Title: How to go from the KS theorem to experimentally testable noncontextuality inequalities

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Abstract: The purpose of this talk is twofold: one, to acquaint the wider community working mostly on Bell-Kochen-Specker contextuality with recent work on Spekkens' contextuality that quantitatively demonstrates the sense in which Bell-Kochen-Specker contextuality is subsumed within Spekkens' approach, and two, to argue that one can test for contextuality without appealing to a notion of sharpness which can needlessly restrict the scope of operational theories that could be considered as candidate explanations of experimental data. Testing contextuality in Spekkens' approach therefore extends the range of experimental scenarios in which contextuality can be witnessed, and refines what it means to witness contextuality in the presence of inevitable noise in KS-type experiments. We will see this for both KS-uncolourability based logical contradiction type proofs of the KS theorem a la Kochen-Specker and statistical proofs on KS-colourable scenarios a la KCBS or Yu-Oh. While Bell-KS contextuality can be mathematically understood as an instance of the classical marginal problem, the same is not true of Spekkens' contextuality. The latter reduces to the classical marginal problem only under very specific conditions, being more general otherwise. All in all, we will argue that all you really need is Leibniz, i.e. identity of indiscernables, to make sense of contextuality in the most general context.

How to go from the Kochen-Specker theorem to experimentally testable noncontextuality inequalities

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Background

Friends, humans, contextuality men and women,
lend me your ears:[1]
I come here not to praise Spekkens,
I speak not to disprove what Adan spoke,
But here I am to speak what I do know.

---

[1]And your laughs and your cheers!

The framework: Spekkens 2005

Contextuality for preparations, transformations, and unsharp measurements

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The Bell-Kochen-Specker theorem establishes the impossibility of a noncontextual hidden variable model of quantum theory, or equivalently, that quantum theory is contextual. In this paper, an operational definition of contextuality is introduced which generalizes the standard notion in three ways: (i) it applies to arbitrary operational theories rather than just quantum theory, (ii) it applies to arbitrary experimental procedures rather than just sharp measurements, and (iii) it applies to a broad class of ontological models of quantum theory rather than just deterministic hidden variable models. We derive three no-go theorems for ontological models, each based on an assumption of noncontextuality for a different sort of experimental procedure; one for preparation procedures, another for unsharp measurement procedures (that is, measurement procedures associated with positive-operator valued measures), and a third for transformation procedures. All three proofs apply to two-dimensional Hilbert spaces, and are therefore stronger than traditional proofs of contextuality.

This talk

What does the Spekkens framework say for Kochen-Specker type experimental scenarios?

1. We know that KS-noncontextuality and POVMs don't work well together. The classical vs. quantum vs. post-quantum hierarchy for correlations breaks down.

2. Option 1: Hold on to your notion of classicality and restrict the set of measurements to those where it is meaningful.

3. Option 2: Broaden your notion of classicality[2] and do not place restrictions on the set of measurements.

We take the latter approach. The revised notion of classicality is noncontextuality à la Spekkens. Can handle trivial POVMs.

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[2]Trivial POVMs are "classical", after all: they cannot distinguish between different input states.

Prior work: Kunjwal-Spekkens 2015

From the Kochen-Specker Theorem to Noncontextuality Inequalities without Assuming Determinism

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(Received 18 June 2015; published 9 September 2015)

The Kochen-Specker theorem demonstrates that it is not possible to reproduce the predictions of quantum theory in terms of a hidden variable model where the hidden variables assign a value to every projector deterministically and noncontextually. A noncontextual value assignment to a projector is one that does not depend on which other projectors—the context—are measured together with it. Using a generalization of the notion of noncontextuality that applies to both measurements and preparations, we propose a scheme for deriving inequalities that test whether a given set of experimental statistics is consistent with a noncontextual model. Unlike previous inequalities inspired by the Kochen-Specker theorem, we do not assume that the value assignments are deterministic and therefore in the face of a violation of our inequality, the possibility of salvaging noncontextuality by abandoning determinism is no longer an option. Our approach is operational in the sense that it does not presume quantum theory: a violation of our inequality implies the impossibility of a noncontextual model for *any* operational theory that can account for the experimental observations, including any successor to quantum theory.

Related work

- Deriving robust noncontextuality inequalities from algebraic proofs of the Kochen-Specker theorem: the Peres-Mermin square, A. Krishna, R. W. Spekkens, E. Wolfe, arXiv:1704.01153 [quant-ph].

- Irreducible noncontextuality inequalities from KS-uncolourability based proofs of the KS theorem, RK (forthcoming).

- Translating proofs of the Kochen-Specker theorem without KS-uncolourability into noise-robust noncontextuality inequalities, RK and R. W. Spekkens (forthcoming).

- From the CSW framework to robust noncontextuality inequalities via hypergraph invariants, RK (forthcoming).

An experiment and its descriptions

A Prepare-and-Measure Experiment

$$m \in V_M$$

Measurement

$$M \in \mathbb{M}$$

$$P_{(S,s)}$$

$$s \in V_S$$

Source

$$S \in \mathbb{S}$$
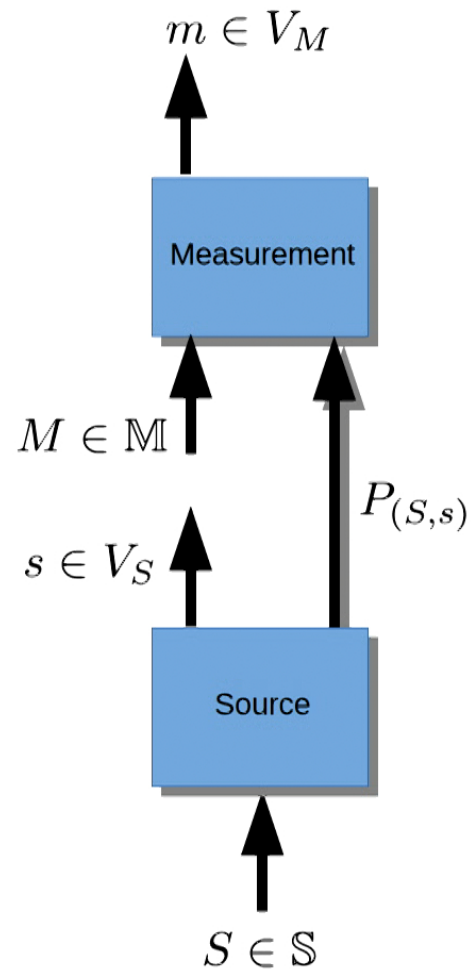
Description I: Operational theory[3]

Operational primitives; utilitarian goals; care about predictions:

$$p(m, s | M, S) \in [0, 1], \tag{1}$$

e.g., operational quantum theory.

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[3]Operational-probabilistic theory à la Giulio

Description II: Ontological model

Ontological primitives (λ); explanatory goals; care about explanations:

$$p(m, s | M, S) = \sum_{\lambda \in \Lambda} \xi(m | M, \lambda) \mu(\lambda, s | S), \qquad (2)$$

e.g., Bohmian mechanics.

Things we are assuming are always possible *in an experiment*

- ► That preparation procedures can be mixed probabilistically to *define* a new preparation procedure, e.g., coarse graining over the outcomes of a source setting S, i.e.,
  $$[\top|S] \equiv \sum_{s \in V_S} p(s|S)[s|S].$$

- ► That measurement events from a given measurement procedure M can be post-processed to *define* new measurement procedures, e.g., given the set of measurement events of M, $\{[m|M]\}_m$, one can define a new mmt M':

$$[m'|M'] \equiv \sum_m \mathrm{pr}(m'|m)[m|M], \qquad (3)$$

where $\mathrm{pr}(m'|m) \in [0, 1]$.

How these things are represented in a *description* of the experiment

Operational[4] description:

► Preparations:

$$\forall [m|M] : \mathrm{pr}(m|M,S) \equiv \sum_{s \in V_S} p(s|S)\mathrm{pr}(m|M,S,s),$$

$$\text{or} \quad \sum_{s \in V_S} \mathrm{pr}(m,s|M,S). \tag{4}$$

► Measurements:

$$\forall [s|S] : \mathrm{pr}(m',s|M',S) \equiv \sum_{m} \mathrm{pr}(m'|m)\mathrm{pr}(m,s|M,S). \tag{5}$$

.

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[4]Operational-probabilistic

What we will *not* presume in the ontological model

Assumption of **outcome determinism**: for any $[m|M]$,
$$\xi(m|M, \lambda) \in \{0, 1\} \forall \lambda \in \Lambda.$$

Features of the operational description necessary to define noncontextuality

Operational equivalence

Preparations

- Source events are operationally equivalent ($[s|S] \simeq [s'|S']$) if no measurement event can distinguish them.

- Source settings (or just "Sources") are operationally equivalent ($S \simeq S'$) when, ignoring their outcomes, no measurement event can distinguish them.

$[s|S] \simeq [s'|S']$ if

$$p(m, s|M, S) = p(m, s'|M, S') \quad \forall [m|M]. \qquad (8)$$

$S \simeq S'$, if $\forall [m|M]$

$$\sum_{s \in V_S} p(m, s|M, S) = \sum_{s' \in V_{S'}} p(m, s'|M, S') \qquad (9)$$

Measurements

Measurement events are operationally equivalent $([m|M] \simeq [m'|M'])$ if no source event can distinguish them, i.e.,

$$\forall [s|S] : p(m, s|M, S) = p(m', s|M', S). \qquad (10)$$

What is a 'context'?

Any distinction between operationally equivalent procedures.

Examples

Preparation contexts: Different realizations of a given quantum state, e.g., different convex decompositions,

$$\frac{I}{2} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|,$$

or different purifications,

$$\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|_{AB} = \mathrm{Tr}_C |\phi\rangle\langle\phi|_{AC}, \text{etc.}$$

Measurement contexts: Different realizations of a given POVM or a POVM element, e.g., same projector appearing in different measurement bases, joint measurability contexts for a given POVM, or even different ways of implementing a fair coin flip measurement.[5]

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[5]Mazurek et. al., Nature Communications 7:11780 (2016)

Defining noncontextuality as an inference from the operational description to the ontological description

Noncontextuality: identity of indiscernibles

If there exists no operational way to distinguish two things, then they must be physically identical.[6]

- ▶ Measurement noncontextuality:

$$[m|M] \simeq [m'|M'] \Rightarrow \xi(m|M, \lambda) = \xi(m'|M', \lambda) \quad \forall \lambda \in \Lambda$$

- ▶ Preparation noncontextuality:

$$[s|S] \simeq [s'|S'] \Rightarrow \mu(\lambda, s|S) = \mu(\lambda, s'|S') \quad \forall \lambda \in \Lambda,$$

$$S \simeq S' \Rightarrow \mu(\lambda|S) = \mu(\lambda|S') \quad \forall \lambda \in \Lambda.$$

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[6]Equivalently: if two things are non-identical, or physically distinct, then there must exist an operational way to distinguish them.

Note that we do not need a definition of "sharp" measurements in our operational description. All measurements are treated on an equal footing.

The Kochen-Specker theorem

Logic of the Kochen-Specker theorem

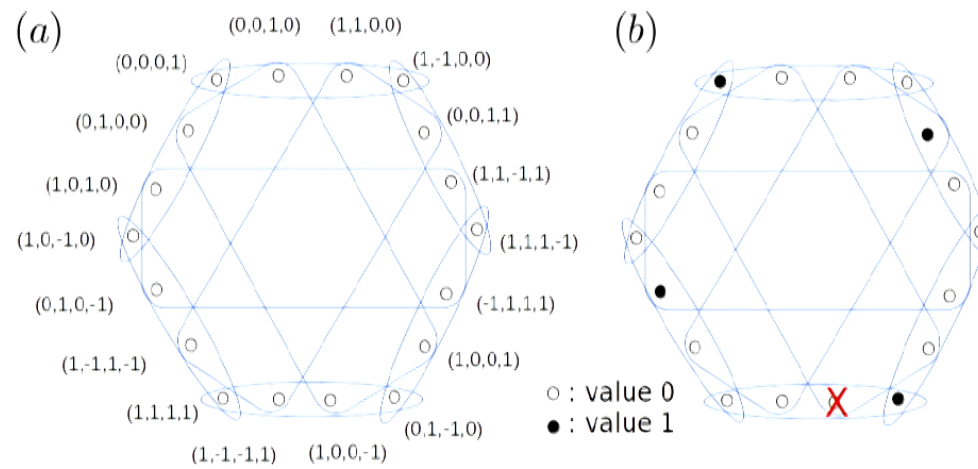For proofs based on KS-uncolourable scenarios,

Operational equivalences between measurements

$+$ Measurement noncontextuality

$+$ Outcome determinism

$\rightarrow$ Contradiction.

The contradiction disappears if outcome determinism is relaxed.

Example: CEGA

We use a proof of the KS theorem[7] with 18 rays in $\mathbb{C}^4$:

- ▶ (a) $p(\Pi|\mathrm{basis}, \rho) = p(\Pi|\mathrm{basis}', \rho) = \mathrm{Tr}(\rho\Pi)$ for all ρ.
- ▶ (b) KS-noncontextuality:
  $\xi(\Pi|\mathrm{basis}, \lambda) = \xi(\Pi|\mathrm{basis}', \lambda) \in \{0, 1\}$ for all $\lambda \in \Lambda$.
- ▶ Let the $\{0, 1\}$ values assigned by a λ be labelled $\{w_1, \ldots, w_{18}\}$.

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[7] A. Cabello, J. Estebaranz, and G. Garcia-Alcaine, Bell-Kochen-Specker theorem: A proof with 18 vectors, Physics Letters A 212, 183 (1996).

$$
\begin{aligned}
w_1 + w_2 + w_3 + w_4 &= 1 \\
w_4 + w_5 + w_6 + w_7 &= 1 \\
w_7 + w_8 + w_9 + w_{10} &= 1 \\
w_{10} + w_{11} + w_{12} + w_{13} &= 1 \\
w_{13} + w_{14} + w_{15} + w_{16} &= 1 \\
w_{16} + w_{17} + w_{18} + w_1 &= 1 \\
w_{18} + w_2 + w_9 + w_{11} &= 1 \\
w_3 + w_5 + w_{12} + w_{14} &= 1 \\
w_6 + w_8 + w_{15} + w_{17} &= 1.
\end{aligned}
$$

$$
2 \sum_{i=1}^{18} w_i = 9. \tag{11}
$$

No $\{0, 1\}$-valued solution.

Not all (measurement) contexts can be made deterministic by any noncontextual assignment of probabilities.

Logic of the Kochen-Specker theorem

For proofs based on KS-colourable scenarios:

Operational equivalences between measurements

$+$ Measurement noncontextuality

$+$ Outcome determinism

$\rightarrow$ Upper bound on strength of measurement-measurement correlations on *any* input state

This can be rewritten as a contradiction:

Operational equivalences between measurements

$+$ Measurement noncontextuality

$+$ Outcome determinism

$+$ Strength of measurement-measurement correlations on *at least one* input state exceeding a certain bound

$\rightarrow$ Contradiction.

Again, the contradiction disappears if outcome determinism is relaxed.

Example: KCBS, PRL 101, 020403 (2008)
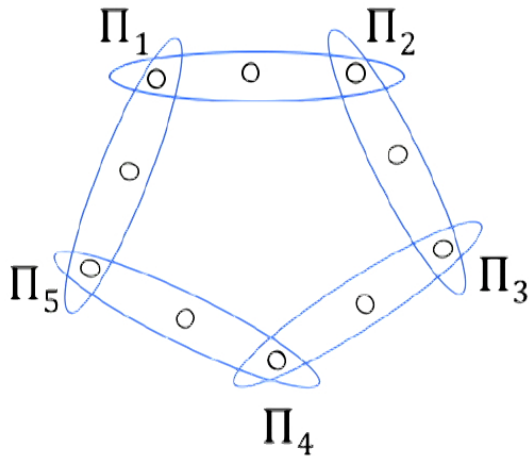
- ▶ (a) $p(\Pi|\mathrm{basis}, \rho) = p(\Pi|\mathrm{basis'}, \rho) = \mathrm{Tr}(\rho\Pi)$ for all ρ. Hence, we denote these by $p(\Pi|\rho)$.

- ▶ (b) KS-noncontextuality:
  $\xi(\Pi|\mathrm{basis}, \lambda) = \xi(\Pi|\mathrm{basis'}, \lambda) \in \{0, 1\}$ for all $\lambda \in \Lambda$. Hence, we denoted these by $\xi(\Pi|\lambda)$.

- ▶ Empirical adequacy: $p(\Pi|\rho) = \sum_\lambda \xi(\Pi|\lambda)\mu(\lambda|\rho)$.

- Let the $\{0, 1\}$ values assigned noncontextually by λ be labelled $\{w_1(\lambda), \dots, w_5(\lambda)\}$, where $w_i(\lambda) \equiv \xi(\Pi_i|\lambda)$.

- Normalization constraints: $p(\Pi_i|\rho) + p(\Pi_{i\oplus 1}|\rho) \leq 1 \quad \forall \rho$, for all $i \in [5]$ (addition mod 5).

- KCBS inequality:
$$R(\rho) \equiv \sum_{i=1}^{5} p(\Pi_i|\rho) = \sum_{i=1}^{5} w_i(\lambda)\mu(\lambda|\rho) \leq 2(\equiv r_{KS}).$$
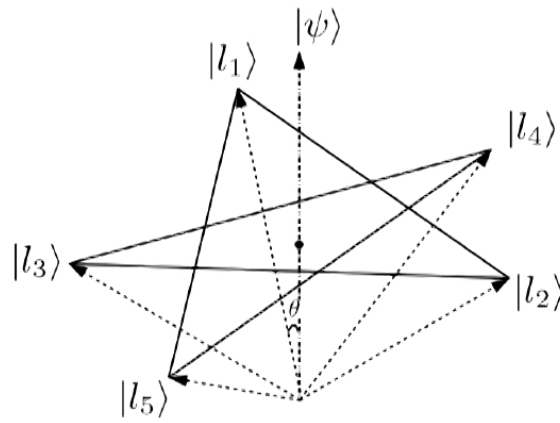
- Hence,

$$\mathrm{mmt\ opequivs+MNC+OD}+R(\rho) > r_{KS} \to \mathrm{contradiction}, \tag{12}$$

- or, equivalently,

$$\mathrm{mmt\ opequivs+MNC+OD} \to R \leq r_{KS}. \tag{13}$$

- Maximum violation ($R(\rho) = \sqrt{5}$) happens for a particular choice of $\rho = |\psi\rangle\langle\psi|$ for a given set of $\{\Pi_i\}_{i=1}^5$. If OD is relaxed, we can have $R = 2.5 > 2$ without a contradiction.

Example: Yu-Oh, PRL 108, 030402 (2012)

Same logic as KCBS, with one difference: for a fixed choice of projectors, it turns out that $R(\rho) \equiv \text{constant} > r_{KS}$ for all choices of ρ. Yu-Oh is therefore *quantum* state-independent, but KCBS isn't. However, logically, they fall into the same category of proof of the KS theorem from KS-colourable scenarios as KCBS.

KS-uncolourability vs. state independence

- ▶ The distinction we draw is between KS-uncolourability based proofs of the KS theorem vs. proofs of KS theorem on KS-colourable scenarios, which *does not* correspond to the usual distinction between state-independent vs. state-dependent proofs of KS theorem.
- ▶ Yu-Oh is quantum state-independent but it is based on a KS-colourable scenario. It is intermediate between KCBS and 18 ray from the quantum point of view.

How Spekkens recovers Kochen-Specker

- ▶ Kochen-Specker: The operational theory is operational *quantum* theory and measurements are projective.

- ▶ Then:

$$\Rightarrow \quad \begin{array}{l} \text{Preparation noncontextuality} \\ \\ \text{Measurement noncontextuality AND} \\ \\ \text{Outcome determinism, i.e., KS-noncontextuality.}^8 \end{array}$$

- ▶ Any proof of KS-contextuality is therefore a proof of preparation contextuality for quantum theory, but not conversely.

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[8] See Leifer-Maroney, arXiv:1208.5132 [quant-ph], for a proof via maximal psi-epistemicity and RK, arXiv:1612.07250 [quant-ph], for a direct proof.

How Spekkens extends Kochen-Specker

- ▶ If the measurements are nonprojective, the KS framework is not applicable. Spekkens provides a proper extension to this case.[9]

- ▶ The Spekkens framework can handle novel joint measurability structures such as the Specker scenario with pairwise compatible POVMs that are triplewise incompatible. These do not occur when restricting to PVMs.[10]

- ▶ If one does not presume operational quantum theory, Spekkens still provides noise-robust criteria for contextuality. (This talk.)

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[9] R. W. Spekkens, The Status of Determinism in Proofs of the Impossibility of a Noncontextual Model of Quantum Theory, Found. Phys. 44, 1125 (2014).
[10] RK, C. Heunen, T. Fritz, Quantum realization of arbitrary joint measurability structures, Phys. Rev. A 89, 052126 (2014)

Key idea

Focus on the operational features that imply outcome determinism for a set of measurements, assuming nothing about "sharpness" (however defined). These features correspond to perfect source-measurement correlations.

Mmt-mmt correlations vs. source-mmt correlations

Mmt-mmt correlations, Corr_{mm}

Compatible sets of measurements are carried out on the same input state and the strength of correlations between the measurements for this input state is the quantity of interest:[11] a function of $p(m_1, m_2, \ldots, m_k | M_1, M_2, \ldots, M_k, S, s)$ for fixed $[s|S]$.

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[11] This is the quantity of interest in Bell-KS scenarios.

KS-uncolourable scenarios
(Bounds on Corr_{sm})

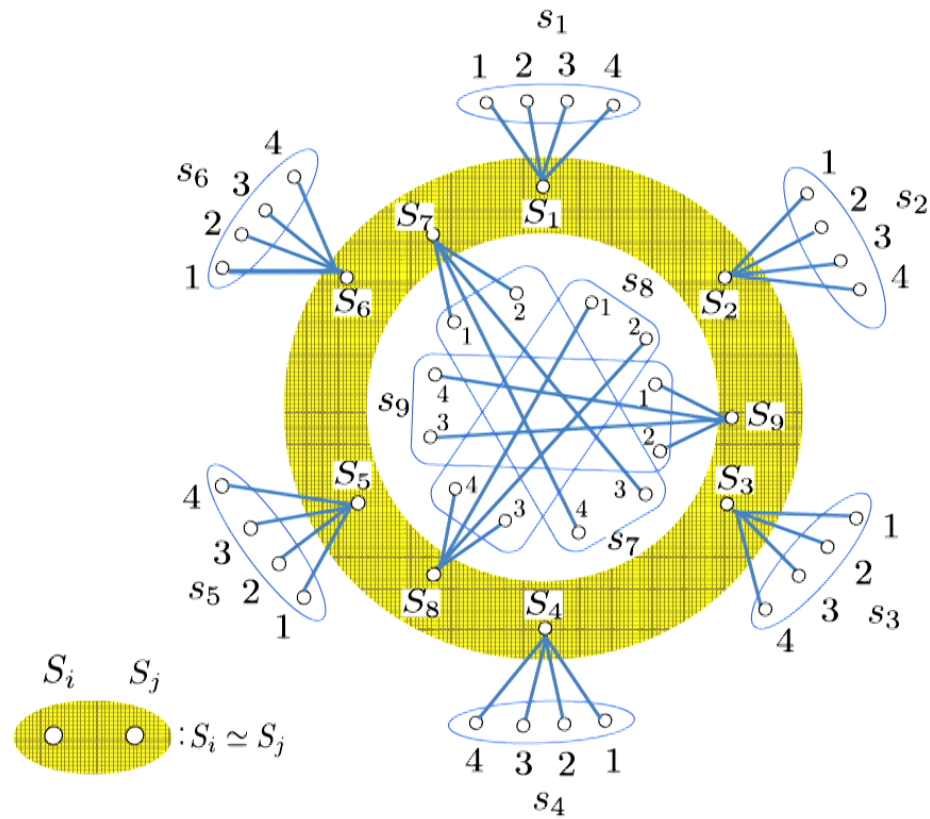**Source-mmt correlations,** Corr_{sm}

For every measurement event of interest, identify a source event that makes the measurement event as likely as possible i.e., for $[m|M]$, maximize $p(m|M, S, s)$ over source events $[s|S]$.

Operational equivalences between measurement events

$$[m_i|M_i] \quad [m_j|M_j]$$

$$: [m_i|M_i] \simeq [m_j|M_j]$$
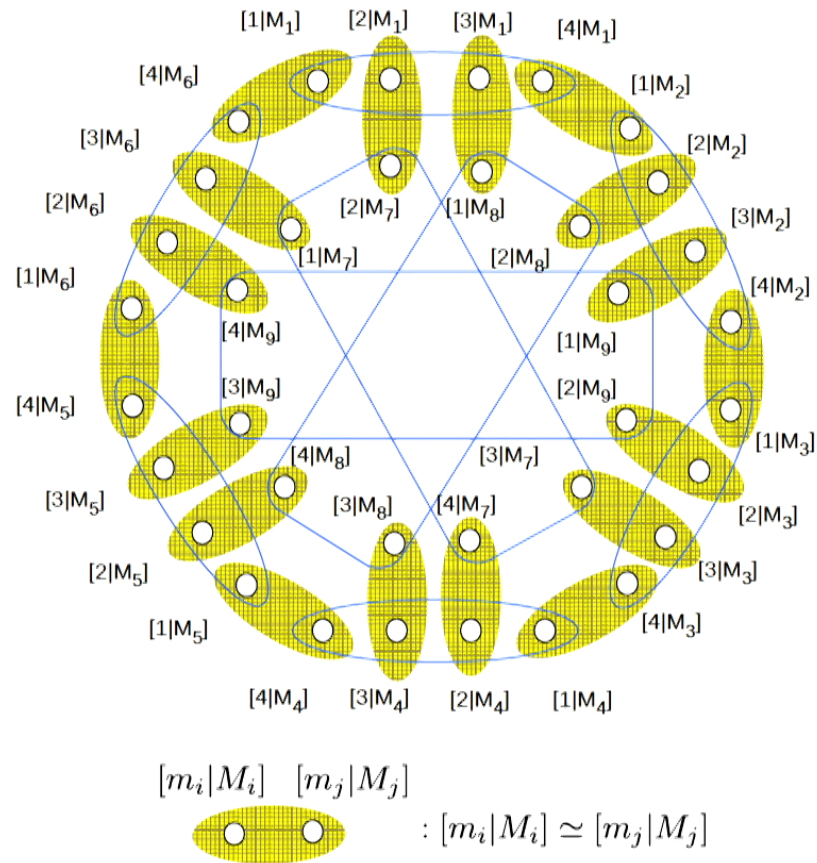
Operational equivalences between sources

Justifying outcome determinism

Operational equivalences between sources

\+   Preparation noncontextuality

\+   Perfect source-measurement correlations

$\rightarrow$ Outcome determinism over the support of $\nu(\lambda) \equiv \mu(\lambda|S_i)\forall i,$

$$(14)$$

where the operational equivalences are $S_i \simeq S_{i'}$ for all $i, i' \in [n]$, preparation noncontextuality requires $\mu(\lambda|S_i) = \mu(\lambda|S_{i'}) \equiv \nu(\lambda)$ for all $i, i' \in [n]$, perfect source-measurement correlations means that $p(m_i|M_i, S_i, s_i) = \delta_{m_i,s_i}$ for all $i \in [n]$, and outcome determinism over the support of $\nu(\lambda)$ means $\xi(m|M, \lambda) \in \{0, 1\}$ for all $\lambda \in \Lambda$ such that $\nu(\lambda) > 0$.

A no-go theorem for noncontextuality

$$
\begin{aligned}
& \text{Operational equivalences} \\
+ \quad & \text{Noncontextuality} \\
+ \quad & \text{Perfect source-measurement correlations} \\
\rightarrow \quad & \text{Contradiction}
\end{aligned}
\tag{15}
$$

Hence,

$$
\begin{aligned}
& \text{Operational equivalences} \\
+ \quad & \text{Noncontextuality} \\
\rightarrow \quad & \text{Imperfect source-measurement correlations}
\end{aligned}
\tag{16}
$$

Hence, noncontextuality bounds the following quantity:

$$\mathrm{Corr}_{sm} \equiv \sum_{i=1}^{n} q_i \sum_{m_i, s_i} \delta_{m_i, s_i} p(m_i, s_i | M_i, S_i), \qquad (17)$$

for arbitrary $q_i \geq 0$ for all i such that $\sum_i q_i = 1$. Clearly, $\mathrm{Corr}_{sm} \in [0, 1]$ and $\mathrm{Corr}_{sm} = 1$ if and only if source-measurement correlations are perfect.

Bounding Corr_{sm}

$$\mathrm{Corr}_{sm} \leq \max_{\lambda \in \Lambda} \sum_{i=1}^{n} q_i \zeta(M_i, \lambda) < 1, \tag{18}$$

for *some* choices of q_i, where Λ can be identified with the set of vertices of the "no-signalling"/"no-disturbance" polytope defined by a measurement noncontextual assignment of probabilities to the hypergraph, and we have $\zeta(M_i, \lambda) \equiv \max_{m_i} \xi(m_i | M_i, \lambda)$.
In particular, **there exists a nontrivial upper bound when $q_i > 0$ for all i, just from the KS-uncolourability.**[12]

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[12]RK and RWS, Phys. Rev. Lett. 115, 110403 (2015).

It turns out that there exists a nontrivial upper bound on Corr_{sm}, i.e., $\sum_{i=1}^{n} q_i \zeta(M_i, \lambda) < 1$, even when $q_i = 0$ for some of the measurement contexts. That is,

$$\mathrm{Corr}_{sm}^{\text{subset}} \equiv \sum_{i=1}^{N} q_{r_i} \sum_{m_{r_i}, s_{r_i}} \delta_{m_{r_i}, s_{r_i}} p(m_{r_i}, s_{r_i} | M_{r_i}, S_{r_i})$$

for an N-context subset $(N < n)$, where $q_{r_i} > 0$ and $\sum_{i=1}^{N} q_{r_i} = 1.$[13]

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[13]A. Krishna, Experimentally Testable Noncontextuality Inequalities Via Fourier-Motzkin Elimination (2015), MSc thesis at `http://hdl.handle.net/10012/9752`.

Obtaining all the noncontextuality inequalities then amounts to identifying all the $\{q_i\}_{i=1}^n$ such that $\mathrm{Corr}_{sm} < 1$. And identifying all such $\{q_i\}$ amounts to identifying the minimal independent subsets ("irrMISCs") of the set of all measurement contexts admitting a nontrivial upper bound when $\{q_i\}$ is supported entirely on them.[14]

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[14]See my forthcoming paper, "Irreducible noncontextuality inequalities from KS-uncolourability based proofs of the KS theorem."

Example: 18 ray

$$\mathrm{Corr}_{sm} \leq \frac{5}{6}, \tag{19}$$

where $q_i = \frac{1}{9}$ for all $i \in [9]$.

KS-colourable scenarios

(Tradeoff between Corr_{mm} and Corr_{sm}.)

Introduce Corr_{sm} and recall justification for OD

> Operational equivalences between sources
>
> \+   Preparation noncontextuality
>
> \+   Perfect source-measurement correlations
>
> $\rightarrow$ Outcome determinism over the support of $\nu(\lambda) \equiv \mu(\lambda|S_i) \forall i$

A new contradiction

$\quad$ Operational equivalences

$+\quad$ Noncontextuality

$+\quad$ Perfect source-mmt correlations, namely, $\mathrm{Corr}_{sm} = 1$

$+\quad$ Strength of measurement-measurement correlations
on *at least one* input state exceeding a certain bound,
namely, $\mathrm{Corr}_{mm}([s_*|S_*]) > c_{KS}$

$\rightarrow\quad$ Contradiction.

A noncontextuality inequality

$$
\begin{aligned}
&\quad \text{Operational equivalences} \\
+\ &\quad \text{Noncontextuality} \\
\rightarrow\ &\quad \text{Tradeoff between } \mathrm{Corr}_{sm} \text{ and } \mathrm{Corr}_{mm}([s_*|S_*])
\end{aligned}
$$

Robust KCBS

Recalling that in the quantum case, $R(\rho) \equiv \sum_{i=1}^{5} p(\Pi_i|\rho)$, we define the operational quantity

$$\mathrm{Corr}_{mm}([s_*|S_*]) \equiv \sum_{i=1}^{5} p(m_i = 0, s_*|M_i, S_*), \qquad (20)$$

where $M_i \equiv \{[m_i|M_i]\}_{m_i=0}^{2}$, with operational equivalences $[2|M_i] \simeq [0|M_{i\oplus 1}]$ for all $i \in [5]$. Here, $[m_i = 0|M_i]$ is the operational counterpart of Π_i for all $i \in [5]$. The operational counterpart of ρ is $[s_*|S_*]$ and that of $R(\rho)$ is

$$R([s_*|S_*]) \equiv \frac{\mathrm{Corr}_{mm}([s_*|S_*])}{p(s_*|S_*)} \qquad (21)$$

Sources

1. We have 5 source settings S_i, $i \in [5]$, such that each S_i has 3 possible outcomes $s_i \in \{0, 1, 2\}$. These sources are such that $S_1 \simeq S_2 \simeq \cdots \simeq S_5 \simeq S_*$, where S_* is an additional special source setting with one of its (at least 2) outcomes labelled s_*.

2. We define

$$\mathrm{Corr}_{sm} \equiv \sum_{i=1}^{5} q_i \sum_{m_i, s_i} \delta_{m_i, s_i} p(m_i, s_i | M_i, S_i) \qquad (22)$$

for the source-mmt correlations.

Robust noncontextuality inequality in KCBS scenario

$$R[s_* | S_*] + \mathrm{Corr}_{sm} \leq 3.$$

Ideal quantum case

In the ideal quantum case, for all $i \in [5]$, we have
$[m_i = 0 | M_i] \equiv \Pi_i = |l_i\rangle\langle l_i|$, $[s_i = 0 | S_i] \equiv |l_i\rangle\langle l_i|$, $[s_* | S_*] \equiv |\psi\rangle\langle\psi|$, with

$$\frac{1}{3} \sum_{s_i = 0}^{2} [s_i | S_i] \simeq \frac{1}{3} [s_* | S_*] + \frac{2}{3} [\bar{s}_* | S_*], \quad \forall i \in [5]. \tag{23}$$

Thus, $\mathrm{Corr}_{sm} = 1$, $R[s_* | S_*] = \sqrt{5}$, and $R[s_* | S_*] + \mathrm{Corr}_{sm} \leq 3$ is violated. Read differently, $\mathrm{Corr}_{sm} = 1$ provides operational grounds to believe in the noncontextuality inequality $R[s_* | S_*] \leq 2$ (which agrees with the usual KCBS inequality and is violated by this construction).

Trivial POVMs are not pathological

On the other hand, if the measurements and sources are noisy, we have $\mathrm{Corr}_{sm} < 1$ and we are left with

$$R[s_*|S_*] \leq 3 - \mathrm{Corr}_{sm} \tag{24}$$

In the noisy quantum case, for trivial POVMs $[m_i = 0|M_i] \equiv \frac{I}{2} \forall i \in [5]$, we have $\mathrm{Corr}_{sm} = \frac{1}{2}$ and $R[s_*|S_*] = \frac{5}{2}$, and the noncontextuality inequality $R[s_*|S_*] + \mathrm{Corr}_{sm} \leq 3$ is trivially satisfied. $R[s_*|S_*] = \frac{5}{2}$ warrants surprise, i.e., a proof of contextuality, if and only if $\mathrm{Corr}_{sm} > \frac{1}{2}$, i.e., the POVMs are not trivial.

What happens to the classical/quantum/GPT hierarchy?

- ▶ Note that in the standard framework (CSW or AFLS), the hierarchy already collapses if POVMs are allowed, since trivial POVMs can realize any probabilistic model.

- ▶ In our approach, however, instead of this collapse of the hierarchy – which renders these approaches unsuitable for POVMs – we have a quantification of the tradeoff between the mmt-mmt correlations and source-mmt correlations and, therefore, the hierarchy still obtains as long as the POVMs are not trivial. The bounds on mmt-mmt correlations shift by an amount depending on the source-mmt correlations.

Takeaway

	Traditional Bell-KS approaches	Spekkens' approach
Type of context	1) ONB contexts 2) Compatibility contexts	Includes more types of contexts, for both preps and mmts.
Assumptions	MNC and OD (or at least Factorizability)	MNC and PNC (and resp. convex mixtures etc.)
Quantity of interest	Mmt-mmt correlations for a fixed input state	Also includes source-mmt correlations
Type of inequalities	Constraints on mmt-mmt corr from the classical marginal problem	More refined approach: tradeoff b/w mmt-mmt corr and source-mmt corr
KS-uncolourability proofs	Logical contradiction, no ineqs on mmt-mmt corr needed.	Robust inequality bounding source-mmt corr. No mmt-mmt corr needed.

Essential reading

Contextuality beyond the Kochen-Specker theorem,
arXiv:1612.07250 [quant-ph]

In conclusion
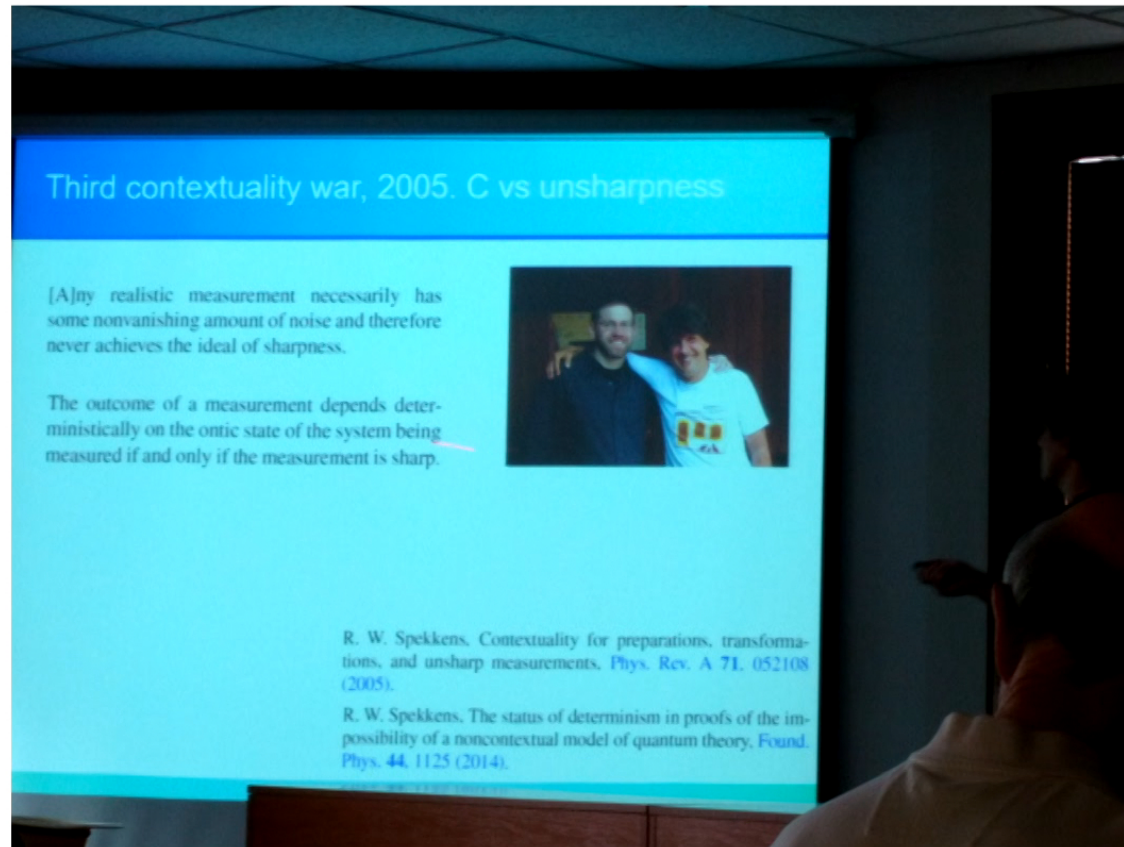
War is not the answer. Universal noncontextuality is.

But if you still insist

I've got just the guy