Title: Topological defects and higher-categorical structures

Date: Aug 01, 2017 03:00 PM

URL: http://pirsa.org/17080003

Abstract: I will discuss some (higher-)categorical structures present in three-dimensional topological field theories that include topological defects of any codimension. The emphasis will be on two topics:
(1) For Reshetikhin-Turaev type theories, regarded as 3-2-1-extended TFTs, I will explain why codimension-1 boundaries and defects form bicategories of module categories over suitable fusion categories.
In the case of defects separating three-dimensional regions supporting the same theory, the relevant fusion category A is the modular tensor category underlying that theory, while for defects separating two theories of Turaec-Viro type with underlying fusion categories A_1 and A_2, respectively, A is the the Deligne product $A_1 \boxtimes A_2^{op}$.
(2) I will indicate the building blocks of a generalization of the TV-BW state-sum construction to theories with defects. Making use of ends and coends, various aspects of this construction can be formulated without requiring semisimplicity.

TOPOLOGICAL DEFECTS AND HIGHER-CATEGORICAL STRUCTURES

KARLSTADS UNIVERSITET

PI

1_8_2017

Motivation

THEME : *3-d TFT with defects of any codimension*

POSSIBLE MOTIVATIONS :

- ☞ TFT with substructures / on stratified spaces
- ☞ gapped interfaces /
  topological line defects in 2+1-dimensional topological orders

Motivation

Topological defects

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POSSIBLE MOTIVATIONS :

- ☞ TFT with substructures / on stratified spaces
- ☞ gapped interfaces /
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- ☞ defects in general quantum field theory
- ☞ applications to 2-d conformal field theory

Warmup: defects in QFT

Defects in QFT

- codimension-1 defect QFT_1 | QFT_2

  = interface separating region supporting QFT_1 from region supporting QFT_2

  - natural part of the structure of a quantum field theory
  - physical boundaries as special case

    QFT_1 |

Defects in QFT

Topological defects

☞ codimension-1 defect QFT_1 | QFT_2

= interface separating region supporting QFT_1 from region supporting QFT_2

⇝ natural part of the structure of a quantum field theory

⇝ physical boundaries as special case

☞ *topological* defect : correlators do not change when deforming the defect

without crossing other substructures

☞ natural wish list for topological defects :

⇝ codimension-2 defects DEF_1 | DEF_2 etc

⇝ allows for natural formulation in terms of higher categories

⇝ dual defect via orientation reversal

⇝ fusion product of defects via moving two defects to coincidence

⇝ transparent defect as unit for fusion product of defects between equal phases

⇝ categories with monoidal and rigid structures

Symmetries from defects

wish list continued :

☞ subclass : *invertible* topological defects : $D \otimes D^{\vee} \cong \mathbf{1} \cong D^{\vee} \otimes D$

Symmetries from defects

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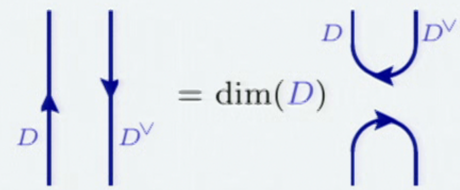
- basic property :



drawn for $d=2$

$\dim(D) = \pm 1$

⇝ identity of correlators when applied locally in any configuration of fields & defects

- invertible defects form a group under fusion
- act on all data of the theory as a symmetry group

Symmetries from defects

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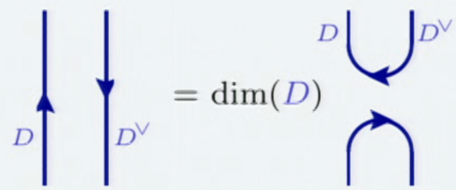

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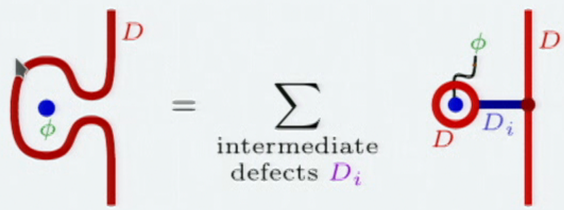
- wrapping of a topological defect around a bulk field :



i.e. bulk field turned into disorder field(s)

Symmetries from defects

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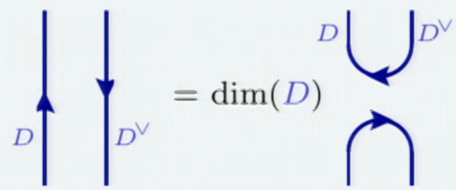

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- subclass : *duality* defects :

  additional wrapping with dual defect turns disorder field back to bulk field
  - happens if and only if $D \otimes D^{\vee} =$ direct sum of invertible defects
  - furnishes order-disorder duality
  - again action on all field theoretic quantities

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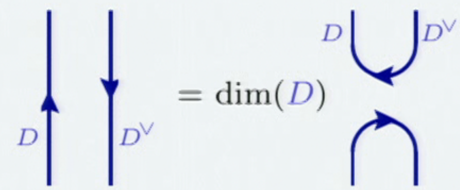

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known to be true for 2-d RCFT :

- defects form a rigid monoidal category
- symmetries and order-disorder dualities

F-Runkel-Schweigert 2002

F-Fjelstad-Runkel-Schweigert 2008

F-Fröhlich-Runkel-Schweigert 2007

(I) RT-type TFT with defects

Extended 3-d TFT

DEFINITION — **Cobordism bicategory**

- monoidal bicategory $\mathcal{C}obord_{3,2,1}$:
  - objects $=$ closed oriented 1-manifolds S
  - 1-morphisms $=$ spans $S \rightarrow M \leftarrow S'$
    with M oriented 2-manifold with boundary $\partial M = -S \sqcup S'$
  - 2-morphisms $=$ 3-manifolds with corners up to diffeomorphisms
  - tensor product $=$ disjoint union

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  - tensor product = disjoint union

DEFINITION — **2-vector spaces**

- monoidal bicategory $2\text{-}\mathcal{V}ect$:
  - objects = semisimple finite $\Bbbk$-linear abelian categories
  - 1-morphisms = $\Bbbk$-linear functors
  - 2-morphisms = $\Bbbk$-linear natural transformations
  - tensor product = Deligne product $\boxtimes$

Extended 3-d TFT

DEFINITION — **Extended 3-d TFT**

☞ 3-2-1 extended oriented topological field theory

$:=$ symmetric monoidal 2-functor $\mathbf{tft}_{3,2,1} : \mathcal{C}obord_{3,2,1} \longrightarrow 2\text{-}\mathcal{V}ect$

Extended 3-d TFT

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COMMENT

in more detail :

- closed oriented 1-manifold $S \longmapsto$ linear category $\mathbf{tft}(S)$
  - in particular for the empty 1-manifold : $\mathbf{tft}(\emptyset) = \mathcal{V}ect$
- span $S \rightarrow M \leftarrow S' \longmapsto$ linear functor $\mathbf{tft}(S) \xrightarrow{\mathbf{tft}(M)} \mathbf{tft}(S')$
  - in particular for closed 2-manifolds M :

    linear functor $\mathbf{tft}(M): \mathcal{V}ect \longrightarrow \mathcal{V}ect$ (thus vector space $\mathbf{tft}(M)(\Bbbk)$)
  - in particular for the empty 2-manifold : $\mathbf{tft}(\emptyset) = \Bbbk$
- 3-manifold with corners $\longmapsto$ linear natural transformation
  - in particular for closed 3-manifolds : linear map from $\Bbbk$ to $\Bbbk$

    (thus a number / an invariant)

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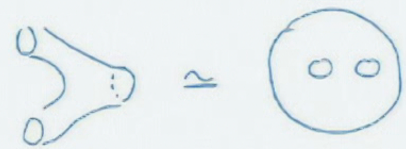
COMMENT

☞ category $\mathbf{tft}(\mathbb{S}^1)$ for the circle $\mathbb{S}^1$ is monoidal

↝ tensor product $\otimes$
furnished by pair-of-pants 1-morphism



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COMMENT

☞ category $\mathbf{tft}(\mathbb{S}^1)$ for the circle $\mathbb{S}^1$ is braided monoidal

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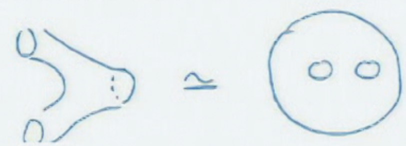
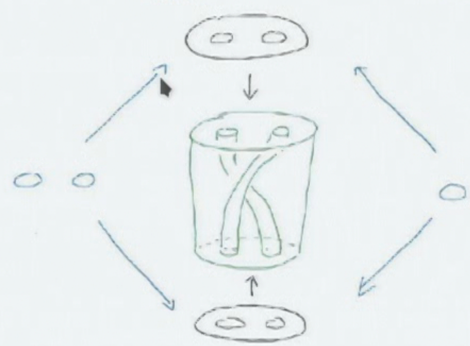


↝ braiding $\otimes \Longrightarrow \otimes^{\mathrm{op}}$
furnished by 2-morphism



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INFORMAL DEFINITION — Defect-ccobordism bicategory

☞ monoidal bicategory $\mathcal{C}obord^{\partial}_{\mathbf{3,2,1}}$:

- objects $=$ closed oriented 1-manifolds with marked points
- 1-morphisms $=$ spans with embedded marked 1-manifolds
- 2-morphisms $=$ 3-manifolds with corners up to diffeomorphisms with ...
- tensor product $=$ disjoint union

Extended 3-d TFT

DEFINITION — **Extended 3-d TFT**

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DEFINITION — **Extended 3-d TFT with defects**

☞ 3-2-1 extended oriented topological field theory with defects

:= symmetric monoidal 2-functor $\mathbf{tft}^{\partial}_{3,2,1} : \mathcal{C}obord^{\partial}_{3,2,1} \longrightarrow 2\text{-}\mathcal{V}ect$

RT-type TFT with defects

☞ Reshetikhin-Turaev - type TFT :

- input : a modular tensor category $\mathcal{C}$

RT-type TFT with defects

- Reshetikhin-Turaev - type TFT :
  - input : a modular tensor category $\mathcal{C}$
  - Wilson lines (ribbons) in three-manifolds labeled by objects of $\mathcal{C}$
  - insertions on Wilson lines / junctions labeled by morphisms of $\mathcal{C}$
  - 2-d cut-and-paste boundaries on which Wilson lines can end
  - state space for cut-and-paste boundary $=$ morphisms space $\mathrm{Hom}_{\mathcal{C}}(X, \mathbf{1})$

RT-type TFT with defects

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- RT-type TFT with boundaries and defects : replace $\mathcal{C}obord_{3,2,1}$ by $\mathcal{C}obord^{\partial}_{3,2,1}$

RT-type TFT with defects

Topological defects

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- RT-type TFT with boundaries and defects : replace $\mathcal{C}obord_{3,2,1}$ by $\mathcal{C}obord^{\partial}_{3,2,1}$
  - in particular three-manifolds with physical boundary and / or surface defects
  - 3-d bulk regions labeled by modular tensor categories $\mathcal{C}_1, \mathcal{C}_2, \ldots$
    (bulk Wilson lines in such a region labeled by objects of $\mathcal{C}_i$)
  - boundary Wilson lines and defect Wilson lines
  - several layers of insertions and of junctions

RT-type TFT with defects

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- RT-type TFT with boundaries and defects : replace $\mathcal{C}obord_{3,2,1}$ by $\mathcal{C}obord^{\partial}_{3,2,1}$
- Final goal : construct symmetric monoidal 2-functor $\mathcal{C}obord^{\partial}_{3,2,1} \longrightarrow 2\text{-}\mathcal{V}ect$

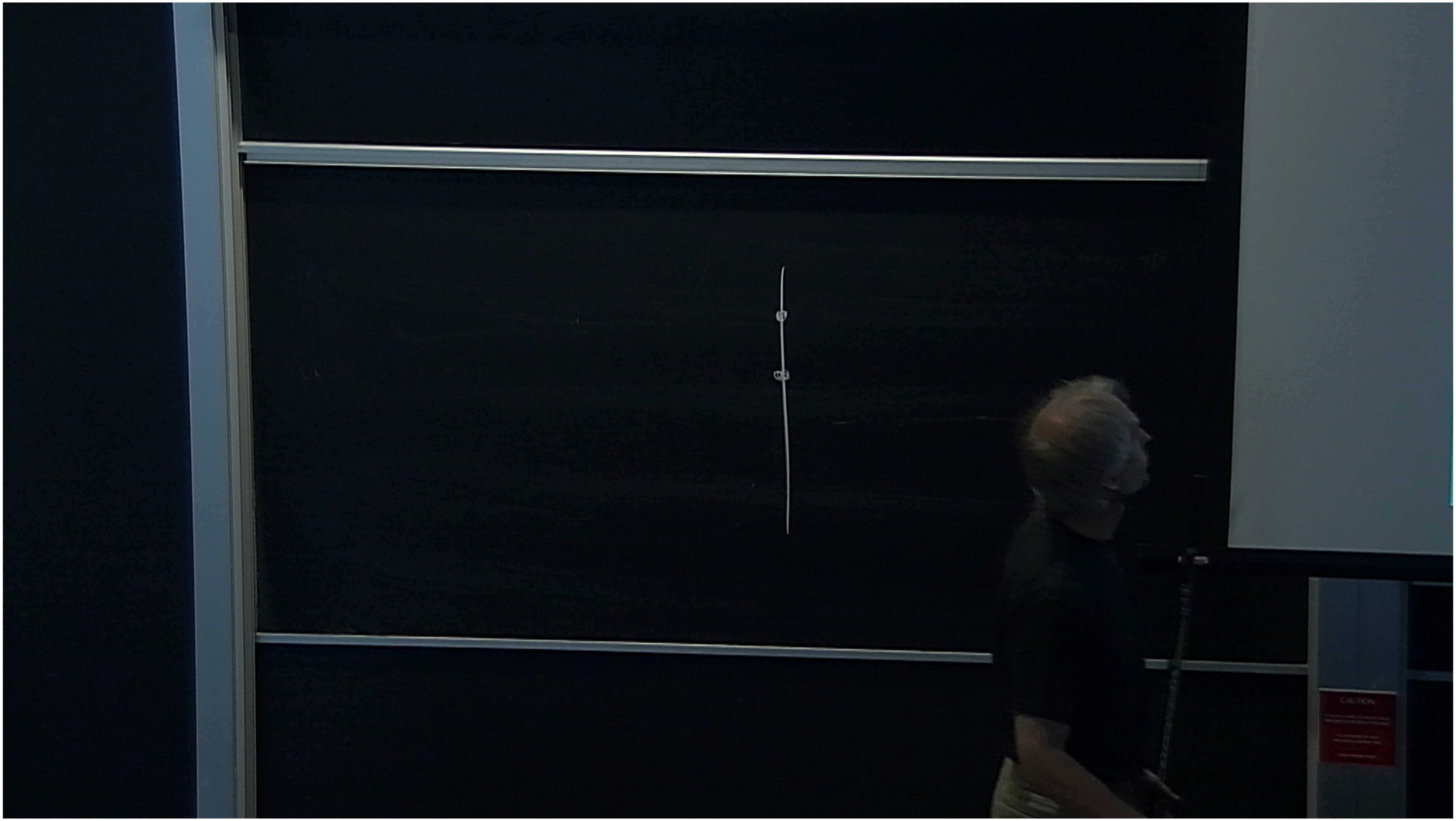
  in particular :
  - determine labels for physical boundaries / for surface defects
  - determine labels for boundary and defect Wilson lines and for insertions
- Conjecture : *these fit together to form bicategories of module categories*

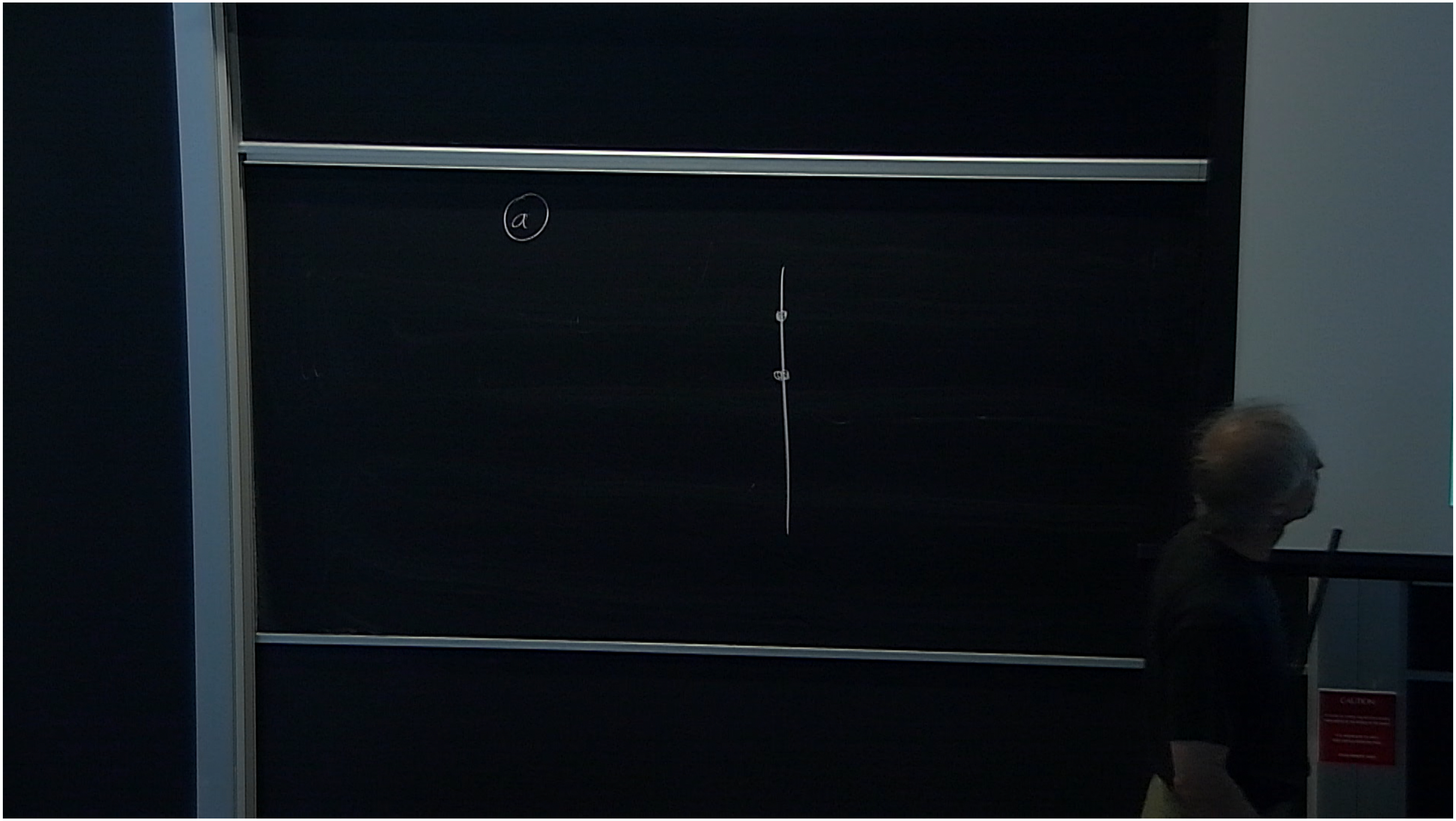
F-SCHWEIGERT-VALENTINO 2013

Labels for boundaries

- assume boundary "a" to some bulk region labeled by a modular tensor cateory $\mathcal{C}$
  - can contain boundary Wilson lines
  - Wilson line can contain insertions
  - such insertions can be composed

$\rightsquigarrow$ category $\mathcal{W}_a$ of Wilson lines on boundary a

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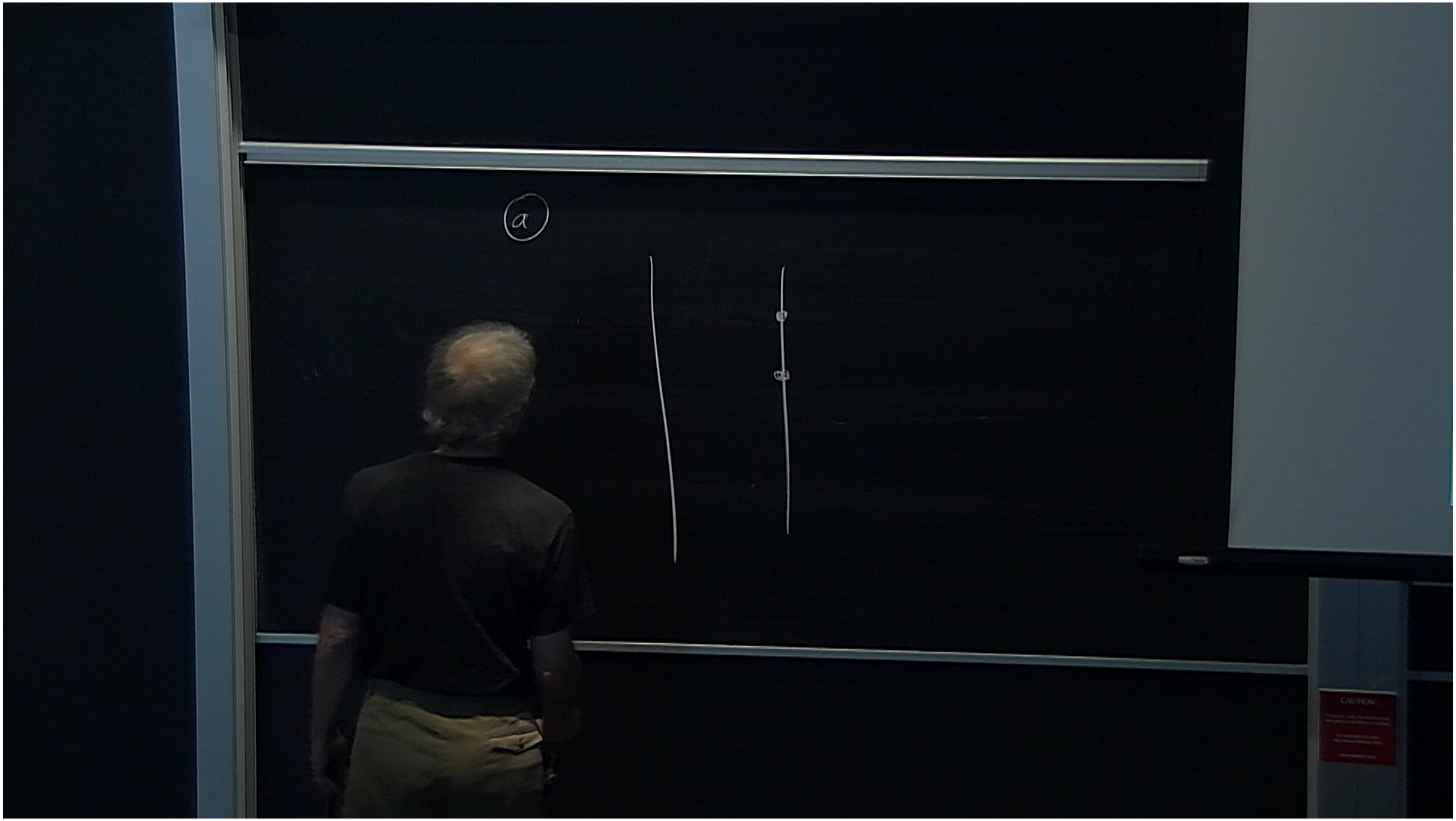
Labels for boundaries

- assume boundary "a" to some bulk region labeled by a modular tensor cateory $\mathcal{C}$
  - can contain boundary Wilson lines
  - Wilson line can contain insertions
  - such insertions can be composed
  - boundary Wilson lines can be fused and can be deformed

$\rightsquigarrow$ rigid monoidal category $\mathcal{W}_a$ of Wilson lines on boundary a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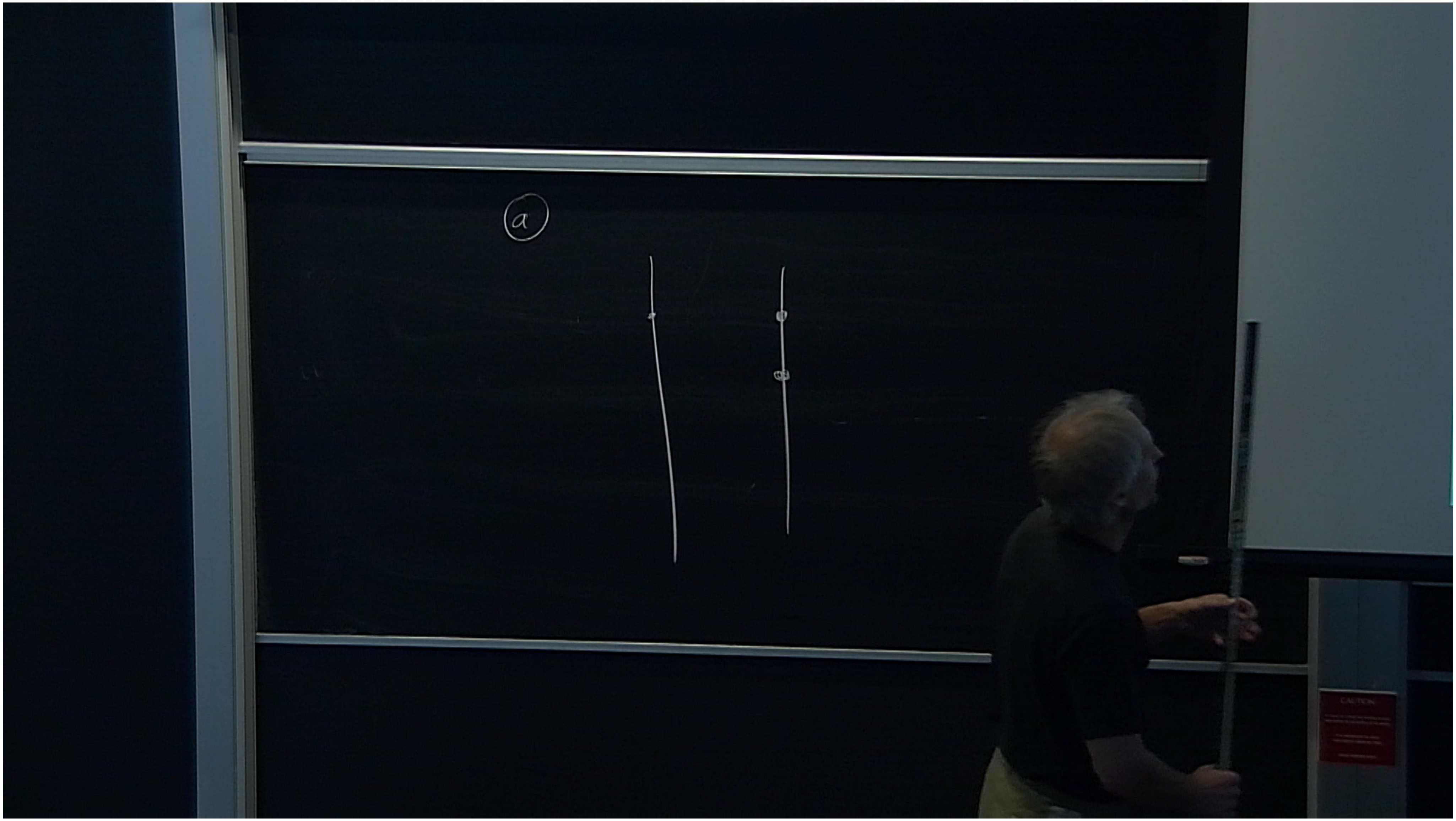
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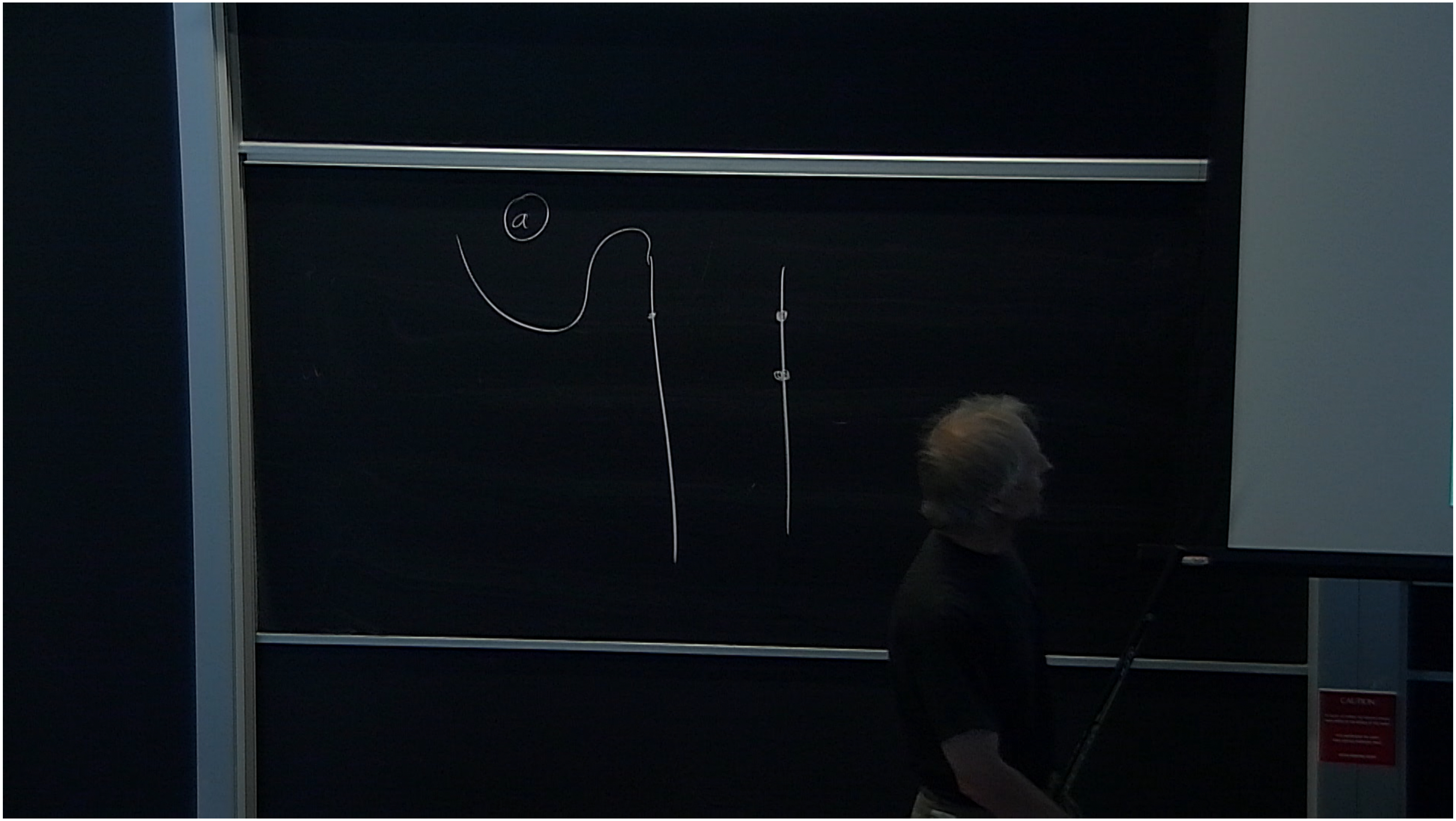
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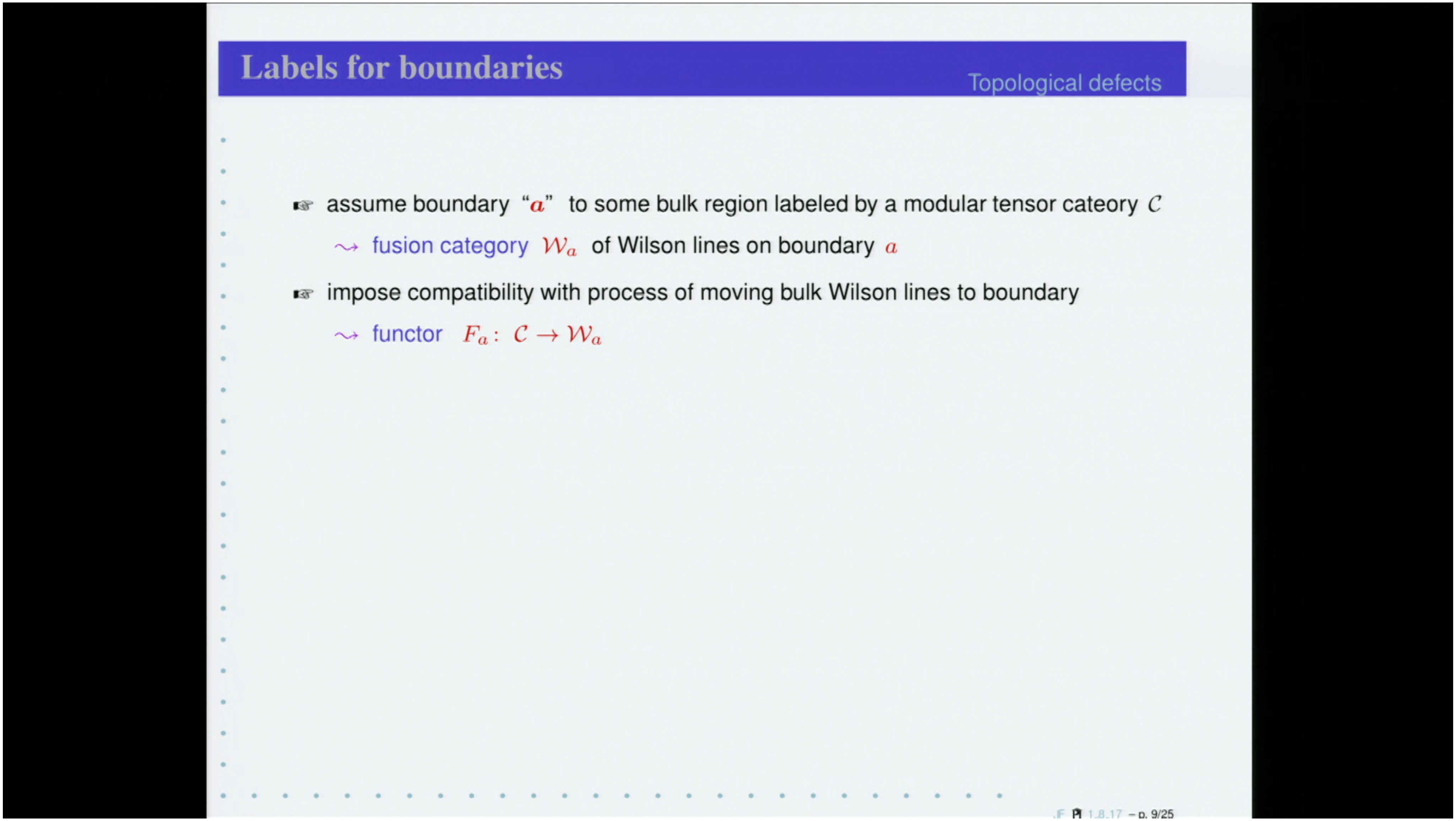
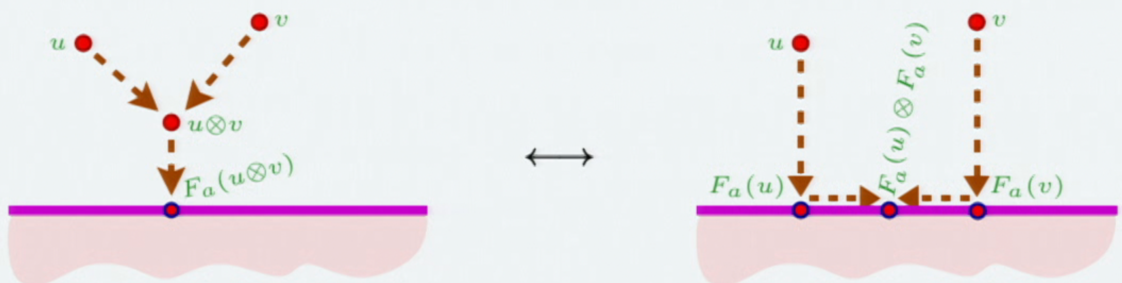

Labels for boundaries
Topological defects
☞ assume boundary "a" to some bulk region labeled by a modular tensor cateory $\mathcal{C}$
⇝ fusion category $\mathcal{W}_a$ of Wilson lines on boundary a
☞ impose compatibility with process of moving bulk Wilson lines to boundary
⇝ functor $F_a:\ \mathcal{C} \to \mathcal{W}_a$
– p. 9/25

Labels for boundaries

- assume boundary "a" to some bulk region labeled by a modular tensor cateory $\mathcal{C}$
  - $\rightsquigarrow$ fusion category $\mathcal{W}_a$ of Wilson lines on boundary a
- impose compatibility with process of moving bulk Wilson lines to boundary
  - $\rightsquigarrow$ functor $F_a\colon\ \mathcal{C} \to \mathcal{W}_a$
- impose compatibility of fusion in bulk and in boundary



$\rightsquigarrow$ monoidal structure $F_a(u \otimes_{\mathcal{C}} v) \xrightarrow{\cong} F_a(u) \otimes_{\mathcal{W}_a} F_a(v)$ coherently

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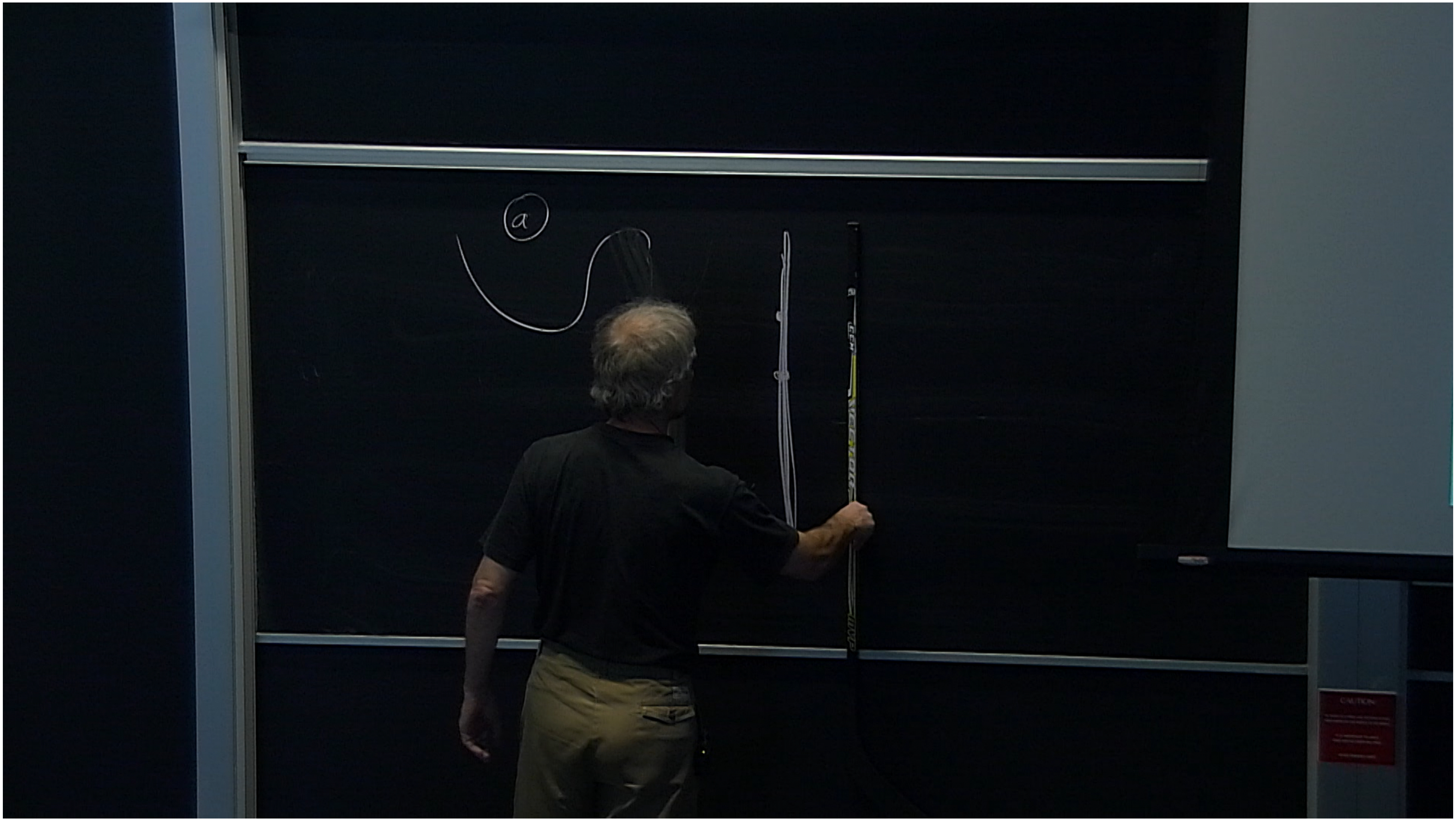
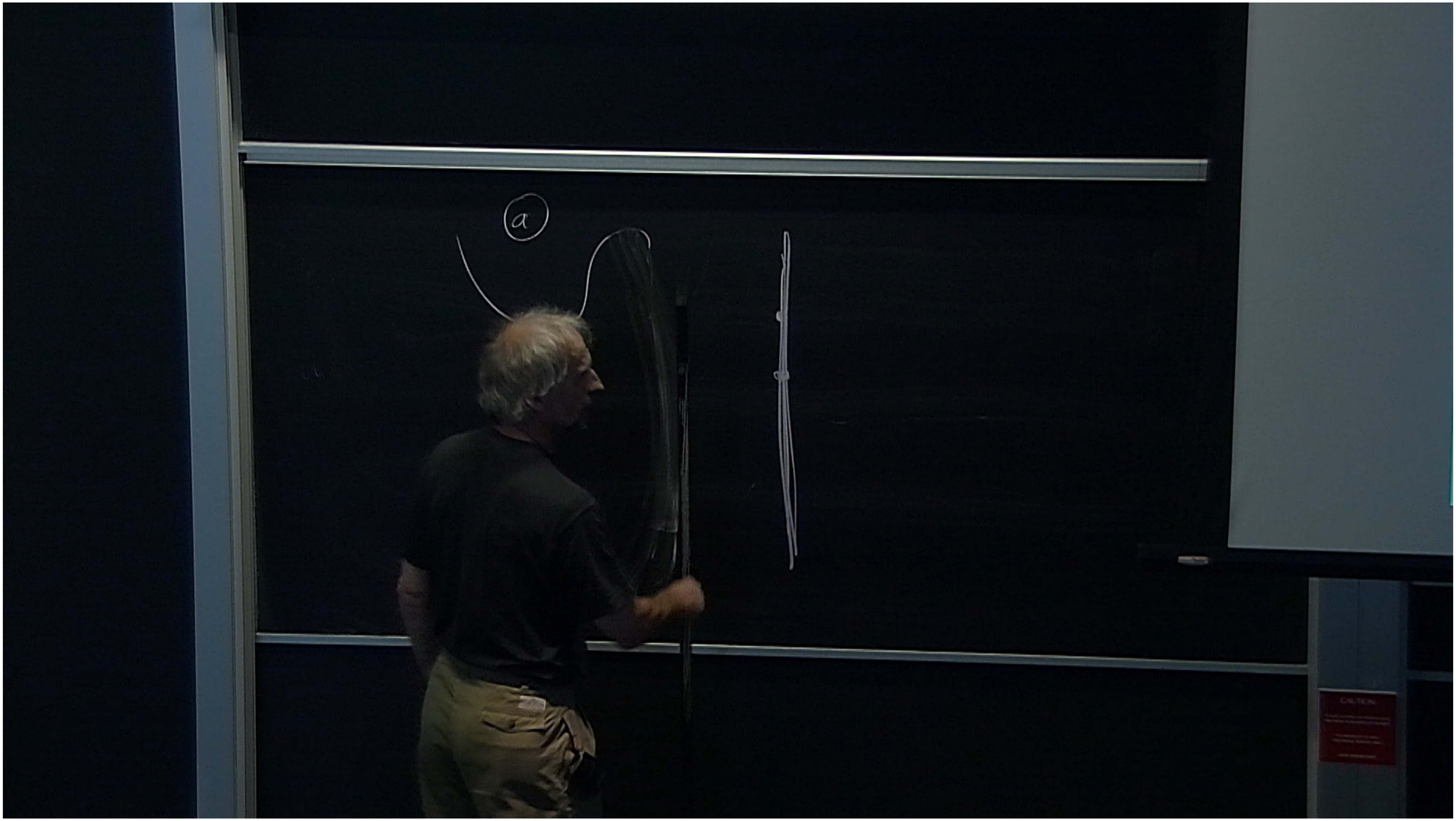
  - $\rightsquigarrow$ monoidal structure on F_a
- impose independence from details of bulk-to-boundary process

$\rightsquigarrow$ central structure $F_a(u) \otimes_{\mathcal{W}_a} x \xrightarrow{\cong} x \otimes_{\mathcal{W}_a} F_a(u)$ coherently

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  - $\rightsquigarrow$ central structure $F_a(u) \otimes_{\mathcal{W}_a} X \xrightarrow{\cong} X \otimes_{\mathcal{W}_a} F_a(u)$

equivalently: choice of lift

$$\begin{array}{ccc} & & \mathcal{Z}(\mathcal{W}_a) \\ & \overset{\widetilde{F}_a}{\nearrow} & \downarrow \text{forget} \\ \mathcal{C} & \xrightarrow[F_a]{} & \mathcal{W}_a \end{array}$$

to Drinfeld center of $\mathcal{W}_a$

$\widetilde{F}_a$ fully faithful

DAVYDOV-MÜGER-NIKSHYCH-OSTRIK 2013

Labels for boundaries

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  $\rightsquigarrow$ monoidal structure on F_a
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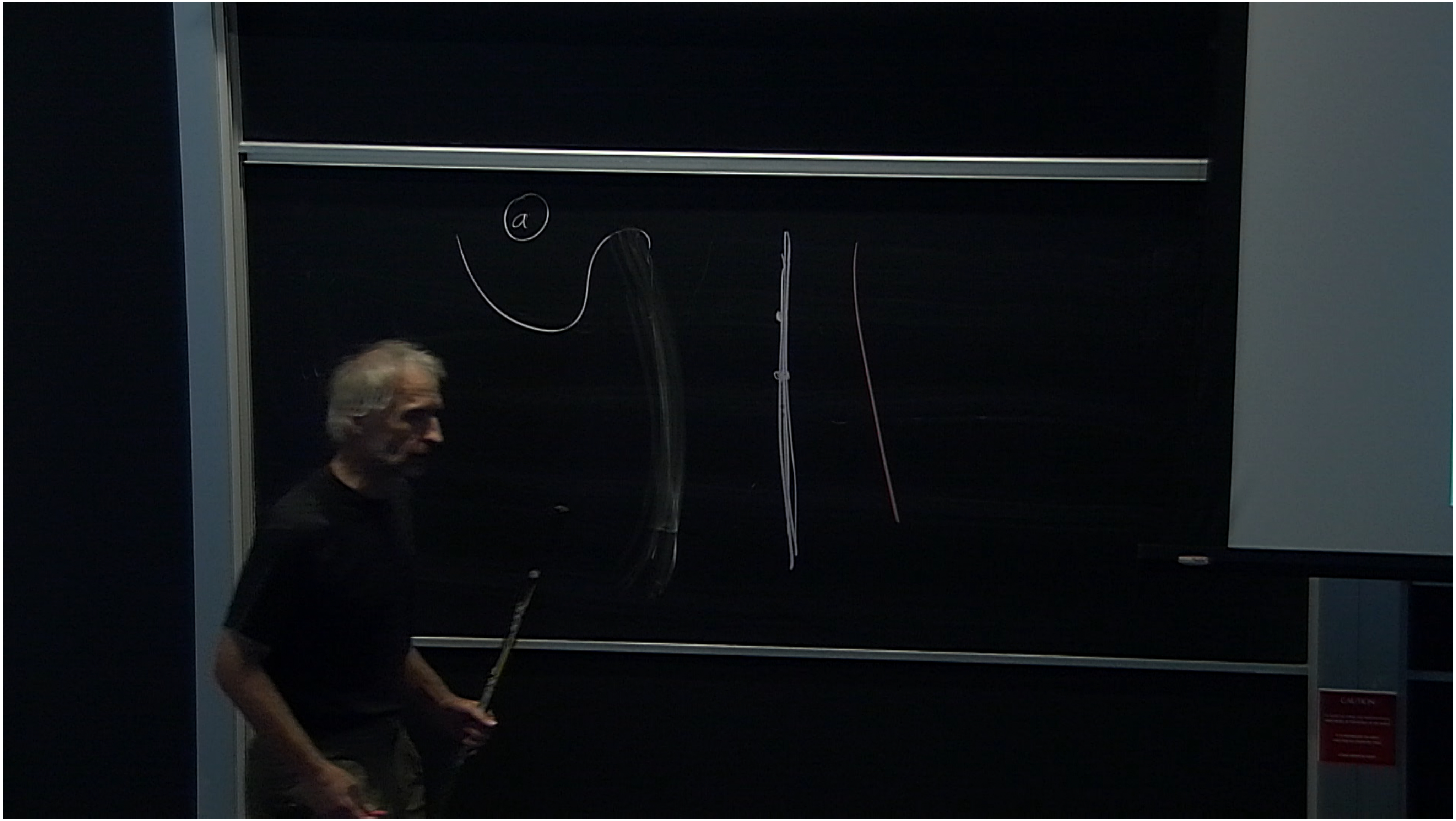
  $\rightsquigarrow$ central structure on F_a
- postulate naturality :

  only reason for being able to consistently move boundary Wilson line $Y \in \mathcal{W}_a$ past any $X \in \mathcal{W}_a$ should be that $Y = F_a(u)$ for some $u \in \mathcal{C}$

  $\rightsquigarrow$ essentially surjective $\rightsquigarrow$ braided equivalence $\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)$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In short: compatible boundary condition for bulk region $\mathcal{C}$

$=$ *Witt trivialization* $\widetilde{F}_a:\ \mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)$ for some fusion category $\mathcal{W}_a$

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Bicategory of boundary conditions

☞ thus : for any given boundary condition a : $\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)$

↯ in particular obstruction : no compatible boundary condition unless $[\mathcal{C}] = 0$ in Witt group of modular tensor categories

Bicategory of boundary conditions

Topological defects

- thus : for any given boundary condition a : $\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)$
  - in particular obstruction : no compatible boundary condition unless $[\mathcal{C}] = 0$ in Witt group of modular tensor categories
- for any other boundary condition b :

  another fusion category $\mathcal{W}_b$ of Wilson lines

Bicategory of boundary conditions

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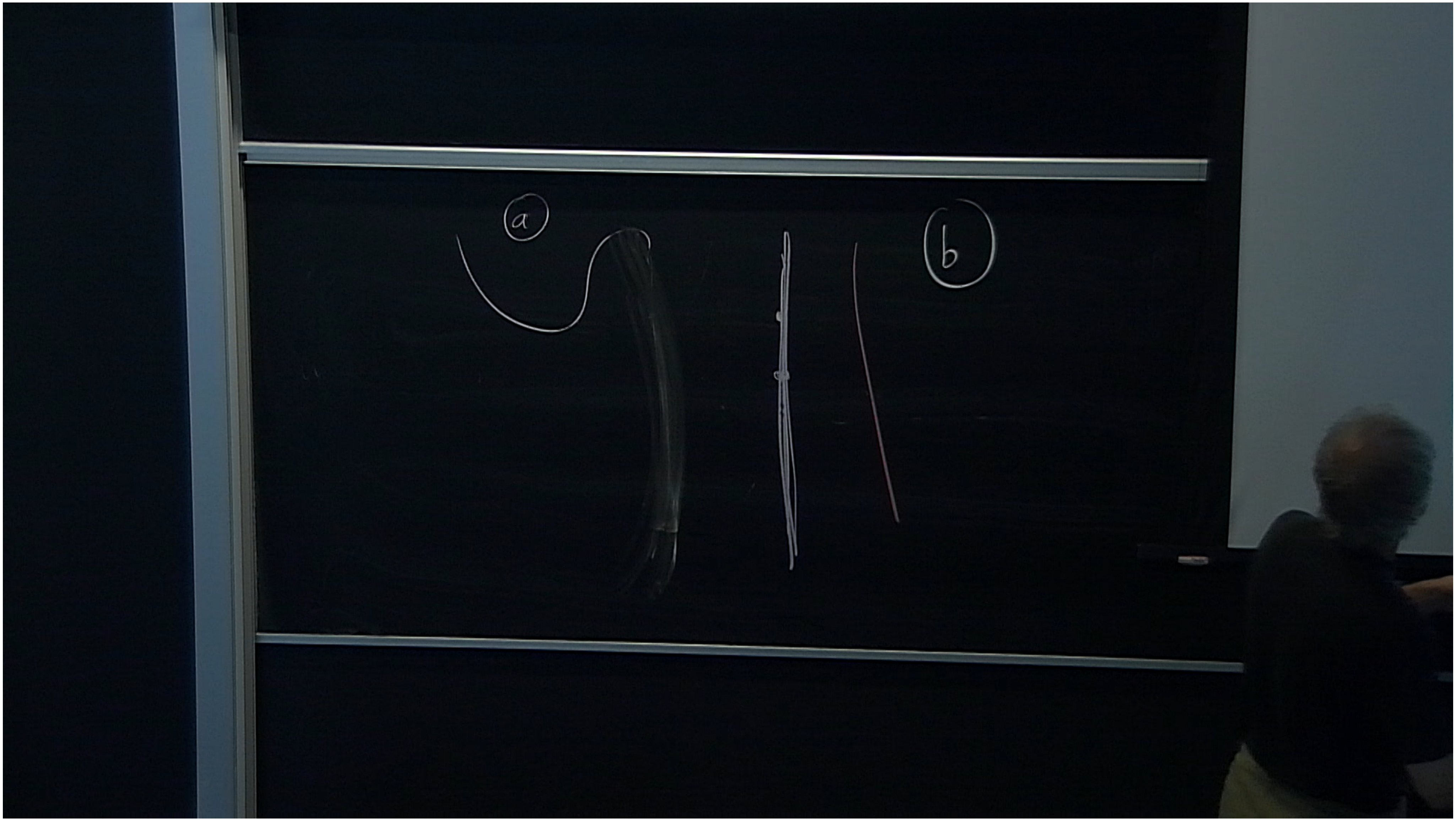
  - in particular obstruction : no compatible boundary condition unless $[\mathcal{C}] = 0$ in Witt group of modular tensor categories
- for any other boundary condition b :
  - also category $\mathcal{W}_{a,b}$ of Wilson lines separating boundary region labeled a from region labeled b
  - fusion of Wilson lines in region a $\rightsquigarrow$ functor $\mathcal{W}_a \times \mathcal{W}_{a,b} \longrightarrow \mathcal{W}_{a,b}$

    $\rightsquigarrow$ $\mathcal{W}_{a,b}$ left module category over $\mathcal{W}_a$
  - likewise : $\mathcal{W}_{a,b}$ right module category over $\mathcal{W}_b$


a
b

Bicategory of boundary conditions
Topological defects
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$\rightsquigarrow$ $\mathcal{W}_{a,b}$ left module category over $\mathcal{W}_a$
likewise: $\mathcal{W}_{a,b}$ right module category over $\mathcal{W}_b$
a
b

Bicategory of boundary conditions

- thus : for any given boundary condition a : $\boxed{\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)}$
  - in particular obstruction : no compatible boundary condition unless $[\mathcal{C}] = 0$ in Witt group of modular tensor categories
- for any other boundary condition b :
  - also category $\mathcal{W}_{a,b}$ of Wilson lines separating boundary region labeled a from region labeled b
  - fusion of Wilson lines in region a $\rightsquigarrow$ functor $\mathcal{W}_a \times \mathcal{W}_{a,b} \longrightarrow \mathcal{W}_{a,b}$

    $\rightsquigarrow$ $\mathcal{W}_{a,b}$ left module category over $\mathcal{W}_a$
  - likewise : $\mathcal{W}_{a,b}$ right module category over $\mathcal{W}_b$
  - but also : $\mathcal{W}_{a,b}$ right module category over $\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b})$ ← module endofunctors

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- *impose* naturality: $\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b}) \simeq \mathcal{W}_b$

  consistency check: $\mathcal{Z}(\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b})) \simeq \mathcal{Z}(\mathcal{W}_a)$ canonically SCHAUENBURG 2001

Bicategory of boundary conditions

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- *impose* naturality: $\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b}) \simeq \mathcal{W}_b$

  $\Longrightarrow$ can work with a single *reference boundary condition*
- Conjecture: *boundary conditions for $\mathcal{C}$ form the bicategory $\mathcal{W}_a$-$\mathcal{M}od$* of module categories over a fusion category $\mathcal{W}_a$ satisfying $\mathcal{Z}(\mathcal{W}_a) \simeq \mathcal{C}$

Bicategory of boundary conditions

Topological defects

COMMENT

☞ boundary conditions given by $\mathcal{W}_a$-$\mathcal{M}od$

$\Longrightarrow$

$\mathcal{W}_{b,c} \simeq \mathcal{F}un_{\mathcal{W}_a}(\mathcal{W}_b, \mathcal{W}_c)$ for any pair b, c of boundary conditions

Bicategory of boundary conditions

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COMMENT

☞ via $\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a) \xrightarrow{\text{forget}} \mathcal{W}_a$

any $\mathcal{W}_a$-module $\mathcal{M}$ has natural structure of $\mathcal{C}$-module

but not every $\mathcal{C}$-module over a Witt-trivial $\mathcal{C}$ gives a boundary condition

illustration: $\mathcal{C} = \mathcal{Z}(\mathcal{V}ect(\mathbb{Z}_2))$ (toric code)

- 6 inequivalent indecomposable $\mathcal{C}$-modules
- 2 inequivalent indecomposable $\mathcal{V}ect(\mathbb{Z}_2)$-modules
- 2 elementary boundary conditions

BRAVYI-KITAEV 2001

Bicategory of surface defects

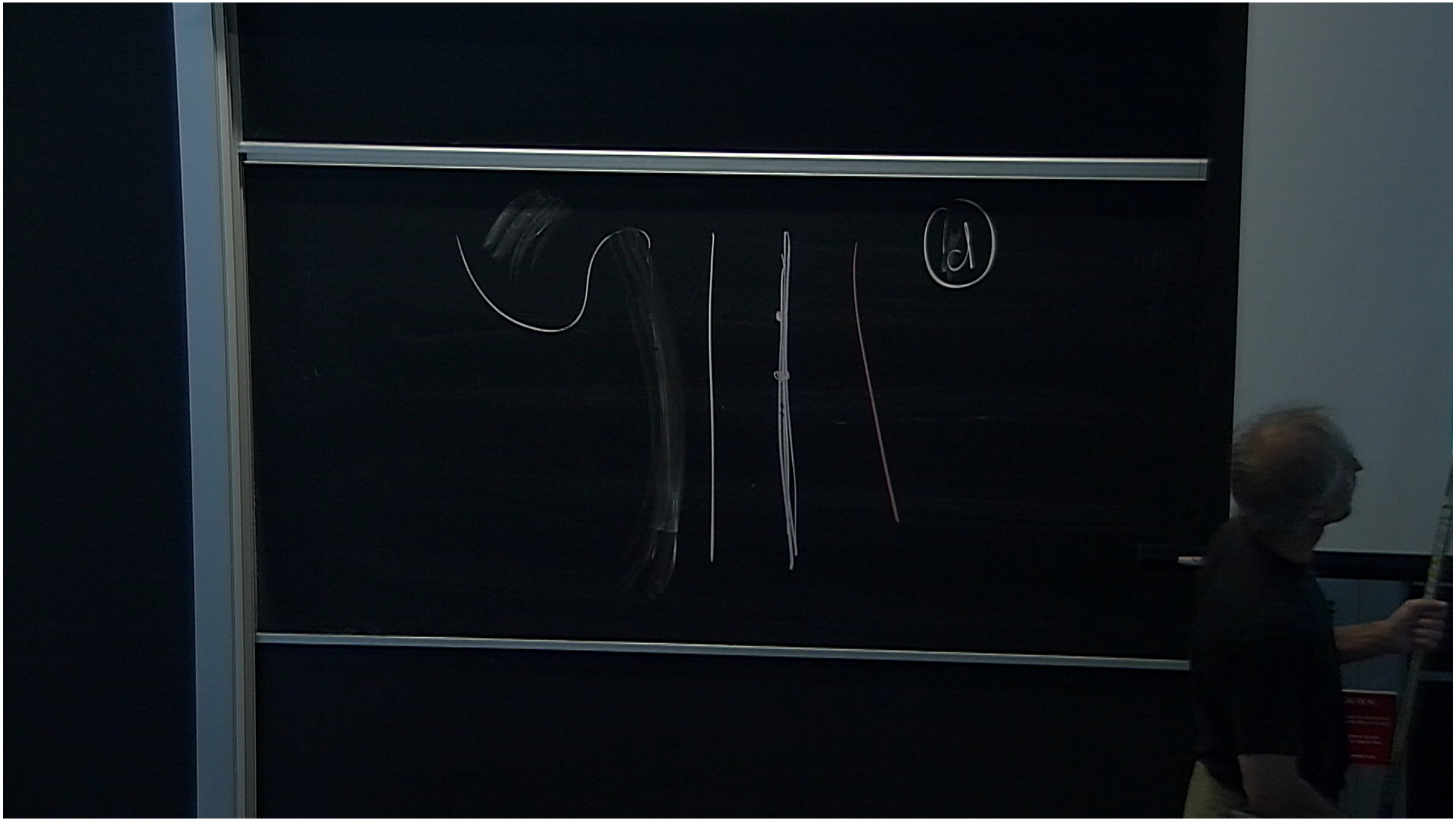
Topological defects

- analysis for surface defects analogous :
  - defect d separating bulk regions labeled by $\mathcal{C}_1$ and $\mathcal{C}_2$
  - two monoidal functors $\mathcal{C}_1 \longrightarrow \mathcal{W}_d$ and $\mathcal{C}_2^{\mathrm{rev}} \to \mathcal{W}_d$ to fusion category $\mathcal{W}_d$
  - combine to central functor $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \to \mathcal{W}_d$

inverse braiding

Deligne product

JF PI 1.8.17 – p. 12/25



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  - combine to central functor $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \rightarrow \mathcal{W}_d$
  - naturality $\rightsquigarrow$ braided equivalence $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_d)$
  - obstruction: no defects between $\mathcal{C}_1$ and $\mathcal{C}_2$ unless $[\mathcal{C}_1] = [\mathcal{C}_2]$ in Witt group

Bicategory of surface defects

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- conclude: *defects separating* $\mathcal{C}_1$ *from* $\mathcal{C}_2$ *form bicategory* $\mathcal{W}_d$*-*$\mathcal{M}od$
  of module categories over a fusion category $\mathcal{W}_d$ satisfying $\mathcal{Z}(\mathcal{W}_d) \simeq \mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}}$

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EXAMPLE

- canonical Witt trivialization $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C})$ ($\mathcal{C}$ modular)
  - defects separating $\mathcal{C}$ from itself $=$ $\mathcal{C}$-modules
  - regular $\mathcal{C}$-module $(\mathcal{C}, \otimes)$ $\rightsquigarrow$ transparent defect $\mathcal{T}$
  - $\mathcal{T}$ serves as monoidal unit for fusion of surface defects
  - Wilson lines separating $\mathcal{T}$ from itself $=$ ordinary Wilson lines

Bicategory of surface defects

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EXAMPLE

- Turaev-Viro / Barrett-Westbury case: $\mathcal{C}_1 = \mathcal{Z}(\mathcal{A}_1)$ and $\mathcal{C}_2 = \mathcal{Z}(\mathcal{A}_2)$
  - $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{A}_1) \boxtimes \mathcal{Z}(\mathcal{A}_2^{\mathrm{op}}) \simeq \mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2^{\mathrm{op}})$
  - thus defects separating $\mathcal{C}_1$ from $\mathcal{C}_2$
    form bicategory $\boxed{\mathcal{A}_1\text{-}\mathcal{A}_2\text{-}\mathcal{B}imod}$ cp KITAEV-KONG 2012

Bicategory of surface defects

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EXAMPLE

- special case of TV-BW: Dijkgraaf-Witten theories

EXAMPLE

- RT TFT's for multi-layer 2+1-dimensional topological orders

Defects in TFTs for multi-layer systems

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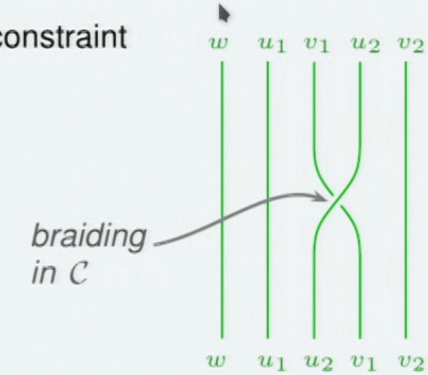
- classification of modules over a generic modular tensor category $\mathcal{D}$ out of reach
  — even finding *any* indecomposable $\mathcal{D}$-module besides $(\mathcal{D}, \otimes)$ can be hard
- TFT for N-layer system : modular tensor category $\mathcal{D} = \mathcal{C}^{\boxtimes N}$
  with $\mathcal{C}$ modular tensor category for each single layer
- *generic* non-trivial right $\mathcal{D}$-module category : $\mathcal{P} \equiv \mathcal{P}_{\mathcal{D}} := (\mathcal{C}, \triangleleft, \alpha)$
  with $w \triangleleft (u_1 \boxtimes \cdots \boxtimes u_N) = w \otimes u_1 \otimes \cdots \otimes u_N$
  and mixed associativity constraint



for $N = 2$

categorification of fact that *commutative* ring R is $R \otimes_{\mathbb{Z}} R$-module

Defects in TFTs for multi-layer systems

Topological defects

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- $\mathcal{D}$-module $\mathcal{P}$ realizable as category $A_\mathcal{P}$-mod of left $A_\mathcal{P}$-modules in $\mathcal{D}$
  - $A_\mathcal{P} = \bigoplus_{i\in I_\mathcal{C}} S_i^\vee \boxtimes S_i$ as object
  - algebra structure directly determined by fusion of simple objects in $\mathcal{C}$
  - $A_\mathcal{P}$ is symmetric special Frobenius and Azumaya
- for A Azumaya $\Psi_A := (\alpha_A^+)^{-1} \circ \alpha_A^-$
  describes transmission of bulk Wilson lines through surface defect A-mod

  $\alpha^+_{A_\mathcal{P}}(u \boxtimes v) \cong \alpha^-_{A_\mathcal{P}}(v \boxtimes u)$ by direct calculation

  $\Longrightarrow$ transmission of bulk Wilson lines through $\mathcal{P}$ permutes the layers
- braided induction for tensor products : $\Psi_{A_1\otimes A_2} = \Psi_{A_1} \circ \Psi_{A_2}$
  as monoidal functors if $A_{1,2}$ Azumaya

  $\Longrightarrow$ $A_\mathcal{P} \otimes A_\mathcal{P}$ Morita equivalent to $\mathbf{1}_\mathcal{D}$
  - fusion rules : $\mathcal{T} \boxtimes_\mathcal{D} \mathcal{P} \simeq \mathcal{P}$ and $\mathcal{P} \boxtimes_\mathcal{D} \mathcal{P} \simeq \mathcal{T}$
  - categories of Wilson lines : $\mathcal{F}un_\mathcal{D}(\mathcal{T},\mathcal{P}) \simeq \mathcal{C} \simeq \mathcal{F}un_\mathcal{D}(\mathcal{P},\mathcal{T})$
    $\mathcal{E}nd_\mathcal{D}(\mathcal{T}) \simeq \mathcal{D} \simeq \mathcal{E}nd_\mathcal{D}(\mathcal{P})$

Defects in Dijkgraaf-Witten theories

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- Dijkgraaf-Witten theory :
  - $\mathcal{C} = D^\omega(G)$-mod $\simeq \mathcal{Z}(\mathcal{V}ect(G)^\omega)$ G finite group $\omega \in Z^3(G, \mathbb{C}^\times)$
  - ω gives holonomy on closed three-manifolds $\rightsquigarrow$ topological bulk Lagrangian
  - two-step gauge-theoretic construction :

$$\mathit{Cobord}_{3,2,1} \xrightarrow{\widetilde{\mathrm{Bun}}} \mathit{SpanGrpd}$$ spans of groupoids

(II) State sums with defects

Framework

- Goal : construction of TFT admitting defects as well as boundaries
- thus obstruction must vanish : $\mathcal{D} \simeq \mathcal{Z}(\mathcal{A})$

  for some spherical fusion category $\mathcal{A}$

Framework

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- thus obstruction must vanish : $\mathcal{D} \simeq \mathcal{Z}(\mathcal{A})$
  for some finite tensor category $\mathcal{A}$

INFORMAL DEFINITION — **Finite tensor category**

- finite tensor category $=$ fusion category minus semisimplicity

i.e.

- $\mathcal{A}$ not necessarily semisimple
- do not require that $\mathcal{A}$ admits a pivotal structure
- in particular do not fix a pivotal or spherical structure

- categorical tools :
  - ends and coends
  - monads (monad $=$ algebra in $\mathcal{E}nd(\mathcal{B})$ for $\mathcal{B}$ not necessarily monoidal)

Framework

- Goal : construction of TFT admitting defects as well as boundaries
- thus obstruction must vanish : $\mathcal{D} \simeq \mathcal{Z}(\mathcal{A})$
  for some finite tensor category $\mathcal{A}$
- Thus : not necessarily semisimple finite $\Bbbk$-linear abelian categories
  - as objects of $2\text{-}\mathcal{V}ect$
  - as decoration data for state sum construction generalizing TV-BW
- Limitations :
  - requires further structure on manifolds
  - 3-d part not yet understood

Framework

- ☞ Goal : construction of TFT admitting defects as well as boundaries
- ☞ thus obstruction must vanish : $\mathcal{D} \simeq \mathcal{Z}(\mathcal{A})$
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- ☞ Geometric framework :
  - ⇝ *combed* oriented manifolds
  - ⇝ stratifications with CW structure

Framework

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- **Goal**: construction of TFT admitting defects as well as boundaries
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- Geometric framework:
  - *combed* oriented manifolds
  - stratifications with CW structure

    e.g. for surfaces: polygonal complex

INFORMAL DEFINITION — Polygonal complex

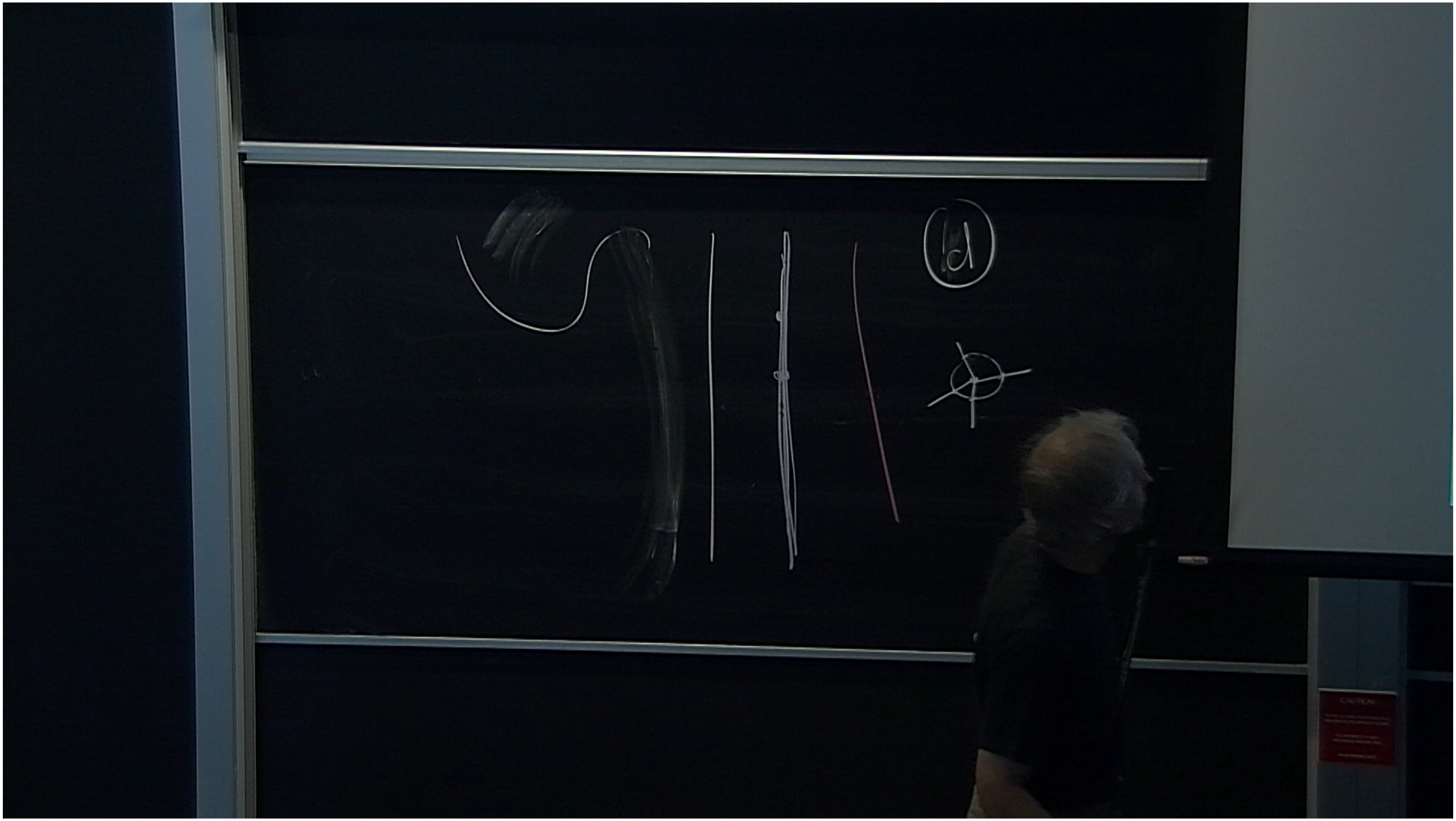
- polygonal complex := CW complex with
  - 2-skeleton a collection of polygons
  - edges of the polygons identified by homeomorphisms
  - vertices of the polygons possibly identified by further equivalences

KUPERBERG 1996

Framework

some details for surfaces Σ:

- polygonal complex
- replace each 0-cell v by small circle $S(v)$ around v intersecting adjacent 1-cells transversally
- fix a global orientation of Σ
- each 0- and 1-cell endowed with their own orientation
- orientation of 2-cells determined by orientation of Σ
- choose auxiliary metric g
  mathematically inessential but simplifies description
  e.g. can represent orientation of 1-cell v by unit tangent field along v
- with help of the normal w.r.t. g edges and circles $S(v)$ acquire a 2-orientation
- w.l.o.g. assume all transverse intersections orthogonal w.r.t. g



Framework

Topological defects

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Decorations

Topological defects

Algebraic framework : decorations :

☞ to 3-cell (*phase*) assign finite tensor category $\mathcal{A}$

Decorations

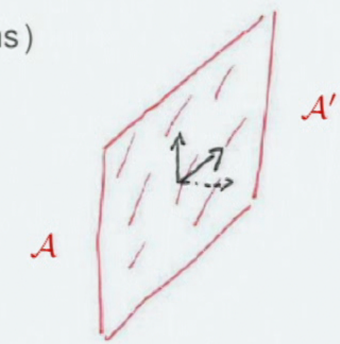
Topological defects

Algebraic framework : decorations :

- to 3-cell (*phase*) assign finite tensor category $\mathcal{A}$
- to 2-cell (*surface defect*) assign finite bimodule category $\mathcal{B} \equiv {}_{\mathcal{A}}\mathcal{B}_{\mathcal{A}'}$

  up to dualities (via orientations)

  and up to twisting



- distinguished case : *transparent* defect for $\mathcal{A}' = \mathcal{A}$
  given by regular bimodule $\mathcal{A} = {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$
- physical *boundary* : $\mathcal{A} = \mathcal{V}ect$ or $\mathcal{A}' = \mathcal{V}ect$

Decorations

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up to dualities (via orientations)

and up to twisting :

- can twist each of the actions by a power of the left / right double dual functor

  $\rightsquigarrow$ module structures labeled by $\mathbb{Z}$

  (all equivalent in presence of pivotal structure)

- can switch between left module structure on $\mathcal{B}$ and right module structure on $\mathcal{B}^{\mathrm{op}}$
- these various module structures related by multiple change of edge orientation
- to keep track : intersection of vertex and edge $\frac{1}{2}\mathbb{Z}$-valued rather than $\mathbb{Z}_2$ (orientation)
- can be done with the help of a *combing* :

  vector field with prescribed isolated singularities KUPERBERG 1996

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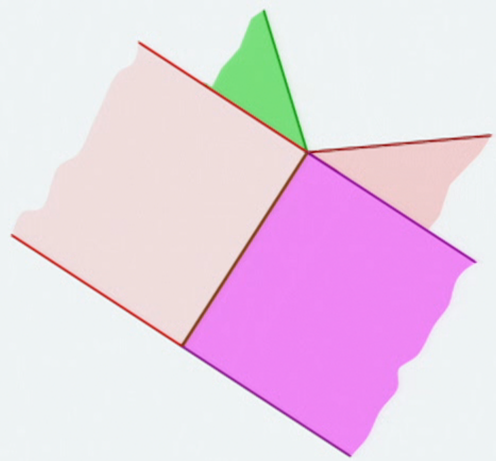

⇝ in standard TV : Wilson line (or rather : ribbon) labeled by object of $\mathcal{Z}(\mathcal{A})$

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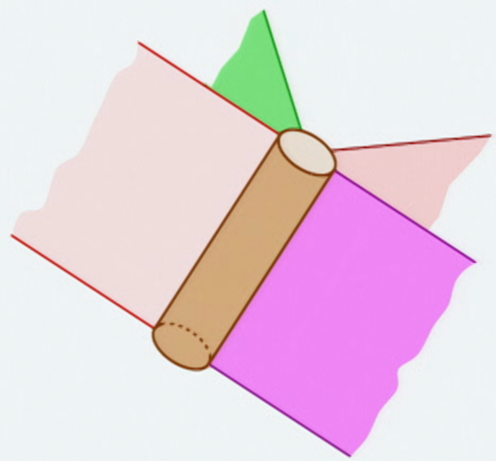
  - ⇝ replace 1-cell by small cylinder



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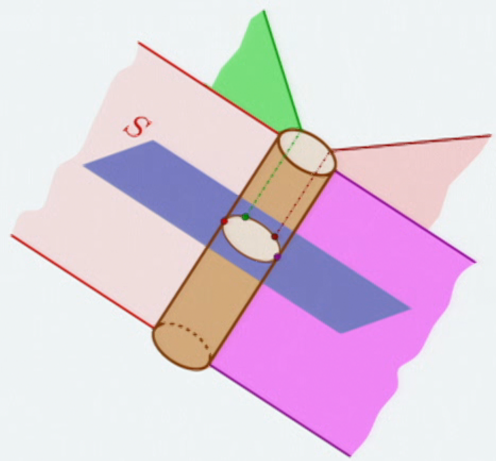

Decorations

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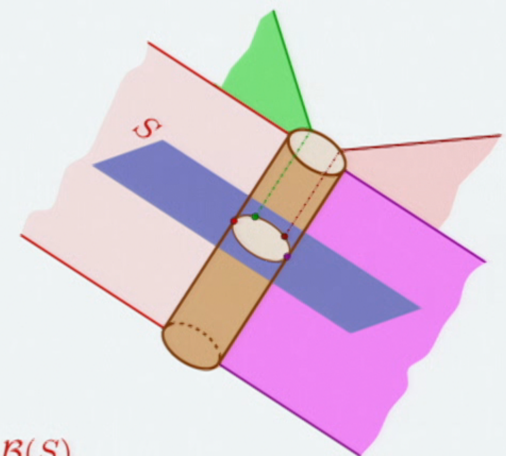

- ☞ to determine $\mathcal{C}$:
  - replace 1-cell by small cylinder
  - consider cross section : decorated 1-manifold S
  - define $\mathcal{C}(S)$ as a category of *balancings* on a certain bimodule category $\mathcal{B}(S)$

Decorations

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DEFINITION — **Balancing**

- for $\mathcal{A}$ monoidal category , $\mathcal{B}$ an $\mathcal{A}$-bimodule and $b \in \mathcal{B}$:

balancing for b $:=$ natural family $(\sigma_a : \, a.b \longrightarrow b.a)$ s.t.

$$\begin{array}{ccc} (a \otimes a') \, . \, b & \xrightarrow{\sigma_{a\otimes a'}} & b \, . \, (a \otimes a') \\ & {\scriptstyle \mathrm{id}_a \otimes \sigma_{a'}} \searrow \quad \swarrow {\scriptstyle \sigma_a \otimes \mathrm{id}_{a'}} & \\ & a \, . \, b \, . \, a' & \end{array}$$

commutes for all $a \, , a'$

Decorations

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DEFINITION — **Balancing**

- $\mathcal{Z}_{\mathcal{A}}(\mathcal{B})$:= category of objects with a balancing

COMMENTS

- for $\mathcal{B} = \mathcal{A}$ get Drinfeld center : $\mathcal{Z}_{\mathcal{A}}(\mathcal{A}) = \mathcal{Z}(\mathcal{A})$
- for $\mathcal{A}$ having right duals :
  - σ_a isomorphism
  - $\mathcal{Z}_{\mathcal{A}}(\mathcal{B}) \simeq Z_{\mathcal{A}}\text{-mod}(\mathcal{B})$ with $Z_{\mathcal{A}}$ the monad $Z_{\mathcal{A}} : \mathcal{B} \to \mathcal{B}$, $b \mapsto \int^{a \in \mathcal{A}} a^{\vee} . b . a$

Decorations

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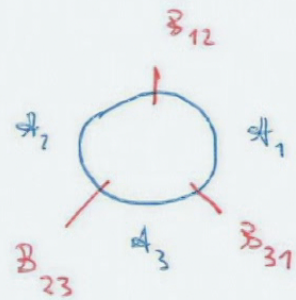

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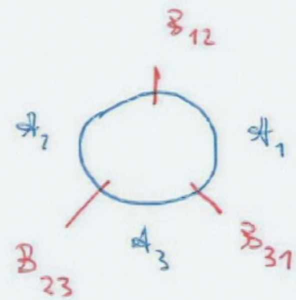
  - cross section gives decorated 1-manifold S
  - $\mathcal{B}(S) := \boxtimes_{v \in S} \mathcal{B}_v$
  - on $\mathcal{B}(S)$ have commuting monads (distributive laws $Z_{\mathcal{A}_i} \circ Z_{\mathcal{A}_j} = Z_{\mathcal{A}_j} \circ Z_{\mathcal{A}_i}$)
    $\rightsquigarrow$ monad Z on $\mathcal{B}(S)$
  - define $\boxed{\mathcal{C}(S) := Z\text{-mod}(\mathcal{B}(S))}$



Decorations

Algebraic framework : decorations :

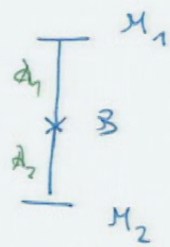
- to 3-cell (*phase*) assign finite tensor category $\mathcal{A}$
- to 2-cell (*surface defect*) assign finite bimodule category $\mathcal{B} \equiv {}_{\mathcal{A}}\mathcal{B}_{\mathcal{A}'}$
- to 1-cell (*generalized Wilson line*) assign finite category $\mathcal{C}$

- to determine $\mathcal{C}$:

COMMENTS

- transparent case $\mathcal{A}_i = \mathcal{A}$ and $\mathcal{B}_\ell = {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ $\Longrightarrow$ $\mathcal{C}(S) = \mathcal{Z}(\mathcal{A})$
- comprises notions of category-valued trace and relative Deligne product

  e.g. $\mathcal{C}(\mathrm{I}) = \mathcal{M}_1 \boxtimes_{\mathcal{A}_1} \mathcal{B} \boxtimes_{\mathcal{A}_2} \mathcal{M}_2$ for $\mathrm{I} =$



Dijkgraaf-Witten theories

EXAMPLE

Dijkgraaf-Witten theory for (G,ω)

$=$ TV-BW theory for fusion category $(G\text{-}\mathcal{V}ect)^{\omega}$

- *indecomposable* bimodule categories classified by subgroup $H \le G\times G$ and 2-cocycle θ satisfying $\mathrm{d}\theta = p_1^*\omega\cdot(p_2^*\omega)^{-1}$

  OSTRIK 2003

- reproduce category for circle $\mathbb{S}^1$ with one vertex lebeled by (H,θ) :
  $G\times G$ -graded vector spaces with twisted $G\times H$-action

- furnishes realization of category-valued trace of the bimodule category

  $\mathcal{F}$-SCHAUMANN-SCHWEIGERT 2017

Conformal blocks

☞ Important task :

construct spaces of conformal blocks / state spaces for any surface Σ

Conformal blocks

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construct spaces of conformal blocks / state spaces for any surface Σ

☞ Ingredients :

- Σ endowed with structure of finite polygonal complex , orientations ......

☞ a guiding principle : generalizes a theory of flat connections

- 1-cells (bimodule categories $\mathcal{B}_e$) supply dynamical degrees of freedom
- each 2-cell supplies a flatness condition

just like in tensor network models

Conformal blocks

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construct spaces of conformal blocks / state spaces for any surface Σ

☞ Ingredients :

- Σ endowed with structure of finite polygonal complex , orientations ......
- replace each vertex $v \in \Sigma$ by decorated 1-manifold S_v
- conformal blocks given by functor

$$Bl_\Sigma : \boxtimes_{v \in \partial\Sigma_{\text{in}}} \mathcal{C}(S_v) \longrightarrow \boxtimes_{v \in \partial\Sigma_{\text{out}}} \mathcal{C}(S_v)$$

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  - functor must be left exact for compatibility with $\sqcup$ / $\boxtimes$
    (parallel formulation with right exact functors)

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- Step 1 of construction: *pre-blocks*
  - left exact functor $pBl\colon \boxtimes_{v\in\Sigma}\mathcal{B}(S_v) \longrightarrow \mathcal{V}ect$ defined as a state sum
  - insight:

    state sum is a coend: $$\boldsymbol{pBl_\Sigma = \boxtimes_{e\in\Sigma}\big(\oint^{b\in\mathcal{B}_e} \mathrm{Hom}(-\,,b_e^\# \boxtimes b_e\big)}$$

    $$\boxtimes_{v\in\Sigma}\mathcal{B}(S_v) \simeq \boxtimes_{e\in\Sigma}(\mathcal{B}_e^\# \boxtimes \mathcal{B}_e)$$

Conformal blocks

EXAMPLE

☞ disk with one incoming and one outgoing 0-cell :

- 1-cells labeled by bimodule categories $\mathcal{N}$ and $\mathcal{M}$
- 0-cells labeled by $\mathcal{N}^{\mathrm{op}} \boxtimes \mathcal{M}$
- trade bimodules for functor categories (Eilenberg-Watts calculus)

$$\mathcal{R}ex(\mathcal{N},\mathcal{M}) \xrightarrow{\simeq} \mathcal{N}^{\mathrm{op}} \boxtimes \mathcal{M}$$
$$F \longmapsto \int_{n\in\mathcal{N}} F(n) \boxtimes m$$
$$\mathrm{Hom}_{\mathcal{N}}(-,\overline{n})^* \otimes m \longleftarrow \overline{n} \boxtimes m$$

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- specify insertions $\overline{n}_1 \boxtimes m_1$, $\overline{n}_2 \boxtimes m_2 \in \mathcal{N}^{\mathrm{op}} \boxtimes \mathcal{M}$
- state sum $\int^{n\in\mathcal{N}}\!\!\int^{m\in\mathcal{M}} \mathrm{Hom}(\overline{n}_1 \boxtimes m_1 \boxtimes n_2 \boxtimes \overline{m}_2, \overline{n} \boxtimes m \boxtimes n \boxtimes \overline{m})$

$$\cong \mathrm{Hom}_{\mathcal{N}}(n_2,n_1) \otimes \mathrm{Hom}_{\mathcal{M}}(m_1,m_2)$$
$$\cong \int_{n\in\mathcal{N}} \mathrm{Hom}(F_1(n),F_2(n)) \cong \mathrm{Nat}(F_1,F_2)$$

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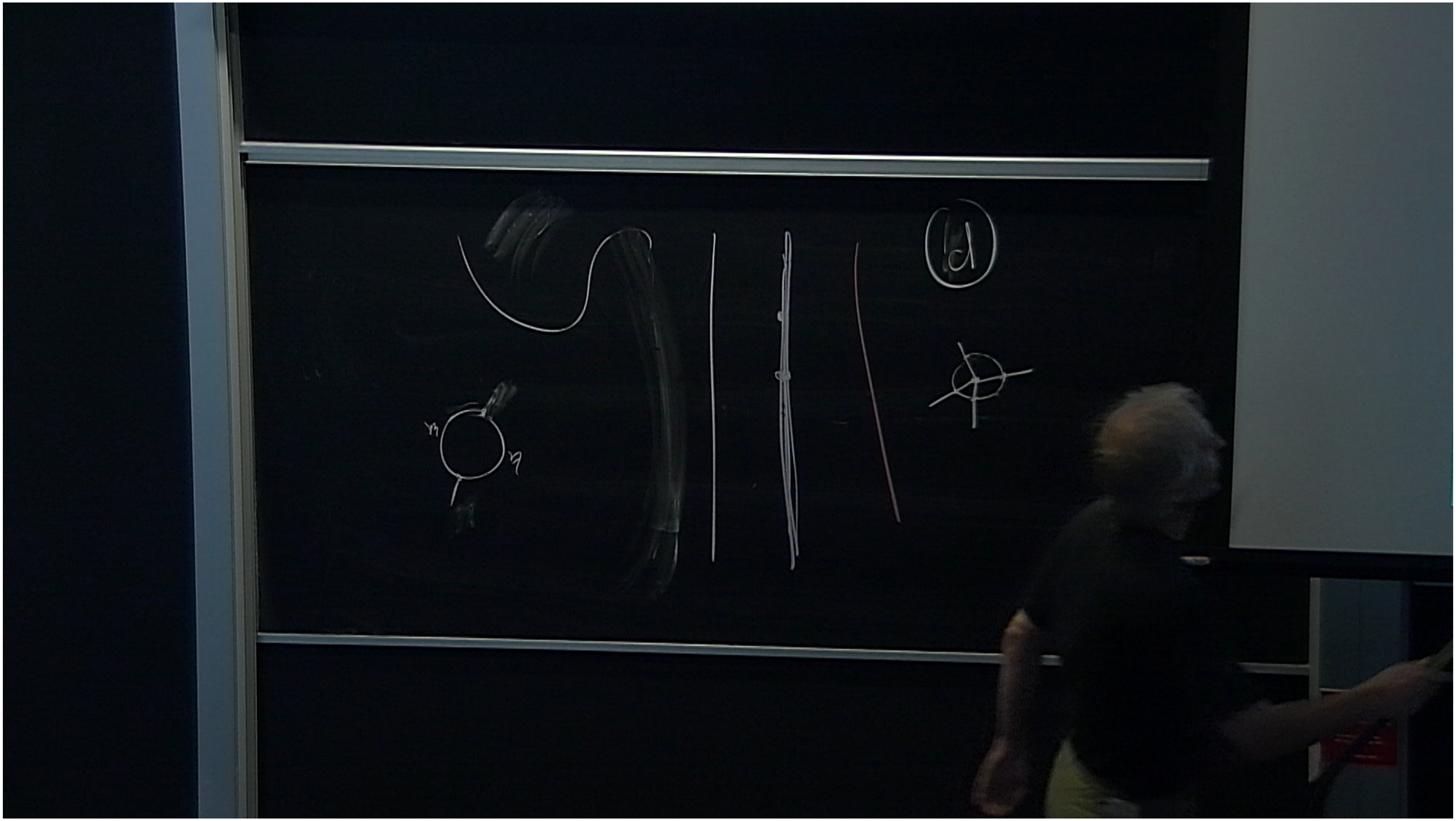

Conformal blocks

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- similarly for more 0-cells
- thus : pre-blocks $=$ natural transformations

Conformal blocks

☞ Step 2 of construction : from pre-blocks to blocks

- ⇝ functors associated to vertices via E-W are *module functors*
- ⇝ impose flatness : equalizers ⇝ *module natural transformations*

THEOREM — Conformal blocks

genus-0 conformal blocks = spaces of module natural transformations

Conformal blocks

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Conformal blocks

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Outlook :

☞ invariance under relevant moves

☞ full description of higher-genus conformal blocks

☞ factorization

☞ 3-manifolds

☞ applications , e.g. to logarithmic conformal field theory