Title: Semisimple Hopf algebras and fusion categories

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Abstract: This talk will be a short introduction to the semisimple Hopf algebras over an algebraically closed field of characteristic 0 and their representation theories. It is intended to outline the main basic results about structure and known methods for the construction of semisimple Hopf algebras: extensions, twisting, Tannakian reconstruction. Basic notions concerning tensor categories will be introduced: braided structures, center construction, fiber functors. Special emphasis is given to the notion of fusion category. At the same time, the relations between these notions and those of the Hopf algebras are studied.

Semisimple Hopf algebras and fusion categories

(Introduction)

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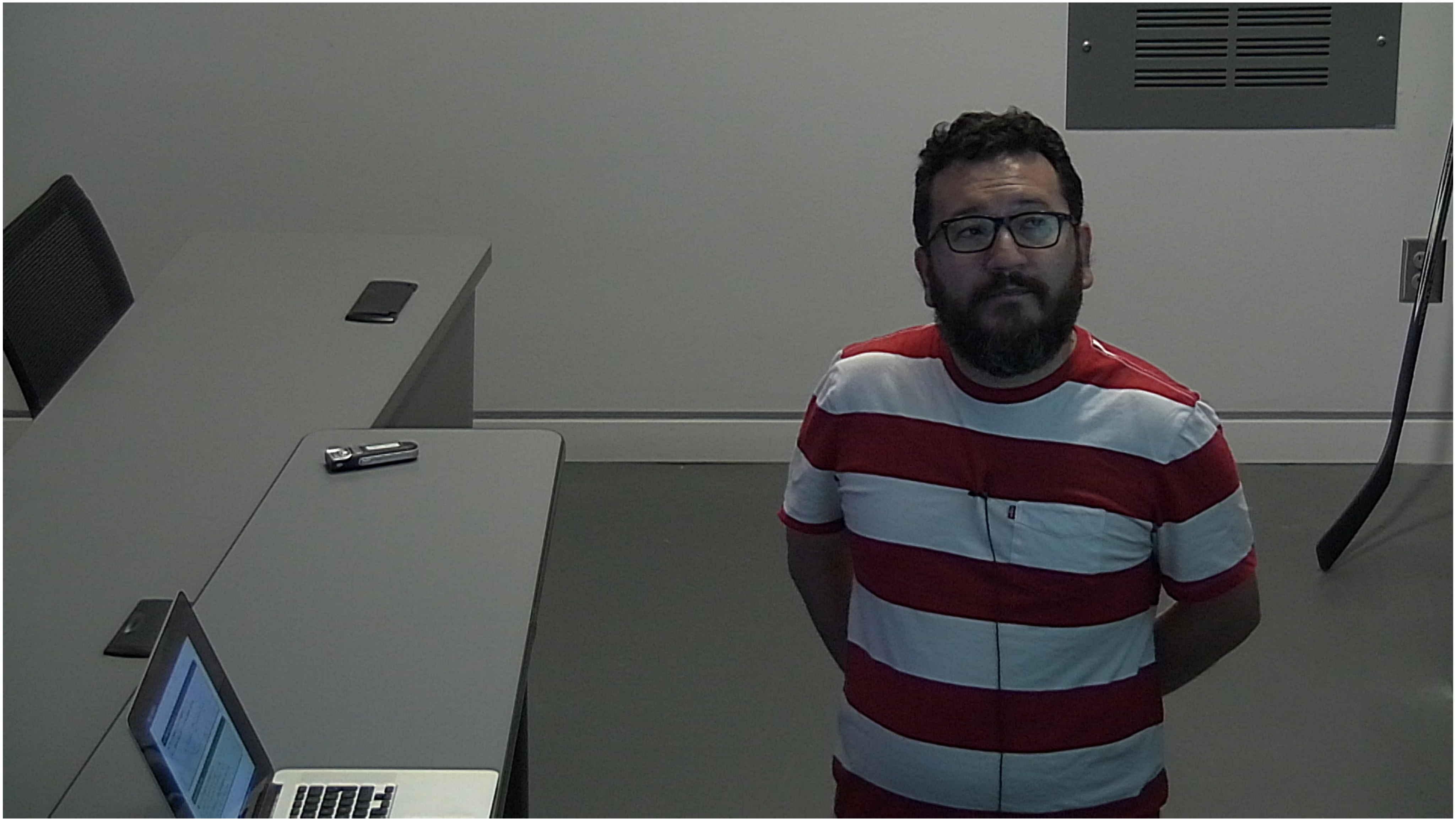
Hopf Algebras in Kitaev's Quantum Double Models
Perimeter Institute
July 31, 2017



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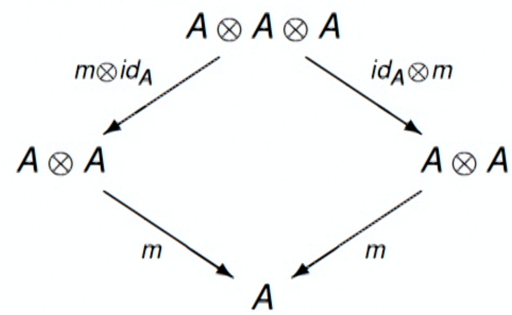
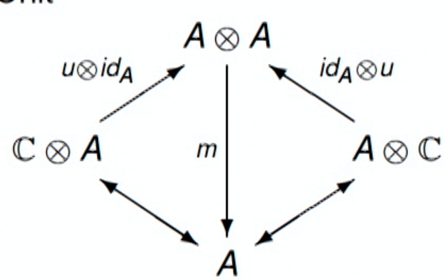
Algebras: (A, m, u)

$m : A \otimes A \to A, u : \mathbb{C} \to A$

Associativity



Unit



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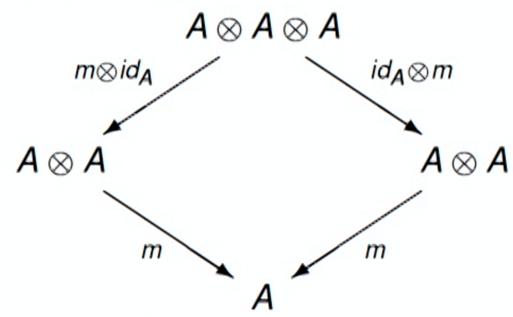

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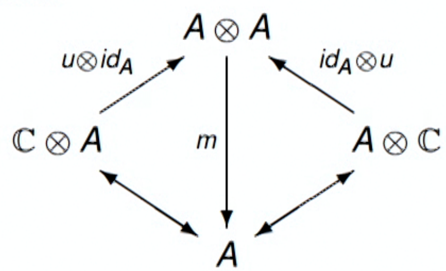

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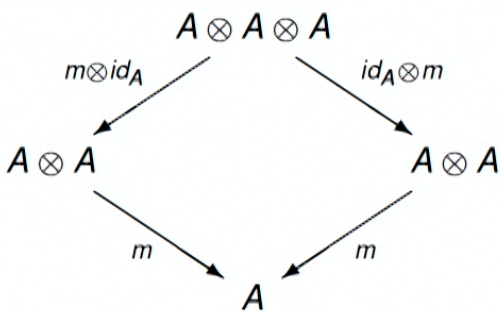
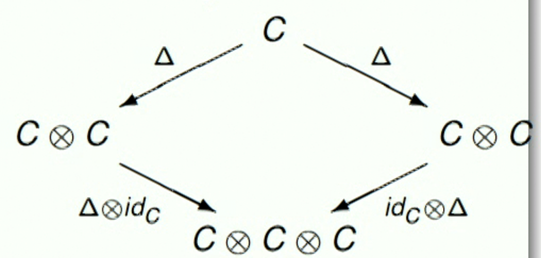


Coassociativity



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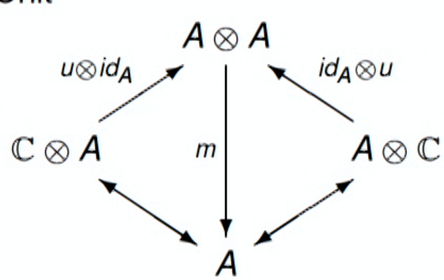
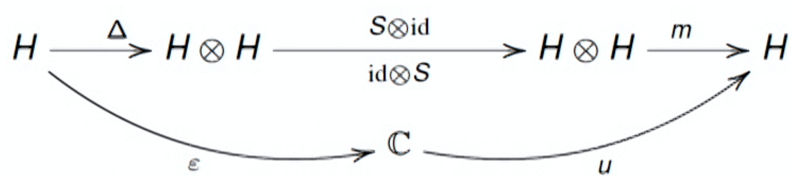


Hopf agebra

Hopf Algebra: $(H, m, u, \Delta, \varepsilon)$

- (H, m, u) algebra, (H, Δ, ε) coalgebra,
- Δ, ε algebra maps, $\exists S : H \to H$ (the antipode) such that



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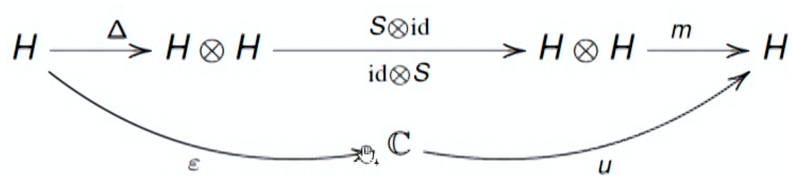
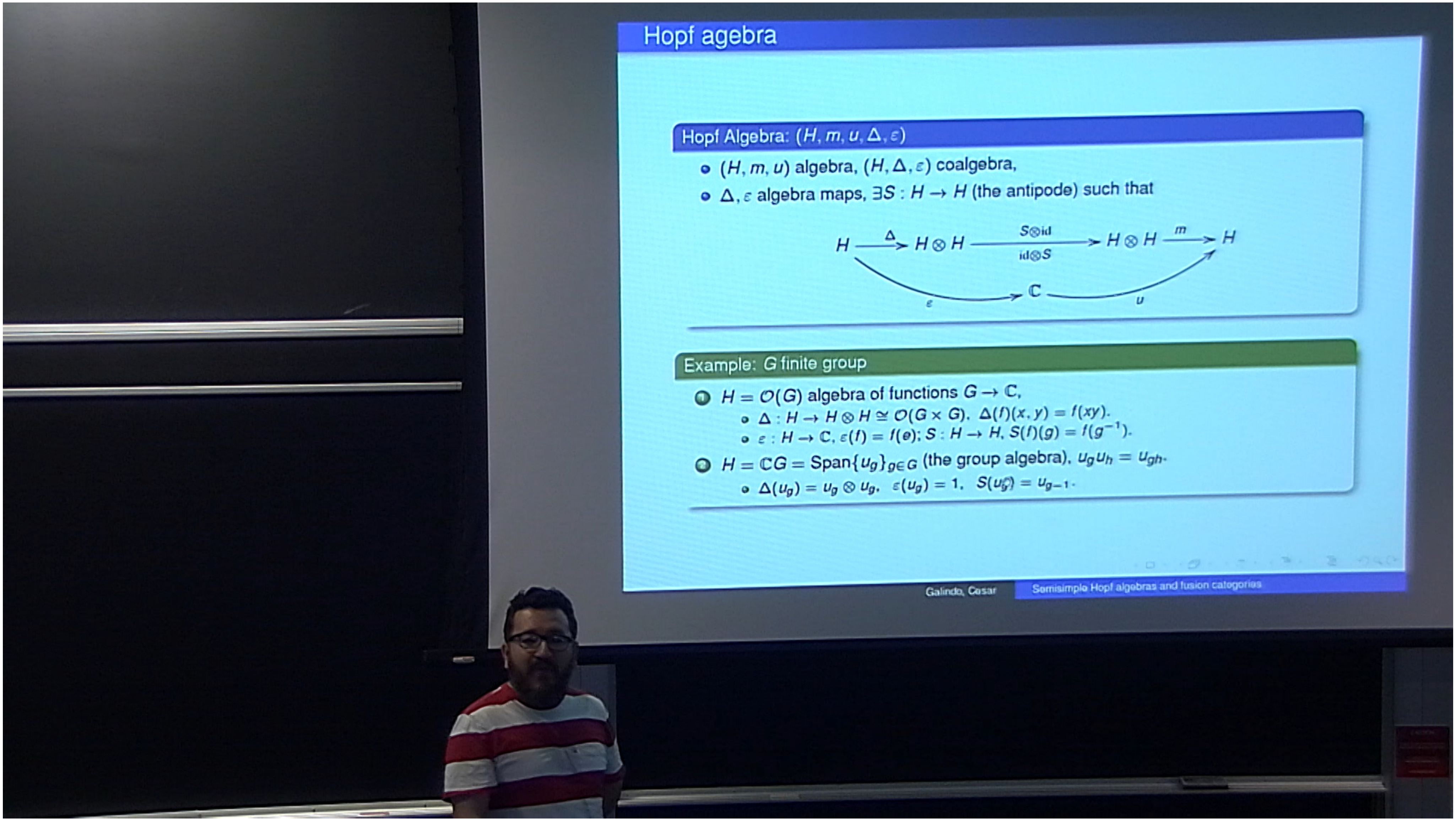


Example: G finite group

1. $H = \mathcal{O}(G)$ algebra of functions $G \to \mathbb{C}$,
   - $\Delta : H \to H \otimes H \cong \mathcal{O}(G \times G)$, $\Delta(f)(x, y) = f(xy)$.
   - $\varepsilon : H \to \mathbb{C}$, $\varepsilon(f) = f(e)$; $S : H \to H$, $S(f)(g) = f(g^{-1})$.
2. $H = \mathbb{C}G = \text{Span}\{u_g\}_{g \in G}$ (the group algebra), $u_g u_h = u_{gh}$.
   - $\Delta(u_g) = u_g \otimes u_g$, $\varepsilon(u_g) = 1$, $S(u_g) = u_{g-1}$.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$H \xrightarrow{\varepsilon} \mathbb{C} \xrightarrow{u} H$
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Semisimple Hopf algebras and fusion categories

Sweedler's notation

If C is a coalgebra, then $\Delta(c) = \sum_i c_i \otimes c^i \in C \otimes C$, $\quad c_i, c^i \in C$. Such an expression is abbreviated as

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}, \qquad \text{or simply} \qquad \Delta(c) = c_1 \otimes c_2.$$

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$$(c_1)_1 \otimes (c_1)_2 \otimes c_2 = c_1 \otimes (c_2)_1 \otimes (c_2)_2,$$

or simply

$$(\Delta \otimes \mathrm{id})\Delta(c) = (\mathrm{id} \otimes \Delta)\Delta(c) := c_1 \otimes c_2 \otimes c_3.$$

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Hopf algebra axioms in Sweedler's notation

Counit

$$\varepsilon(x_1)x_2 = x = x_1\varepsilon(x_2)$$

Δ algebra map

$$(xy)_1 \otimes (xy)_2 = x_1 y_1 \otimes x_2 y_2$$

Antipode

$$S(x_1)x_2 = \epsilon(x)1_H = x_1 S(x_2)$$

Basic invariants of a Hopf algebra H,

- $G(H) = \{x \in H : \Delta(x) = x \otimes x\}$, group of grouplikes.
- $\mathrm{Prim}(H) = \{x \in H : \Delta(x) = x \otimes 1 + 1 \otimes x\}$. Lie algebra of primitive elements.

The flip map

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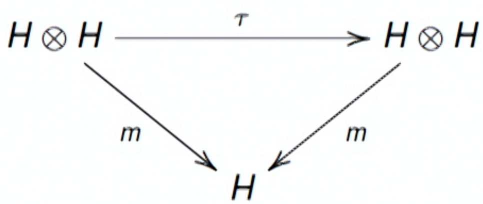
Commutative Hopf algebra



$xy = yx, \quad \forall x, y \in H$

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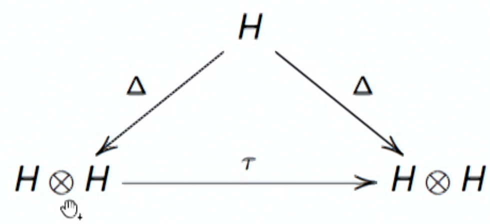

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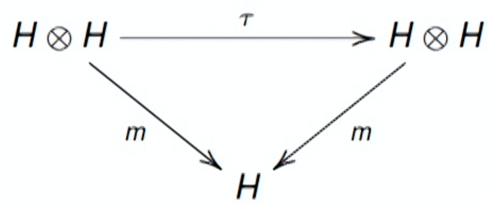

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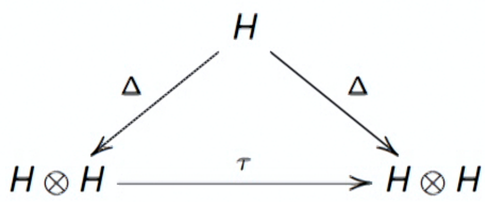


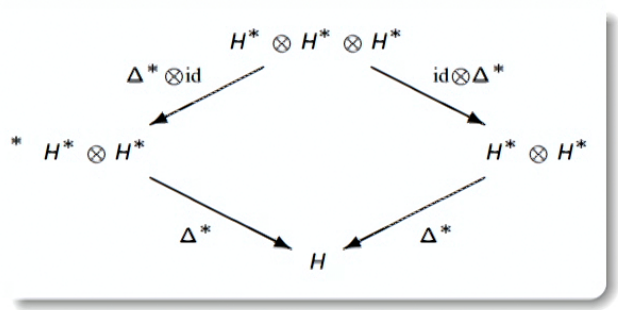
$x_1 \otimes x_2 = x_2 \otimes x_1. \quad \forall x \in H$

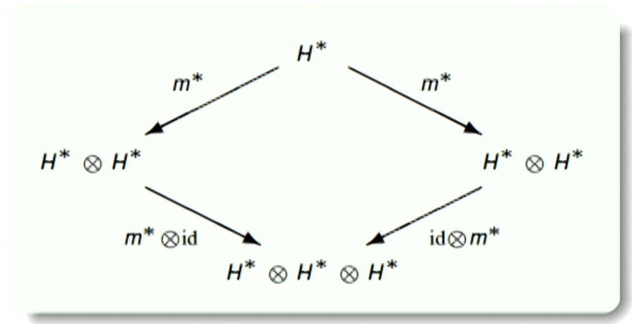
Theorem (Cartier-Konstant, early 60's)

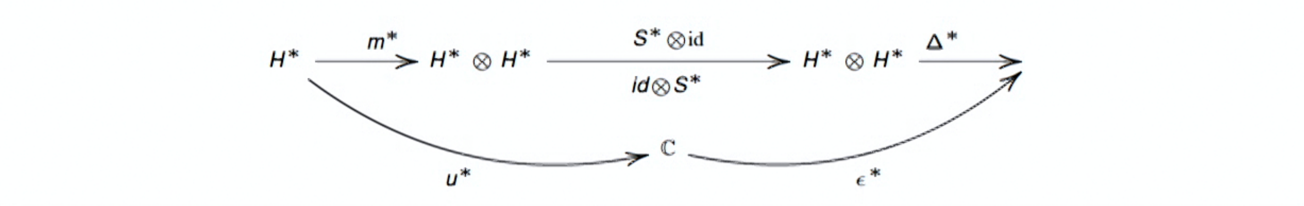
Any cocommutative Hopf algebra is of the form $U(\mathfrak{g})\#\mathbb{C}[G]$.

Duality

If $(H, m, u, \Delta, \epsilon, S)$ is a finite dimensional Hopf algebra, then $(H^*, m^*, u^*, \Delta^*, \epsilon^*, S^*)$ is also a Hopf algebra.







Example: Drinfel'd quantum double of a Hopf algebra

Quasitriangular Hopf algebras (Drinfel'd)

- An R-matrix for H is an invertible element $R \in H \otimes H$ such that $R\Delta(x)R^{-1} = \tau \circ \Delta(x)$ for all $x \in H$.

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}, \qquad (\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12},$$

where $R_{12} = R \otimes 1_H$, $R_{23} = 1_H \otimes R$, $R_{13} = R_1 \otimes 1_H \otimes R_2$.

- A quasitriangular Hopf algebra is a Hopf algebra with an R-matrix.

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Definition (Drinfel'd)

Let H be a finite dimensional Hopf algebra and let H^{*cop} denote the Hopf algebra H^* except with the opposite comultiplication. Then Drinfel'd quantum double $(D(H), R)$ is

- $D(H) = H \otimes H^{*cop}$ as coalgebra,
- H and H^{*cop} are Hopf subalgebras,
- $(\psi \otimes h)(\psi' \otimes h') = \psi'_1(S^{-1}(h_3))\psi'_3(h_1)\psi\psi' \otimes h_2h'$
- if e_i is a basis if H and e^i the dual basisin H^*, the element $R = \sum_i e_i \otimes e^i \in D(H) \otimes D(H)$ is an R-matrix.

Integrals in Hopf algebras

Definition

The spaces of left and right integral in H are defined, respectively, as follows

$$\mathcal{I}_l(H) = \{h \in H : xh = \epsilon(h)x, \forall x \in H\}, \quad \mathcal{I}_r(H) = \{h \in H : hx = \epsilon(h)x, \forall x \in H\},$$

Examples

- $H = \mathbb{C}[G] = \mathcal{O}(G)^*$, (the space of distributions on G). Left integrals in H are left invariant measures.
- if H is finite dimensional. $\lambda \in H^*$ is integral iff $h_1 \langle \phi, h_2 \rangle = \langle \phi, h \rangle 1_H$, for all $h \in H$
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Theorem (Larson-Sweedler, 1969)

Let H finite dimensional Hopf algebra over a field k. Then

- *dim($\mathcal{I}_l(H)$)=dim ($\mathcal{I}_l(H)$)=1.*
- *The antipode S is bijective and $S(\mathcal{I}_l(H)) = \mathcal{I}_r(H)$*

Classical results about semisimple hopf algebra

H is called semisimple if every finite dimensional left H-module is completely reducible.

Theorem (Maschke Theorem for Hopf algebras)

Let H be a finite dimensional Hopf algebra. Then, H is semisimple $\iff \langle \epsilon, \mathcal{I}_l(H) \rangle \neq 0$.

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Theorem (Larson-Radford Theorem, 1989)

Let H be a finite dimensional Hopf algebra. Then,

$$H \text{ is semisimple} \iff S^2 = id_H \iff H^* \text{ is semisimple}$$

- Hence every Hopf subalgebra and quotient of a semisimple Hopf algebra is semisimple.
- The Drinfel'd double for a semisimple Hopf algebra is semisimple.

Examples of Semisimple Hopf algebras

Examples: Abelian extensions (G. I. Kac (1968), M. Takeuchi (1981))

Input

(i) A matched pair of finite groups $(F, G, \triangleleft, \triangleright)$;

(ii) compatible cocycles $(\sigma, \tau) \in Z^2(F, (\mathbb{C}^G)^\times) \times Z^2(G, (\mathbb{C}^G)^\times)$, denoted by

$$\sigma(x,y) = \sum_{s\in G} \sigma_s(x,y)\delta_s, \quad x, y \in F, \quad \tau(s,t) = \sum_{x\in F} \tau_x(s,t)\delta_x, \quad s,t \in G.$$

(i) means $G \xleftarrow{\triangleleft} G \times F \xrightarrow{\triangleright} F$ are (respectively, right and left) actions, such that $s \triangleright xy = (s \triangleright x)((s \triangleleft x) \triangleright y)$ and $st \triangleleft x = (s \triangleleft (t \triangleright x))(t \triangleleft x)$, for all $s, t \in G$, $x, y \in F$.

Hopf algebra $H = \mathbb{C}^G\, {}^{\tau}\!\bowtie_{\sigma} \mathbb{C}F$ on the vector space $\mathbb{C}^G \otimes \mathbb{C}F$ with

$$(\delta_s \# x)(\delta_t \# y) = \delta_{s\triangleleft x, t}\sigma_s(x,y)\delta_s \# xy,$$

$$\Delta(\delta_s \# x) = \sum_{s=ab} \tau_x(a,b)\delta_a \#(b \triangleright x) \otimes \delta_b \# x.$$

Example (continued)

- (Drinfel'd double a finite groups) If $G = F$, $h \triangleleft g = g^{-1}hg$, $g \triangleright h = h$. Then $\mathbb{C}^G \bowtie \mathbb{C}G$ is the twisted Drinfel'd double of $\mathbb{C}G$.
- (Kac-Paljutkin) Matched corresponds with group factorizations of groups $\Sigma = FG$, $F \cap G = \{e\}$. The case of $\Sigma := D_{2n} = \langle r, s : r^n = 1 = s^2, srs = r^{-1} \rangle$. If $n = 4m$, $F = \langle s \rangle$, $G = \langle sr, r^m \rangle$.

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Twisting the comultiplication

- A twist for H is $J \in H \otimes H$ invertible such that

$$(\Delta \otimes \mathrm{id})(J)(J \otimes 1) = (\mathrm{id} \otimes \Delta)(J)(1 \otimes J), \quad (\varepsilon \otimes \mathrm{id})(J) = (\mathrm{id} \otimes \varepsilon)(J) = 1.$$

- A new Hopf algebra H^J (called a twist of H) is constructed over the same algebra $H^J = H$, but with comultiplication $\Delta^J(h) = J^{-1}\Delta(h)J$.
- Twists in $H = \mathbb{C}G$, were classified by Movshe, by classes of pairs (S, α), where $S \subset N$ is subgroup, and $\omega \in H^2(S, \mathbb{C}^\times)$ is a non-degenerate 2-cocycle .
- For instance, if S is abelian, then the twist is given by

$$J = \sum_{\chi, \eta \in \widehat{S}} \alpha(\chi, \lambda) e_\chi \otimes e_\lambda, \qquad \text{where, } e_\chi = \sum_{s \in S} \chi(s^{-1}) \delta_s.$$

Representation theory of a Hopf algebra

If $H \to \mathrm{End}(V), H \to \mathrm{End}(W)$ are H-modules, the composition

$$H \xrightarrow{\Delta} H \otimes H \to \mathrm{End}(V) \otimes \mathrm{End}(W) \to End(V \otimes W),$$

provides $V \otimes W$ an H-module structure.
The natural isomorphisms $(V \otimes W) \otimes Z \cong V \otimes (W \otimes W), (v \otimes w) \otimes z \mapsto v \otimes (w \otimes z)$ are H-module maps.

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the antipode is an algebra anti-isomorphism

$$H \xrightarrow{S} H^{op} \xrightarrow{\text{transpose}} \mathrm{End}(V^*).$$

The evaluation an coevaluation maps ev: $V \otimes V^* \to \mathbb{C}$, coev: $\mathbb{C} \to V^* \otimes V$ are H-module maps.

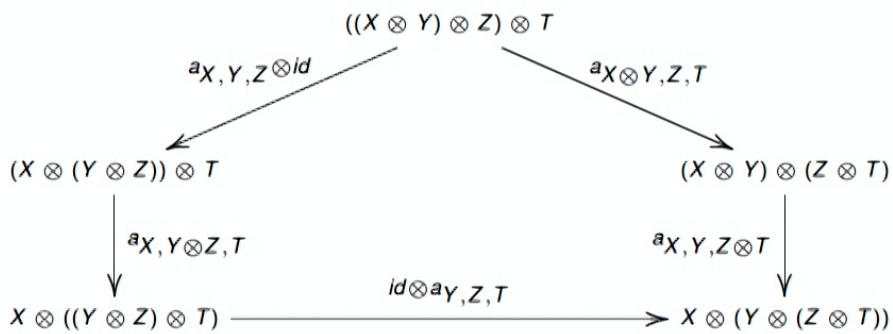
The category Rep(H) of finite dimensional representation of a Hopf algebra is a tensor category.

Tensor categories

Definition (MacLane)

Tensor category: quadruple $(\mathcal{C}, \otimes, a, \mathbf{1})$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is an associativity constraint, $\mathbf{1}$ is the unit object.

1) Pentagon axiom : the diagram commutes, $X, Y, Z, T \in \mathrm{Obj}(\mathcal{C})$.



2) Unit axiom: both functors $\mathbf{1} \otimes (-)$ and $(-) \otimes \mathbf{1}$ are isomorphic to the identity functor.

Rigidity

If $X \in \mathcal{C}$, a right dual is $X^* \in \mathcal{C}$ together with $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$ such that the compositions equal the identities :

$$X \xrightarrow{\mathrm{coev}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$

$$X^* \xrightarrow{\mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^*$$

Rigidity

If $X \in \mathcal{C}$, a right dual is $X^* \in \mathcal{C}$ together with $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$ such that the compositions equal the identities :

$$X \xrightarrow{\mathrm{coev}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes ev_X} X$$

$$X^* \xrightarrow{\mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{ev_X \otimes \mathrm{id}_{X^*}} X^*$$

Example

$\mathcal{C} = \mathrm{Rep}(H)$ finite dimensional modules over a Hopf algebra (no necessarily finite dimensional or semisimple).

Fusion categories

Definition (Etingof, Nikshych, Ostrik)

A fusion category $\mathcal{C}$ is a $\mathbb{C}$-linear semisimple category with finitely many simple objects and $\mathbf{1}$ simple.

Example

$\mathcal{C} = \mathrm{Rep}(H)$ finite dimensional modules over a semisimple Hopf algebra.

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$\mathcal{C}(\mathfrak{g}, q, l)$,

The category of tilting modules of the quantum groups $U_q(\mathfrak{g})$ (q^2 a lth root of unity) module negligible morphisms.
For example:

- $SU(N)_k = \mathcal{C}(\mathfrak{sl}_N, N + k)$,
- $SO(N)_k$,
- $PSU(N)_k \subset SU(N)_k$, for $\gcd(k, N) = 1$.

Definition of fusion category in coordinates

Fusion rules

Let $L = \{\mathbf{1}, a, b, c \ldots\}$ be a set of representatives of isomorphism classes of simple objects.

- There is an involution $* : L \to L$ such that $\mathbf{1}^* = \mathbf{1}$.
- $a \otimes b \cong \bigoplus_c N_{ab}^{k} c$, we have integers $N_{ab}^{c} = \dim(Hom(c, a \otimes b))$, and satisfy

$$N_{\mathbf{1}a}^{b} = \delta_{ab} = N_{a\mathbf{1}}^{b}, \qquad N_{ab}^{\mathbf{1}} = \delta_{a^* b}, \qquad N_{abc}^{u} := \sum_{e} N_{ab}^{e} N_{ec}^{u} = \sum_{e'} N_{ae'}^{u} N_{bc}^{e'}$$

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Fix isomorphism $\sigma_{a,b} : a \otimes b \to a \odot b := \bigoplus_c N_{ab}^c c$ and define $a_{a,b,c}^{\odot}$

$$\begin{array}{ccc}
(a \odot b) \odot c & \xrightarrow{a_{a,b,c}^{\odot}} & a \odot (b \odot c) \\
\uparrow \sigma_{a \odot b, c} & & \uparrow \sigma_{a, b \odot c} \\
(a \odot b) \otimes c & & a \otimes (b \odot c) \\
\uparrow \sigma_{a,b} \otimes c & & \uparrow \mathrm{id}_a \otimes \sigma_{b,c} \\
(a \otimes b) \otimes c & \xrightarrow{a_{a,b,c}} & a \otimes (b \otimes c)
\end{array}$$

F-matrices (6j-symbols)

Denoting by $\begin{bmatrix} c \\ a,b \end{bmatrix}$ the hom space $\mathrm{Hom}(c, a \otimes b)$, and using the isomorphisms $a^{\odot}$

$$F^{d}_{abc} : \mathrm{Hom}_{\mathcal{C}}((a \odot b) \odot c, d) \to \mathrm{Hom}_{\mathcal{C}}(a \odot (b \odot c), d)$$
$$f \mapsto f \circ a^{\odot}_{a,b,c}$$

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F-matrices

The linear maps

$$F\begin{bmatrix} d \\ a,b,c \end{bmatrix} : \bigoplus_{i \in L} \begin{bmatrix} i \\ a,b \end{bmatrix} \otimes_{\mathbb{C}} \begin{bmatrix} d \\ i,c \end{bmatrix} \longrightarrow \bigoplus_{j \in L} \begin{bmatrix} j \\ b,c \end{bmatrix} \otimes_{\mathbb{C}} \begin{bmatrix} d \\ a,j \end{bmatrix}$$

are called the F-matrices and they satisfy the **pentagonal identity (pentagon axiom)**.

Examples

Pointed fusion categories, $\mathcal{C}(G,\omega)$

- $L = G$ (a finite group)
- fusion rules are the product in G
- $F^{d}_{a,b,c} = \omega(a,b,c)\delta_{abc,d}$, so is a function $\omega : G^{\times 3} \to \mathbb{C}^{\times}$
- Pentagon equation is exactly 3-cocycle condition of group cohomology:

$$\omega(a,b,c)\omega(b,c,d)\omega(a,bc,d) = \omega(ab,c,d)\omega(a,b,cd)$$

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Fibonnaci theory

- $L = \{\mathbf{1}, \mathbf{x}\}$
- fusion rules $x^2 = 1 + x$.
- $F^x_{xxx} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & \phi^{-1} \end{pmatrix}$

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Not every fusion rules admit a set of F-matrices

As an example the fusion rules:

- $L_k = \{\mathbf{1}, \mathbf{x}\}$
- $x^2 = 1 + kx$, $k \in \mathbb{Z}^{>0}$

define a fusion category if and only if $k = 1$ (Victor Ostrik).

Fiber functors

A fiber functor is an $\mathbb{C}$-linear exact faithful functor $F : \mathcal{C} \to \text{Vec}$, such that $F(\mathbf{1}) = \mathbb{C}$, with natural isomorphism

$$b_{X,Y} : F(V \otimes W) \to F(V) \otimes_{\mathbb{C}} F(W), \quad X, Y \in \mathcal{C}$$

such that

$$\begin{array}{ccc}
F((X \otimes Y) \otimes Z) & \xrightarrow{b_{X \otimes Y, Z}} & F(X \otimes Y) \otimes_{\mathbb{C}} F(Z) \\
\downarrow {\scriptstyle Fa_{X,Y,Z}} & & \downarrow {\scriptstyle b_{X,Y} \otimes \mathrm{id}} \\
F(X \otimes (Y \otimes Z)) & & (F(X) \otimes_{\mathbb{C}} F(Y)) \otimes_{\mathbb{C}} F(Z) \\
\downarrow {\scriptstyle b_{X, Y \otimes Z}} & & \downarrow {\scriptstyle \simeq} \\
F(X) \otimes_{\mathbb{C}} F(Y \otimes Z) & \xrightarrow{\mathrm{id} \otimes b_{Y,Z}} & F(X) \otimes_{\mathbb{C}} (F(Y) \otimes_{\mathbb{C}} F(Z))
\end{array}$$

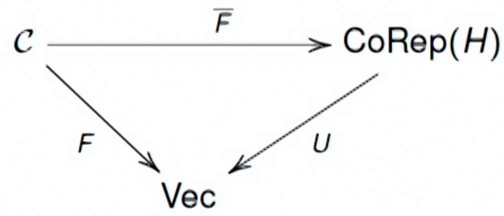
Examples

- H Hopf algebra. The forgetful functor $\text{Rep}(H) \to \text{Vec}$ is a fiber functor.
- $\mathcal{C}(G, \omega)$ admits fiber functors if and only if $0 = [\omega] \in H^3(G, \mathbb{C}^\times)$.

Tannakian-Krein reconstruction

Theorem (Saavedra, Deligne-Milne, Ulbrich)

Let $\mathcal{C}$ be a tensor category and $F := (F, b) : \mathcal{C} \to \mathrm{Vec}$ be a fiber functor. Then there is a Hopf algebra $H := coend(F)$ such that F factorizes through a tensor equivalence $\overline{F} : \mathcal{C} \to \mathrm{CoRep}(H)$ and the diagram



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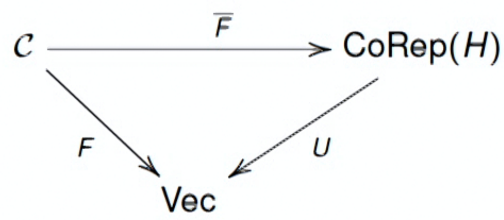

commutes.

Example

If $J \in H \otimes H$ is a twist. The forgetful functor $U : \mathrm{Rep}(H) \to \mathrm{Vec}$, with

$$b_{V,W} : U(V \otimes W) \to U(V) \otimes U(W)$$
$$v \otimes w \mapsto J(v \otimes w),$$

defines a new fiber functor. The associated new Hopf algebra is the twisting H^J.

Group-theoretical semisimple Hopf algebras

Let $\omega \in Z^3(G, \mathbb{C}^\times)$ and $\mathcal{C}(G, \omega)$ the associated pointed fusion category.

- If $F < G$ and $\alpha \in C^2(F, \mathbb{C}^\times)$ such that $d\alpha = \omega_{|F \times F \times F}$, then the twisted group algebra $\mathbb{C}_\alpha F$ is associative in $\mathcal{C}(G, \omega)$.
- The category $\mathcal{C}(G, \omega; F, \alpha)$ of $\mathbb{C}_\alpha F$-bimodules in $\mathcal{C}(G, \omega)$ is a fusion category.

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- These Hopf algebras are called group-theoretical.

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Examples

- $\mathcal{C}(G, \omega, \{e\}, 1) = \mathcal{C}(G, \omega)$.
- $\mathcal{C}(G, 1, G, 1) = \text{Rep}(G)$.
- $\Sigma = GF$ is matched pair, $\mathcal{C}(G, \omega, F, \alpha) = \text{Rep}(\mathbb{C}^G {}^\tau\!\bowtie_\sigma \mathbb{C}F)$.

Let $\mathcal{C}$ be a strict tensor category. A braiding for $\mathcal{C}$ is natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$, such that

$$c_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{id}_Z), \qquad c_{X\otimes Y,Z} = (c_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes c_{Y,Z})$$

$$c_{X,\mathbf{1}} = \mathrm{id}_X = c_{\mathbf{1},X}.$$

Categorical constructions: Drinfeld center

Let $\mathcal{C}$ be a strict tensor category. A braiding for $\mathcal{C}$ is natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$, such that

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$$c_{X,\mathbf{1}} = \mathrm{id}_X = c_{\mathbf{1},X}.$$

Definition

A (strict) braided tensor category $(\mathcal{C}, \sigma)$, is a tensor category with a braiding c.

Example

- If (H, R) is a quasitriangular Hopf algebra, Rep(H) is braided with

$$c_{V,W} : V \otimes_{\mathbb{C}} W \to W \otimes_{\mathbb{C}} V, \quad v \otimes w \mapsto \tau(R(v \otimes w)).$$

- Conversely, if c is a braiding for Rep(H), then $R := \tau(c_{H,H}(1_H \otimes 1_H))$ is an R-matrix.

The braided group representation associated to a Braided tensor category

The braid group on n strands, $\mathcal{B}_n$,

$$\mathcal{B}_n := \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \ \ \sigma_i\sigma_j = \sigma_j\sigma_i \rangle,$$

where $1 \leq i \leq n-2$ in the first relation and in the second relation, $|i-j| > 1$.

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The braid group representation

Let $\mathcal{C}$ be a braided tensor category, then for every object $X \in \mathcal{C}$, the map

$$\mathcal{B}_n \to \mathrm{Aut}_{\mathcal{C}}(X^{\otimes n}), \qquad \sigma_i \mapsto c_i,$$

is a group morphism, where

$$c_i = \mathrm{id}_{X^{\otimes(i-1)}} \otimes c_{X,X} \otimes \mathrm{id}_{X^{\otimes(n-i-1)}}$$

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Example

If (H, R) is quasi-triangular and V is an H-module,

$$c_{V,V} : V \otimes_{\mathbb{C}} V \to V \otimes_{\mathbb{C}} V, \quad v \otimes v' \mapsto \tau(R(v \otimes v')),$$

is a solution of the quantum Yang-Baxter equation (or braid equation).

Drinfel'd center of a tensor category, $\mathcal{Z}(\mathcal{C})$

- Objects $\mathcal{Z}(\mathcal{C})$: pairs $(X, \sigma_{X,-})$, where $X \in Obj(\mathcal{C})$ and $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ is a natural isomorphism, such that

$$\sigma_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes \sigma_{X,Z})(\sigma_{X,Y} \otimes \mathrm{id}_Z), \quad \sigma_{X,\mathbf{1}} = \mathrm{id}_X, \quad \forall, Z \in \mathrm{Obj}(\mathcal{C}).$$

- tensor product: $(X, \sigma_{X,-}) \otimes (Y, \sigma_{Y,-}) = (X \otimes Y, \sigma_{X\otimes Y,-})$, where $\sigma_{X\otimes Y,Z} = (\sigma_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \sigma_{Y,Z})$, for all $Z \in \mathrm{Obj}(\mathcal{C})$.
- The Drinfel'd center is braided with $c_{(\sigma_{X,-}X),(\sigma_{Y,-}Y)} := \sigma_{X,Y}$.

Dijkgraaf-Witten theories.

$\mathcal{Z}(\mathcal{C}(G,\omega))$ are the represention of the twisted Drinfel'd double or Dijkgraaf-Witten theories.

Braided fusion category in coordinates

If $(\mathcal{C}, c)$ is a braided fusion, the maps

$$R^c_{a,b} : \mathrm{Hom}_{\mathcal{C}}(a \otimes b, c) \to \mathrm{Hom}_{\mathcal{C}}(b \otimes a, c)$$
$$f \mapsto f \circ c_{a,b}$$

define

R-matrices

$$R^c_{a,b} : \begin{bmatrix} \mathrm{c} \\ \mathrm{a,b} \end{bmatrix} \to \begin{bmatrix} \mathrm{c} \\ \mathrm{b,a} \end{bmatrix}$$

that satisfy the hexagon equations

$$\sum_{i,j,k} R^i_{a,c} F \begin{bmatrix} \mathrm{j} \\ \mathrm{b,a,c} \end{bmatrix} R^k_{a,b} = \sum_{i,j,k} F \begin{bmatrix} \mathrm{i} \\ \mathrm{a,c,b} \end{bmatrix} R^j_{b,c} F \begin{bmatrix} \mathrm{k} \\ \mathrm{a,b,c} \end{bmatrix}$$

$$\sum_{i,j,k} (R^i_{a,c})^{-1} F \begin{bmatrix} \mathrm{j} \\ \mathrm{b,a,c} \end{bmatrix} (R^k_{a,b})^{-1} = \sum_{i,j,k} F \begin{bmatrix} \mathrm{i} \\ \mathrm{a,c,b} \end{bmatrix} (R^j_{b,c})^{-1} F \begin{bmatrix} \mathrm{k} \\ \mathrm{a,b,c} \end{bmatrix}.$$

Example:Pointed braided fusion category

- If $\mathcal{C}(G,\omega)$ has a braid structure then G is abelian
- $R^z_{xy} = c(x,y)\delta_{xy,z}$, so is a function $c: G \times G \to \mathbb{C}^\times$
- Hexagonal equation is exactly the abelian 3-cocycle condition

$$\omega(y,z,x)c(x,yz)\omega(x,y,z) = c(x,z)\omega(y,x,z)c(x,y)$$
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A dictionary of terminologies between anyon theory and braided fusion category theory

Braided fusion categories	***Anyonic system***
simple object	anyon
label	anyon type or anyonic charge
tensor product $a \otimes b$	fusion
fusion rules $a \times b$	fusion rules
triangular space $V^c_{ab} := \mathrm{Hom}(a \otimes b, c)$	fusion/splitting space $\vert axb \to c\rangle$
dual	antiparticle
coevaluation /evaluation	creation/annihilation
mapping class group representations	generalized anyon statistics
nonzero vector in $V(Y)$	ground state vector
unitary F-matrices	recoupling rules
twist $\theta_x = e^{2\pi i s_x}$	topological spin
morphism	physical process or operator
colored braided framed trivalent graphs	anyon trajectories
quantum invariants	topological amplitudes

Universality of Quantum computer associated to a BFC

A fusion category is called "weakly-integral" if the square of the quantum dimension of all simple objects are integers.

Property F Conjecture (Eric Rowell)

Every non weakly-integral braided fusion category is universal for quantum computation.

Theorem (Etingof, Rowell and Witherspoon)

The representation category of the twisted Drinfel'd double of a finite groups is not universal for quantum computing.

Theorem (G, Rowell)

The representation category of the twisted Drinfeld double of a discrete groups is not universal for quantum computing.

Theorem (Nicolas Escobar, G, Zhenghan Wang)

Braiding gapped boundaries of Dijkgraaf-Witten theories alone cannot provide universal gate sets for topological quantum computing with gapped boundaries.

Conjectures:

- Every weakly-integral fusion category is weakly group-theoretical fusion category. (Etingof, Nikshych and Ostrik)
- Every semisimple Hopf algebra admits a unique C^*-Hopf algebra structure. (Nicolás Andruskiewitsch)

Theorem (G., Eric Rowell, Seung-Moon Hong)

Every weakly-group theoretical fusion category admits a unique unitary structure. In particular every weakly group-theoretical semisimple Hopf admits a unique C^*-Hopf algebra structure.

Thanks for listening!