Title: Braided algebra and dual bases of quantum groups

Date: Jul 19, 2017 02:00 PM

URL: http://pirsa.org/17070060

Abstract: <p>The talk is based on my recent work with Ryan Aziz. We find a dual version of a previous double-bosonisation theorem whereby each finite-dimensional braided-Hopf algebra in the category of corepresentations of a coquasitriangular Hopf algebra gives a new larger coquasitriangular Hopf algebra, for example taking c_q[SL_2] to c_q[SL_3] for these quantum groups reduced at certain odd roots of unity. As an application we find new generators for c_q[SL2] with the remarkable property that their monomials are essentially a dual basis to the standard PBW basis of the reduced quantum enveloping algebra u_q(sl2). This allows one to calculate Fourier transform and other results for such quantum groups. Our method also works for even roots of unity where we obtain new finite-dimensional quantum groups, including an 8-dimensional one at q=-1. Our method
can be used to construct many other new finite-dimensional quasitriangular Hopf algebras and their duals that could be fed into applications in quantum gravity and quantum computing.</p>

BRAIDED ALGEBRA AND DUAL BASES

Shahn Majid (QMUL)

- with Ryan Aziz arXiv:1703.03456 (math.QA)
- S.M. `A quantum groups primer' CUP 2002

Famous quantum groups from 1980's but root unity versions $q^n = 1$, n odd

- $u_q(sl_2)$ $E^n = F^n = 0, \quad K^n = 1, \quad KEK^{-1} = q^{-2}E, \quad KFK^{-1} = q^2F, \quad [E,F] = K - K^{-1}$

$$\Delta K = K \otimes K, \quad \Delta F = F \otimes 1 + K^{-1} \otimes F, \quad \Delta E = E \otimes K + 1 \otimes E$$

$$\mathcal{R} = \frac{1}{n} \sum_{r,a,b=0}^{n-1} \frac{(-1)^r q^{-2ab}}{[r]_{q^{-2}}} F^r K^a \otimes E^r K^b \in u_q(sl_2) \otimes u_q(sl_2)$$

quasitriangular structure

=> Braided category of repns
=> knot invariants and TQFT

- $c_q[SL_2]$ $ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad cb = bc$

$$a^n = 1 = d^n, \quad b^n = 0 = c^n \qquad ad - q^{-1}bc = 1, \quad da - ad = (q - q^{-1})bc$$

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

coproduct

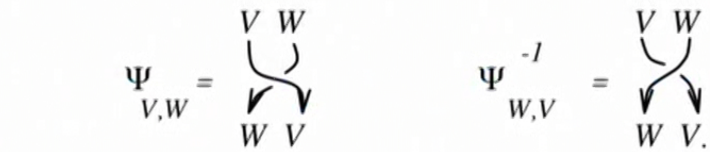
=> For generic q, quantise Poisson-Lie group SU2
=> example of `noncommutative geometry'

Problem: find basis of $c_q[SL_2]$ dual to PBW basis $\{F^i K^j E^k\}_{0 \leq i,j,k < n}$ of $u_q(sl_2)$

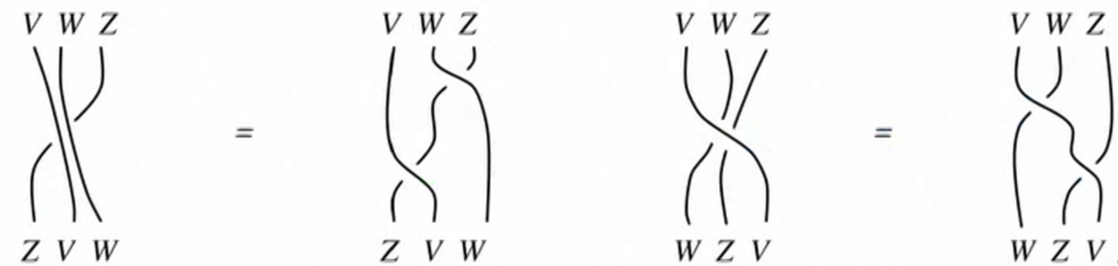
Braided category

$$\mathcal{C}, \otimes, \Psi : \otimes \to \overset{\mathrm{op}}{\otimes}, \Phi : (\ \otimes(\ \otimes\ )) \to ((\ \otimes\ )\otimes\ )$$

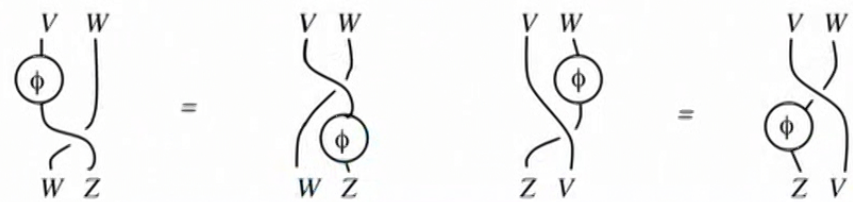
Diagrammatic notation



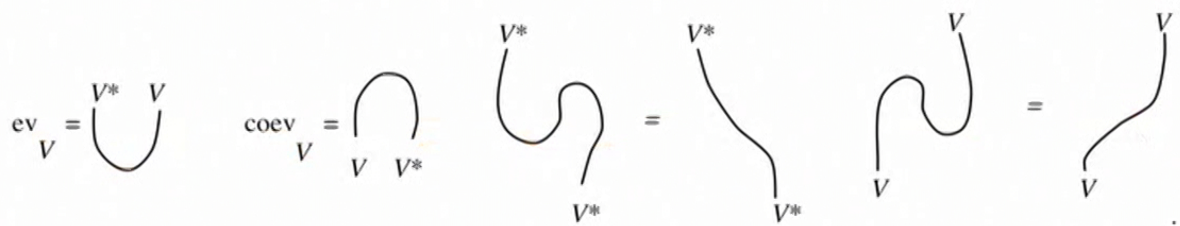
Hexagons



Functoriality



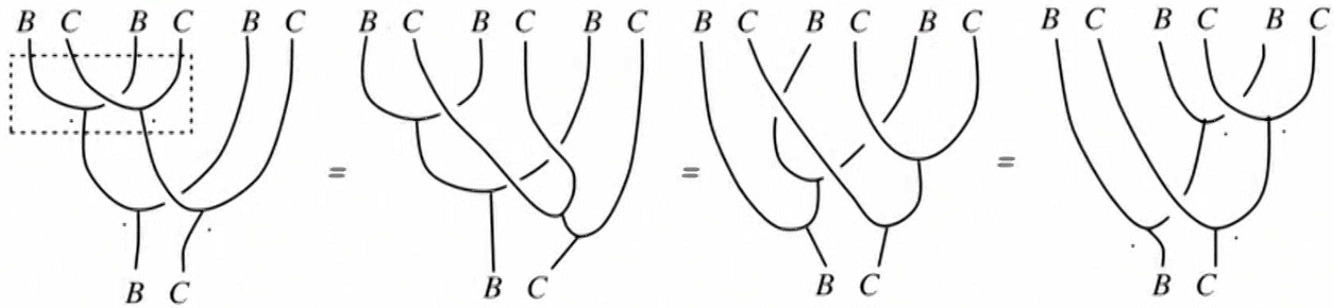
Left dual (rigidity)



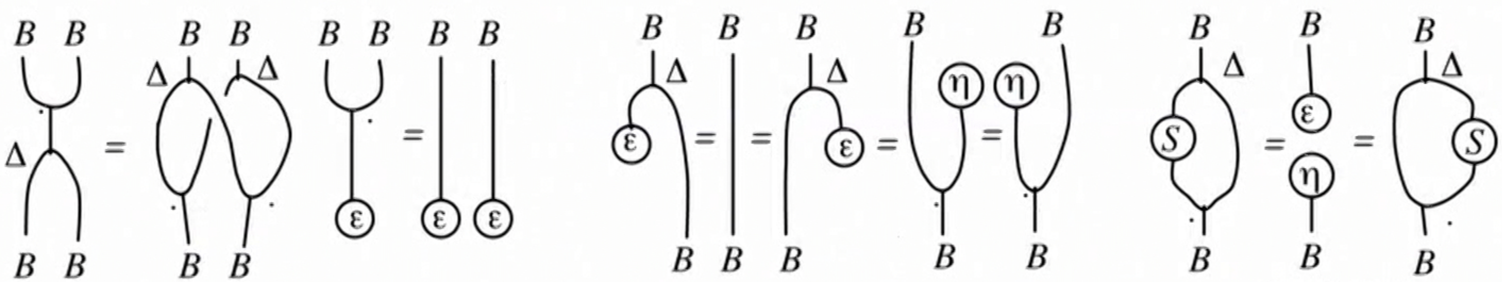
Braided algebra $\cdot : B \otimes B \to B, \quad \eta : \underline{1} \to B$

SM 1990's

Lemma: in a braided category there is a tensor product of two algebras

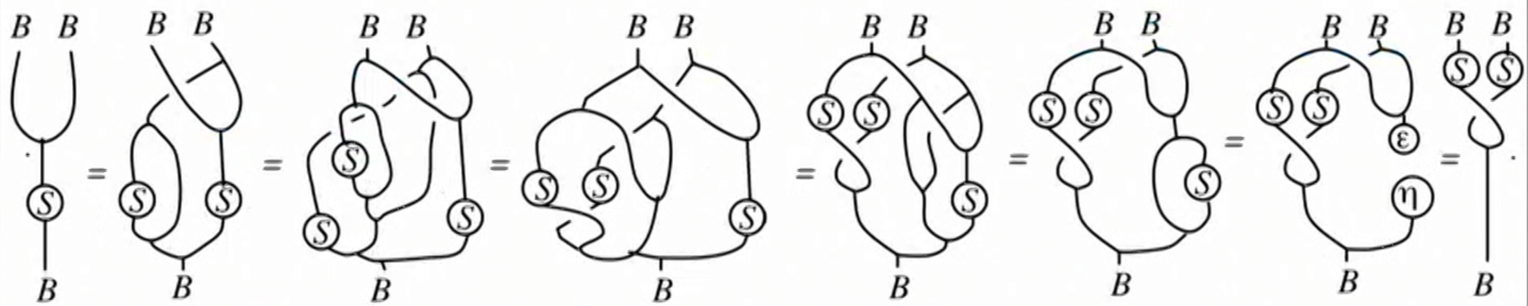


Braided-Hopf algebra $\Delta : B \to B \underline{\otimes} B, \quad \epsilon : B \to \underline{1} \quad S : B \to B$

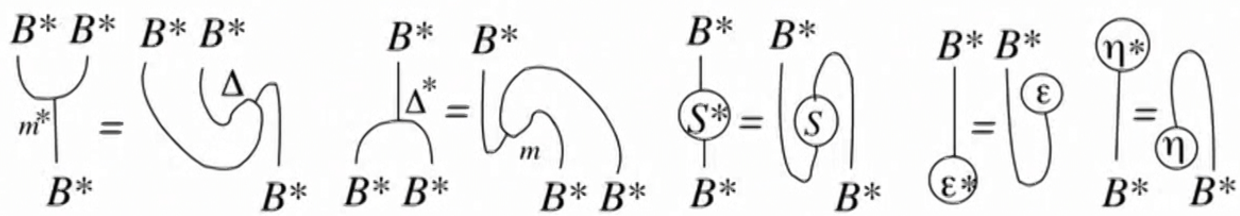


homomorphism co/unity antipode (inversion)

Lemma:

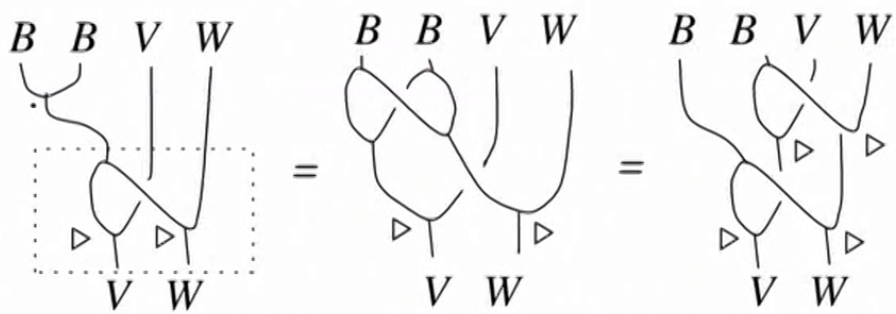


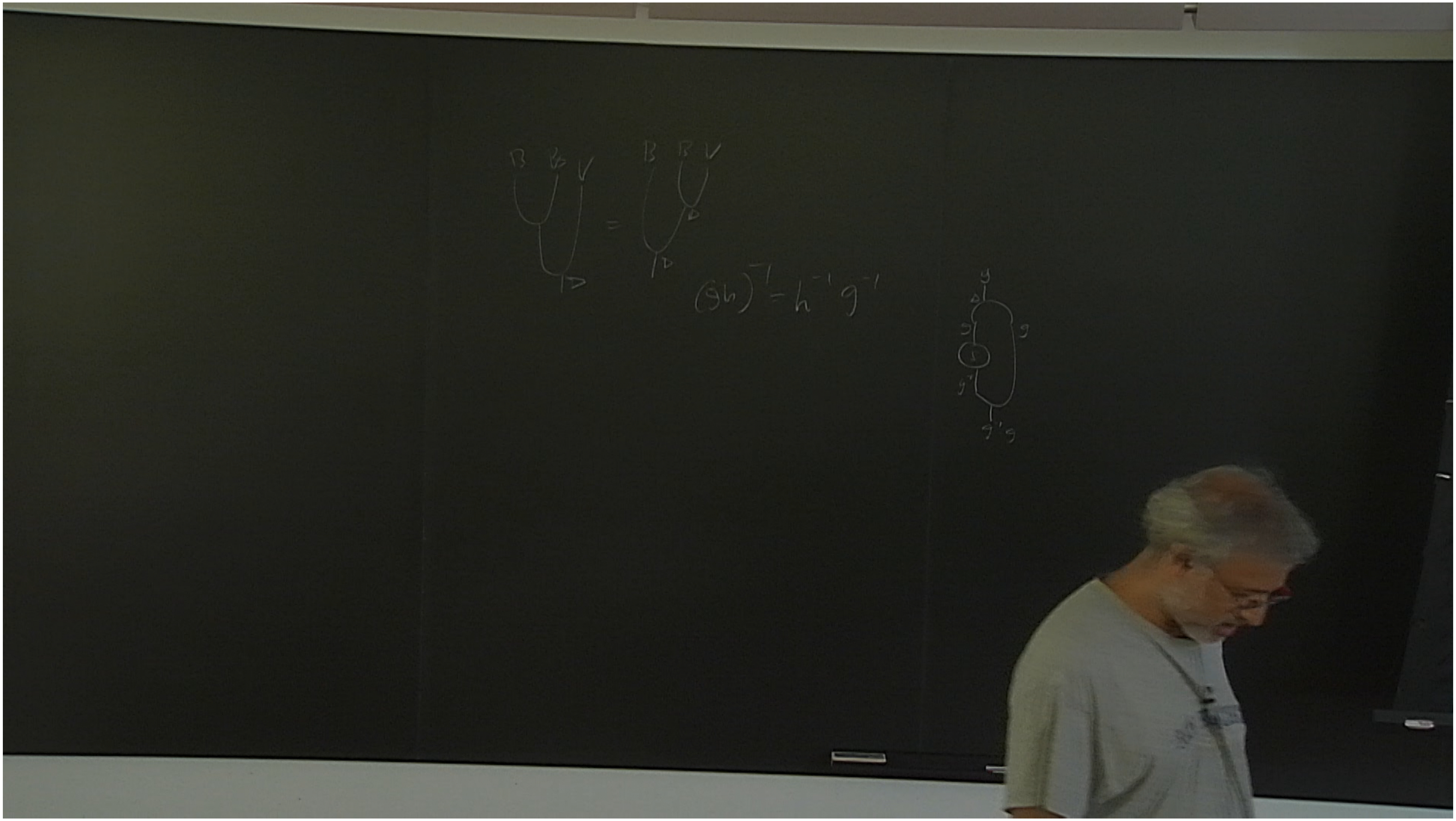
Dual:

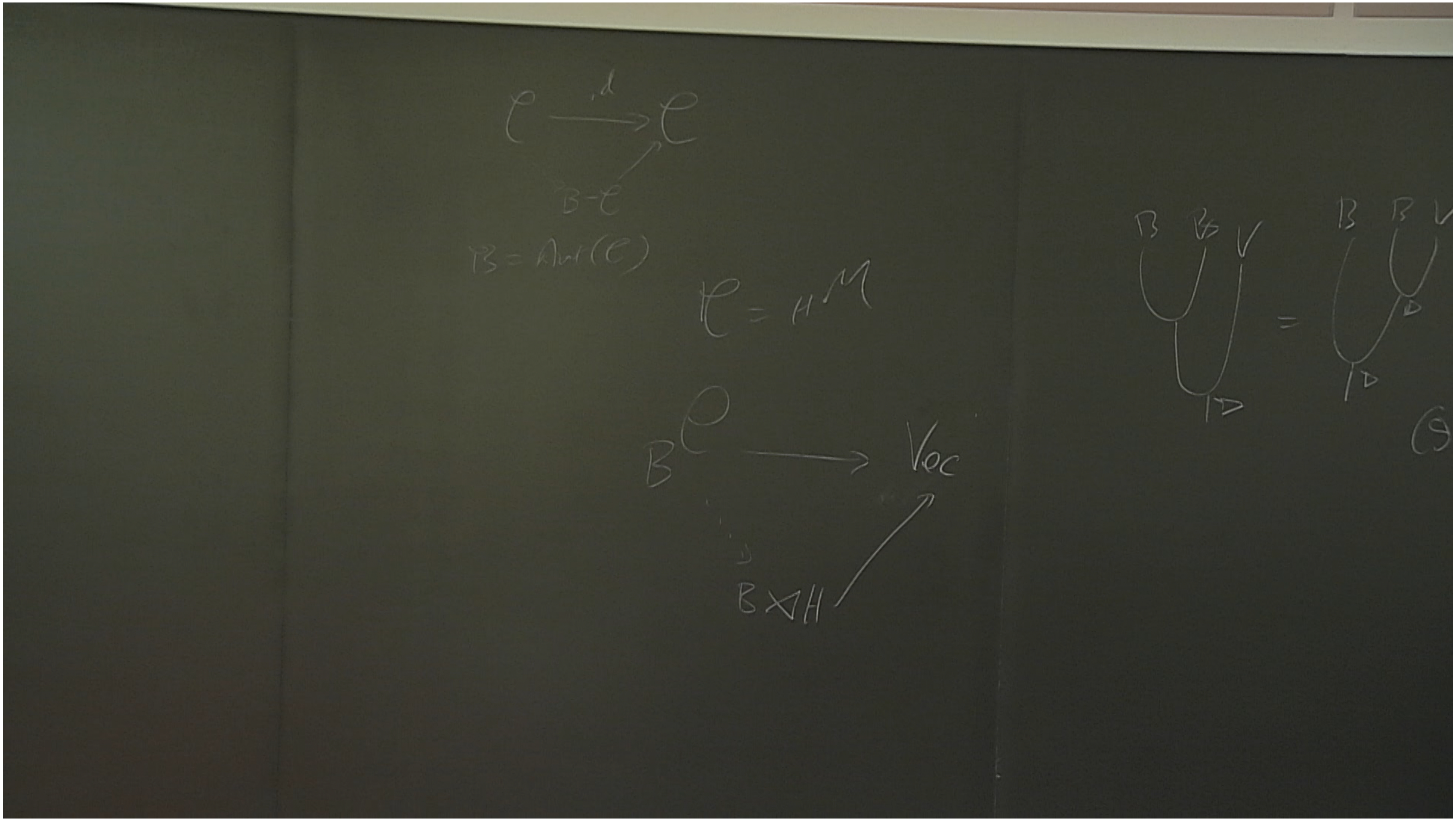


Tensor product of B-modules

=> monoidal category $_B\mathcal{C}$

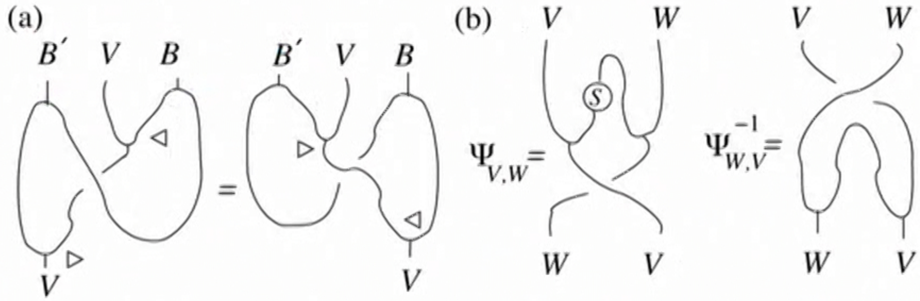



B B V
B B V
=
$(gh)^{-1} = h^{-1}g^{-1}$
g
$g^{-1}g$


id
B-C
Vec
B⋊H

Thm; *B'* dually paired with *B*, category ${}_{B'}\mathcal{C}_B$ of crossed *B'*-*B*-bimodules is another braided category

cf Bespalov, Drabant, SM mid 1990s



=> inductive construction of quantum groups by `double bosonisation' SM '95

H quasitri. Hopf alg. ⇒ $\mathcal{C}=\mathcal{M}_H$ braided cat right *H*-mod ⇒ ($B\in\mathcal{C}$) ${}_{B'}\mathcal{C}_B$ new braided cat ⇒ (Tannaka-Krein) $B^{*\underline{\mathrm{cop}}}{\bowtie}H{\bowtie}B$ new quasitri. Hopf alg.

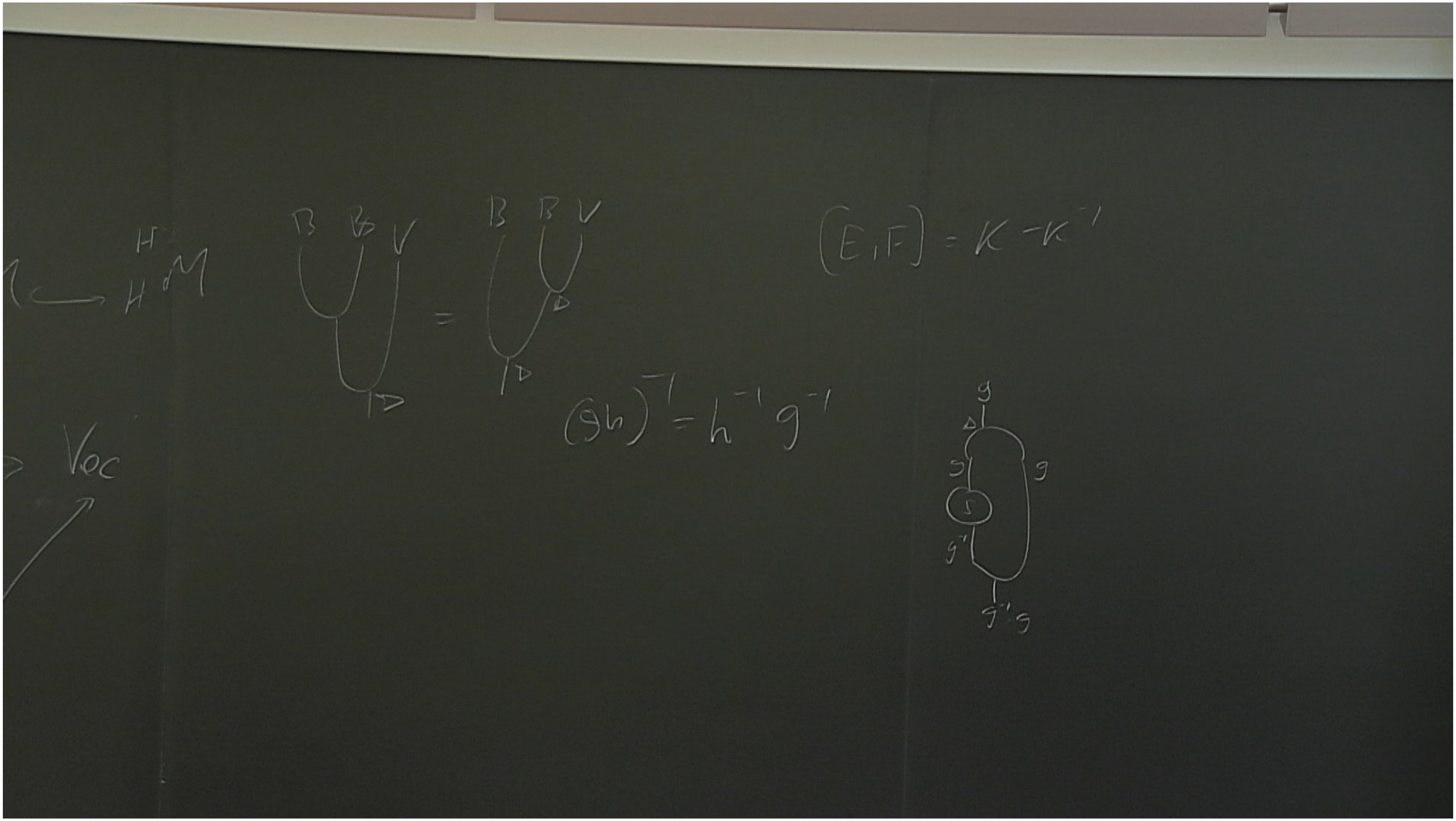
$$bc=(\mathcal{R}_1^{(2)}\triangleright c_{\overline{(2)}})\mathcal{R}_2^{(2)}\mathcal{R}_1^{-(1)}(b_{\underline{(2)}}\triangleleft\mathcal{R}_2^{-(1)})\langle\mathcal{R}_1^{(1)}\triangleright c_{\overline{(1)}},b_{\underline{(1)}}\triangleleft\mathcal{R}_2^{(1)}\rangle$$
$$\langle\mathcal{R}_1^{-(2)}\triangleright\bar{S}c_{\overline{(3)}},b_{\underline{(3)}}\triangleleft\mathcal{R}_2^{-(2)}\rangle$$

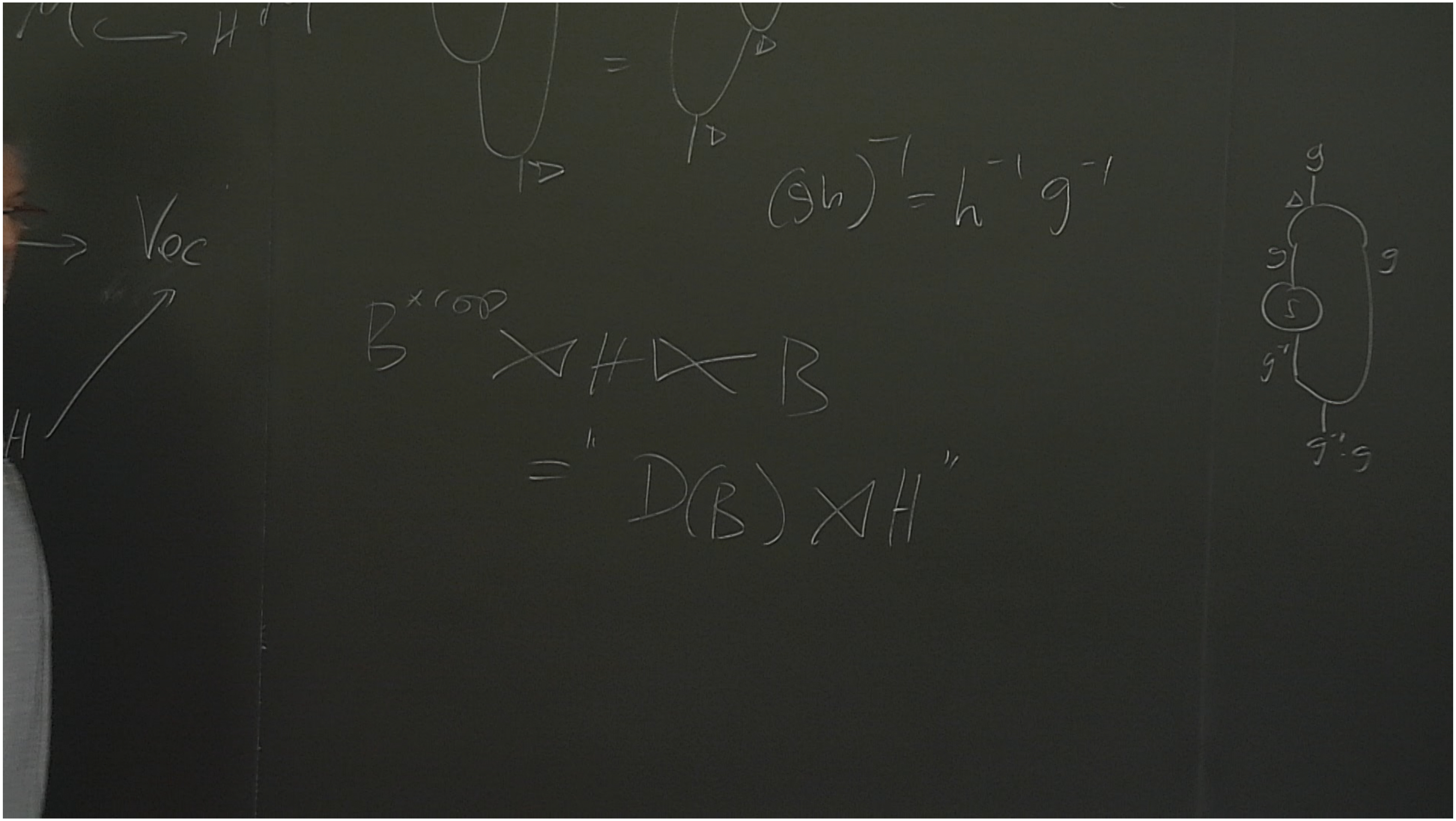
cross relations between $B^{*\underline{\mathrm{cop}}}, B$

$$\mathcal{R}_{new}=\overline{\exp}.\mathcal{R}$$

$$\overline{\exp}=\sum f^a\otimes\underline{S}e_a$$

e_a is basis of B and f^a is dual basis


B B V
B B V
=
[E,F] = K - K^-1
(gh)^-1 = h^-1 g^-1
Voc



Vec
H
(gh)^-1 = h^-1 g^-1
B^*cop ⋈ H ⋉ B
= "D(B) ⋊ H"
g
Δ
g
g
S
g^-1
g^-1·g

Example

$H = \mathbb{C}_q\mathbb{Z}_n = \mathbb{C}[K]/(K^n - 1)$

$\mathcal{R}_K = \frac{1}{n}\sum_{a,b=0}^{n-1} q^{-ab} K^a \otimes K^b$

$\mathcal{C} = \mathbb{Z}_n -$ graded v. spaces

$V = \oplus_i V_i, \ v \in V_i <=> |v| = i$

$\Psi_{V,W}(v \otimes w) = q^{|v||w|} w \otimes v$

$B = \mathbb{C}[E]/(E^n) \in \mathcal{C} \qquad E \triangleleft K = qE$

$\underline{\Delta} E = E \otimes 1 + 1 \otimes E \qquad (|E| = 1)$

$B^{*\underline{\mathrm{cop}}} \bowtie\!\!\triangleleft H \triangleright\!\!\bowtie B = u_q(sl_2)$

$E^n = F^n = 0, \quad K^n = 1, \quad KEK^{-1} = q^{-1}E, \quad KFK^{-1} = qF, \quad [E,F] = K - K^{-1}$

$\Delta K = K \otimes K, \quad \Delta F = F \otimes 1 + K^{-1} \otimes F, \quad \Delta E = E \otimes K + 1 \otimes E$

$$\mathcal{R}_{u_q(sl_2)} = \frac{1}{n}\sum_{r,a,b=0}^{n-1} \frac{(-1)^r q^{-ab}}{[r]_{q^{-1}}!} F^r K^a \otimes E^r K^b \qquad [r]_q = \frac{1-q^r}{1-q}$$

$u_q(sl_2) \cong u_{q^{-m}}(sl_2)$ if $n=2m+1$

=> new quasitri. Hopf algebras for n even

R.A./S.M. arXiv:1703.03456

e.g. $u_{-1}(sl_2)$ 8-diml self-dual strictly quasitri. $\mathcal{R} = (1 \otimes 1 - F \otimes E)\mathcal{R}_K$

$E^2 = F^2 = 0, \quad K^2 = 1, \quad EF = FE, \quad KE = -EK, \quad KF = -FK$

Next iteration e.g. $n=2m+1$ odd such that soln to $\alpha^2 = m(m-1) \mod n$

$$H = \widetilde{\mathfrak{u}_q(sl_2)} = u_{q^{-m}}(sl_2) \otimes \mathbb{C}_{q^\alpha}[g, g^{-1}]/(g^n - 1) \Rightarrow \mathcal{C} = \mathcal{M}_H$$

e.g. B=reduced quantum plane $\quad e_2 e_1 = q^{-m} e_1 e_2 \qquad e_i \triangleleft g = q^\alpha e_i.$

$$e_1^n = e_2^n = 0, \quad e_2 e_1 = q^{-m} e_1 e_2, \quad \underline{\Delta}(e_i) = e_i \otimes 1 + 1 \otimes e_i, \quad \underline{\epsilon}(e_i) = 0, \quad \underline{S}(e_i) = -e_i$$

$$\Psi(e_i \otimes e_i) = q e_i \otimes e_i, \quad \Psi(e_1 \otimes e_2) = q^{-m} e_2 \otimes e_1, \quad \Psi(e_2 \otimes e_1) = q^{-m} e_1 \otimes e_2 + (q-1) e_2 \otimes e_1$$

$\Rightarrow B^{*\underline{\mathrm{cop}}} \bowtie H \ltimes B = \mathfrak{u}_q(sl_3) \cong u_{q^{-m}}(sl_3)$ if $\alpha > 0$ (n=11,13,23,37…)

$\cong u'_{q^{-1}}(sl_3) \otimes \mathbb{C}\mathbb{Z}_3$ if $\alpha = 0 (n = 3)$

$u'_{q^{-1}}(sl_3) = u_{q^{-1}}(sl_3)/(K_2 - K_1)$

is result starting from $H= u_{q^{-1}}(sl_2)$
9-diml, quasitriangular

Many other choices
=> tree of quantum groups
(includes Dynkin classif)

$\mathbb{C}\mathbb{Z}_n \Rightarrow \mathfrak{u}_q(sl_2) \Rightarrow \mathfrak{u}_q(sl_3) \Rightarrow$

use other braided Hopf algebra B

Dual version `codouble-bosonisation'

A coquasitri. Hopf alg. $(A, \mathcal{R} : A \otimes A \to k)$ → ${}^A\mathcal{M}$ braided cat left A-comod → ($B \in \mathcal{C}$) → ${}_{B'}\mathcal{C}_B$ new braided cat → $B^{\underline{\mathrm{op}}} \bowtie A \bowtie B^*$ new coquasitri. Hopf alg.

Initial example → $\mathcal{C} = \mathbb{Z}_n$ − graded v. spaces (same category) → ${}_{B'}\mathcal{C}_B$

$A = \mathbb{C}_q[t, t^{-1}]/(t^n - 1)$

$B = \mathbb{C}[X]/(X^n)$

coaction $X \mapsto t \otimes X$

→ $\mathfrak{c}_q[SL_2]$ $\quad X^n = Y^n = 0, \quad t^n = 1, \quad YX = XY, \quad Xt = qtX, \quad Yt = qtY$

$$\Delta(t) = \frac{t}{q-1}\left(q\frac{1-((q-1)Y\otimes X)^{n-1}}{1-(q-1)Y\otimes X} - \frac{1-((1-q^{-1})Y\otimes X)^{n-1}}{1-(1-q^{-1})Y\otimes X}\right)t$$

$$\Delta(X) = X \otimes 1 + t\left(\frac{1-((q-1)Y\otimes X)^{n-1}}{1-(q-1)Y\otimes X}\right)X \qquad \Delta(Y) = 1 \otimes Y + Y\left(\frac{1-((1-q^{-1})Y\otimes X)^{n-1}}{1-(1-q^{-1})Y\otimes X}\right)t$$

$\mathfrak{c}_q[SL_2] \cong c_{q^{-m}}[SL_2]$ if $n=2m+1$ $\quad X = \frac{bd^{-1}}{q^{-m}+1}, \quad t = d^{-2}, \quad Y = \frac{d^{-1}c}{q^m - 1}$

=> new quasitri. Hopf algebras for n even $\qquad \mathfrak{c}_{-1}[SL_2] \cong \mathfrak{u}_{-1}(sl_2)$

Thm $A = H^*$ => $B^{\underline{op}} \bowtie A \bowtie B^* = (B^{*\underline{cop}} \bowtie H \bowtie B)^*$

Cor $\mathfrak{c}_q[SL_2]^* \cong \mathfrak{u}_q(sl_2)$ and $\{X^i t^j Y^k\}_{0 \le i,j,k \le n-1}$ is a `dual basis'

$$\langle X^i t^j Y^k, F^{i'} K^{j'} E^{k'} \rangle = \delta_{i,i'} \delta_{k,k'} q^{jj'} [i]_{q^{-1}}! [k]_q!$$

Fourier Transform right integral structure $\int : H \to k$, $\int^* : A = H^* \to k$, $\mu = \int(\int^*)$

$$\mathcal{F}(h) = \sum_a \left(\int e_a h\right) f^a, \quad \mathcal{F}^*(\phi) = \sum_a e_a \left(\int^* \phi f^a\right), \quad \mathcal{F}^* \circ \mathcal{F} = \mu S$$

e_a *is basis of* H, f^a *is the dual basis*

$\mathcal{F} : \mathfrak{c}_q[SL_2] \to \mathfrak{u}_q(sl_2)$ *is given by*

$$\mathcal{F}(X^a t^b Y^c) = \sum_{l=0}^{n-1} \frac{q^{-(l+a)(1-b)+b(n-1-c)}}{n[n-1-a]_{q^{-1}}! [n-1-c]_q!} F^{n-1-a} K^l E^{n-1-c}$$

$$\mu = \frac{q^{-1}}{n[n-1]_{q^{-1}}! [n-1]_q!}$$

Another application (braiding for TQFT)

$$D(H) = H^{*op} \bowtie H \qquad \mathcal{R}_D = \sum_a (f^a \otimes 1) \otimes (1 \otimes e_a)$$

Braided-Hopf Algebra Fourier Transform

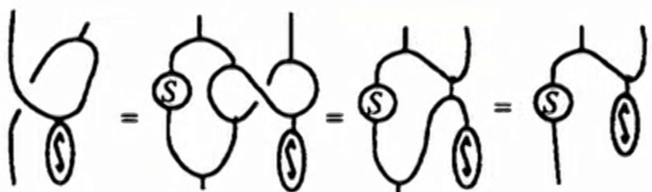
Suppose (1) B has a left integral $\int B \to \underline{1}$ $\quad (\mathrm{id}\otimes \int)\Delta = \eta \otimes \int$

(2) B has a left dual

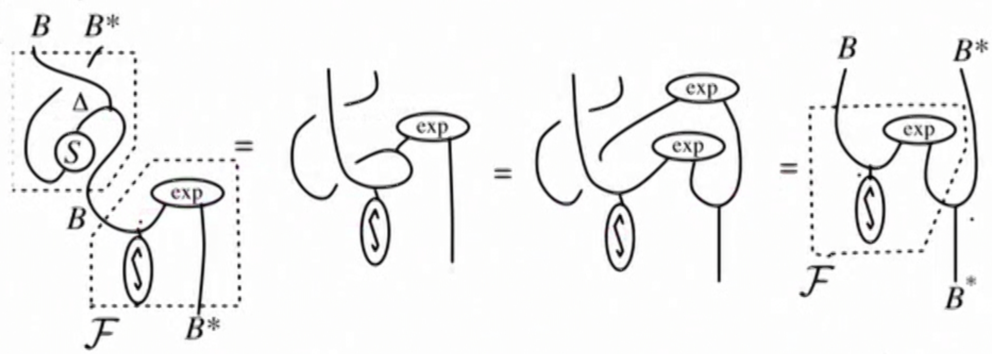
$$\mathrm{ev} = \cup : B^* \otimes B \to \underline{1}, \quad \mathrm{coev} = \cap = \exp : \underline{1} \to B \otimes B^*$$

$\Rightarrow$ braided Fourier transform $\quad \mathcal{F} : B \to B^*, \quad \mathcal{F}\circ \mathrm{Reg} = \cdot(\mathcal{F}\otimes \mathrm{id})$

Proof (Kempf & SM '94)

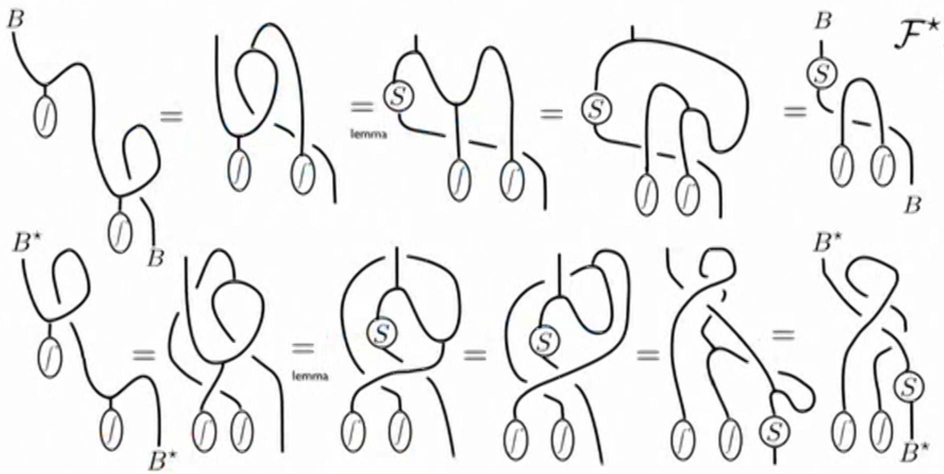


$\Rightarrow$



Suppose B* also has an integral, define $\mathcal{F}^*$ then (new)

$$\mathcal{F}^\star\mathcal{F} = \mu S, \quad \mu := (\int \otimes \int^*)\exp$$



If the integrals are both unimodular and morphisms then $\mathcal{F}\mathcal{F}^\star = \mu S$ *and* $[\mathcal{F}, S] = 0$

Example: $q^{n+1} = 1$ $\mathcal{C} = \mathbb{Z}/(n+1)$-graded spaces, $B = k[x]/(x^{n+1})$

$\Psi(x^m \otimes x^p) = q^{mp} x^p \otimes x^m$ $\Delta x = x \otimes 1 + 1 \otimes x$ $\Rightarrow$ $\int x^m = \delta_{m,n}$

$B^\star = k[y]/(y^{n+1})$ $\mathrm{ev}(y^m \otimes x^p) = \delta_{m,p}[m;q]!$ $\exp = \sum_{m=0}^n x^m \otimes y^m/[m;q]!$

$$\mathcal{F}(x^m) = \int x^m \exp(x \otimes y) = \frac{y^{n-m}}{[n-m;q]!}, \quad \mathcal{F}^\star(y^m) = \frac{q^{(n-m)^2} x^{n-m}}{[n-m;q]!}, \quad \mu = [n;q]!^{-1}$$

$$\mathcal{F}\mathcal{F}^\star = q^{2D+1}\mu S \qquad S\mathcal{F} = \mathcal{F}S q^{2D+1} \qquad D = \text{monomial deg}$$

Application to DG of Quantum Groups

- space of 1-forms, i.e. `differentials' on an algebra A

  Ω^1 A-bimodule $\mathrm{d} : A \to \Omega^1$ derivation

  $\{\sum a\mathrm{d}b\} = \Omega^1$ (surjective) $\ker \mathrm{d} = k.1$ (connected)

- require this to extend to a DGA $\Omega = T_A\Omega^1/\mathcal{I} = \oplus_n \Omega^n, \quad \mathrm{d}^2 = 0$

  $\Longrightarrow \quad H_{\mathrm{dR}}(A)$ quantum de Rham cohomology

- quantum metric $g \in \Omega^1 \underset{A}{\otimes} \Omega^1$ $\wedge(g) = 0$ `quantum symmetric'

  invertible in the sense exists inverse: $(\ ,\ ) : \Omega^1 \underset{A}{\otimes} \Omega^1 \to A$

$$((\ ,\ ) \otimes \mathrm{id})(\omega \otimes g) = \omega = (\mathrm{id} \otimes (\ ,\ ))(g \otimes \omega), \quad \forall \omega \in \Omega^1$$

$$a(\omega, \eta) = (a\omega, \eta), \quad (\omega, \eta)a = (\omega, \eta a)$$

$\Longrightarrow \quad g$ is central `start of noncomm Riemannian geom' (w/ Beggs)

Hodge Operator on Quantum Groups

$\Omega(H)$ is a super Hopf algebra (Brzezinski), S.M. Alg. Repn. Th. 2017

$\Omega \cong H \ltimes \Lambda$ $\quad\Lambda$ a super braided Hopf algebra in $\mathcal{C} = \overset{\times}{\mathcal{M}}{}^H_H = \mathcal{Y}D^H_H$
with primitive generators $\Lambda^1 = H^+/\mathcal{I}$

if Vol , g central and binvariant => $\# = (\mathrm{id} \otimes g \circ \mathcal{F}) : \Omega^n \to \Omega^{top-n}$

Lemma $\mu = \langle \mathrm{Vol}, \mathrm{Vol} \rangle^{-1} \in k^\times$, $\exists \sharp^{-1}$, $\sharp S = (-1)^{top} S \sharp$

g quantum symetric =>

$$S = (-1)^D, \quad \sharp^* = \sharp, \sharp^2 = \mu, \quad \text{on} \quad D = 0, 1, top-1, top$$

$$\exp = 1 \otimes 1 + g + \cdots + g^{(n-1)} + \mu \mathrm{Vol} \otimes \mathrm{Vol}$$

codifferential and Hodge Laplacian

$$\delta := (S\sharp)^{-1} \mathrm{d}(S\sharp), \quad \Box := \mathrm{d}\delta + \delta \mathrm{d}$$

$$\mathrm{d}f = (\partial^a f) e_a, \quad \alpha = \alpha^a e_a, \quad g = g^{ab} e_a \otimes e_b$$

=> $\delta\alpha = \alpha^a \delta e_a + g_{ab} \partial^a \alpha^b, \quad \Box f = (\partial^a f)\delta e_a + g_{ab} \partial^a \partial^b f$

=> canonical Hodge operator:

$$\exp = \sum_{m=0}^{top} \sum_{I,J} e_{i_1} \cdots e_{i_m} ({}_m B)^{-1}_{IJ} \otimes e_{j_1} \cdots e_{j_m}$$

$$\begin{aligned} {}_m B_{IJ} &= \langle e_{i_1} \cdots e_{i_m}, e_{j_1} \cdots e_{j_m} \rangle = \mathrm{ev}(e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_m}, [m, -\tilde{\Psi}]!(e_{j_1} \otimes e_{j_2} \cdots e_{j_m}) \\ &= g_{i_1 p_1} \cdots g_{i_m p_m} [m, -\tilde{\Psi}]!^{p_m \cdots p_2 p_1}_{j_1 j_2 \cdots j_m} \end{aligned}$$

=>

$$\sharp e_a = -q^4 e_a e_b e_c, \quad \sharp e_b = -q^4 e_a e_b e_d, \quad \sharp e_c = q^6 e_a e_c e_d, \quad \sharp e_d = q^4 e_b e_c e_d + \lambda q^4 e_a e_b e_c$$

$$\sharp(e_a e_b) = -q^2 e_a e_b, \quad \sharp(e_a e_c) = q^4 e_a e_c, \quad \sharp(e_a e_d) = q^2 e_b e_c + \lambda q^4 e_a e_d$$

$$\sharp(e_b e_c) = q^4 e_a e_d, \quad \sharp(e_b e_d) = q^4 e_b e_d + (1 - q^4) e_a e_b, \quad \sharp(e_c e_d) = -q^2 e_c e_d$$

$$\sharp(e_a e_b e_c) = -q^2 e_a, \quad \sharp(e_a e_b e_d) = -q^2 e_b \quad \sharp(e_a e_c e_d) = e_c \quad \sharp(e_b e_c e_d) = q^2 e_d + \lambda q^2 e_a$$

$\lambda = 1 - q^{-2}$

=> on $\Omega^D(\mathbb{C}_q[SL_2])$

$$\sharp^2 = q^6, \quad (D \neq 2); \quad (\sharp - q^4)(\sharp + q^2) = 0, \quad (D = 2).$$

up to normalisation obeys the q-Hecke relation