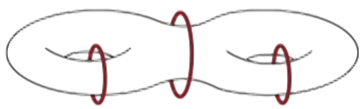
In particular: braiding needs 2D surfaces. How to (conveniently) implement it in (3+1)D?

In addition: number of reasons from quantum gravity.
A new quantum geometry realization:
Hilbert spaces are finite, all observables have discrete spectra, new gauge invariant bases.

Strategy: from (2+1)D TQFT to a (3+1)D theory with line defects

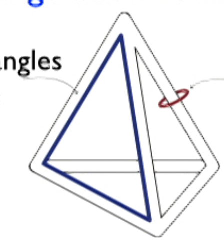
[Delcamp, BD: JMP 2017]

(2+1)D TQFT



assigns degrees of freedom
to non-contractible curves
on a surface

(3+1)D TQFT: 3-sphere with one-skeleton of (tetrahedral) triangulation removed

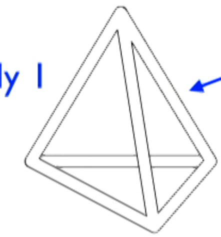


want to assign degrees of
freedom
to curves around edges of
triangulation

Use (2+1) D theory to assign state space to a 3D triangulation.
But impose (contractibility/ flatness) constraints associated to curves around triangles.

Heegaard splitting and diagrams

Handlebody 1

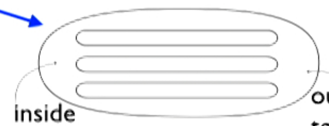


Compact 3D manifold
(e.g. 3-sphere)

≡

Handlebody 1 ∪ Handlebody 2

Handlebody 2

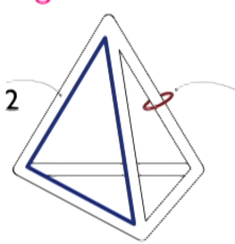


Handlebody 1 ∩ Handlebody 2

≡

Heegaard surface

two-handle cycle (t-cycle):
contractible in handlebody 2



one-handle cycle (e-cycle):
contractible in handlebody 1

A Heegaard diagram is a Heegaard surface decorated with generating basis of one-handle cycles and two-handle cycles.

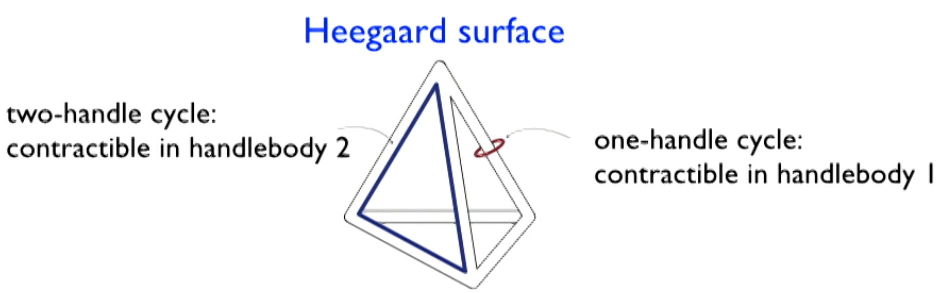
Heegaard diagrams encode uniquely topology of 3D manifold.

Heegaard diagrams

Heegaard diagrams can be constructed from a triangulation of the 3D manifold.

Set of cycles around triangles generates (over-completely) all curves that are contractible even if we do take out the one-skeleton of the triangulation.

Thus it is sufficient to impose flatness constraints for the cycles around the triangles.



Part II: Examples

Strategy

1. Hilbert space, operators and bases for a closed surface.

2. Apply this to a Heegaard surface.

3. Impose constraints for 2-handle cycles and find operators and bases consistent with these constraints.

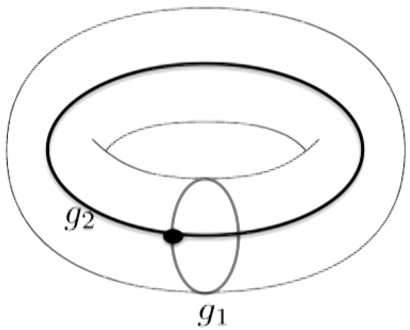
BF theory / Kitaev model

string nets with non-modular fusion category Rep(G)

BF theory / Kitaev model

string nets with non-modular fusion category Rep(G)

Hilbert space:
gauge invariant wave functions of flat G-connection



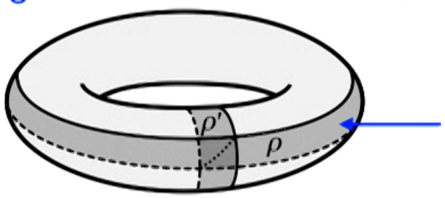
$$\delta_e(g_1 g_2 g_1^{-1} g_2^{-1})\ \psi_{\text{inv}}(g_1, g_2)$$

A basis?

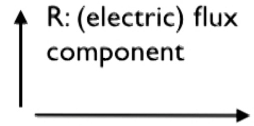
BF theory / Kitaev model

string nets with non-modular fusion category

(2+1)D Hilbert space:
gauge invariant wave functions of flat G-connection



(maximal commuting set of) Operators:
projective Ribbon operators labeled by representations of Drinfel'd double of G $\rho = (C, R)$



BF theory / Kitaev model

string nets with non-modular fusion category Rep(G)

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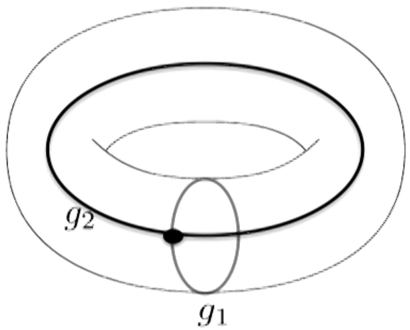
gauge invariant wave functions of flat G-connection



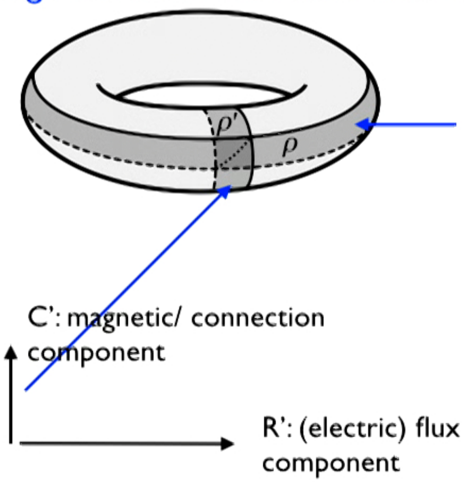
$$\delta_e(g_1 g_2 g_1^{-1} g_2^{-1})\ \psi_{\rm inv}(g_1, g_2)$$

A basis?

BF theory / Kitaev model

string nets with non-modular fusion category

(2+1)D Hilbert space:
gauge invariant wave functions of flat G-connection



(maximal commuting set of) Operators:
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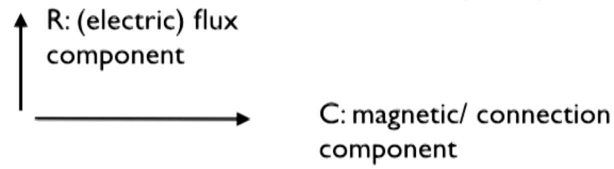
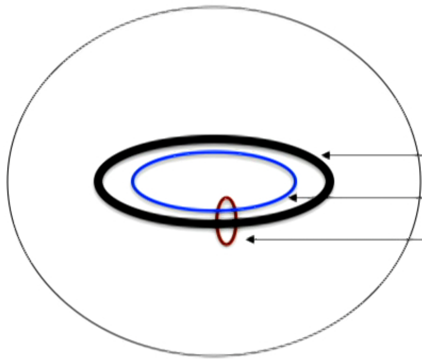


Transformation between basis:
S-matrix: $S_{\rho,\rho'}$

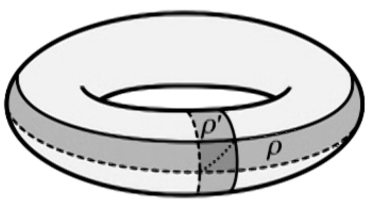
Interpreted as (3+1)D state space



defect

contractible loop, impose $\delta_e(h_t)$

non-contractible loop



Impose constraint or projector:

ρ-basis: C =triv, R =irrep of group G

spin network basis

ρ'-basis: C =arbitrary, sum over R

curvature basis:
new gauge invariant basis

(gauge invariant)
group
Fourier-transform
from S-matrix

Lifting (2+1)D to (3+1)D

[Delcamp, BD: JMP 2017]

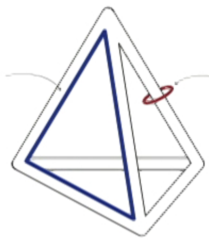
(2+1)D	**(3+1)D**
Fusion basis adjusted to (thickened) triangulation	curvature basis (for 3-sphere)
Fusion basis adjusted to (thickened) dual graph	spin network basis
General fusion basis on Heegard surface	general basis
Ribbon operators along e-cycles	(magnetic) Wilson loop
Ribbon operators along t-cycles	(electric) surface (t'Hooft) operator

Fourier-transform from S-matrix

Generalizations

- line defects along triangulation and
- line defects along dual graph

allow degrees of freedom
associated to triangles/
edges of dual graph



e-cycle measures curvature/
magnetic dof associated to edges

↓

weaken triangle constraints /
projectors

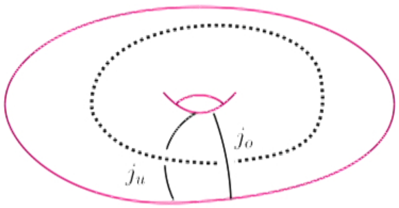
$$\delta_e(h_t) \quad \rightarrow \quad \delta_N(h_t)$$

N: normal subgroup of G

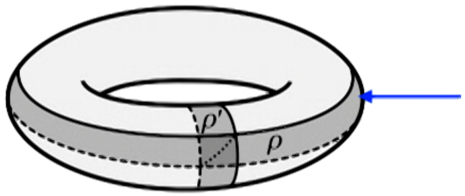
- line defects along triangulation and
- line defects along dual graph

'interaction': (3+1) interpretation?

(2+1)D Turaev-Viro 'code space'/ string nets with modular fusion category C



What to expect?



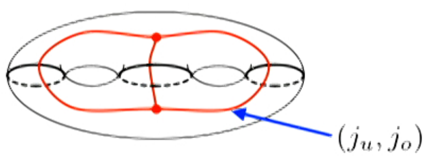
(maximal commuting set of) Operators:
Projective ribbon operators labeled by objects of Drinfel'd centre of C: $C \boxtimes C^{op}$

$\rho = (j_o, j_u)$

Impose constraint or projector:
trivializes one of the copies of the double

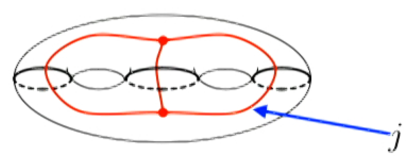
- (q-deformed) spin network basis: definition of Walker-Wang model
- curvature basis: diagonalizes Walker-Wang Hamiltonian

Basis for TV -TQFT



projector

Basis for WRT -TQFT



Quantization of Chern-Simon theory.

classical phase space:
quantum deformed (3+1) lattice Yang Mills = (2+1) Chern-Simons on Heegard surface

[Frolov; Riello]

Hilbert space for (2+1)D Turaev-Viro TQFT

here: for surfaces without punctures

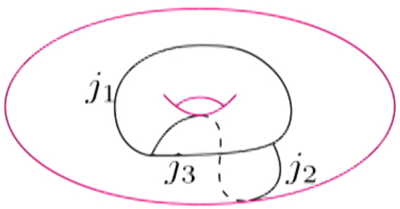
[Levin, Wen; Koenig, Kuperberg, Reichardt; Kirillov; BD, Geiller]

Kinematical (but gauge invariant) Hilbert space:

States based on spin-labelled three-valent graphs with $\mathrm{SU}(2)_\mathrm{k}$ coupling rules imposed on the nodes.

Admissible spins: $j = 0, \frac{1}{2}, 1, \ldots, \frac{\mathrm{k}}{2}$ labelling undirected edges of the graph.

Coupling rules: $i \leq j + k, \quad j \leq i + k, \quad k \leq i + j, \quad i + j + k \in \mathbb{N}, \quad i + j + k \leq \mathrm{k}$

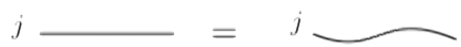


Hilbert space for (2+1)D Turaev-Viro TQFT

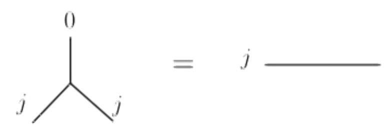
Physical Hilbert space - impose 'flatness' constraints:

Flatness constraint are imposed as equivalence relations between graph states:

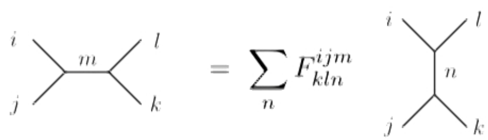
Strands can be (isotopically) deformed.



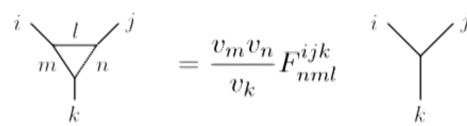
Strands with trivial spin can be omitted.



2-2 Pachner move. Involving the F-symbol.



3-1 Pachner move. Involving the F-symbol.

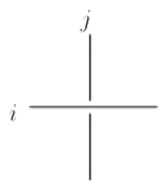


$$v_j = (-1)^j \sqrt{d_j}$$

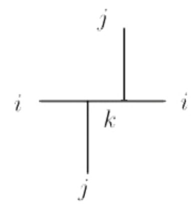
Rather involved now:
Finding a basis of independent states and operators consistent with equivalence relations.
We need a) braiding and b) vacuum strands to define these.

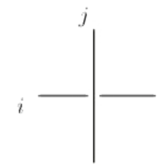
a) Braiding

Strands can cross each other. Such crossings can be resolved using the R-matrix of $\mathrm{SU}(2)_{\mathrm{k}}$.

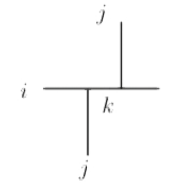


$$= \sum_k \frac{v_k}{v_i v_j} R_k^{ij}$$





$$= \sum_k \frac{v_k}{v_i v_j} (R_k^{ij})^*$$



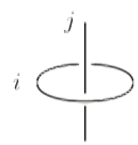
We can thus define the so-called s-matrix as the evaluation of the Hopf link.

(Planar graphs are equivalent to a number times the empty graph. This number is called the evaluation of the planar graph.)

$s_{ij} :=$  gives

$$s_{jk} = (-1)^{2k+2j} \frac{\sin\left(\frac{\pi}{\mathrm{k}+2}(2j+1)(2k+1)\right)}{\sin\left(\frac{\pi}{\mathrm{k}+2}\right)}$$

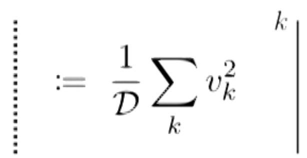
An important identity:



$$= \frac{s_{ij}}{s_{0j}}$$

b) Vacuum strands

Vacuum strands are defined as weighted sum over strands labelled by admissible spins:

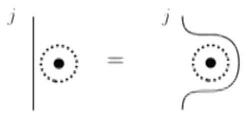


$$v_j = (-1)^j \sqrt{d_j}$$

$$\mathcal{D} := \sqrt{\sum_j v_j^4}$$ total quantum dimension

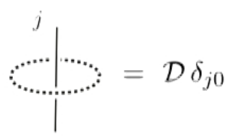
A vacuum loop is similar to a $\delta(g)$ function. Wilson lines (strands) can be deformed across a region enclosed by a vacuum loop.

Sliding property:



Vacuum loops encircling a strand force the associated spin label to be trivial.

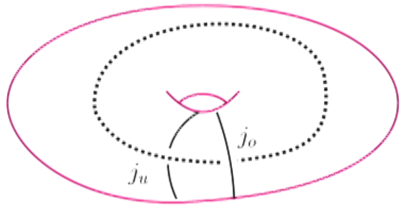
Killing property:



Hilbert space for (2+1)D: Bases

[Kohno 1992; Alagic et al 2010]

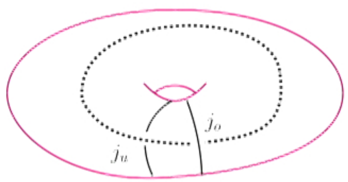
For the torus:



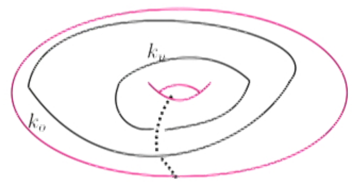
Basis states parametrized by two spins (j_u, j_o) labelling an under- and over-crossing strand.

We will see that this basis diagonalizes over- and under-crossing Wilsonloops parallel to the vacuum loop.

S-transformation (generalized Fourier transformation):



$$= \sum_{k_o, k_u} \mathbb{S}_{j_o j_u, k_o k_u}$$



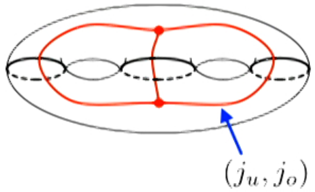
$$\mathbb{S}_{j_o j_u, k_o k_u} = \frac{1}{\mathcal{D}^2} s_{j_o k_o} s_{j_u k_u}$$

Hilbert space for (2+1)D: Bases

[Kohno 1992; Alagic et al 2010]

For g>1 surface:

To each pant decomposition of the surface we can associate a basis.



These bases states include a

- set of vacuum loops
- over-crossing graph (dual to vacuum loops)
- under-crossing graph (dual to vacuum loops).

From (2+1)D to (3+1)D

We discussed:

- choice of basis for (2+1)D Hilbert space
- consistent operators: under- and over-crossing Wilson loops.

For these constructions braiding relations play a very important role.
Using the encoding of a 3D manifold into a Heegaard surface we can export these braiding relations to the (3+1)D theory.

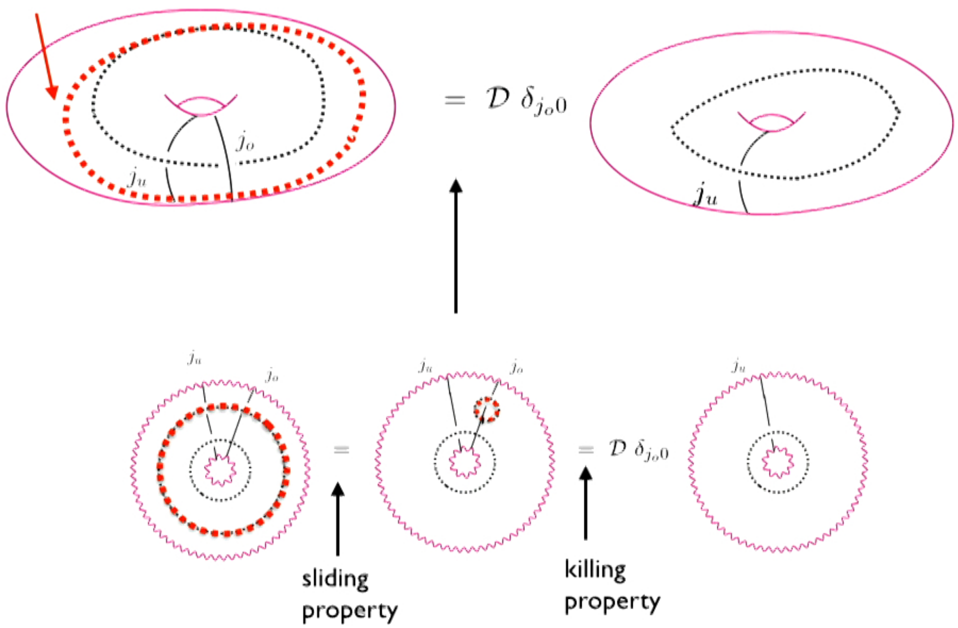
To proceed:

a) Construct bases for Heegaard surface.
b) Impose constraints.
c) Find operators preserving constraints.

Example: defect loop in 3-sphere

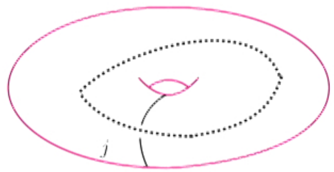
The corresponding Heegaard surface: a torus.
Flatness constraint along equator of this torus.

flatness constraint (over-crossing vacuum loop)
along equator

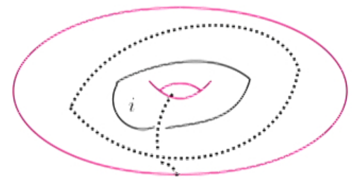


The flatness constraints surpress the over-crossing graph copy.

Example: defect loop in 3-sphere



$$= \sum_i \frac{s_{ji}}{\mathcal{D}}$$



Here over-crossing graph copy given by vacuum loop.

Spin network basis: based on graph dual to curvature defects.

(quantum group) Fourier transform

Curvature basis: based on graph along curvature defects

Diagonalizes (under-crossing) Wilson loop around equator.

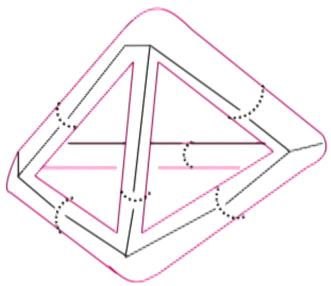
Measure area (of surface spanned by curvature defect).

Diagonalizes (under-crossing) Wilson loops around meridian.

Measures curvature (of curvature defect).

Curvature basis for general 3D triangulation

- Choose pant-decomposition adjusted to the one-skeleton of the triangulation
- After imposing flatness constraints: curvature basis.



Under-crossing graph along one-skeleton of triangulation which can be freely labelled by spins: labels of the curvature basis.
Over-crossing graph given by vacuum loops around triangles.

- (Curvature or Crane-Yetter) vacuum state:
  trivial spins associated to all edges of (triangulation) graph.

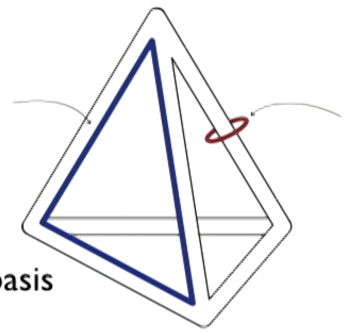
  Non-degenerate vacuum state for all topologies.
  Crane-Yetter invariant is ‘trivial’.

Rather hard to see in Walker-Wang formulation.

[Keyserlingk et al 2013]

Operators for the (3+1)D theory

Under-crossing Wilson loops preserve flatness constraints.



Wilson loops around triangles.

- diagonalized by spin network basis
- measure area of triangles:

Wilson loops around edges.

- diagonalized by curvature basis
- measures curvature around edges

For normalized k-Wilson loop:

$$\frac{\sin\left(\frac{\pi}{k+2}(2j+1)(2k+1)\right)\sin\left(\frac{\pi}{k+2}\right)}{\sin\left(\frac{\pi}{k+2}(2k+1)\right)\sin\left(\frac{\pi}{k+2}(2j+1)\right)} \xrightarrow{k\to\infty} 1-\frac{8}{3}\,j(j+1)\,k(k+1)\,\left(\frac{\pi}{k+2}\right)^2$$

Operators for the (3+1)D theory

Under-crossing Wilson loops encode curvature and area operators.

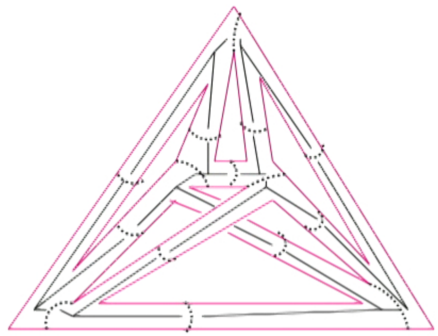
Spectra are discrete and bounded and coincide:

$$\frac{\sin\left(\frac{\pi}{\mathrm{k}+2}(2j+1)(2k+1)\right)\sin\left(\frac{\pi}{\mathrm{k}+2}\right)}{\sin\left(\frac{\pi}{\mathrm{k}+2}(2k+1))\right)\sin\left(\frac{\pi}{\mathrm{k}+2}(2j+1))\right)}$$

A self-dual quantum geometry.

Examples with even more self-duality

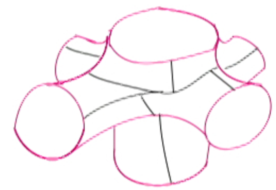
quantum-quantum 4-simplex



Curvature basis for 4-simplex.
(Over-crossing graph copy, which is given by vacuum loops around triangles, is suppressed.)

Spin network basis for 4-simplex.

quantum-quantum 3-torus



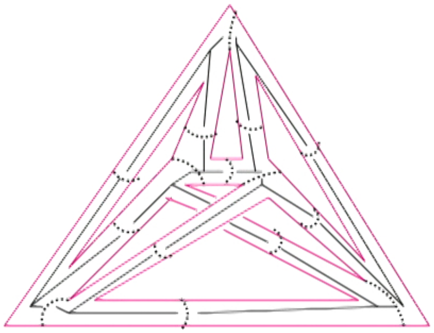
Curvature basis for 3 torus with cubical lattice.
(Over-crossing graph copy and vacuum loops are surpressed.)

Spin network basis for 3-torus.
(With Vacuum loops suppressed)

Two dual bases for the Walker-Wang model.
Here operators: under-crossing Wilson-loops on Heegard surface.

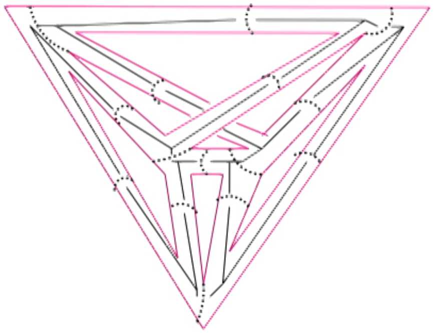
Crane-Yetter simplex amplitude:

vacuum state expanded in SNW basis



Curvature basis:
j=0 for all 15 labels

$$= \sum_j \{15j\}_q$$



Spin network basis:
all allowed labels

Conclusions

- general technique to lift Hilbert spaces and operators for a (2+1)D TQFT to (3+1)D theory with line defects. We discussed in more detail:
  - Turaev-Viro for modular fusion category: Crane-Yetter with curvature defects
  - BF theory (Turaev Viro for Rep(G)): 4D BF theory with curvature defects
- provides a straightforward analyses of excitations and operator (algebra) of (3+1)D theories
- e.g.: generalization of fusion basis to (3+1)D yields an entire family of new bases

Outlook

- generalizations ala [Baerenz, Barrett 2016]
  - weaken flatness constraints for triangles
  - allows for degenerate ground state (non-trivial 4D invariants)
  - introduces torsion degrees of freedom in addition to curvature defects?

- impose a different excitation content
  - start with Dijkgraaf-Witten models
  - allow for torsion defects instead of curvature defects

- consider boundaries and boundary excitations [Keyserlingk et al PRB 2013, ...]
  (in a 'natural' manner: compression bodies)