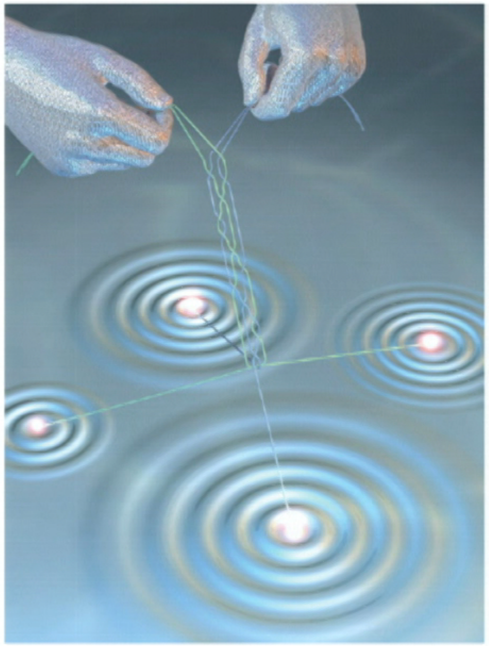
Title: Topological Quantum Computation

Date: Jul 31, 2017 03:00 PM

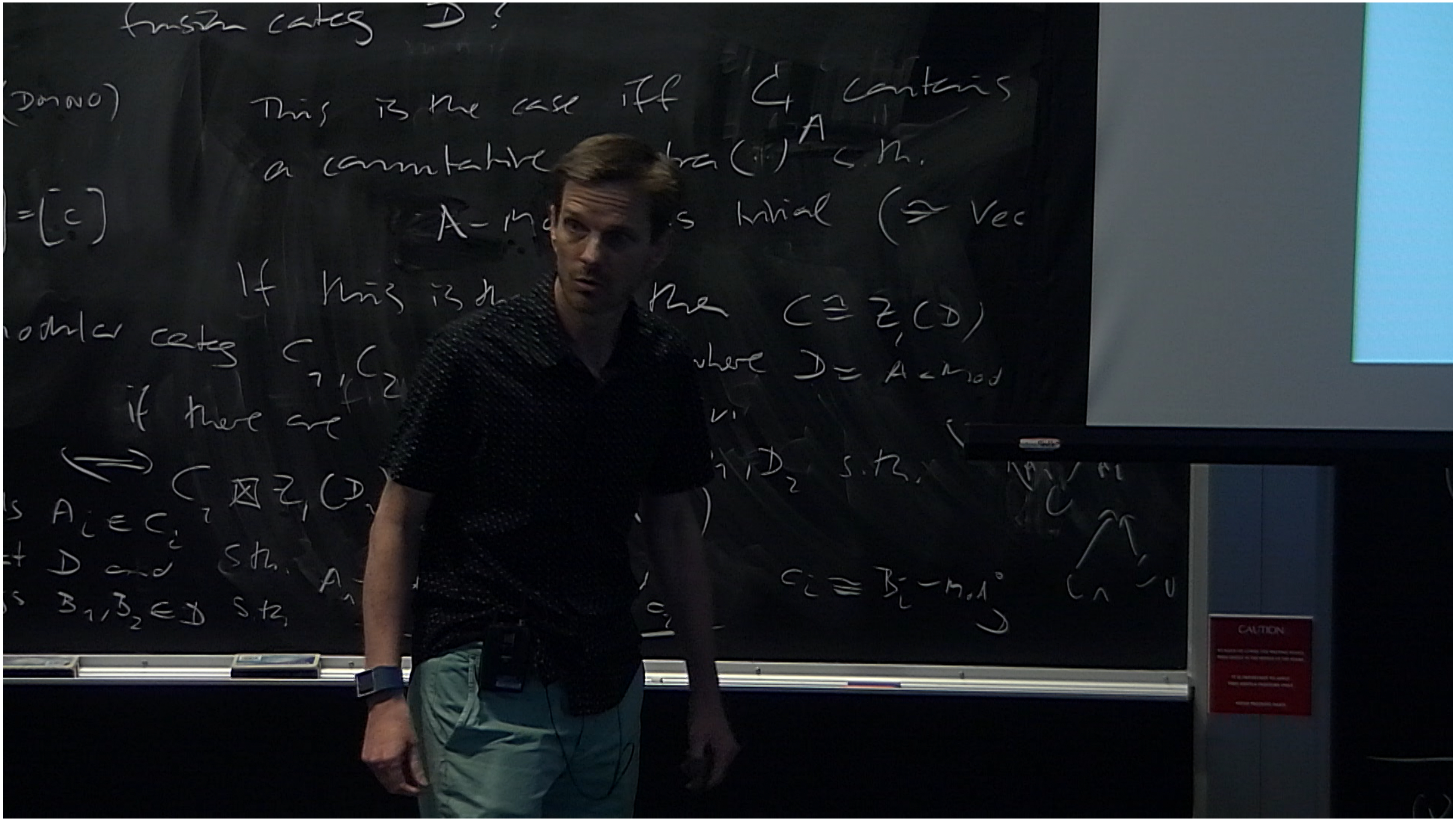
URL: http://pirsa.org/17070065

Abstract: The (Freedman-Kitaev) topological model for quantum computation is an inherently fault-tolerant computation scheme, storing information in topological (rather than local) degrees of freedom with quantum gates typically realized by braiding quasi-particles in two dimensional media. I will give an overview of this model, emphasizing the mathematical aspects.

Topological Quantum Computation



Eric Rowell, PI, July 2017


This is the case iff C contains
a commutative
If this is
where D = A-Mod
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CAUTION

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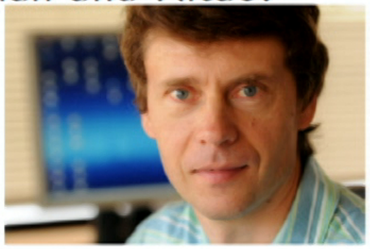

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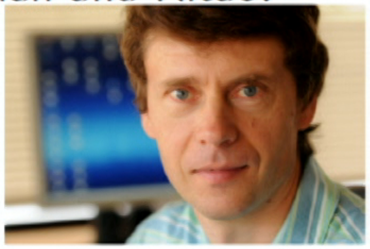

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Exchange Statistics: Anyons

How does the wave function $\psi(z_1, z_2)$ of point-like particles change under $z_1 \leftrightarrow z_2$?

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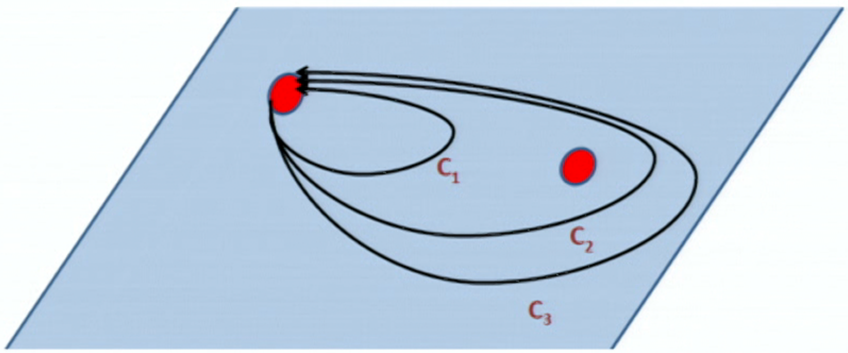

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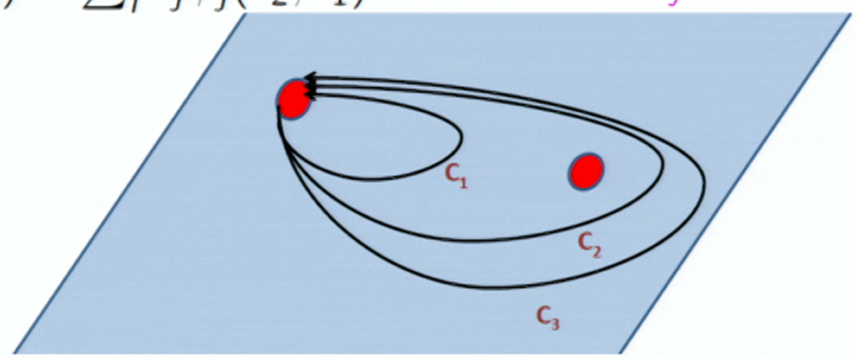

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- Why? $\pi_1(\mathbb{R}^3 \setminus \{z_i\}) = 1$ but $\pi_1(\mathbb{R}^2 \setminus \{z_i\}) = F_n$ Free group.

The hero is the **Braid Group** $\mathcal{B}_n$: $\sigma_1, \ldots, \sigma_{n-1}$ with

(R1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

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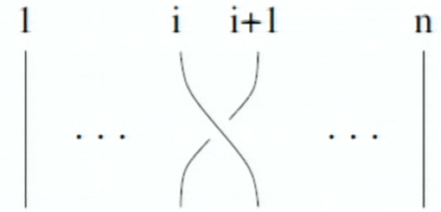

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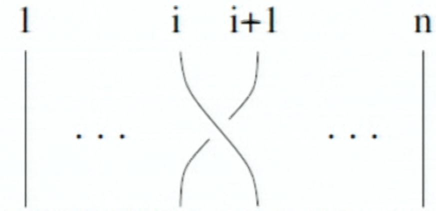

Motions of n points in a disk/Mapping Class Group.

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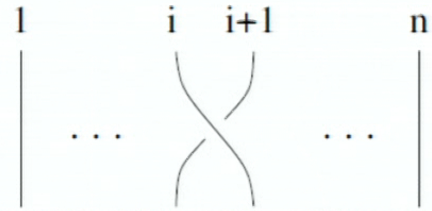

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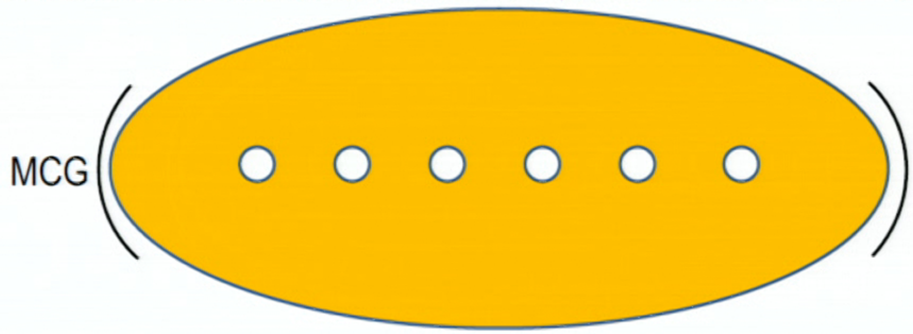
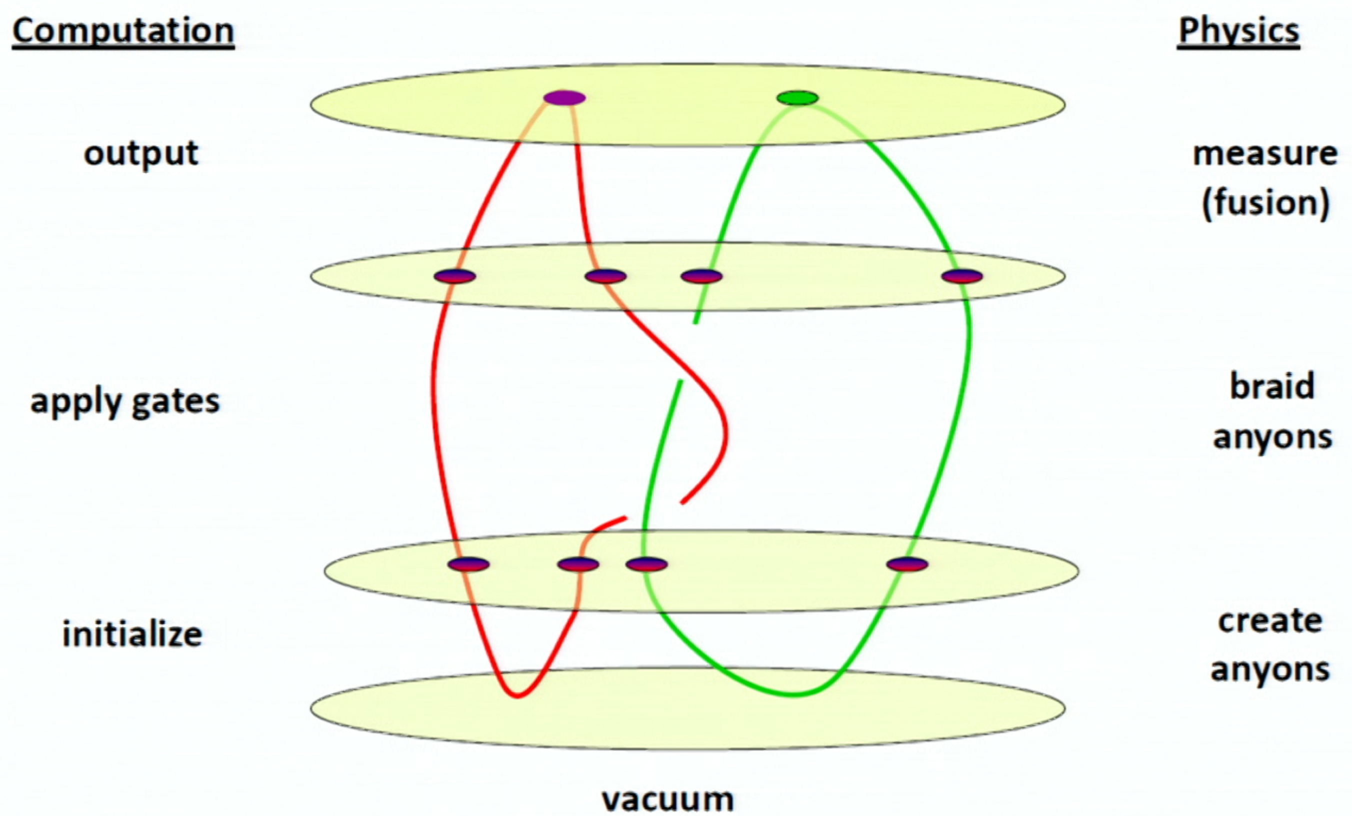


Topological Model



Foundational Questions

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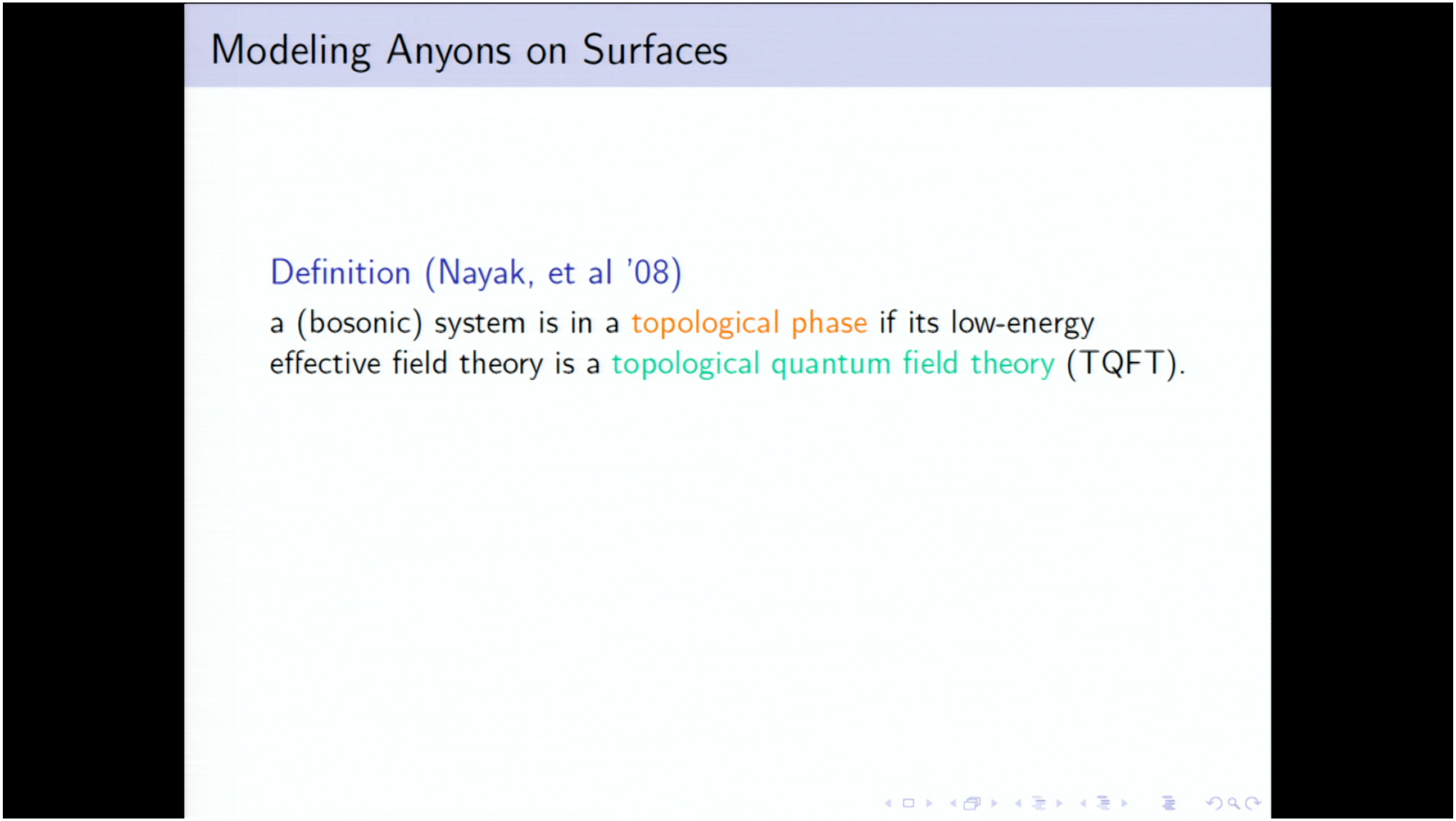
3. How Powerful are TQCs?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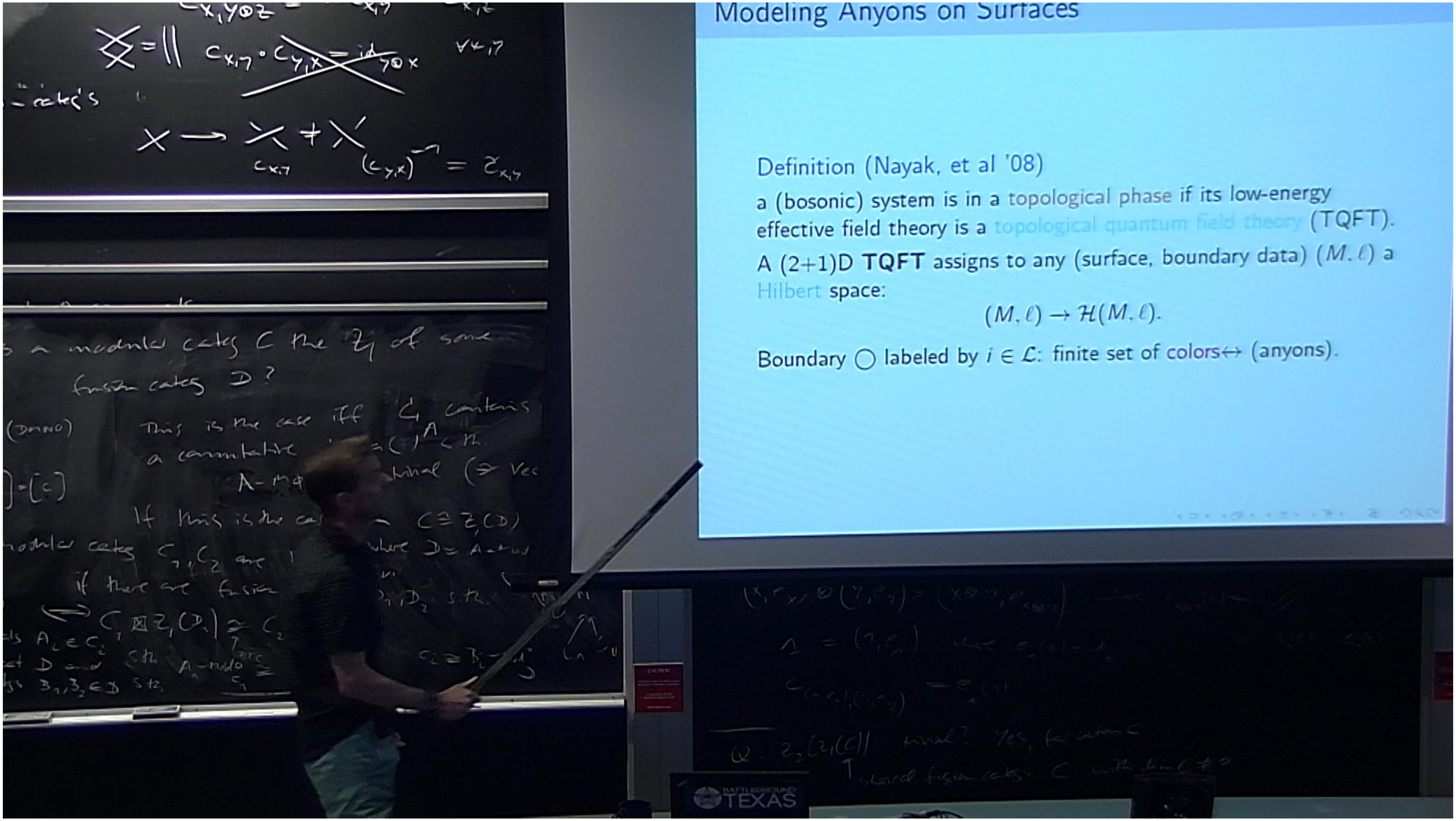
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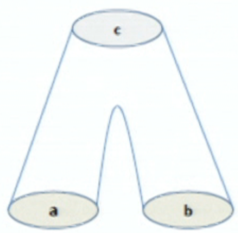

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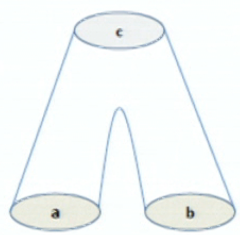
$P :=$ 

$\mathcal{H}(P; a, b, c) = \mathbb{C}^{N(a,b,c)}$ ↖choices!

Compatibility Axioms

Axiom (Disjoint Union)

$$\mathcal{H}[(M_1, \ell_1) \coprod (M_2, \ell_2)] = \mathcal{H}(M_1, \ell_1) \otimes \mathcal{H}(M_2, \ell_2)$$

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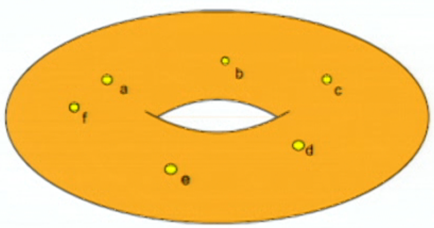

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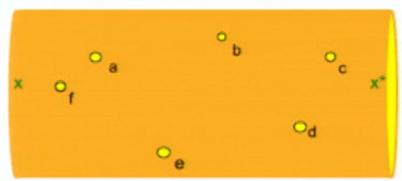

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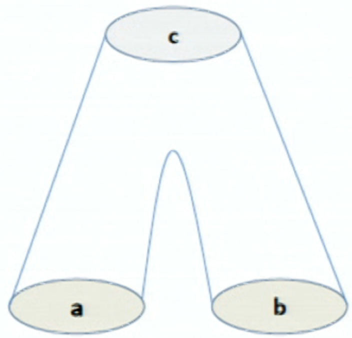
Fusion Channels

The state-space dimension $N(a, b, c)$ of:



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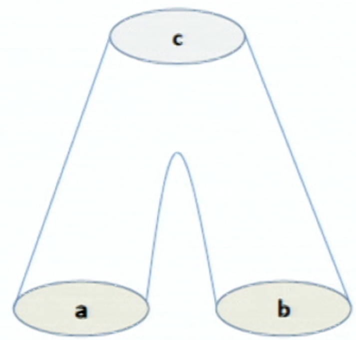

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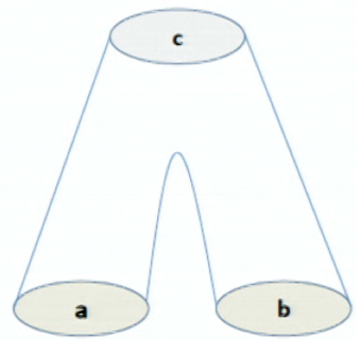

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Principle

The Computational Space $\mathcal{H}_n := \mathcal{H}(D^2; a, \ldots, a)$: the state space of n identical type a anyons in a disk.

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$\mathrm{Rep}(D^\omega G)$ Drinfeld doubles/centers ...

Theorem (Bruillard,Ng,R,Wang)

For fixed k, finitely many models with $|\mathcal{L}| = k$

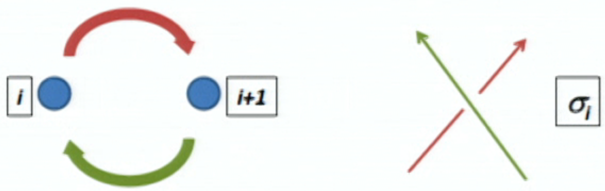
Anyon Model $\leftrightarrow$ Modular Category

anyonic system	Modular Category
anyon types $x \in \mathcal{L}$	simple X
vacuum $0 \in \mathcal{L}$	$\mathbf{1}$
x^* antiparticle	dual X^*
$\mathcal{H}(P; x, y, z)$ state space	$\mathrm{Hom}(X \otimes Y, Z)$
particle exchange	braiding $c_{X,X}$
Locality	Gluing Axiom
Entanglement	Disjoint Union Axiom
anyon types distinguishable	$\det(S) \neq 0$
topological spin	θ_X
n anyon state space	$\mathrm{End}(X^{\otimes n})$

Topological Circuits and Fault-Tolerance

Fix anyon a

- $\sigma_i \in \mathcal{B}_n$ acts on $\mathcal{H}_n = \mathrm{End}(X_a^{\otimes n})$ by particle exchange



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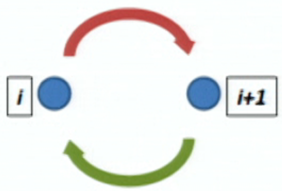

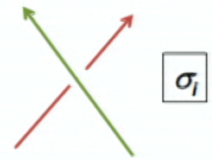

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Information is de-localized$\rightarrow$ **Fault-Tolerance** as errors are local.

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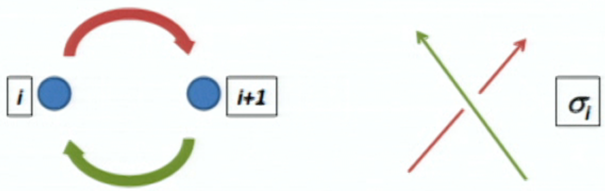

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quantum gates $\rho_a(\sigma_i)$ are non-local, topological operations.

Simulating TQCs on QCM

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- Set $V = \bigoplus_{(a,b,c)\in\mathcal{L}^3} \mathcal{H}(P; a, b, c)$ and $W_n = V^{\otimes(n-1)}$
- TQFT axioms (gluing, disjoint union) imply:

$$\mathcal{H}_n \oplus \mathcal{H}_n^{\perp} = W_n$$

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Freedman, Kitaev & Wang showed TQCs have hidden locality: Let $U(\beta) \in \mathbf{U}(\mathcal{H}_n)$ be a braiding quantum circuit.

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- Set $V = \bigoplus_{(a,b,c)\in\mathcal{L}^3} \mathcal{H}(P; a, b, c)$ and $W_n = V^{\otimes(n-1)}$
- TQFT axioms (gluing, disjoint union) imply:

$$\mathcal{H}_n \oplus \mathcal{H}_n^{\perp} = W_n$$

Remark

V can be quite large and $U(\beta)$ only acts on the subspace $\mathcal{H}_n$, non-computational space $\mathcal{H}_n^{\perp}$ can be large

Local $\mathcal{B}_n$ representations: Yang-Baxter eqn.

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$$v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \xrightarrow{\rho^R(\sigma_i)} v_1 \otimes \cdots \otimes R(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_n$$

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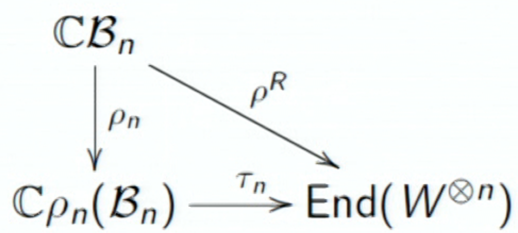

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$$\begin{array}{ccc} \mathbb{C}\mathcal{B}_n & & \\ \downarrow \rho_n & \searrow^{\rho^R} & \\ \mathbb{C}\rho_n(\mathcal{B}_n) & \xrightarrow{\tau_n} & \mathrm{End}(W^{\otimes n}) \end{array}$$

Idea: Push braiding gates inside a braided QCM.

Example Ising $\mathcal{C}(\mathfrak{sl}_2, 4)$

Let $R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

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Question (Quantum Information)

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KEY: $\overline{\rho_\tau(\mathcal{B}_n)} \supset SU(f_n) \times SU(f_{n-1})$, where $\mathcal{H}_n \cong \mathbb{C}^{f_n} \oplus \mathbb{C}^{f_{n-1}}$,

$$f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^2}{2^n\sqrt{5}}$$

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$\dim(\sigma) = \sqrt{2}$ (Ising) while $\dim(\tau) = \frac{1+\sqrt{5}}{2}$ (Fibonacci).

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True for "fundamental" anyons in all quantum group models.

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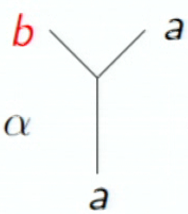

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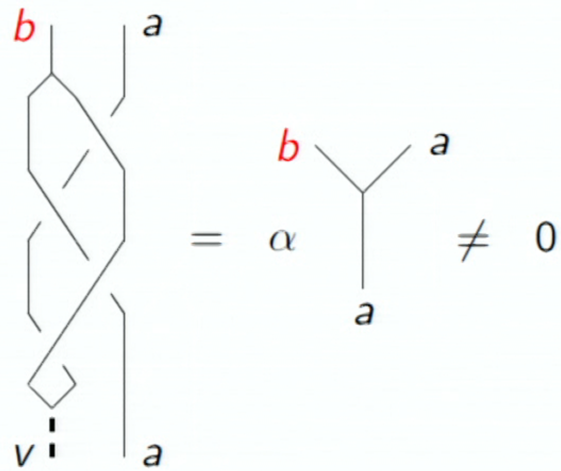

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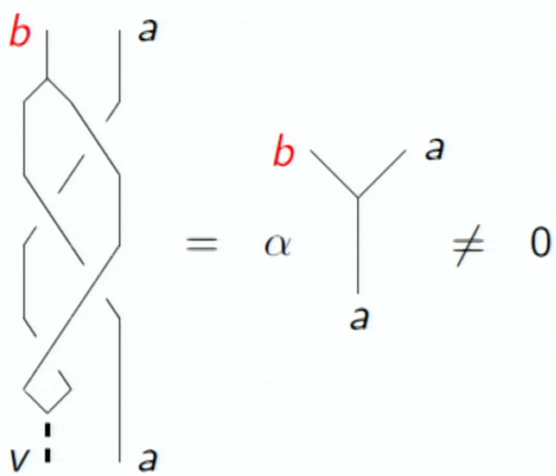

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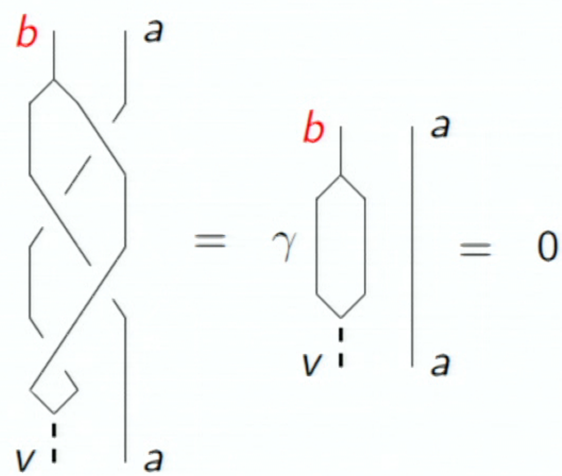

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THANK YOU!

see: arXiv:1705.06206