Title: An Introduction to Hopf Algebra Gauge Theory

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Abstract: A variety of models, especially Kitaev models, quantum Chern-Simons theory, and models from 3d quantum gravity, hint at a kind of lattice gauge theory in which the gauge group is generalized to a Hopf algebra. However, until recently, no general notion of Hopf algebra gauge theory was available. In this self-contained introduction, I will cover background on lattice gauge theory and Hopf algebras, and explain our recent construction of Hopf algebra gauge theory on a ribbon graph (arXiv:1512.03966). The resulting theory parallels ordinary lattice gauge theory, generalizing its structure only as necessary to accommodate more general Hopf algebras. All of the key features of gauge theory, including gauge transformations, connections, holonomy and curvature, and observables, have Hopf algebra analogues, but with a richer structure arising from non-cocommuntativity, the key property distinguishing Hopf algebras from groups. Main results include topological invariance of algebras of observables, and a gauge theoretic derivation of algebras previously obtained in the combinatorial quantization of Chern-Simons theory.

An Introduction to Hopf Algebra Gauge Theory

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(joint work with Catherine Meusburger)

Hopf Algebras in Kitaev's Quantum Double Models

Perimeter Institute

31 July – 4 August 2017

Hopf algebra gauge theory

Goal: Conservative generalization of (lattice) gauge theory from groups to Hopf algebras.

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  Turaev–Viro as regularization of 3d quantum gravity,
  Combinatorial quantization of Chern-Simons theory,
  Other gauge-theory-like models with specific algebras, bases, lattices, etc.
- Kitaev models. (See Catherine Meusburger's talk, up next!)

Strategy

Take lattice gauge theory, and apply the monoidal functor

$$(\mathrm{FinSet}, \times, 1) \to (\mathrm{Vect}, \otimes, \mathbb{C})$$

to everything in sight.

FinSet	**Vect**
sets	vector spaces or better: *coalgebras*
groups	Hopf algebras
group actions	Hopf algebra modules

Then generalize to other fin. dim. Hopf algebras *(conservatively!)*

Result

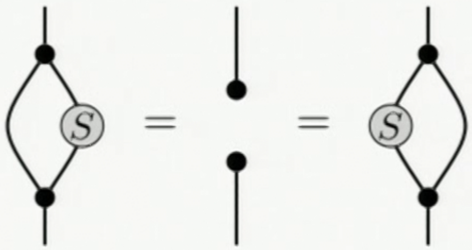
Reproduce Hamiltonian quantum Chern-Simons theory.

- topological invariant: **quantum moduli space** (analog of the moduli space of flat (classical) connections)
- derived “axiomatically” by generalizing gauge theory, rather than quantizing Poisson structures.

Hopf algebras

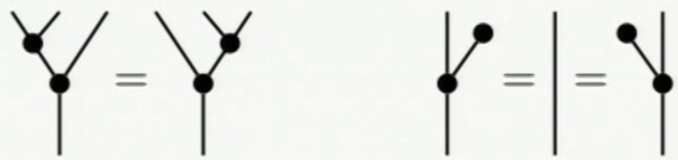
A **Hopf algebra** is a bialgebra H with **antipode** $S\colon H \to H$, drawn as:

Satisfying:

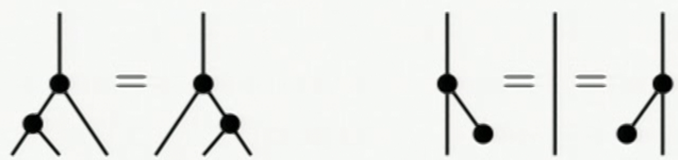


Bialgebras

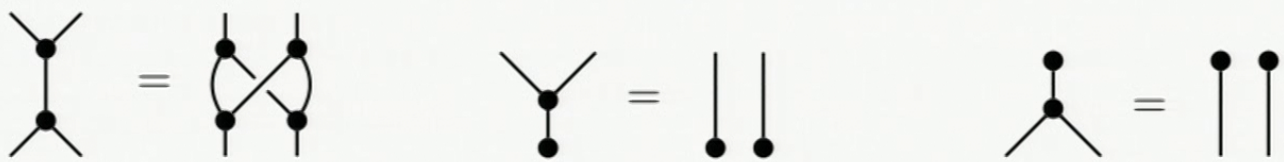
A bialgebra is an algebra:



and a coalgebra:

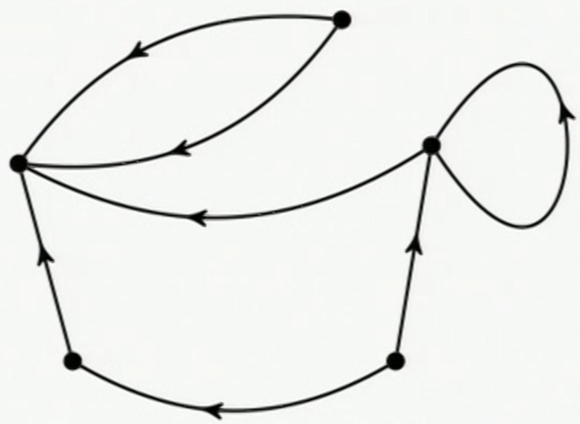


such that:



Lattice gauge theory

Graph with set V of vertices, set E of edges.



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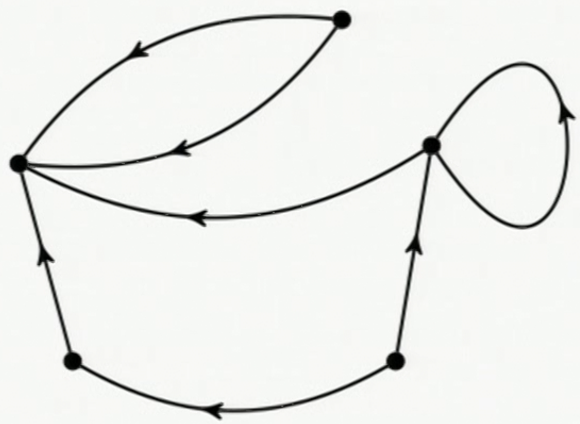

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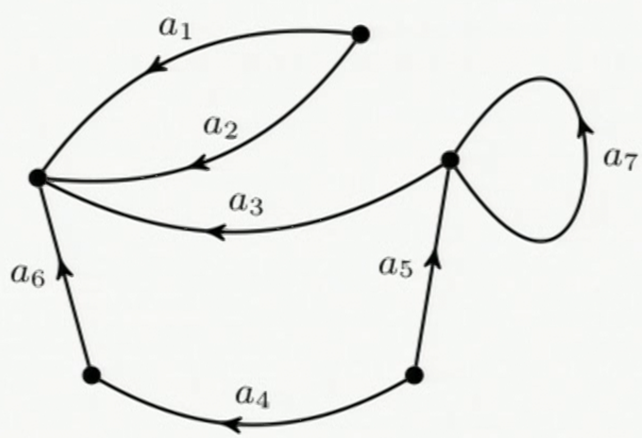

$a_1, \ldots a_7 \in G$. G^E is the **set of connections**

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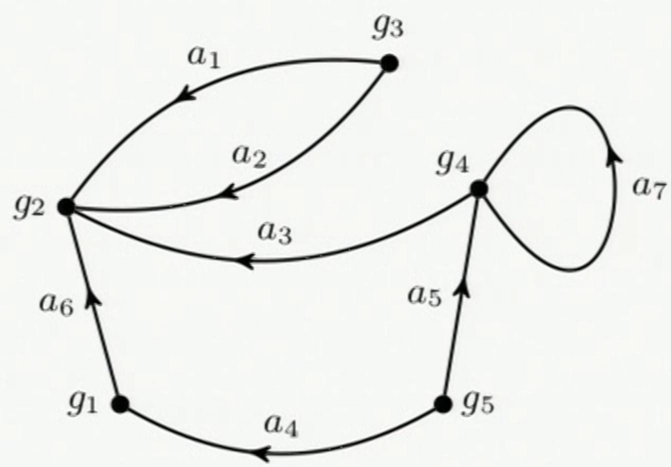

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$g_1, \dots g_5 \in G$. G^V is the **group of gauge transformations**.

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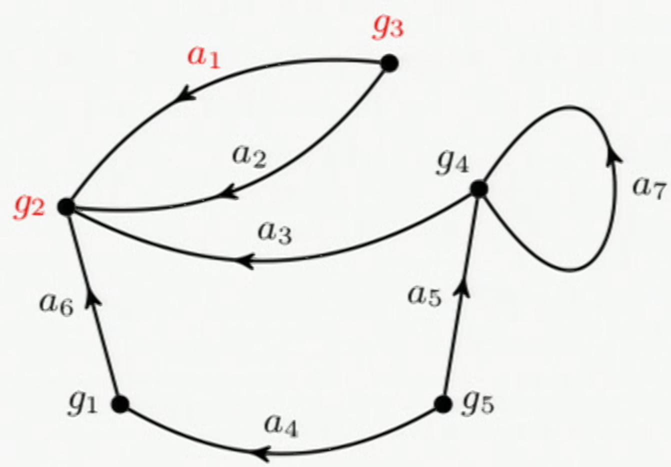

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Action of G^V on G^E: e.g. $a_1 \mapsto g_2 a_1 g_3^{-1}$.

Hopf algebra gauge theory from group gauge theory

Gauge theory for G

Gauge group G
Gauge trans.: $\mathcal{G} = G^V$
Connections: $\mathcal{A} = G^E$
Gauge action:

$$\triangleright : \mathcal{G} \times \mathcal{A} \to \mathcal{A}$$

Functions:

$$\mathcal{A}^* = \{f \colon \mathcal{A} \to \mathbb{C}\} \cong \mathbb{C}[G]^{*\otimes E}$$

Observables: $\mathcal{A}^*_{\text{inv}} \subset \mathcal{A}^*$ with

$$f(g \triangleright a) = f(a)$$

Gauge theory for $\mathbb{C}[G]$

Gauge Hopf algebra $\mathbb{C}[G]$
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Gauge transformations

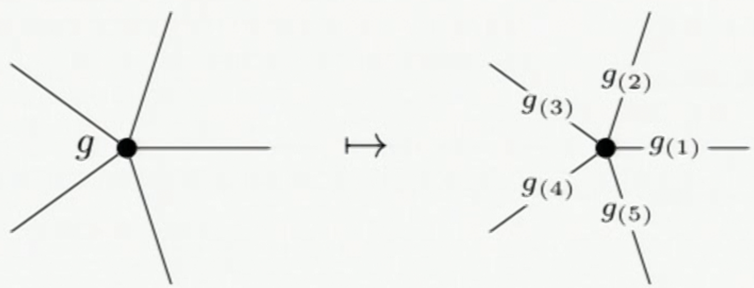
For a Hopf algebra H, the gauge action on connections should be

$$\triangleright \colon H^{\otimes V} \otimes H^{\otimes E} \to H^{\otimes E}$$

To make this *linear*, we need H's **comultiplication**:

$$\Delta \colon H \to H \otimes H$$

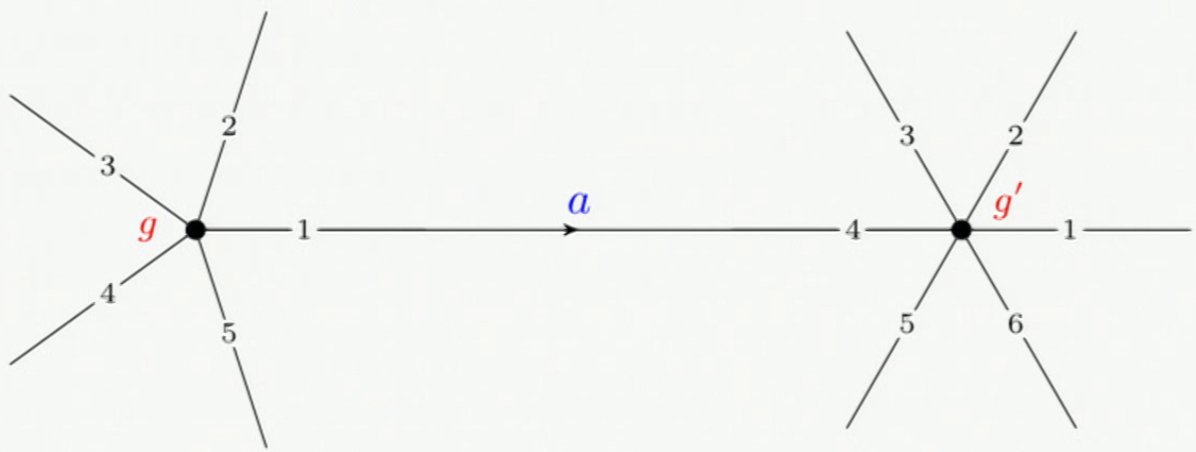
to "duplicate" vertex elements:



If H is not cocommutative, we need a total order at each vertex!

Gauge transformations

Otherwise, copy the gauge action as closely as possible:



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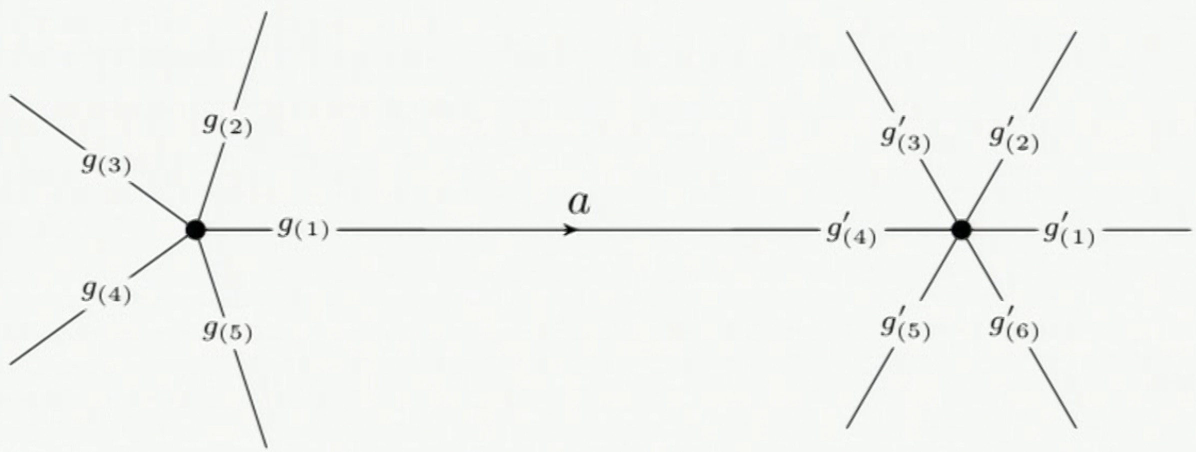


$$\cdots \otimes a \otimes \cdots \quad \mapsto \quad \cdots \otimes g'_{(4)} a S(g_{(1)}) \otimes \cdots$$

Ribbon Graphs

A graph with *cyclically ordered* edge-ends is a **ribbon graph**

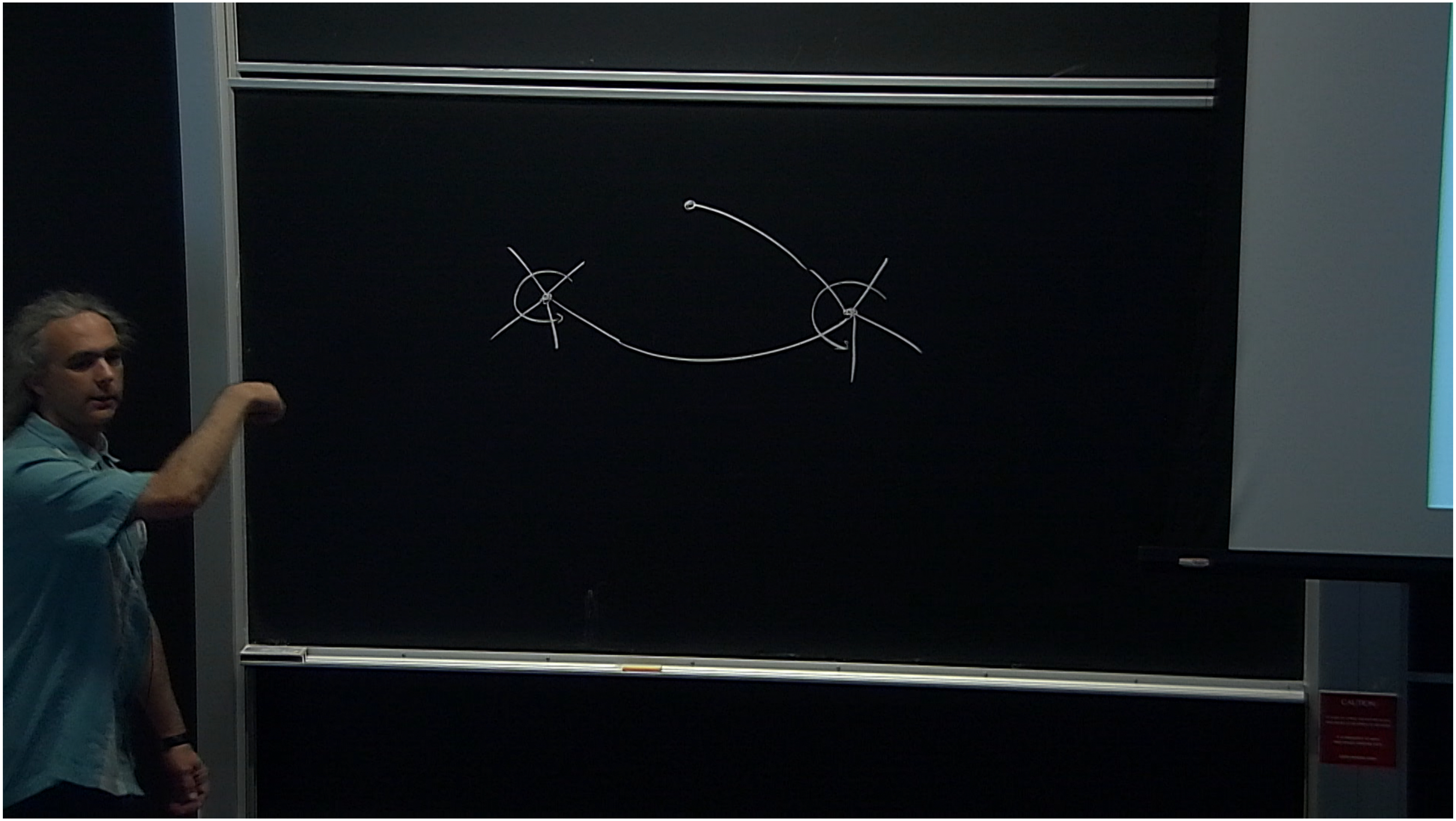
$\Longrightarrow$ surface with boundary.

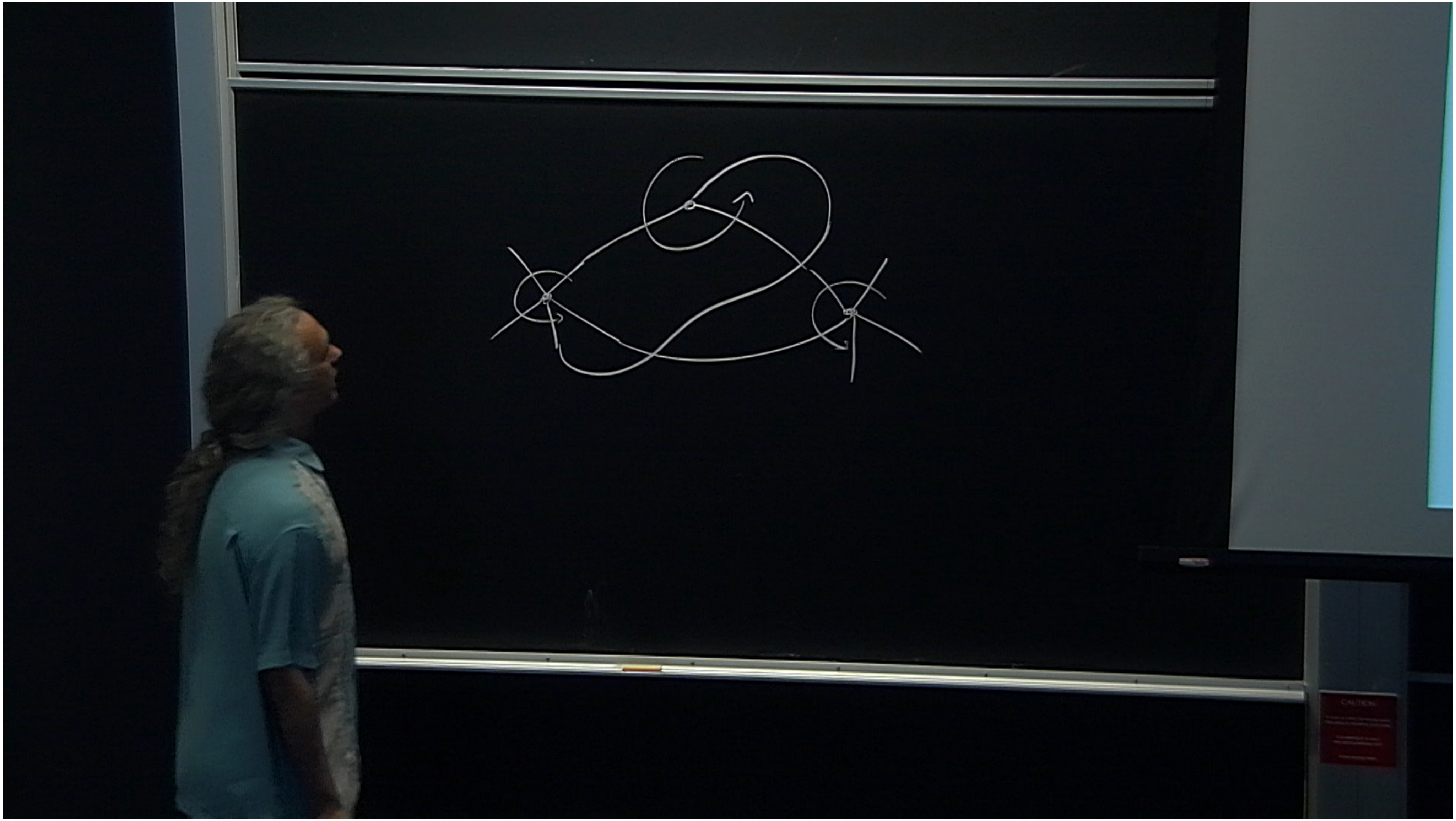
$\Longrightarrow$ closed surface, after sewing discs.

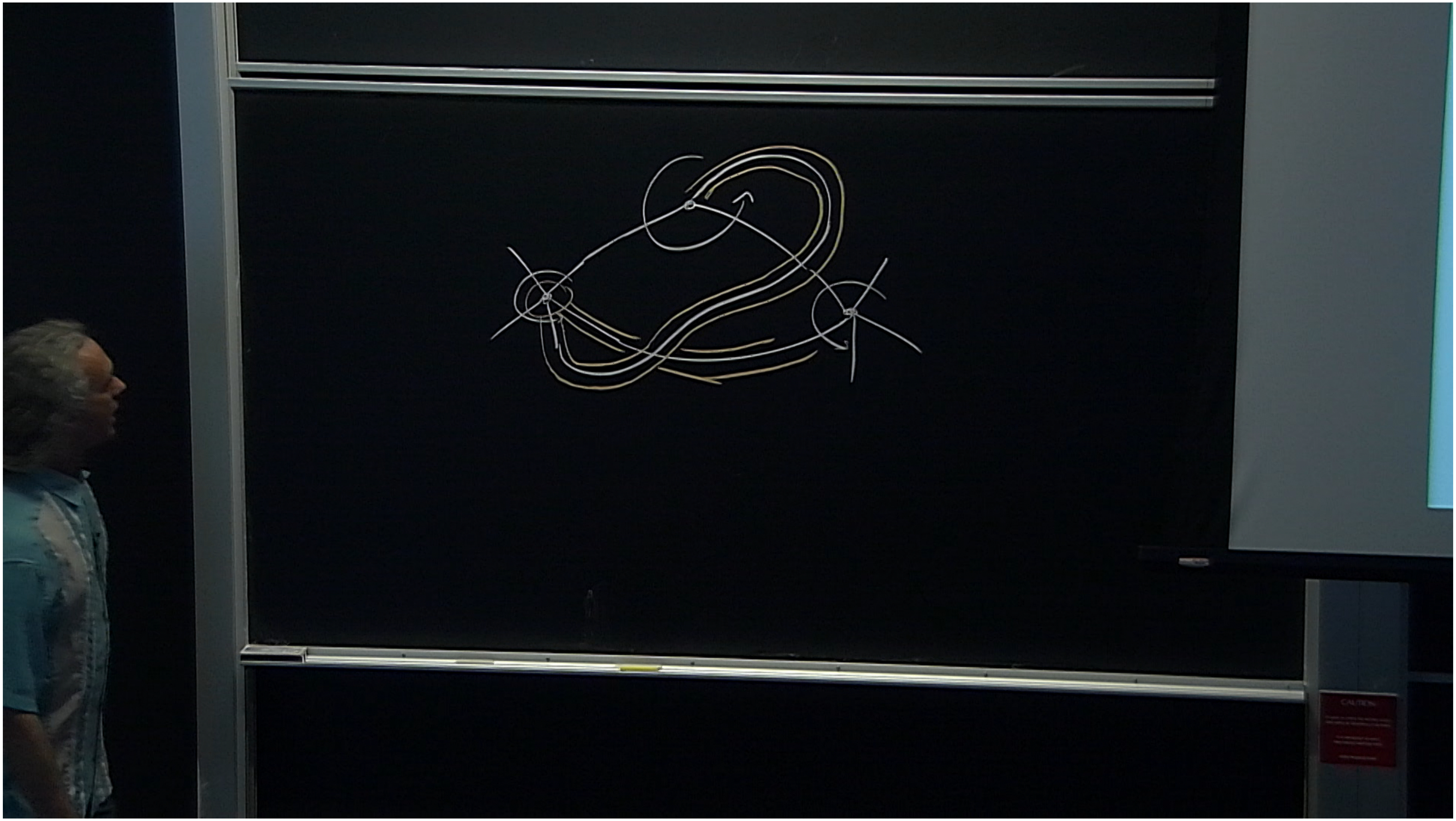
We've got a bit more… A graph with *totally ordered* edge-ends is a **ciliated ribbon graph**

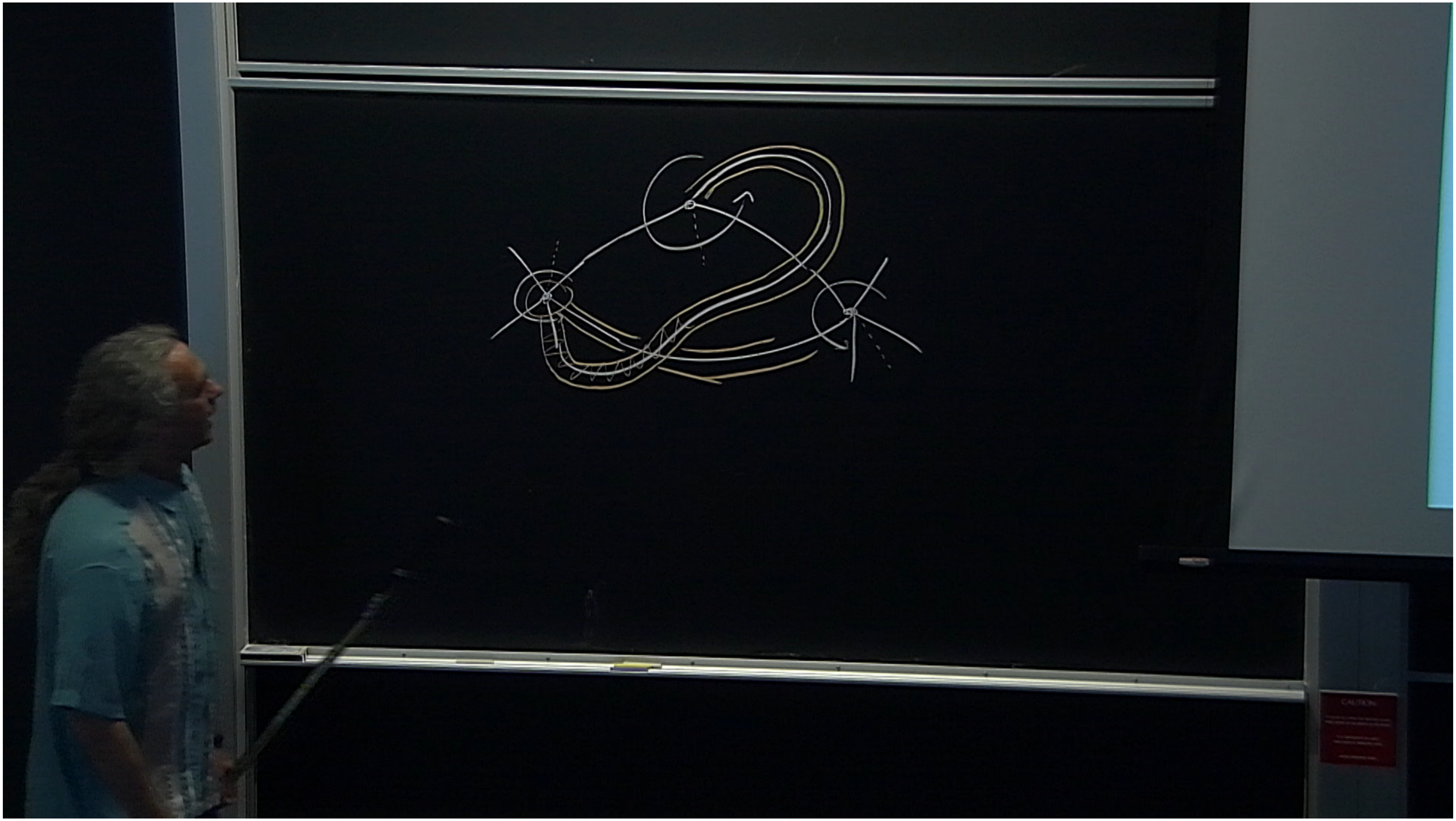
End result is independent of 'ciliation' up to isomorphism, but the cyclic order matters

$\Longrightarrow$ Hopf algebra gauge theory is fundamentally 2-dimensional.









Hopf gauge theory

So far, we've got...

Groups	**Hopf algebras**
Gauge group G	Gauge Hopf algebra H
Graph (V, E)	Ciliated ribbon graph (V, E)
Gauge trans.: $\mathcal{G} = G^V$	Gauge trans.: $\mathcal{G} = H^{\otimes V}$
Connections: $\mathcal{A} = G^E$	Connections: $\mathcal{A} = H^{\otimes E}$
Functions: $\mathcal{A}^* = \mathrm{Fun}(G) = \mathbb{C}[G]^*$	Functions: $\mathcal{A}^* = H^{*\otimes E}$

Observables

Groups

Observables are functions

$$f\colon G^E \to \mathbb{C}$$

that are **gauge invariant**:

$$f(g \triangleright a) = f(a) \qquad \forall g \in G^V$$

Functions form an *algebra* in an obvious way: $\mathcal{A}^* \cong \mathrm{Fun}(G)^{\otimes E}$

Observables form a *subalgebra*.

Hopf algebras

Observables are linear maps

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that are **gauge invariant**:

$$f(g \triangleright a) = \epsilon(g) f(a) \quad \forall g \in H^{\otimes V}$$

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Observables

Groups	Hopf algebras
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Observables form a *subalgebra*.	Observables are **not** a subalgebra, unless H is cocommutative!

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New approach: *generalize* algebra structure on $\mathcal{A}^* \cong H^{*\otimes E}\ldots$

so that observables form a subalgebra.

But first, *why* doesn't the obvious algebra structure work?

Module coalgebras

To get a gauge-invariant *subalgebra* $\mathcal{A}^*_{\text{inv}} \subset \mathcal{A}^*$,
we need the action of $\mathcal{G}$ to preserve the algebra structure of $\mathcal{A}^*$
$\Leftrightarrow$ preserve the coalgebra structure of $\mathcal{A}$.

This means we need $\mathcal{A}$ to be a $\mathcal{G}$-**module coalgebra**:

$$\Delta(h \triangleright a) = \Delta(h) \triangleright \Delta(a) \qquad \epsilon(h \triangleright a) = \epsilon(h)\epsilon(a)$$

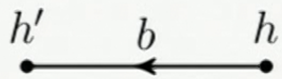
In the group case, this works automatically.

For Hopf algebras it does *NOT* work if we use the tensor product coalgebra structure on $\mathcal{A} \cong H^{\otimes E}$, unless H is cocommutative.

For example ...

Example: Gauge theory "on the edge"

Graph with one edge, two vertices:



$$(h' \otimes h) \triangleright b = h' b S(h)$$

For a module coalgebra, we need:

$$\Delta((h' \otimes h) \triangleright a) \stackrel{?}{=} \Delta(h' \otimes h) \triangleright \Delta(a)$$

However, with the "obvious" coalgebra structure, we find

$$\text{LHS} \qquad h'_{(1)} a_{(1)} S(h_{(2)}) \otimes h'_{(2)} a_{(2)} S(h_{(1)})$$
$$\text{RHS} \qquad h'_{(1)} a_{(1)} S(h_{(1)}) \otimes h'_{(2)} a_{(2)} S(h_{(2)})$$

Fails because S is a coalgebra *anti*homomorphism:

$$\Delta(S(h)) = S(h)_{(1)} \otimes S(h)_{(2)} = S(h_{(2)}) \otimes S(h_{(1)})$$

But S is not the only problem ...

Example: Single vertex

Another example:

Edges all get acted on by gauge transformation at the vertex

But tensor product of module coalgebras is not generally a module coalgebra!

(More problems with ordering of factors in comultiplication...)

Quasitriangular Hopf Algebras

Need to relate Δ with Δ^{op}.

Suggests using a *quasitriangular* Hopf algebra, with R-matrix $R \in H \otimes H$.

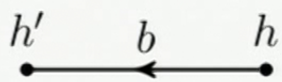
$$\Delta^{\mathrm{op}}(h) = R\Delta(h)R^{-1} \qquad \forall h \in H$$

This helps. For example ...

Gauge theory "on the edge"

Gauge transformations: $\mathcal{G} = H \otimes H$ as a Hopf algebra.

Connections: $\mathcal{A} = H$ as a vector space.
$\mathcal{G}$-module structure:



$$(h' \otimes h) \triangleright b = h' b S(h)$$

Coalgebra structure: (H, δ, ϵ)

$$\delta(a) = \Delta(a) R_{21}$$

This gives a *module coalgebra.*

Functions: Dual of $\mathcal{G}$-module coalgebra structure on $\mathcal{A}$
$\Longrightarrow$ right $\mathcal{G}$-module algebra structure on $\mathcal{A}^*$
$\Longrightarrow$ observables are a *subalgebra.*

Example: Single vertex

Solution is related to Majid's 'braided tensor products' of module (co)algebras ...

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But how do we figure this out systematically?

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Decide on axioms! What should a Hopf algebra gauge theory be like?

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  $\Longrightarrow$ module algebra of functions.

Plan: Axioms for Gauge Theory

Decide on axioms! What should a Hopf algebra gauge theory be like?

- Mimic the gauge action from the group case
- Give a comodule algebra of connections
  $\Longrightarrow$ module algebra of functions.
- Have an algebra of functions that is "local"

Local algebra

Since $\mathcal{A}^* \cong \bigotimes_E K^*$, we have embeddings for edges:

$$\iota_e \colon K^* \to \mathcal{A}^* \qquad \iota_e(\alpha) =: (\alpha)_e$$

and pairs of edges:

$$\iota_{ee'} \colon K^* \otimes K^* \to \mathcal{A}^* \qquad \iota_{ee'}(\alpha) =: (\alpha)_{ee'}$$

Say an algebra structure on $\mathcal{A}^* \cong K^{*\otimes E}$ with unit $1^{\otimes E}$ is **local** if:

(i) each $\iota_e(K^*)$ is a subalgebra of $\mathcal{A}^*$

(ii) each $\iota_{ee'}(K^* \otimes K^*)$ is a subalgebra of $\mathcal{A}^*$

(iii) If $e, e' \in E$ have no common vertex:
$(\alpha)_e \cdot (\beta)_{e'} = (\beta)_{e'} \cdot (\alpha)_e = (\alpha \otimes \beta)_{ee'}$ for all $\alpha, \beta \in K^*$

Hopf Algebra Gauge Theory

(E, V) a ciliated ribbon graph, H a Hopf algebra.

Gauge theory on Γ with values in H consists of:

1 The Hopf algebra $\mathcal{G} = H^{\otimes V}$.

2 The vector space $\mathcal{A} = H^{\otimes E}$, equipped with a coalgebra structure such that the dual algebra structure on $\mathcal{A}^* \cong H^{*\otimes E}$ is *local.*

3 A left $\mathcal{G}$ module structure $\rhd : \mathcal{G} \otimes \mathcal{A} \to \mathcal{A}$ on $\mathcal{A}$ such that:

  (i) $\rhd$ makes $\mathcal{A}$ into a $\mathcal{G}$ module coalgebra,

  (ii) $\rhd$ acts "as expected" for gauge transformations on single edges. That is: if $e \in E$ is not a loop, and $v \in V$ is not an endpoint of e:

$$(h)_v \rhd (a)_e = \epsilon(h)(a)_e$$
$$(h)_{\mathbf{t}(e)} \rhd (a)_e = (hk)_e$$
$$(h)_{\mathbf{s}(e)} \rhd (a)_e = (aS(h))_e.$$

Example: Single vertex

 (thought of as a degenerate 'graph')

For H quasi-triangular, there's an essentially unique algebra structure on $\mathcal{A}^*$ compatible with Hopf algebra gauge theory axioms:

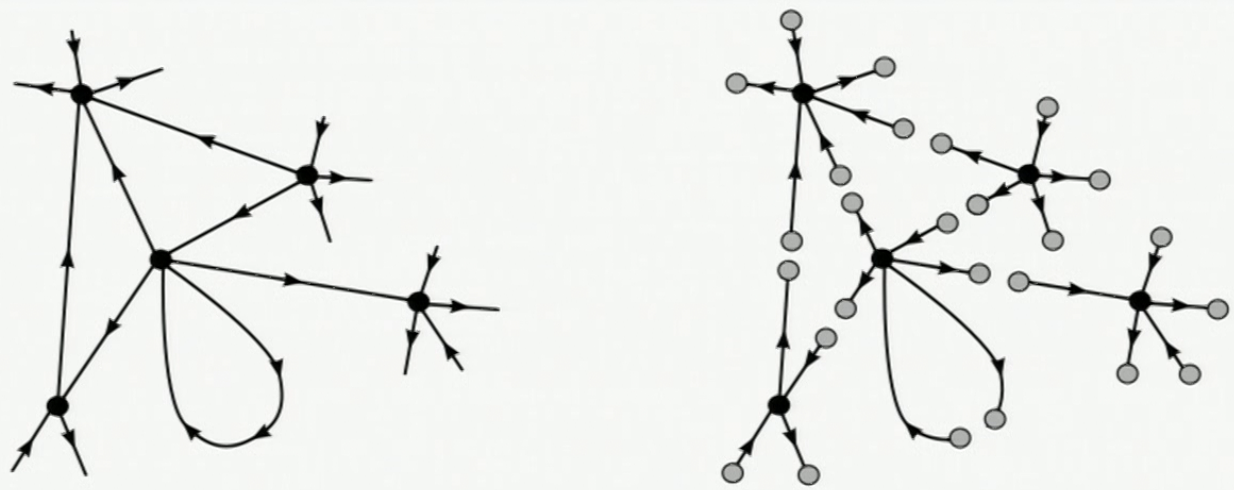
$$(\alpha)_i \cdot (\beta)_j = (\alpha \otimes \beta)_{ij} \qquad i < j$$
$$(\alpha)_i \cdot (\beta)_j = \langle \beta_{(1)} \otimes \alpha_{(1)}, R \rangle \, (\alpha_{(2)} \otimes \beta_{(2)})_{ij} \qquad i > j.$$

and for $i = j$, we have two choices:

$$(\alpha)_i \cdot (\beta)_i = (\alpha\beta)_i \qquad \text{“normal”}$$
$$(\alpha)_i \cdot (\beta)_i = \langle \beta_{(1)} \otimes \alpha_{(1)}, R \rangle \, (\beta_{(2)}\alpha_{(2)})_i \qquad \text{“twisted”}$$

independently for each edge end. (Reversing arrows requires *semisimple*, or more generally, *ribbon* Hopf alg.)

Strategy: Dissect the graph! (Locality lets us do this)



For each edge, one half-edge is "normal", and the other is "twisted". Comultiplication in H^* gives an injective linear map

$$G^* : \mathcal{A}^* \to \bigotimes_{v \in V} \mathcal{A}_v^*$$

Theorem: The image of G^* is a subalgebra and a $K^{\otimes V}$-submodule of $\bigotimes_v \mathcal{A}_v^*$. Pulling back this structure makes $\mathcal{A}^* := K^{*\otimes E}$ into the algebra of functions for a Hopf algebra gauge theory.

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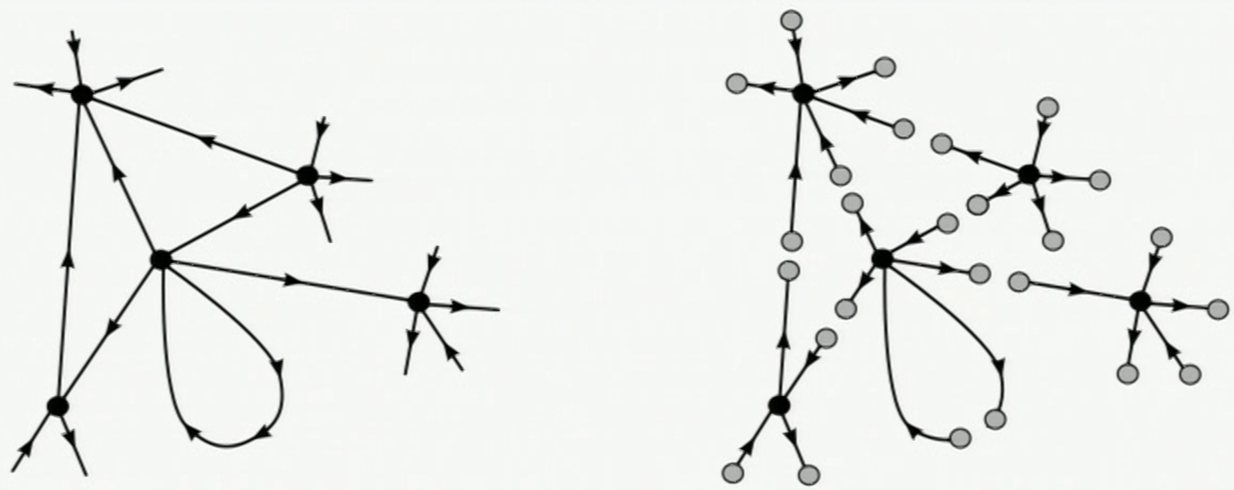

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Results

- Hopf gauge theory determined by axioms: locality, module (co)algebra, and expected local gauge action.
- In any Hopf algebra gauge theory $\mathcal{A}^*_{inv} \subset \mathcal{A}^*$ is a subalgebra, the **algebra of observables**.
- Examples:
  - $H = \mathbb{C}[G] \implies$ Lattice gauge theory for G.
  - $H = D(H)$, single edge $\implies$ Heisenberg double of H
  - $H = D(H)$, single looped edge $\implies D(H)$
- Algebra of functions coincides with the "lattice algebra" from combinatorial quantization of Chern-Simons theory. [Alekseev, Grosse, Schomerus 94], [Buffenoir, Roche 95]
- Topological invariant of the surface with boundary obtained from the ribbon graph.

Holonomy

If H is semisimple, then Hopf algebra gauge theory has a **holonomy functor**:

$$\mathrm{Hol}\colon \mathcal{P} \to \hom(H^{\otimes E}, H)$$

$\mathcal{P}$ is the **path groupoid** of the graph:

- objects: vertices
- morphisms: equivalence classes of edge-paths.

$\hom(H^{\otimes E}, H)$ is an **algebra** with multiplication

$$f \cdot g = m \circ (f \otimes g) \circ \Delta_{\otimes}$$

Associative algebra $\Leftrightarrow$ linear category with one object.

Curvature

- Holonomy around a face is **curvature**. A connection is flat if curvature at every face is 1.
- A Haar integral in H^* gives rise to a projector

$$P_{\text{flat}} : \mathcal{A}^*_{\text{inv}} \to \mathcal{A}^*_{\text{inv}}$$

Image of P_{flat} is the **quantum moduli space** [Alekseev, Grosse, Schomerus '94], [Buffenoir Roche '95], [Meusburger, W]

- Topological invariant of the closed surface obtained from the ribbon graph. (Quantum analog of $\hom(\pi_1, G)/G$).

More information (and references)

C. Meusburger, D. K. Wise, Hopf algebra gauge theory on a ribbon graph, arXiv:1512.03966

Catherine's talk, after the coffee break.