Title: The Hopf C*-algebraic quantum double models - symmetries beyond group theory

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Abstract: I will give an introduction to the Kitaev quantum double models for Hopf C*-algebras. To this end I will introduce a graphical tensor-network notation to represent the algebraic objects and axioms. Using this notation I will then present the vertex- and plaquette symmetries of the model and discuss their interaction and the excitation structure they give rise to.

The Hopf C*-algebraic quantum double models - symmetries beyond group theory

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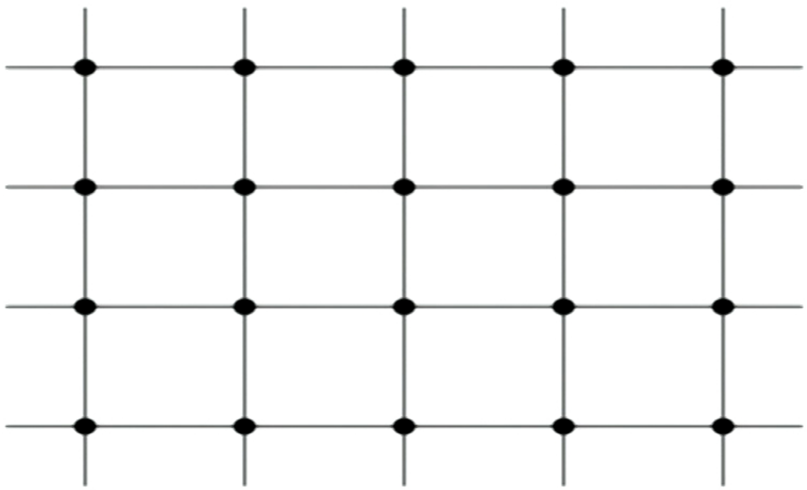
18.07.2016

Outline

- String-net picture of the toric code
- Hopf C*-algebras in tensor network notation
- The plaquette and vertex symmetries
- Excitations and quantum double algebra
- String- and ribbon operators

String-net picture of the toric code

- *Groundstate*: superposition of all closed loop configurations



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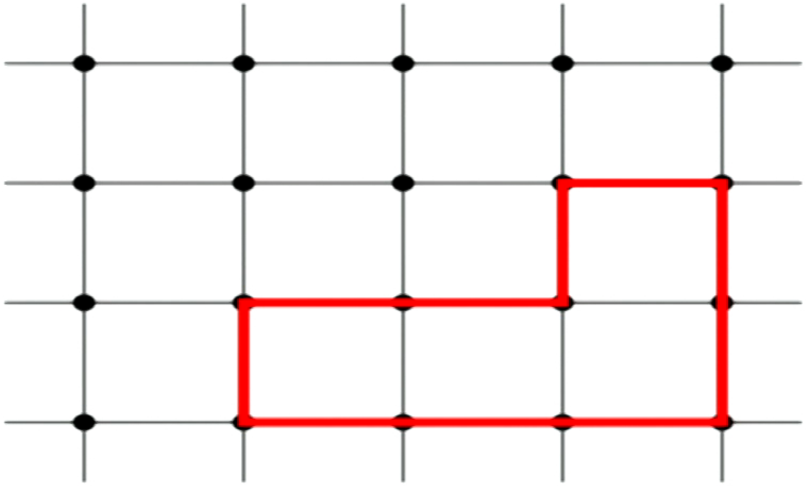

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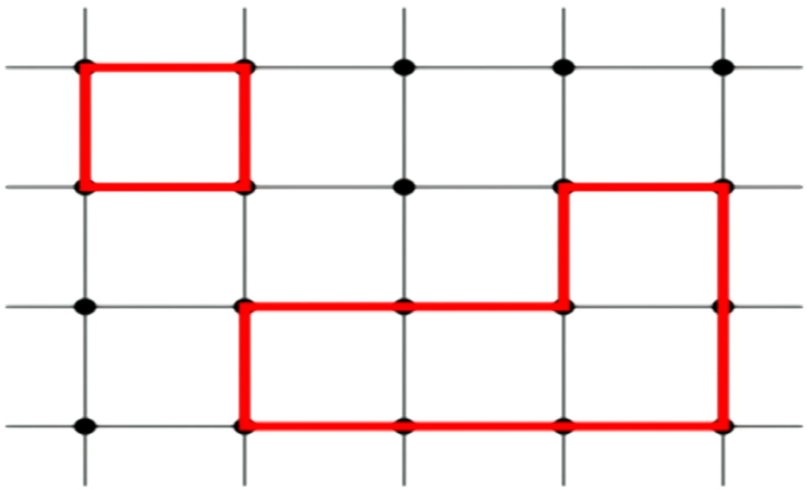

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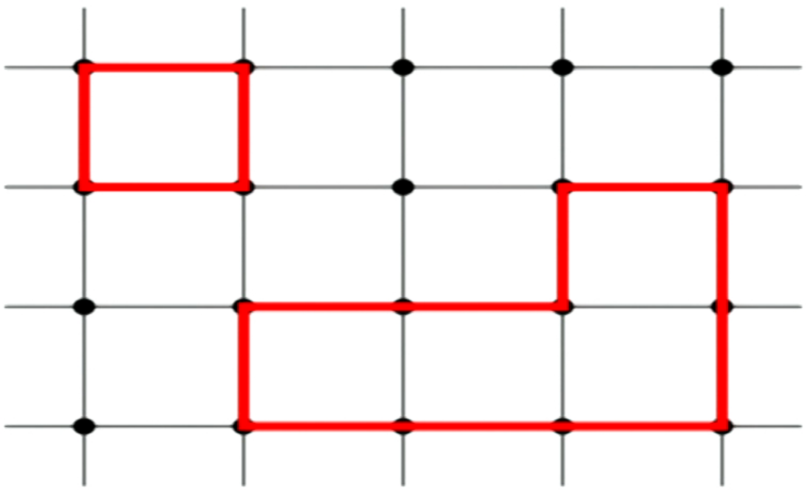

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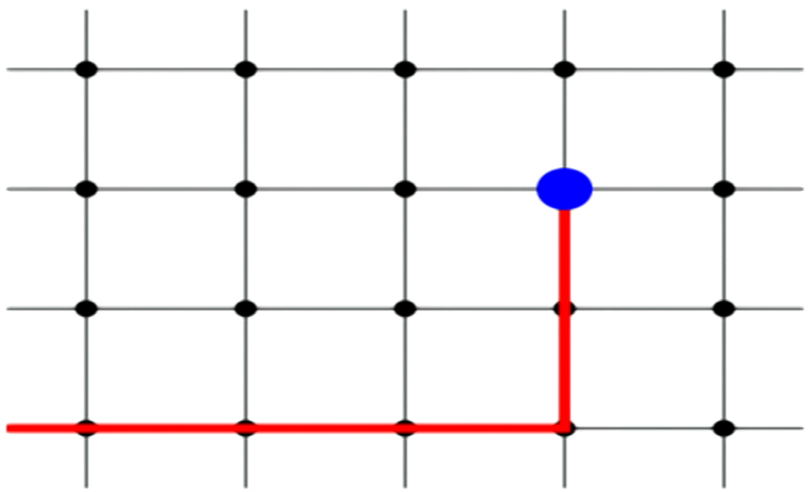

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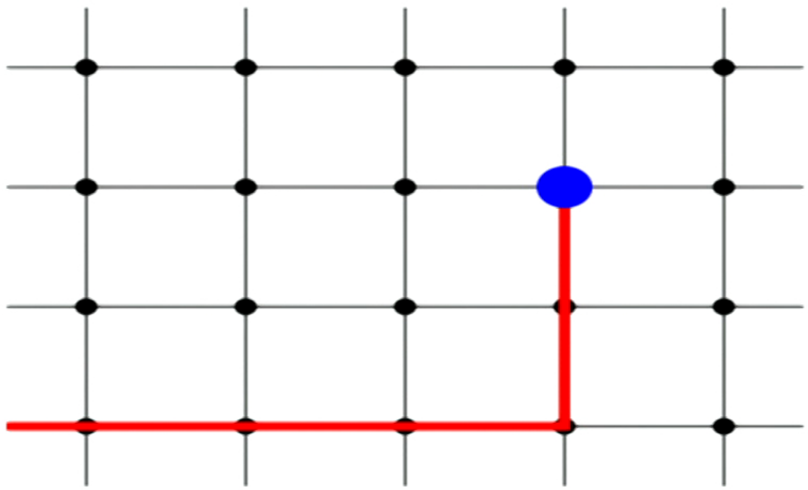

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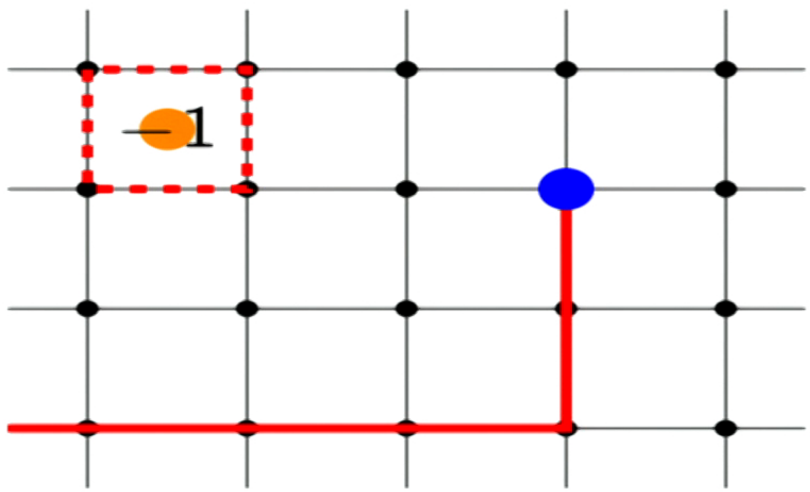

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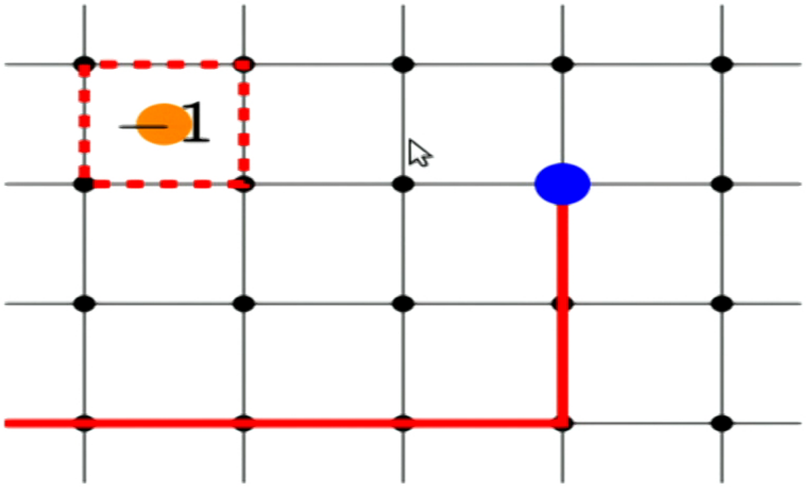

Generalisations

Toric code: Z_2

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↓

String-net models:
Unitary fusion categories

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equivalent

Quantum doubles:
Weak Hopf C*-algebras

Hopf C*-algebras – objects

$\eta : \mathbb{C}^1 \to \mathbb{C}^H$

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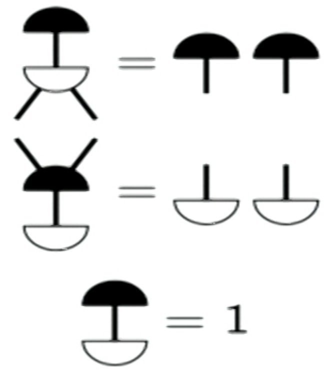

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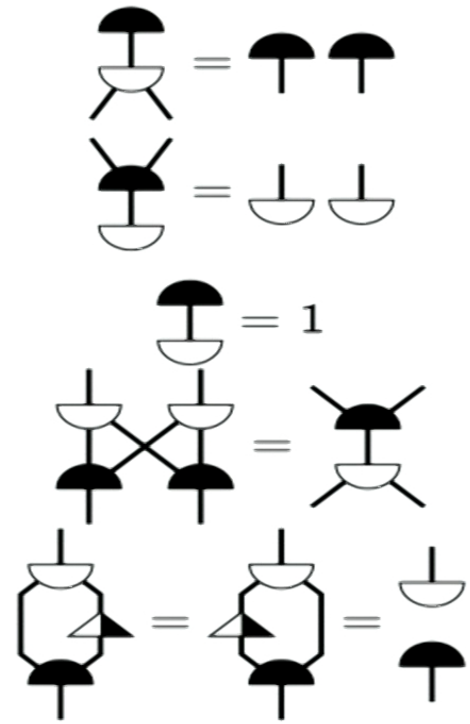

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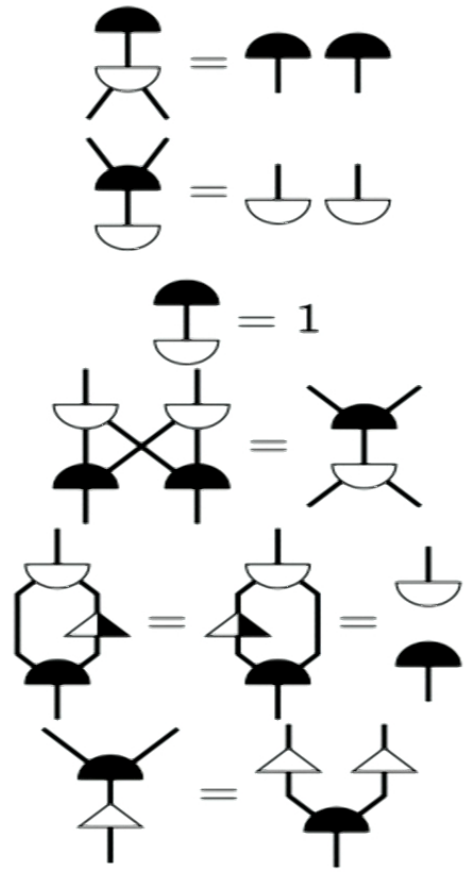
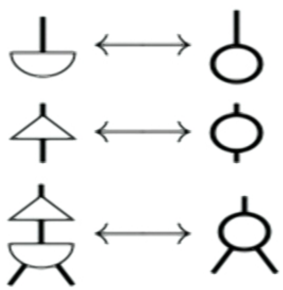


Hopf C*-algebras – cyclic form



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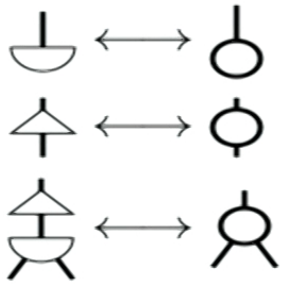

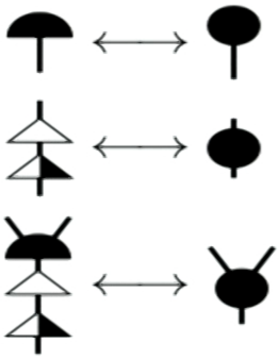

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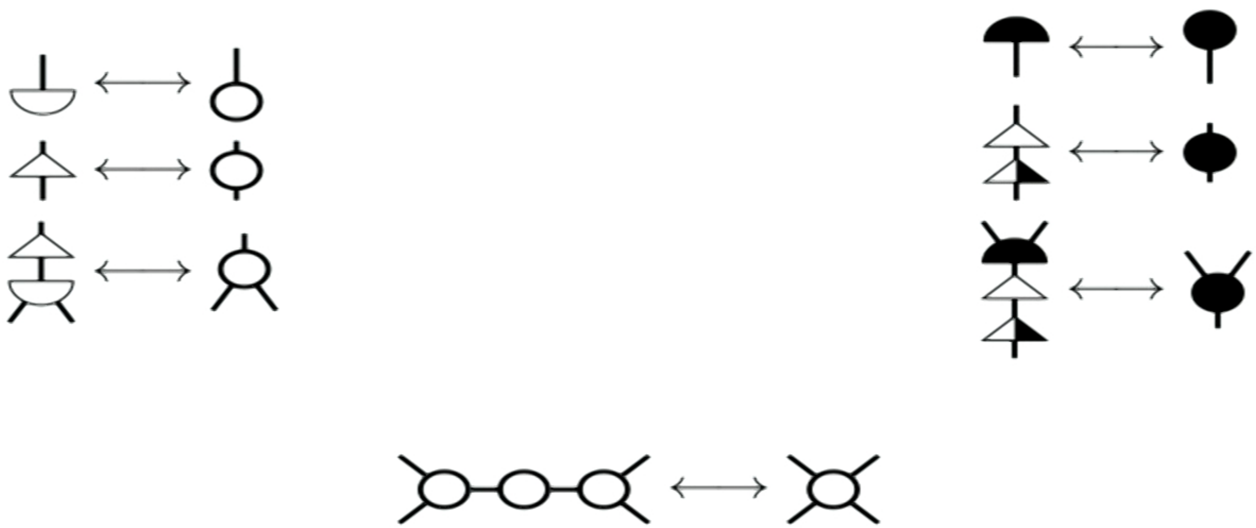

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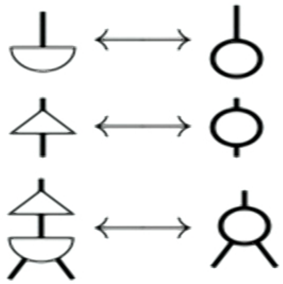

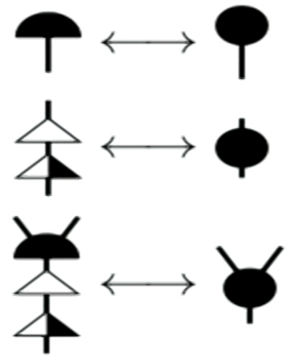

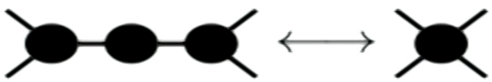

Hopf C*-algebras as symmetries

○ symmetry operations | ● particles

Hopf C*-algebras as symmetries

○	symmetry operations	●	particles
○—	trivial symmetry operation for all particles	●—	trivial particle for all symmetry operations

Hopf C*-algebras as symmetries

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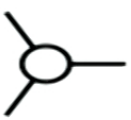
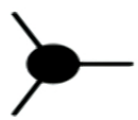
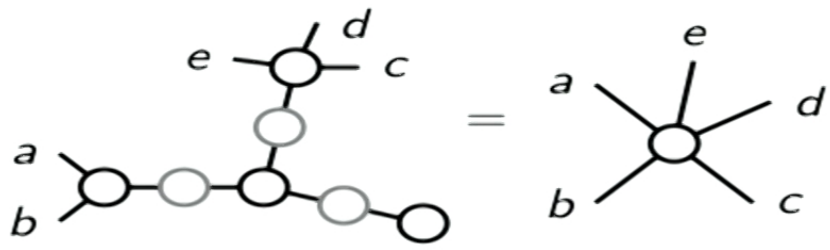
	concatenation of symmetry operations copying of a particle		fusion of particles copying a symmetry

Hopf C*-algebras – cyclic form

- 

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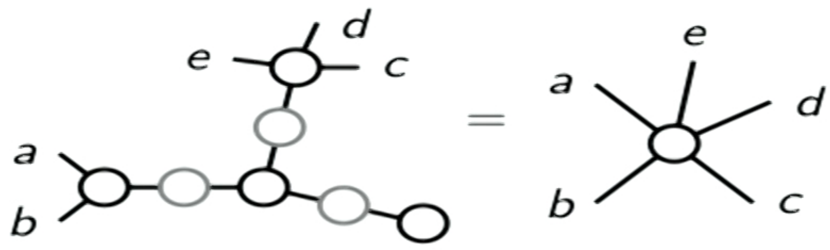
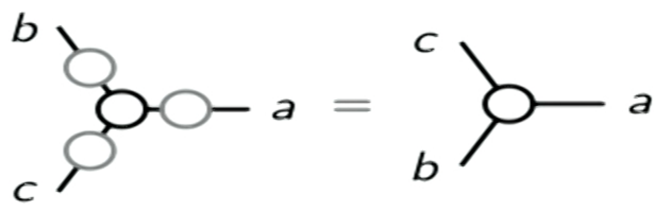
- 

- 

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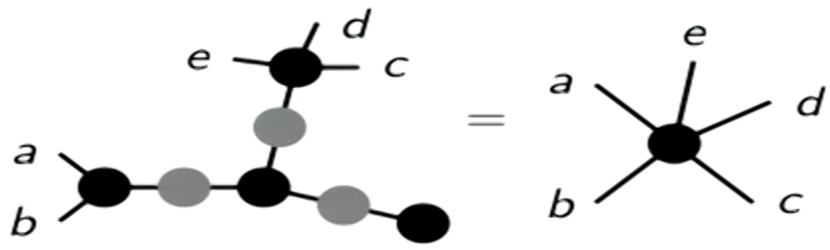
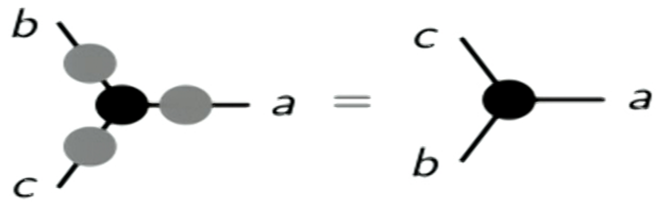
- 

- 

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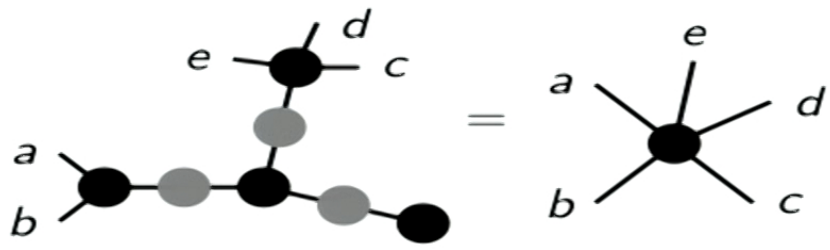
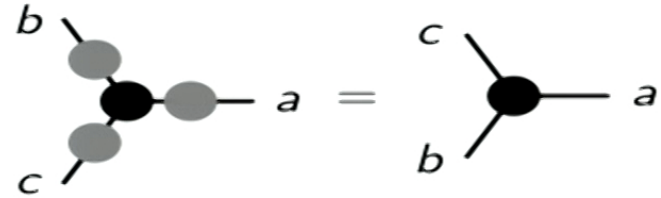
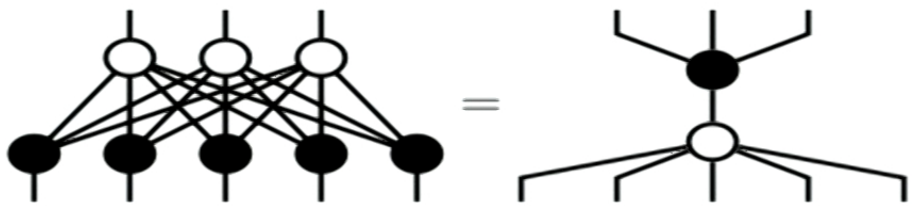
- 

- 

- 

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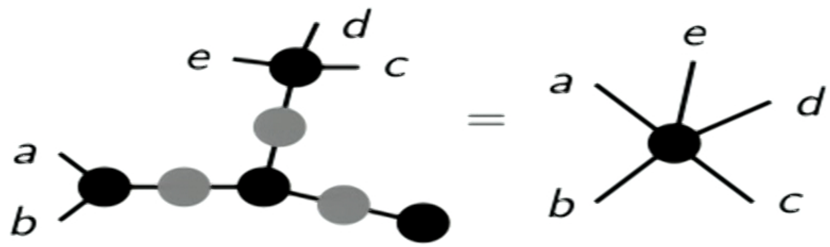
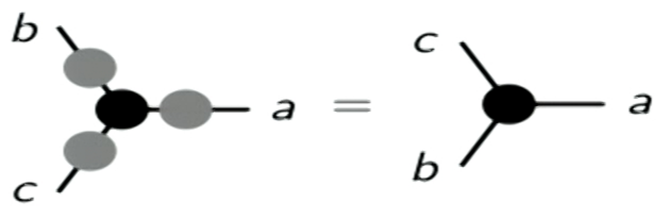
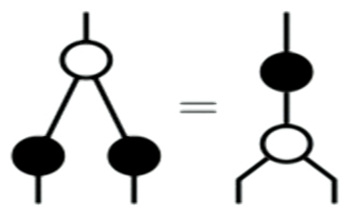
- 

- 



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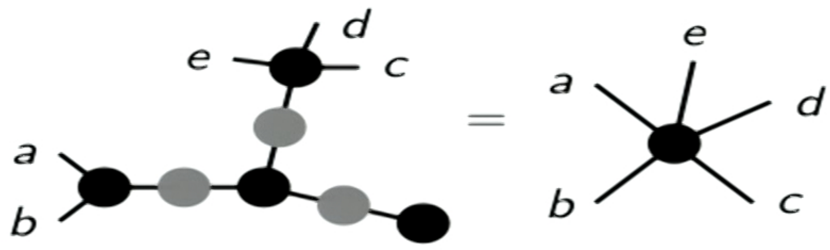
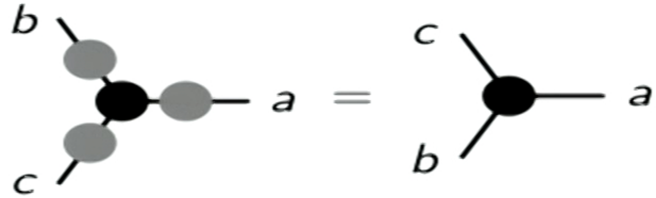
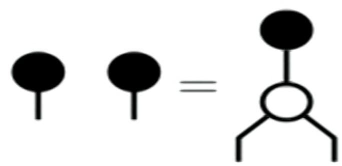
- 

- 



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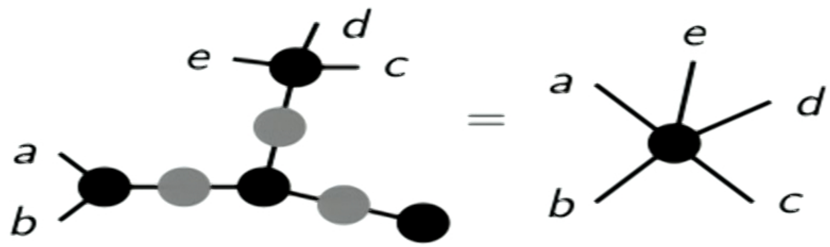
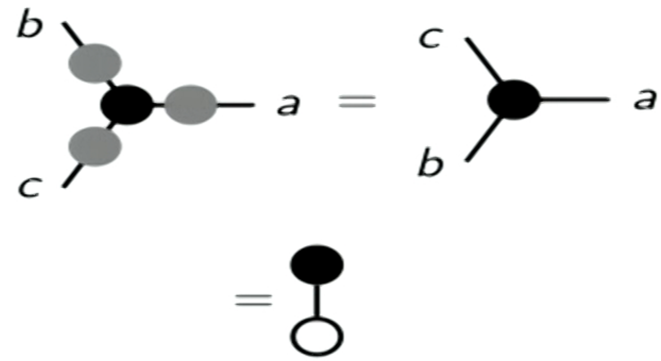
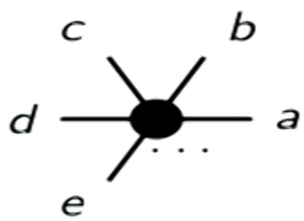
- 

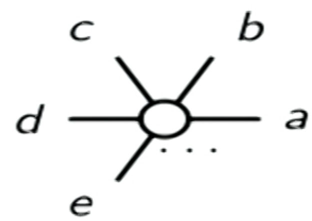
- 

Hopf C*-algebras as quantum groups

- Every group defines Hopf C*-algebra



$= \delta_{a,b,c,d,e,\ldots}$



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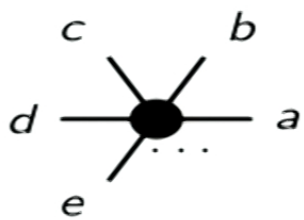

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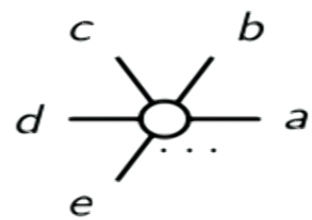

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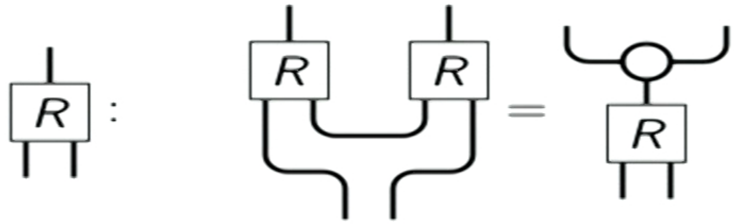
- Hopf C*-algebra axioms can be seen as linearisation of group axioms

Hopf C*-algebras – representations

- Representation: homomorphism to matrix algebra



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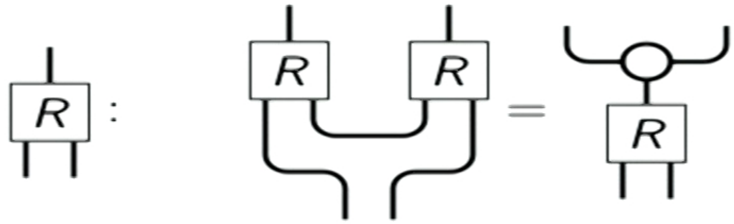

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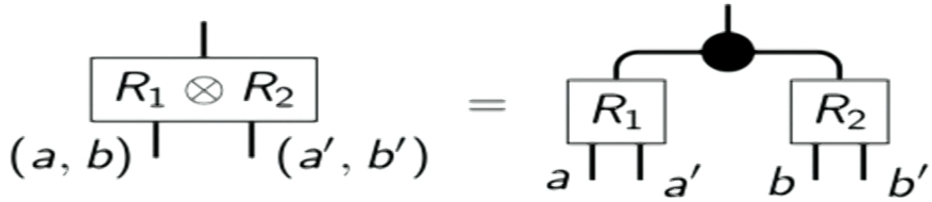

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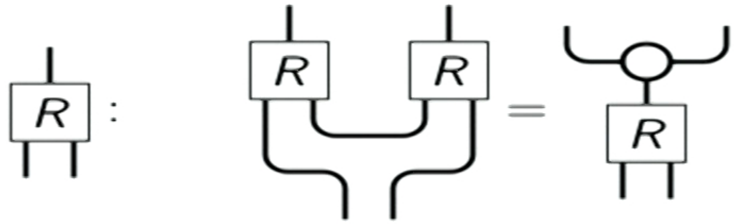

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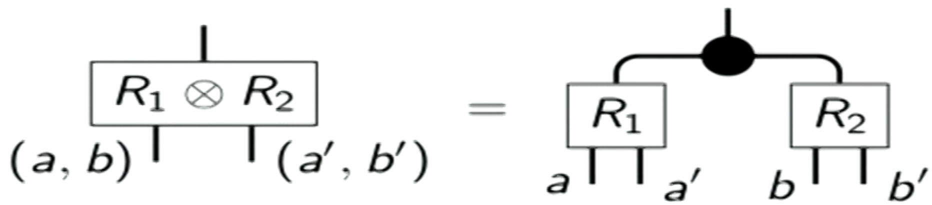

- Left- and right-regular representation:

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- State space is representation of global Hopf C*-algebra $\mathcal{G} := \bigotimes_{s \in S} G_s$
- Decompose state space into *irreducible subspaces* under action of $\mathcal{G}$:

$$H = \bigoplus_{\alpha, m} V^{\alpha, m}, \text{ where } \alpha = (\alpha_1, ..., \alpha_S) \text{ and } m = 1, ..., M_\alpha$$

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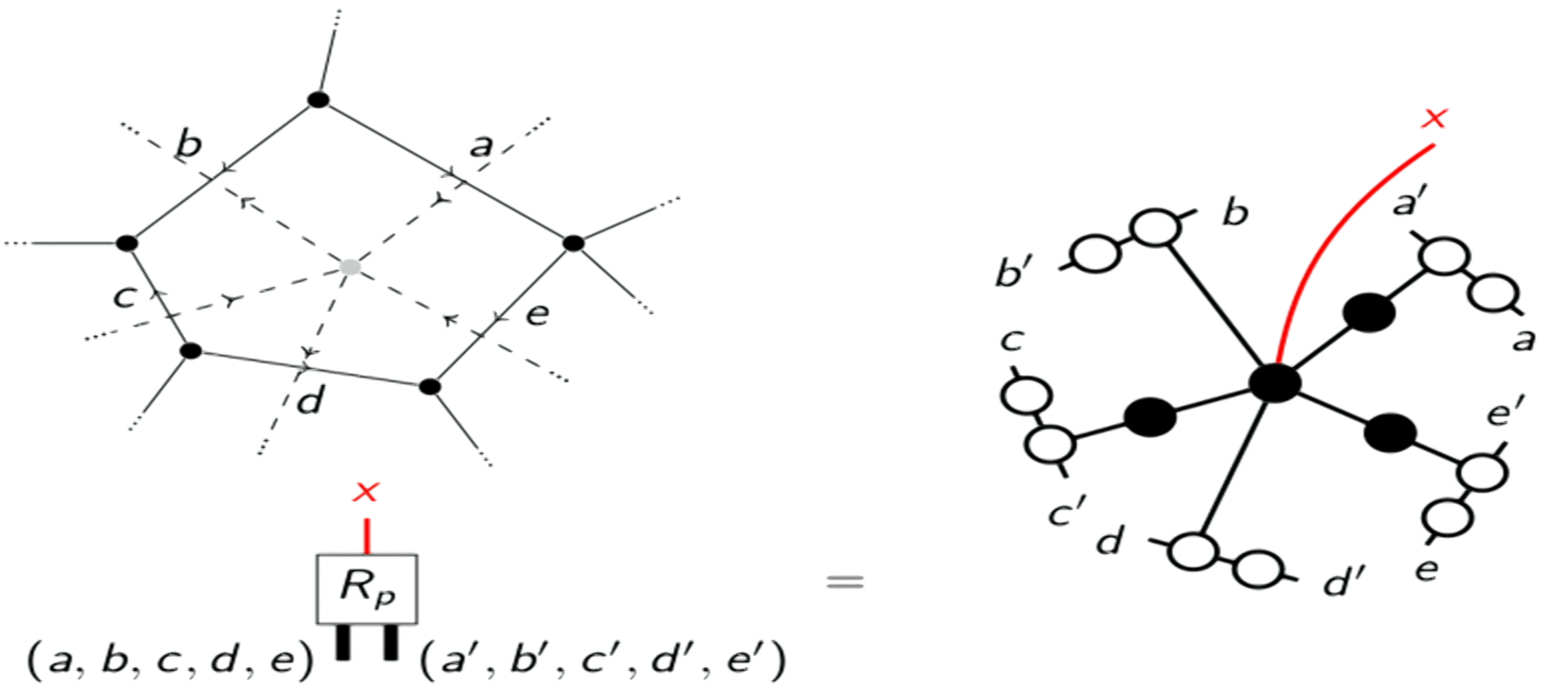

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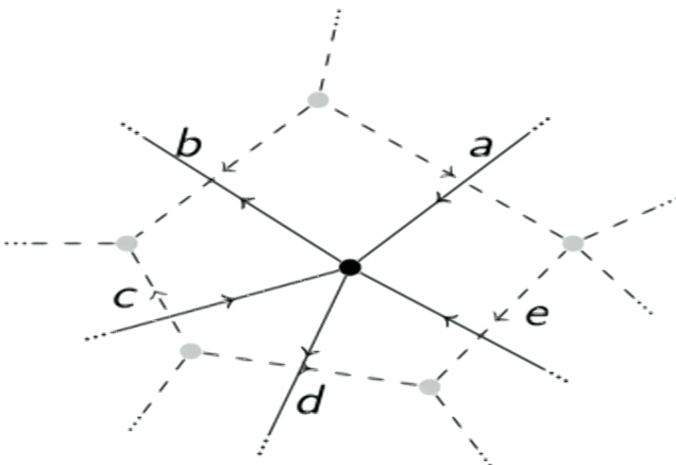
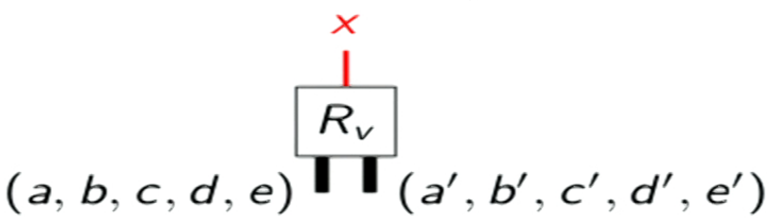
- Vertex symmetry: representation of dual of G





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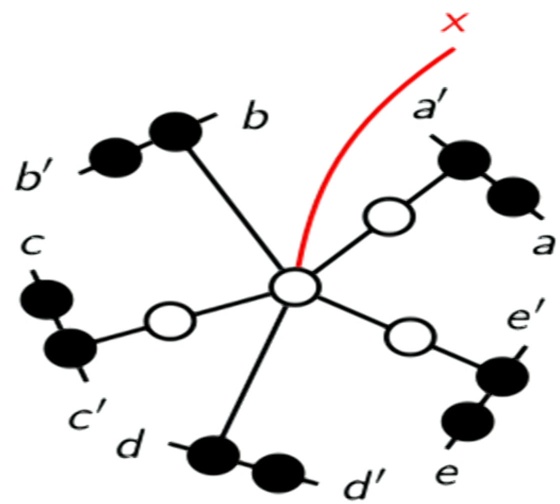

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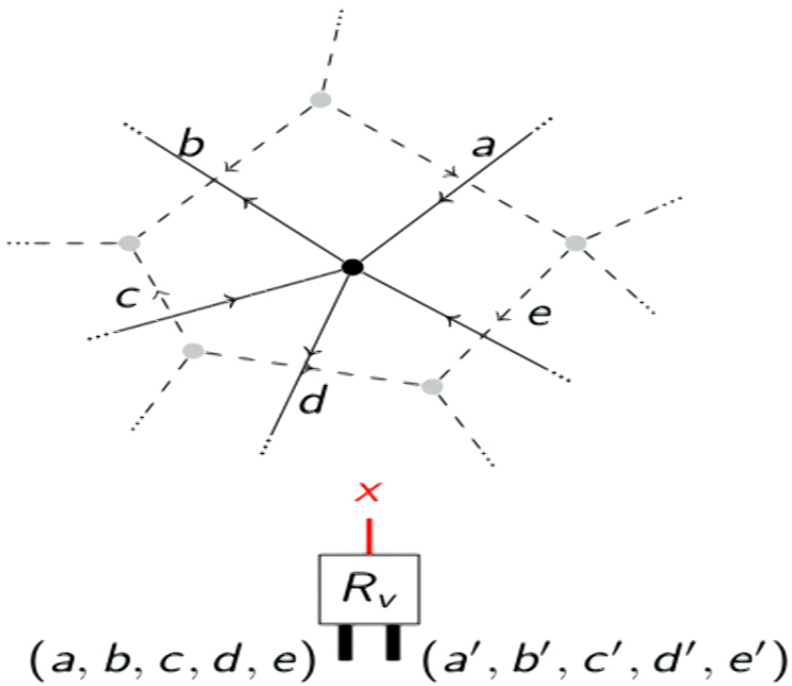

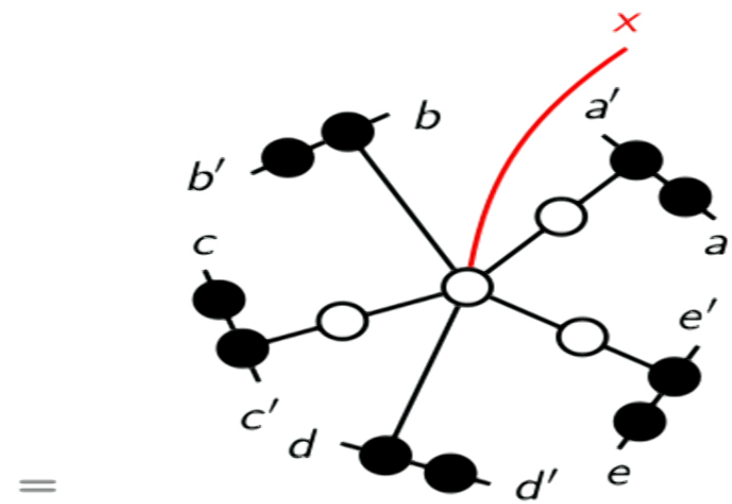

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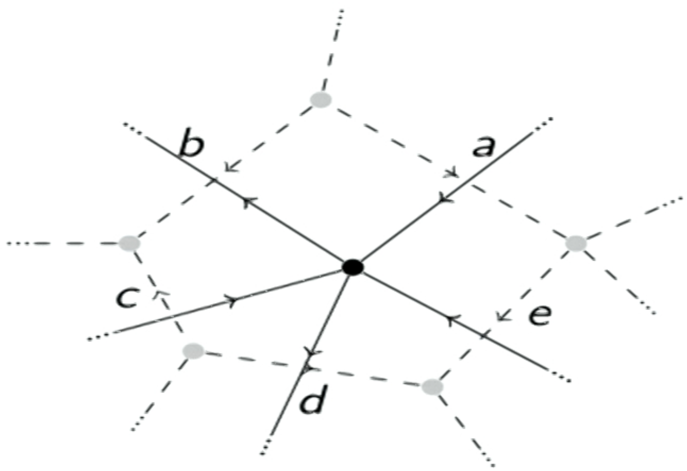

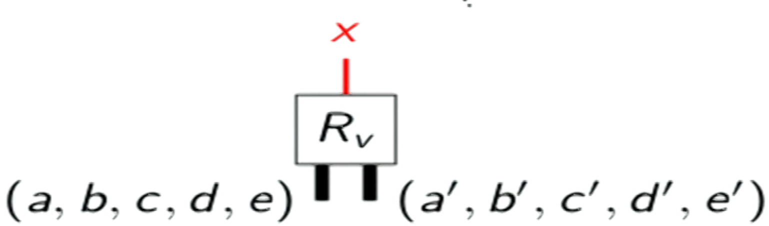

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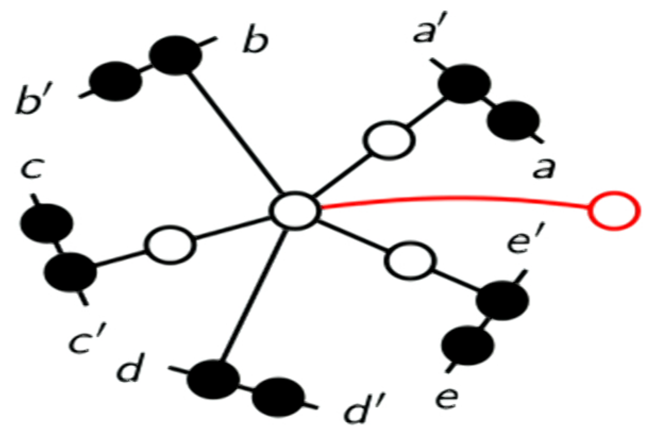

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- Plaquette symmetry: representation of Hopf C*-algebra G
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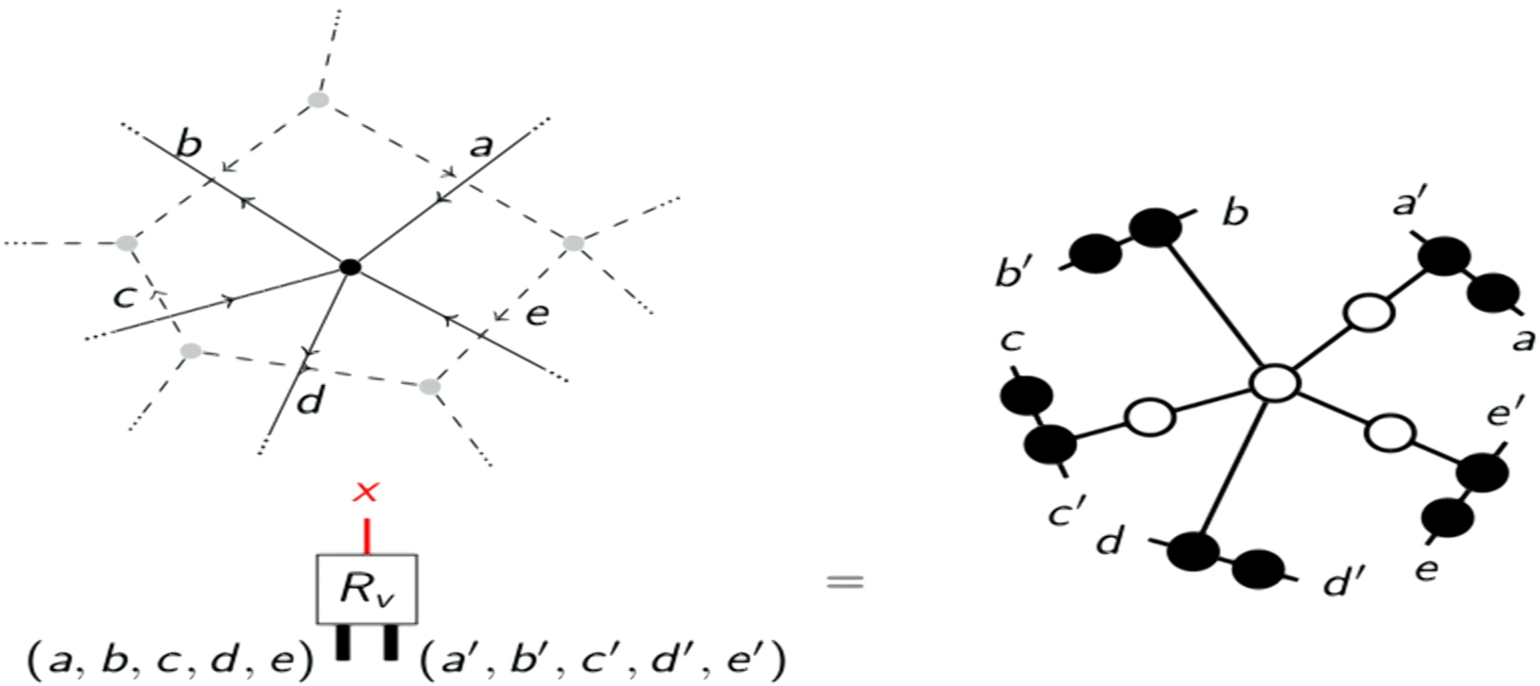
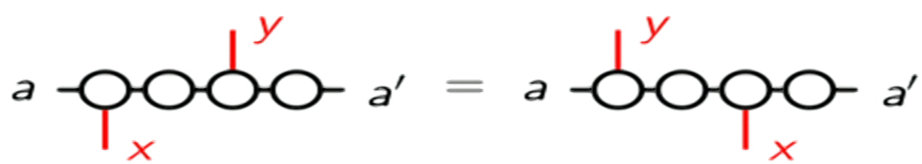


Vertex- and plaquette symmetries – commutation properties

- Vertex-vertex and plaquette-plaquette pairs commute



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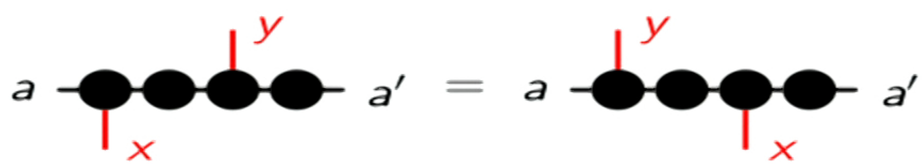
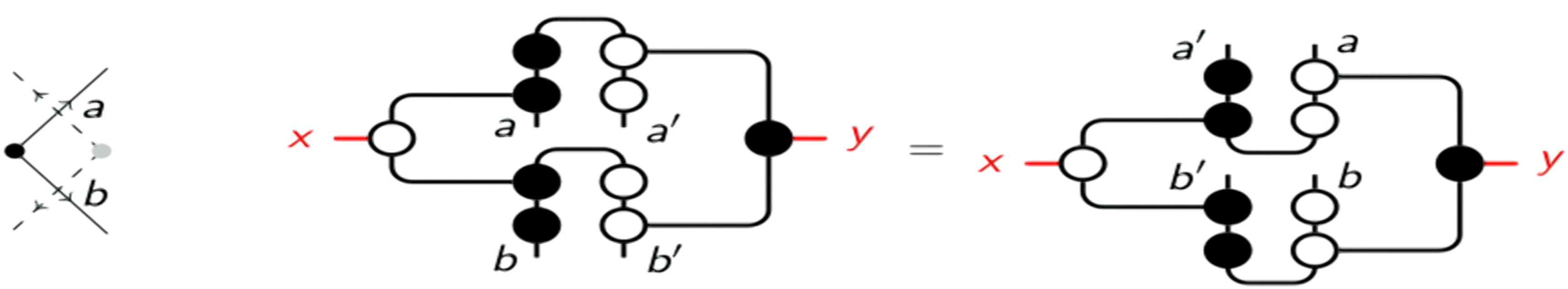
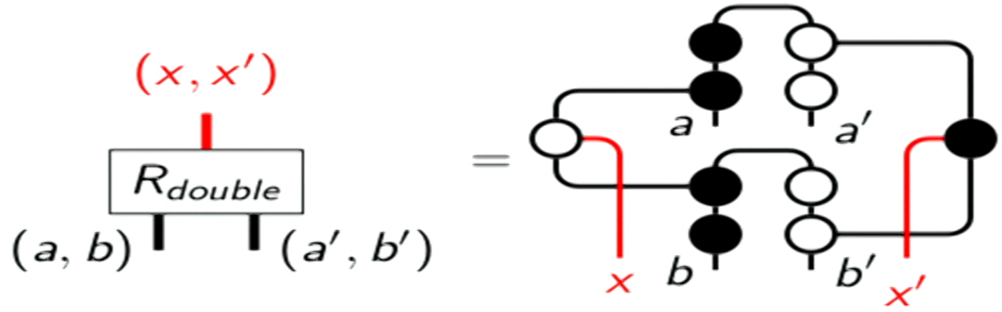


- Neighboring plaquette-vertex pairs commute depending on location of red index



Plaquette and vertex term – together

- Consider product of plaquette- and vertex term



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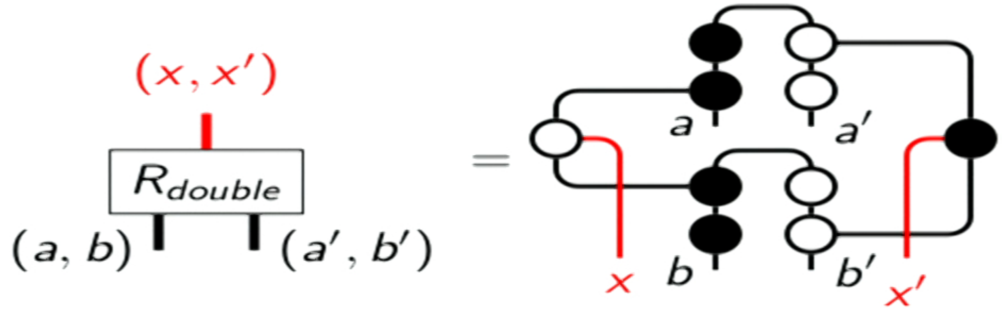
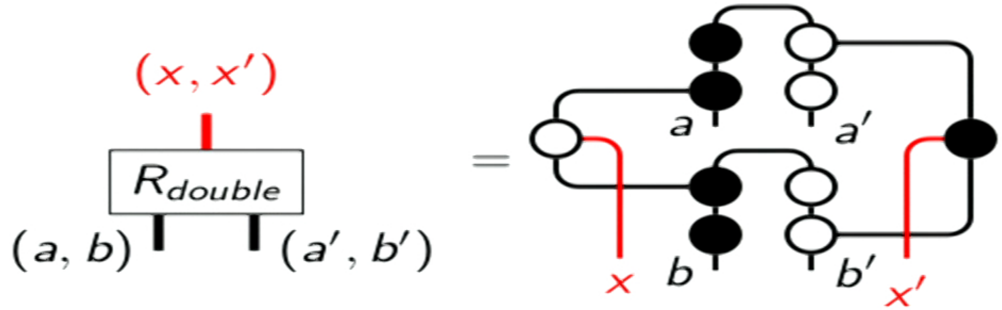


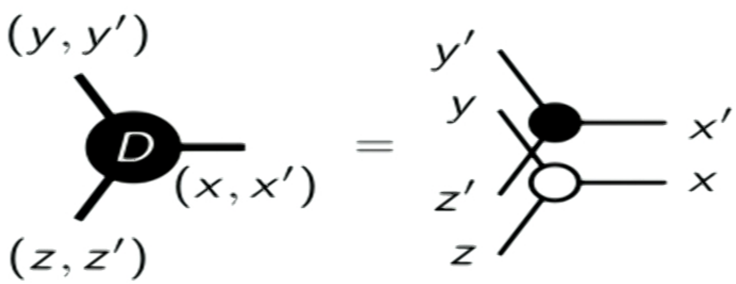
- Do not commute $\Rightarrow R_{double}$ is not a representation of the tensor product of G and its dual

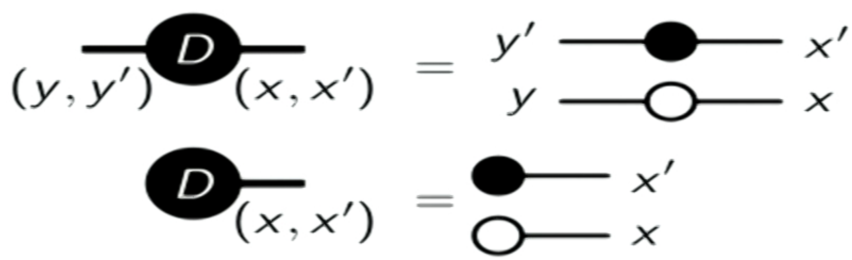
Plaquette and vertex term – together

- Consider product of plaquette- and vertex term



- Do not commute $\Rightarrow R_{double}$ is not a representation of the tensor product of G and its dual
- But: it is representation of the *quantum double algebra* (or *Drinfeld double*) $D(G)$

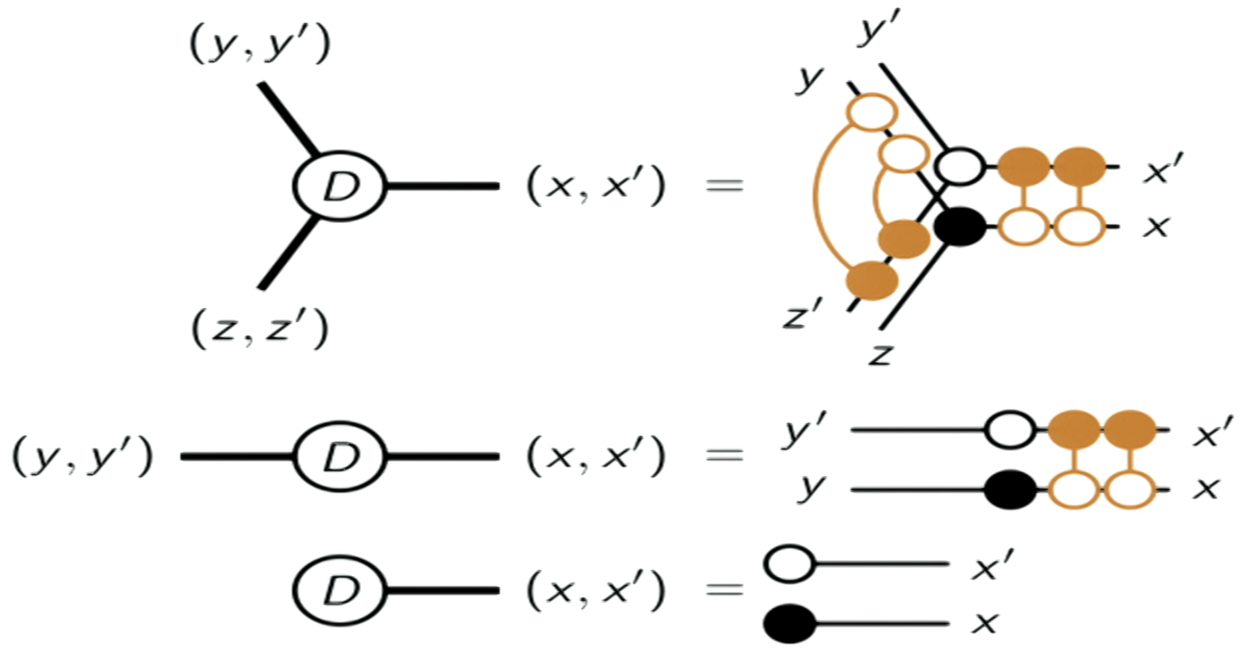





Do not commute ⇒ P double is not a representation of
product of G and its dual
Rather, it is representation of the quantum double
double) D(G)

The quantum double algebra

- Quantum double algebra is bi-crossed product:



The quantum double algebra

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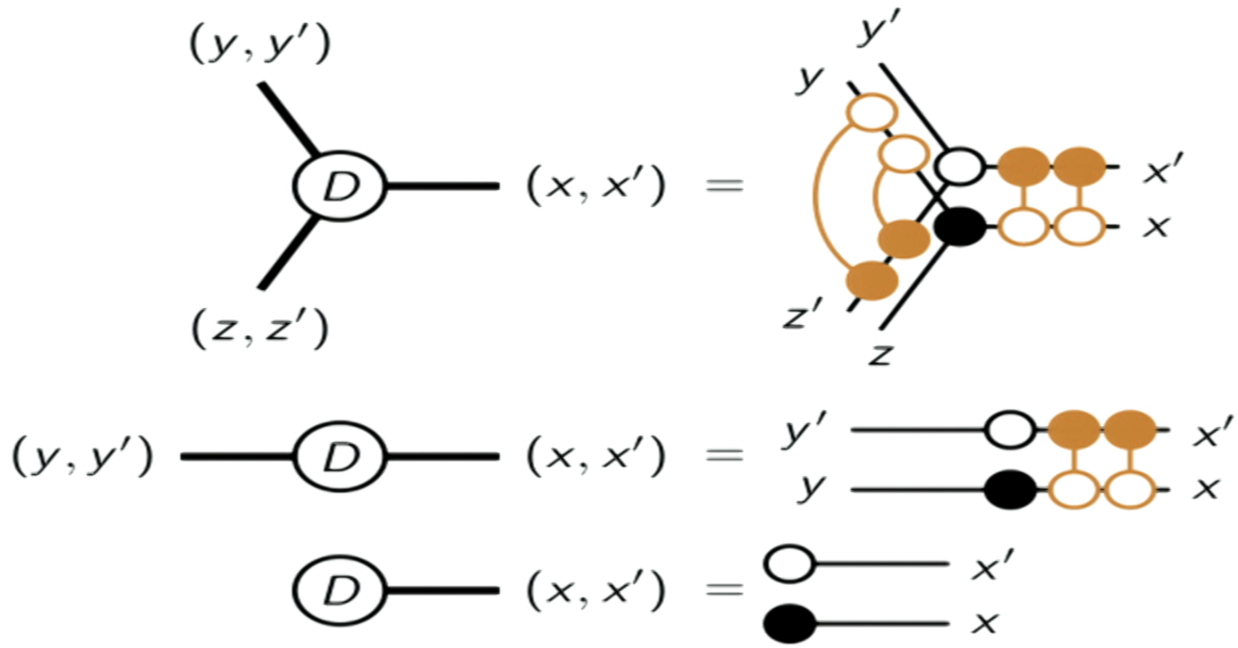

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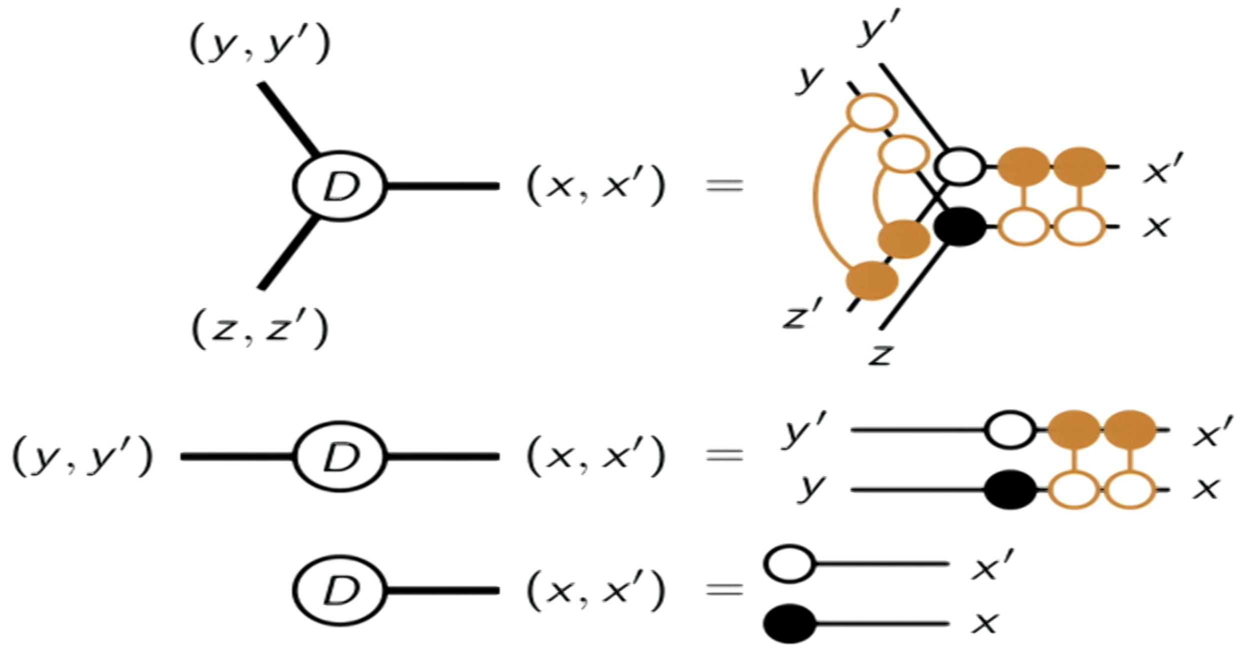

- Quasi-triangular $\Rightarrow$ admits braiding

The quantum double algebra

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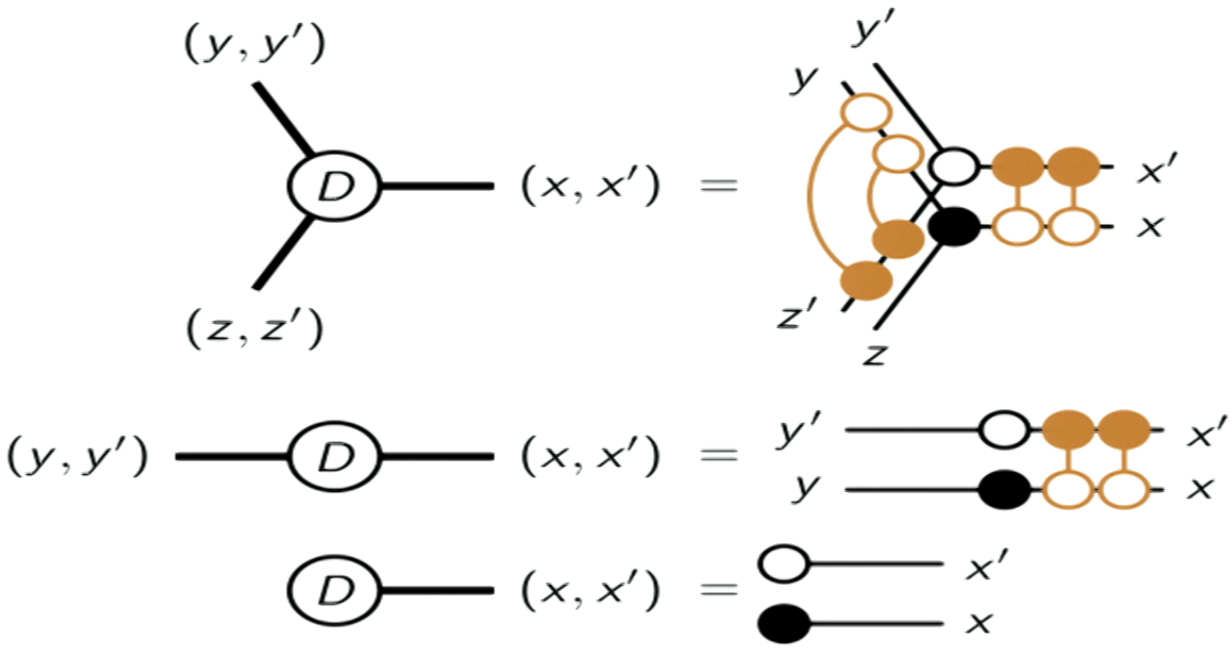

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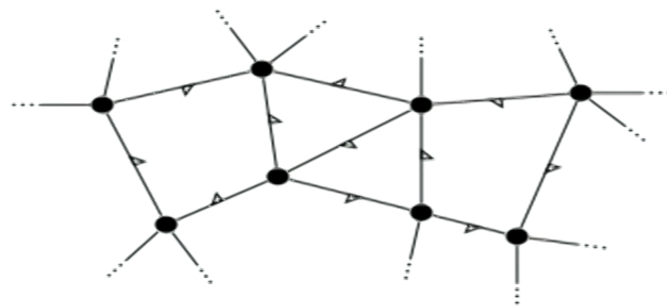
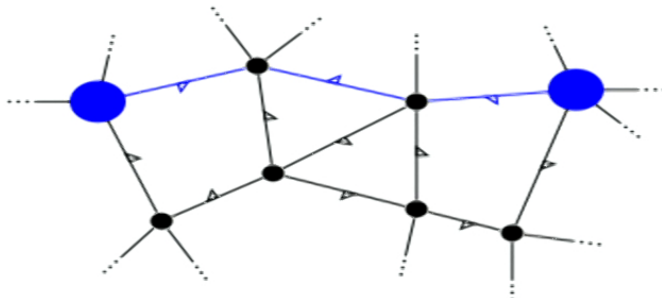
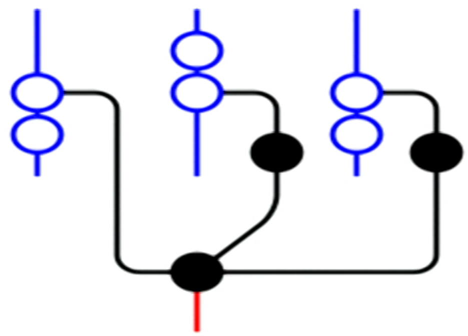
Open string and ribbon operators

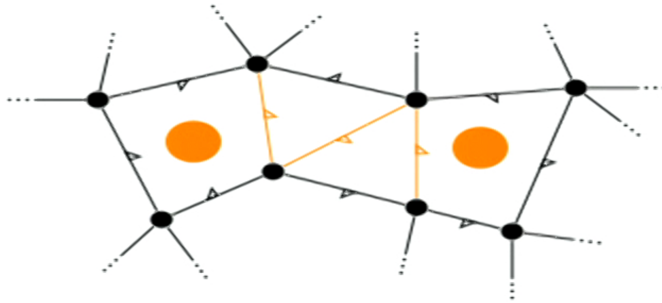


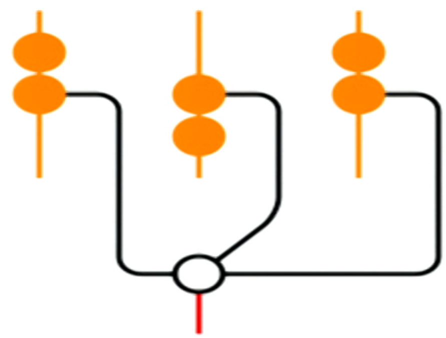
Open string and ribbon operators





Open string and ribbon operators





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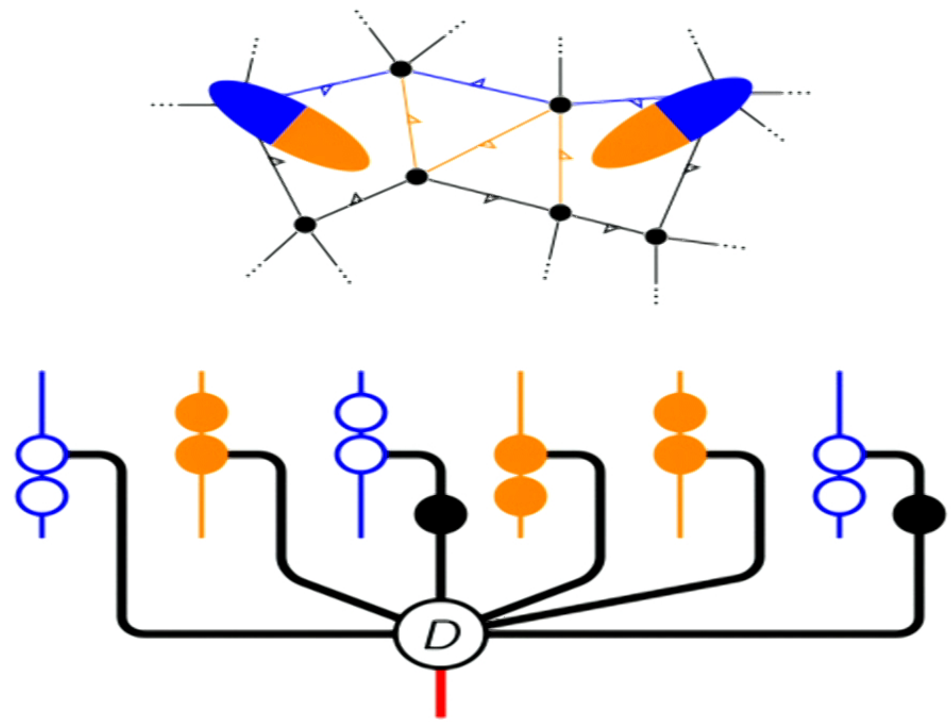
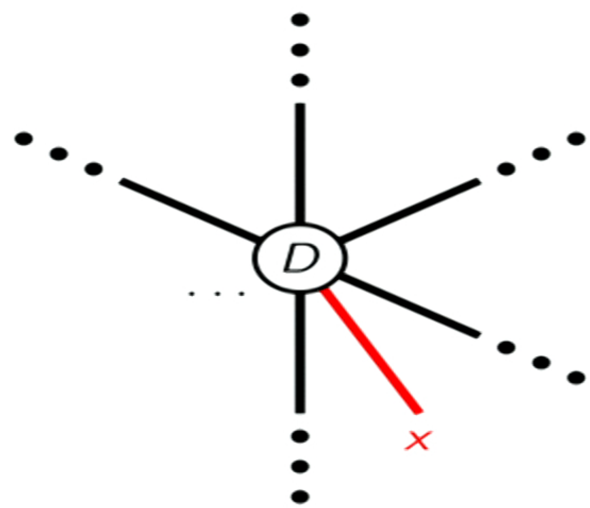


Closed string- and ribbon operators

- Closed ribbon operator should be nowhere visible



Closed string- and ribbon operators

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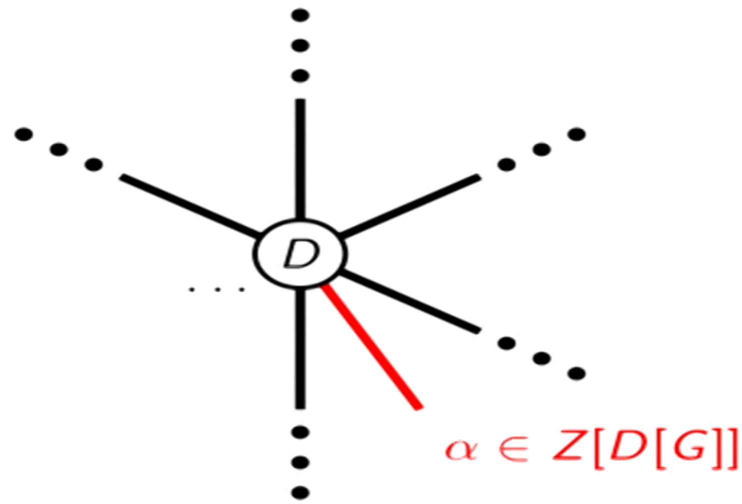
- $\Rightarrow$ Put element of the center $\Rightarrow$ commutes through



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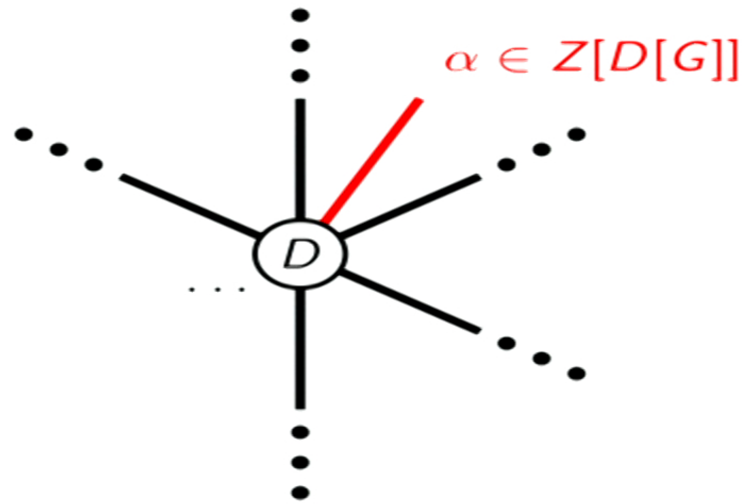

Thanks for your attention!