Title: Experimental state and measurement tomography for generalised probabilistic theories: bounding deviations from quantum theory via noncontextuality inequality violations

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Abstract: In order to perform foundational experiments testing the correctness of quantum mechanics, one requires data analysis tools that do not assume quantum theory. We introduce a quantum-free tomography technique that fits experimental data to a set of states and measurement effects in a generalised probabilistic theory (GPT). (This is in contrast to quantum tomography, which fits data to sets of density operators and POVM elements.) We perform an experiment on the polarization degree of freedom of single photons, and find GPT descriptions of the states and measurements in our experiment. We gather data for a large number of preparation and measurement procedures in order to map out the spaces of allowed GPT states and measurement effects, and we bound their possible deviation from quantum theory. Our GPT tomography method allows us to bound the extent to which nature might be more or less contextual than quantum theory, as measured by the maximum achievable violation of a particular noncontextuality inequality. We find that the maximal violation is confined to lie between 1.2Â±0.1% less than and 1.3Â±0.1% greater than the quantum prediction.
Coauthors: Matthew Pusey, Robert Spekkens, Kevin Resch

Akaike information criterion

- Criterion for model selection

$$AIC = -2\log\mathcal{L} + 2n$$

- Lower AIC value implies higher relative model likelihood
  - Trade-off between not underfitting and overfitting
- For normally-distributed, independent errors

$$\mathcal{L} = e^{-\chi^2}$$

$$AIC(k) = 2\chi^2(k) + 2n(k)$$

Akaike information criterion

Model no.	AIC value
1	AIC_1
2	AIC_2
3	AIC_3
$\vdots$	$\vdots$

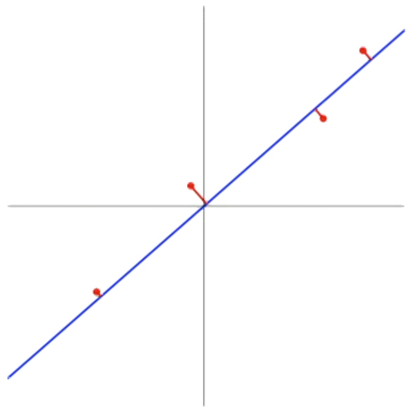
Good for comparing models to each other

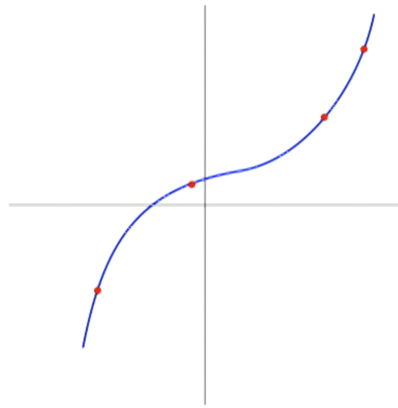
Akaike information criterion

Model no.	AIC value
1	AIC_1
2	AIC_2
3	AIC_3
⋮	⋮

Good for comparing models to each other

$$\text{likelihood}(k) = \frac{\exp\{-\frac{1}{2}(AIC(k) - AIC_{min})\}}{\sum_k \exp\{-\frac{1}{2}(AIC(k) - AIC_{min})\}}$$





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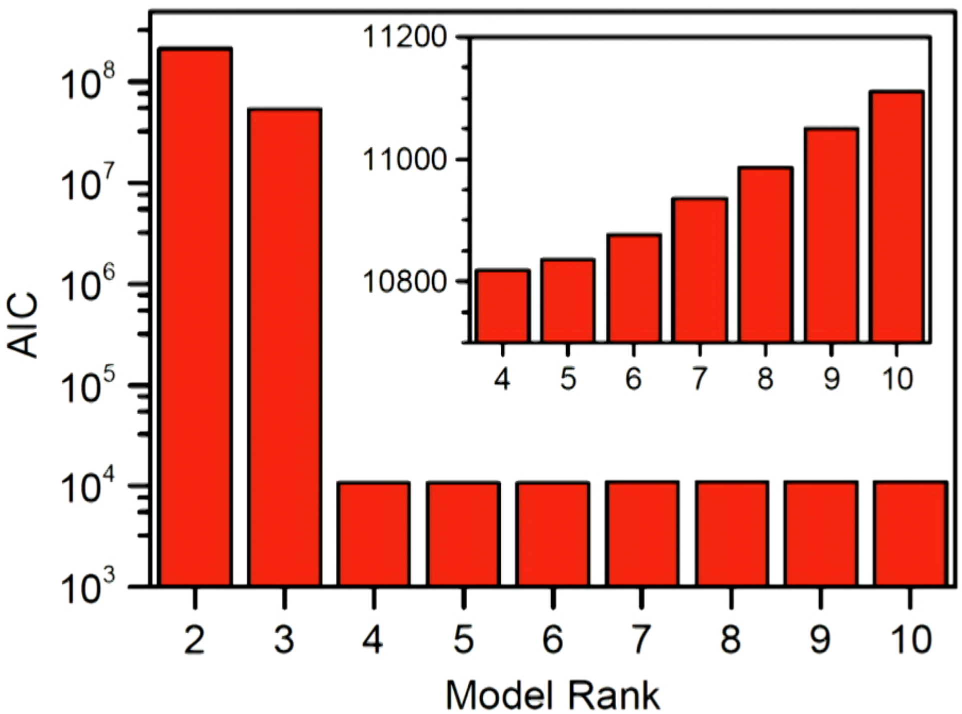
Akaike information criterion



$$AIC(k) = 2\chi^2(k) + 2n(k)$$

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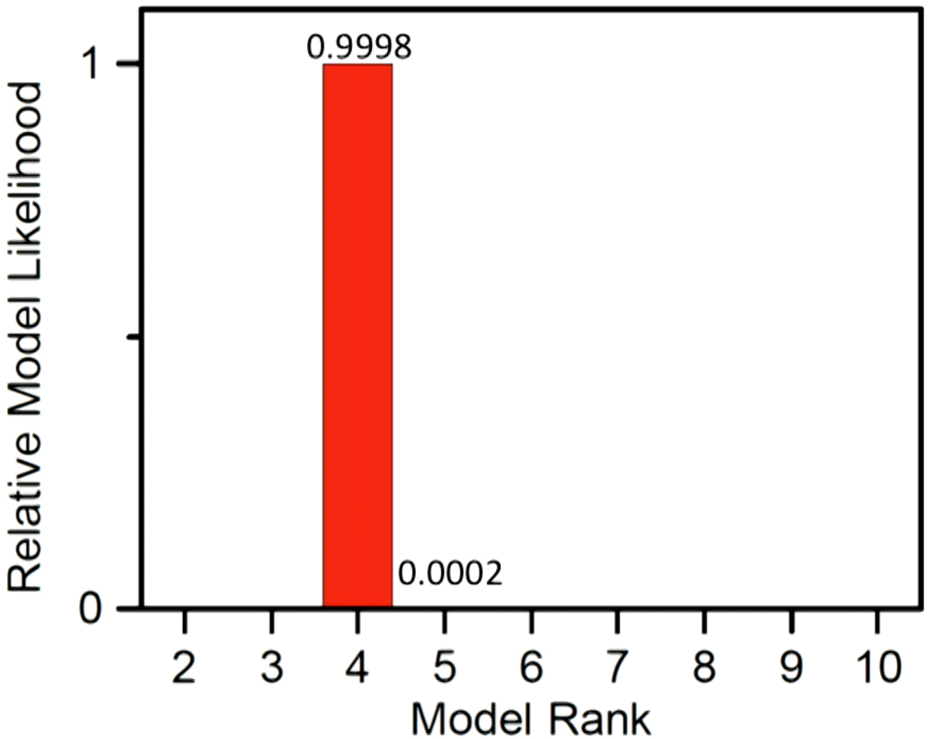
Akaike information criterion



$$AIC(k) = 2\chi^2(k) + 2n(k)$$

$$\text{likelihood}(k) = \frac{\exp\{-\frac{1}{2}(AIC(k) - AIC_{min})\}}{\sum_k \exp\{-\frac{1}{2}(AIC(k) - AIC_{min})\}}$$

Use k = 4

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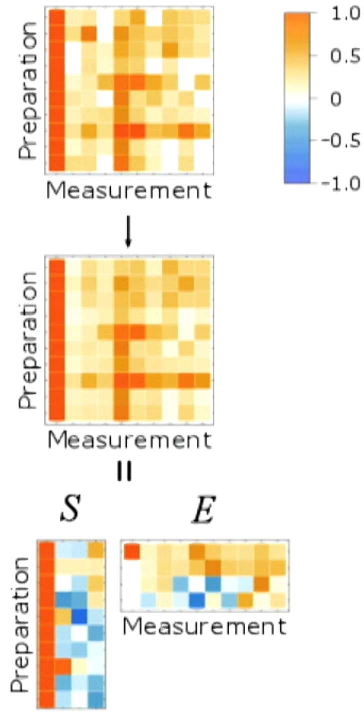
Experiment

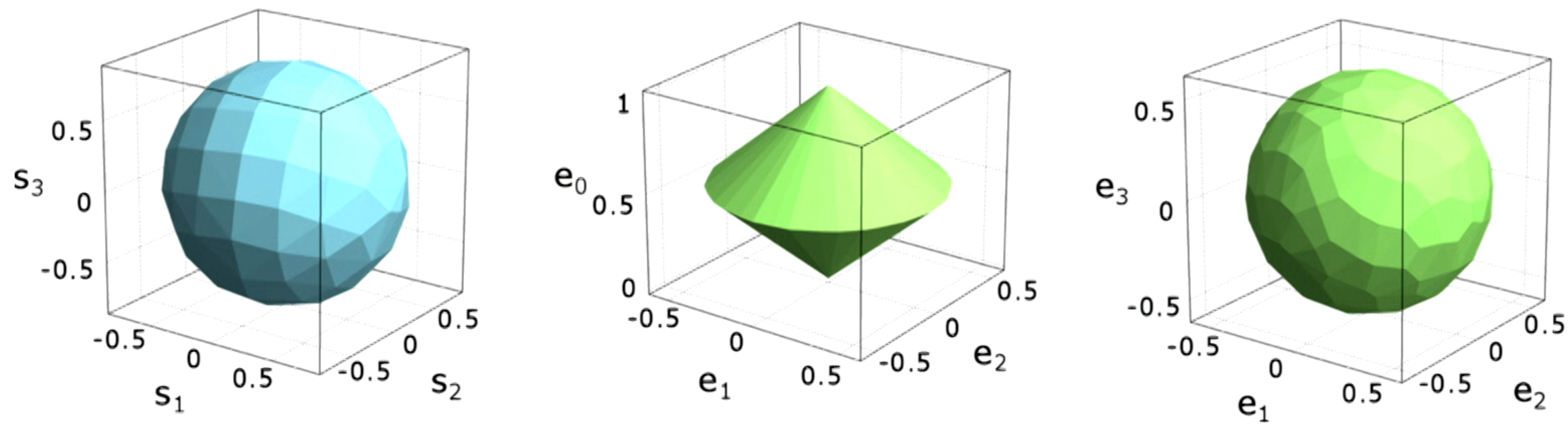
Raw data

Fit data to
rank-k model

Decompose



Measured state and effect spaces



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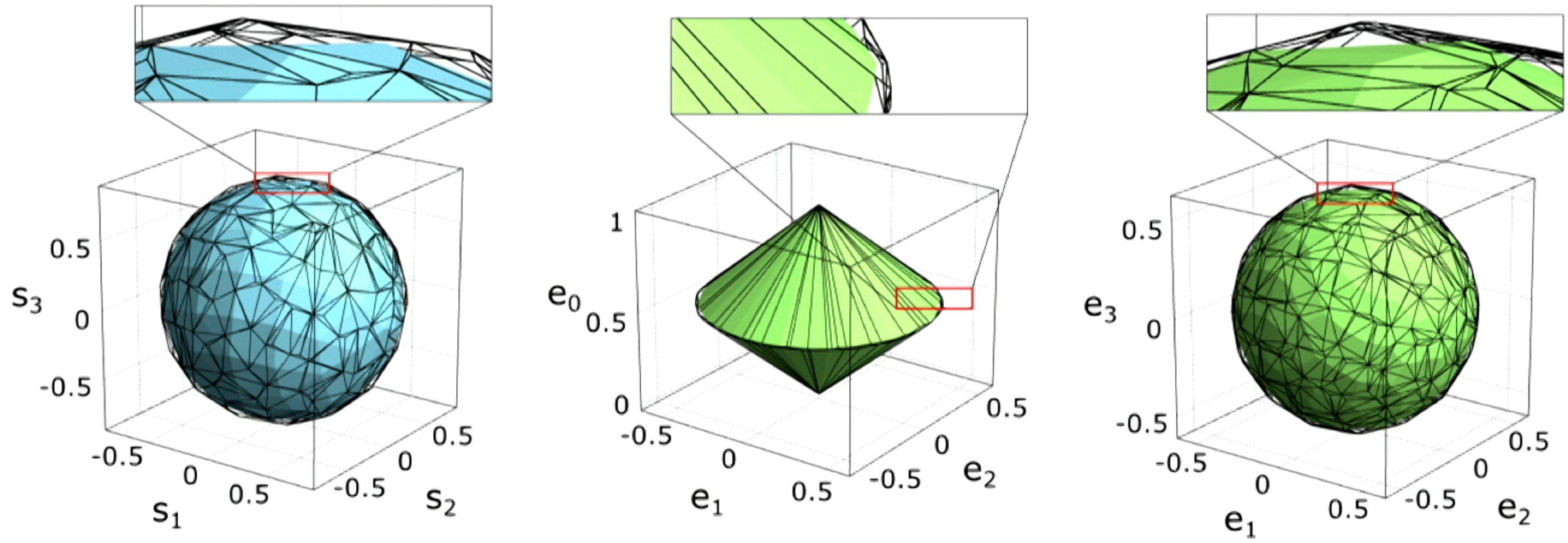
Measured state and effect spaces



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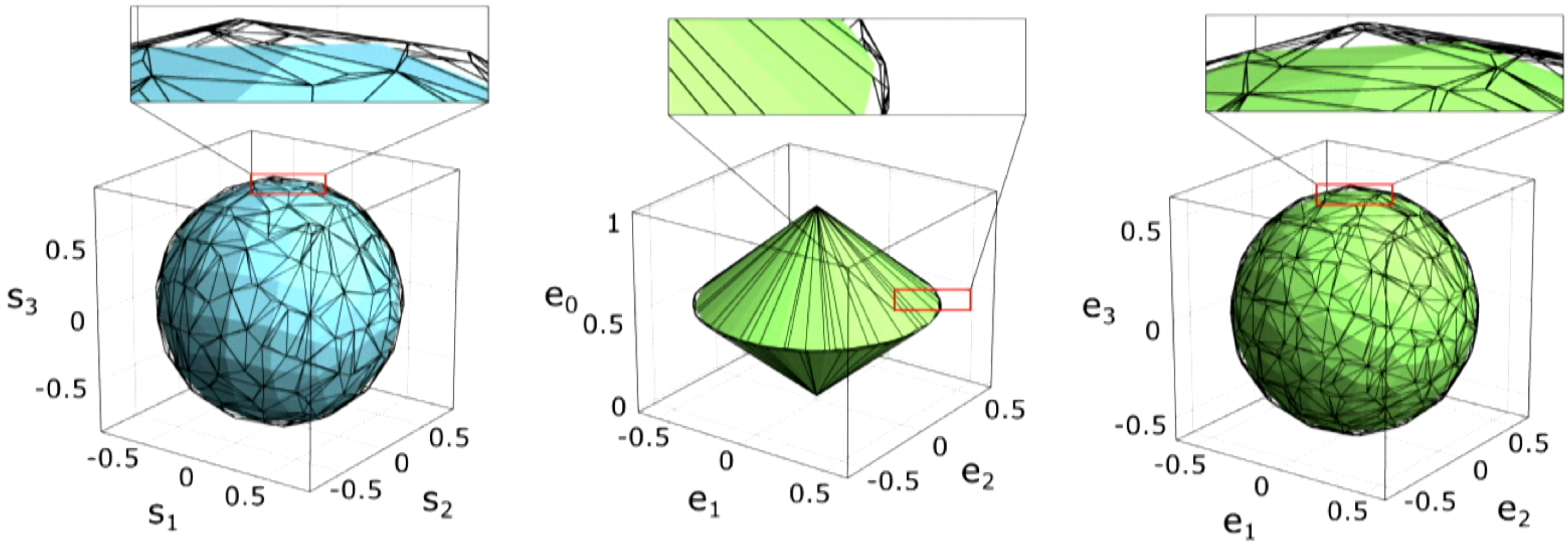
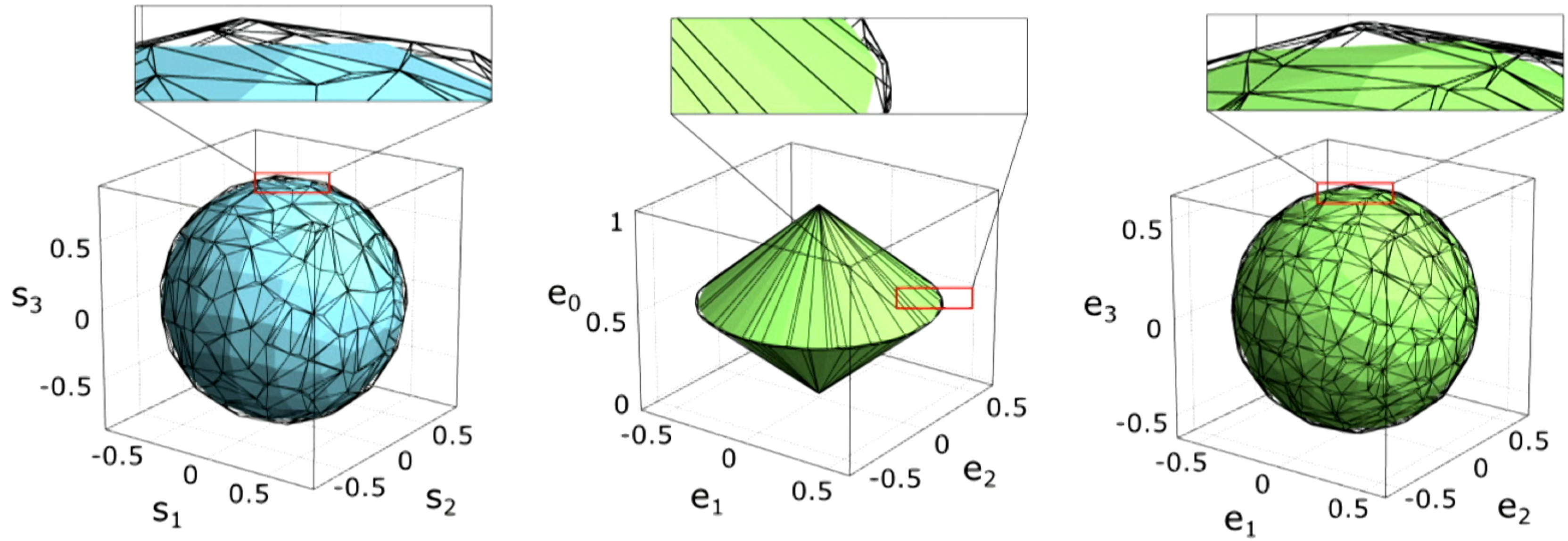
Measured state and effect spaces



True state space S_{true} lies between S and E_{dual}

True effect space lies between E and dual of S_{true}

Measured state and effect spaces



True state space S_{true} lies between S and E_{dual}

True effect space lies between E and dual of S_{true}

More preparations and measurements could increase volume ratio

→ Noise prevents reconstructing QT exactly

$$V_S/V_{E_{dual}} = 0.91267 \pm 0.00001$$

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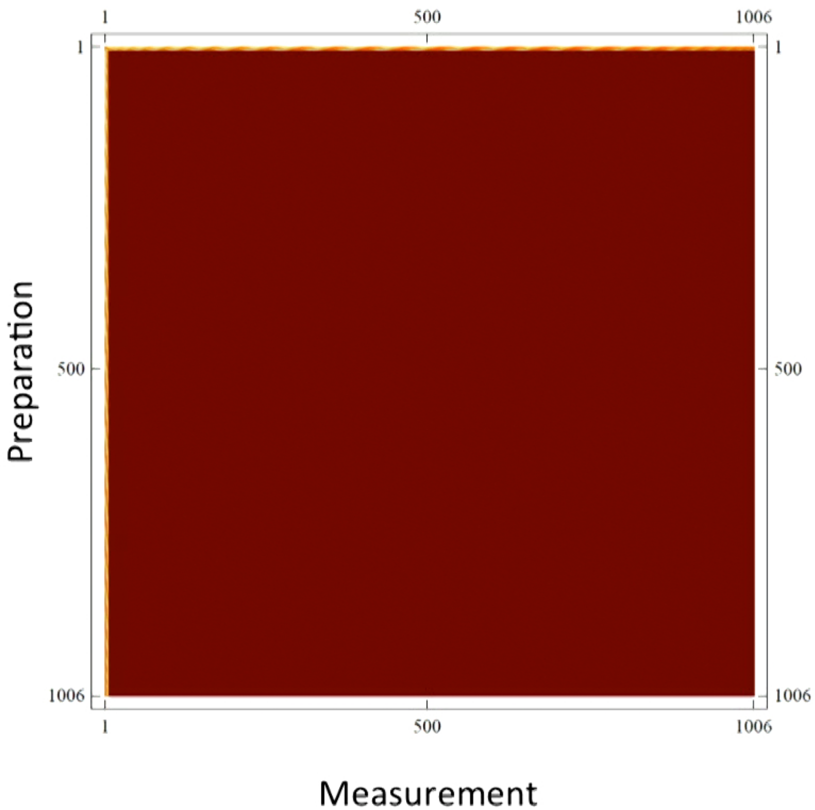
1000 preparations and measurements

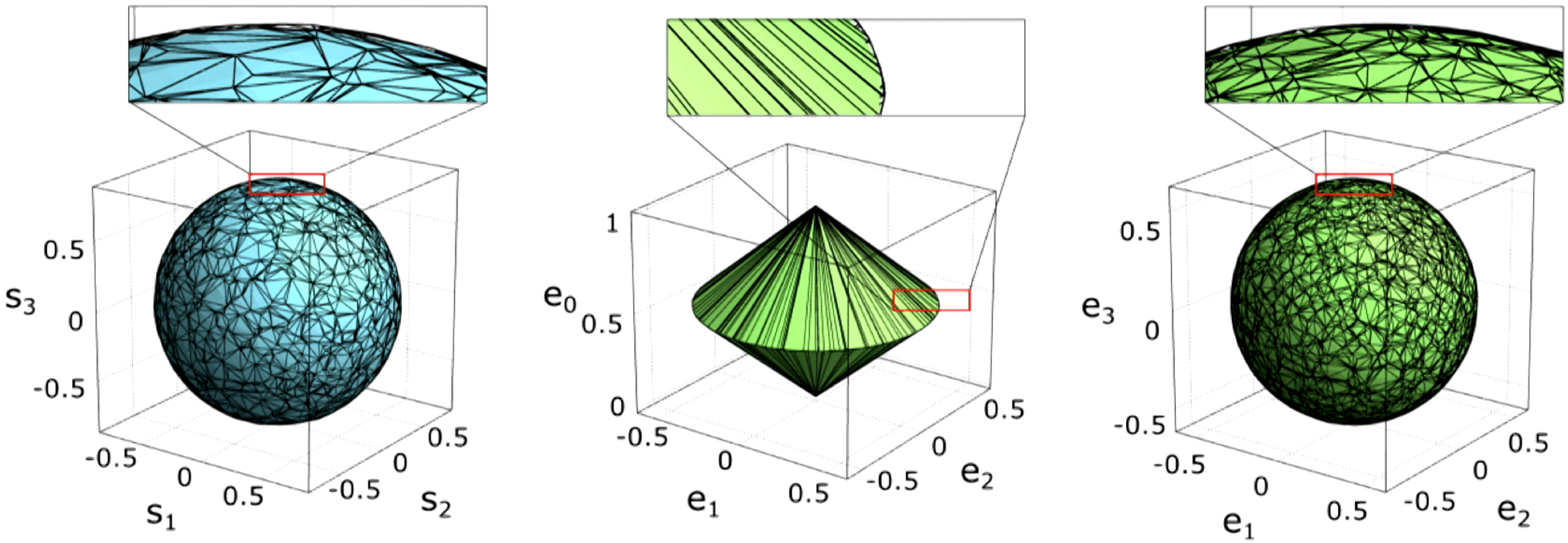


- 1006 measurements on 6 states
- 6 measurements on 1006 states
- If data is rank 4 → can predict empty 1000x1000 entries

- Chose 6 measurements (vs 4) to allow possibility of revealing additional dimensionality

More preparations and measurements

$$V_S/V_{E_{dual}} = 0.968 \pm 0.001$$



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GPT tomography

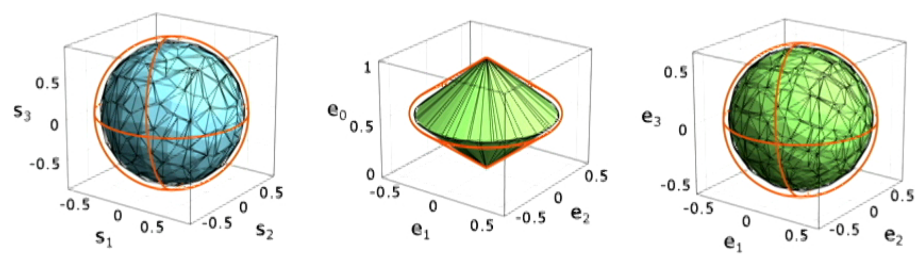
- Put states and measurements on equal footing (self-consistent)

GPT tomography

- Put states and measurements on equal footing (self-consistent)
- Dimension inferred from data
- Directly extracted qubit-like state and measurement spaces without invoking QM
- Placed a small upper bound on possible deviation from QM

Outline

- GPT framework
- GPT tomography method and application to an experiment
- Noncontextuality inequality



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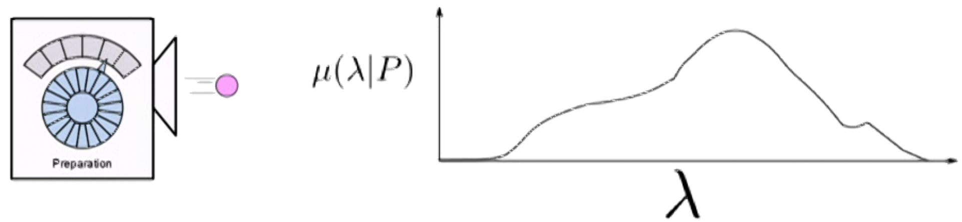
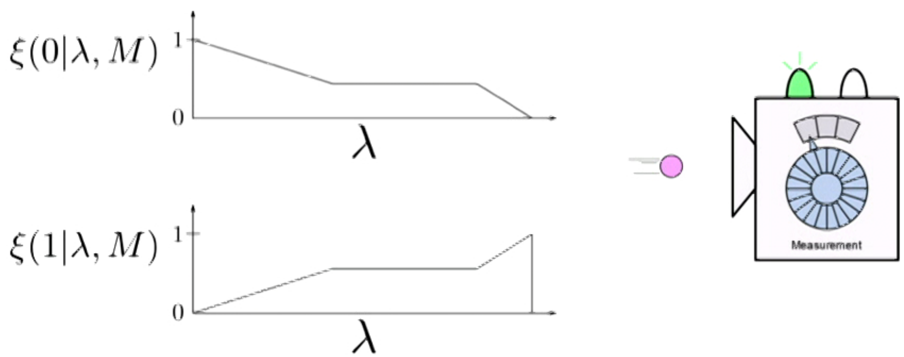
Ontological models framework

- Preparations



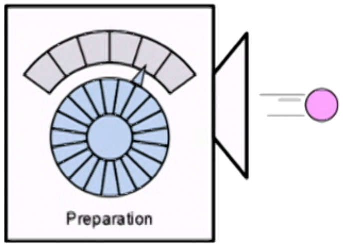
- Measurements



- Probabilities $p(X|M,P) = \int d\lambda\, \mu(\lambda|P)\, \xi(X|\lambda, M)$

Operational equivalence

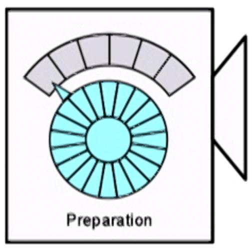
P



If

$$p(X|P,M) = p(X|P',M) \quad \forall\ X,\ M$$

P'



then P and P' are operationally equivalent

[Spekkens, PRA **71**, 052108 (2005)]

Noncontextuality

Operational equivalence implies equivalence in the ontological model

$$p(X|P,M) = p(X|P',M) \qquad \forall\ X,\ M$$
$$\implies \mu(\lambda|P) = \mu(\lambda|P')$$

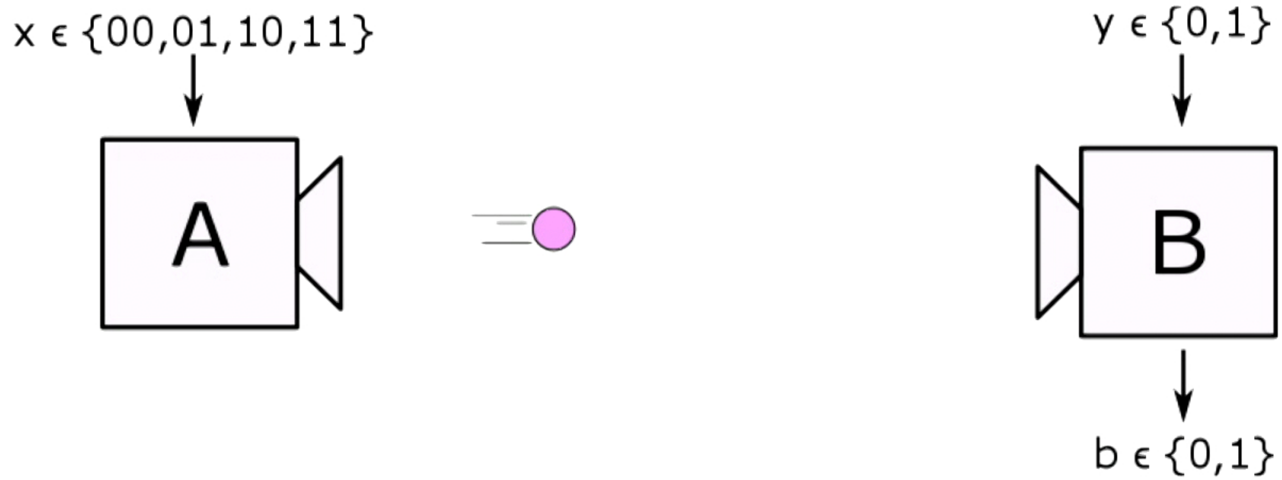
Preparation noncontextuality

$$p(X|P,M) = p(X|P,M') \qquad \forall\ X,\ P$$
$$\implies \xi(X|\lambda,M) = \xi(X|\lambda,M')$$

Measurement noncontextuality

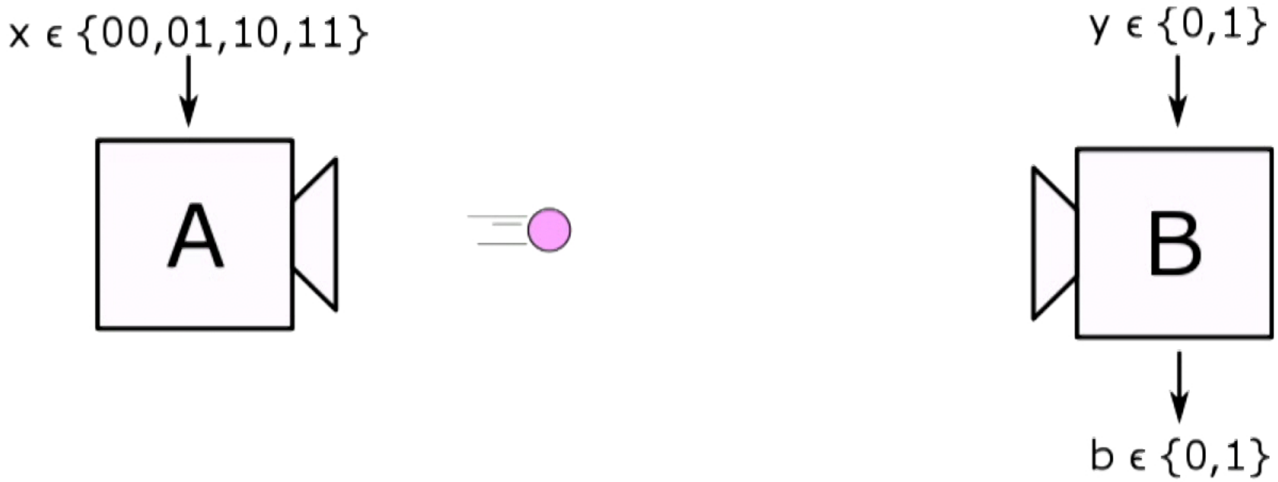
[Spekkens, PRA **71**, 052108 (2005)]

Parity-oblivious multiplexing “game”



[SBKTP, PRL **102**, 010401 (2009)] 89

Parity-oblivious multiplexing "game"

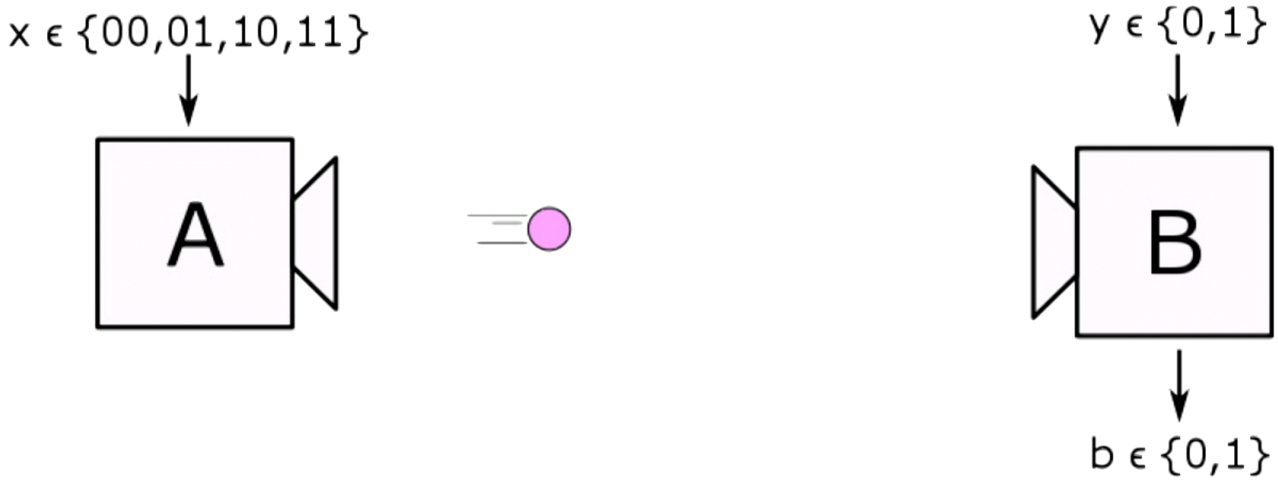


- Bob guesses *y*-th bit of Alice's input
  - Bob is not allowed to learn anything about the parity of Alice's input

$$p(\text{success}) = \frac{1}{8} \sum_{x,y} p(b = x_y | x, y)$$

[SBKTP, PRL **102**, 010401 (2009)] 91

Classical strategy



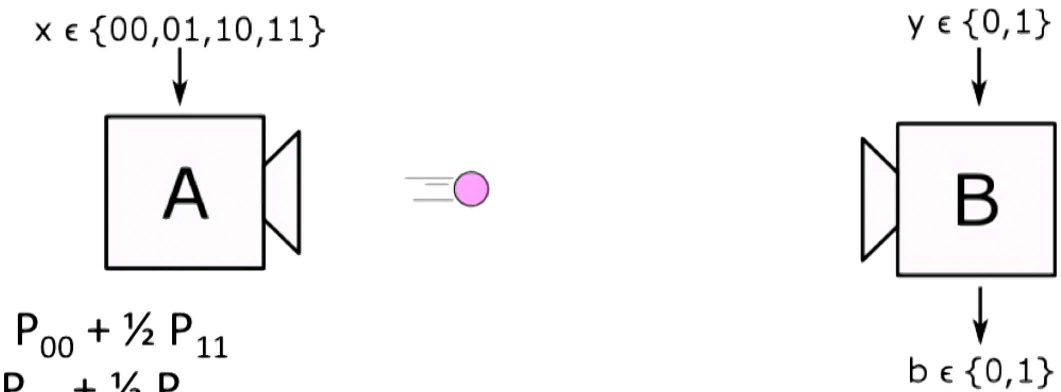
- Alice encodes x_0 and sends it to Bob

$$p(\text{success}) \leq \frac{3}{4}$$

[SBKTP, PRL **102**, 010401 (2009)] 92

A noncontextuality inequality

Implement one of 4 preparations:
P_{00}, P_{01}, P_{10}, P_{11}



Implement one of 2 measurements:
M_0, M_1

$P_{even} = ½\ P_{00} + ½\ P_{11}$
$P_{odd} = ½\ P_{01} + ½\ P_{10}$

Parity obliviousness $\Longrightarrow p(b|P_{\text{even}}, M) = p(b|P_{\text{odd}}, M), \quad \forall b,\ M$

Prep. noncontextuality $\Longrightarrow p(\lambda|P_{\text{even}}) = p(\lambda|P_{\text{odd}})$

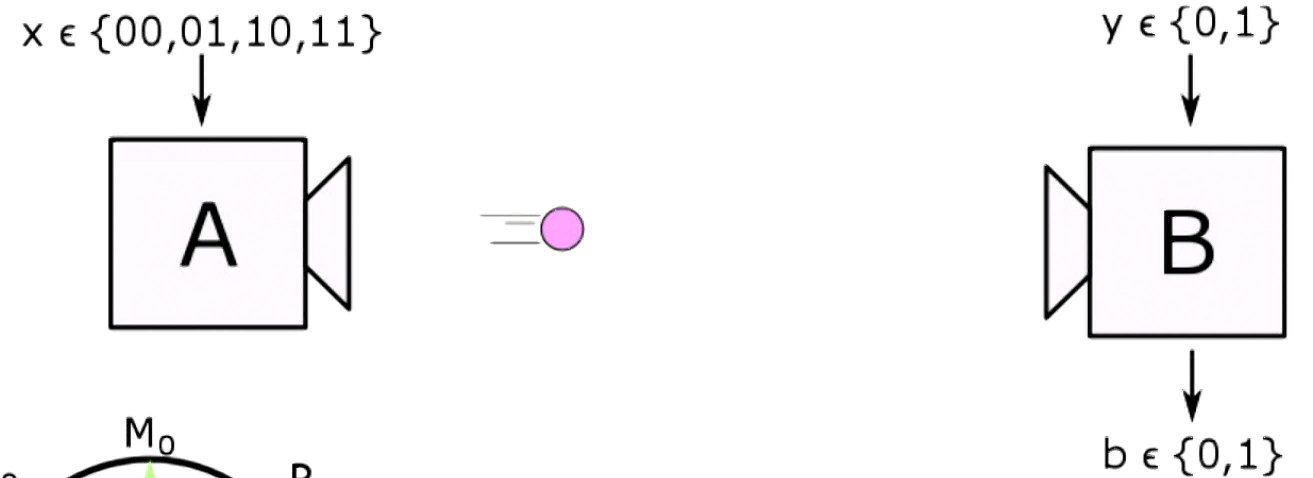
$$\frac{p(P_{\text{even}}|\lambda)\cancel{p(P_{\text{even}})}}{\cancel{p(\lambda)}} = \frac{p(P_{\text{odd}}|\lambda)\cancel{p(P_{\text{odd}})}}{\cancel{p(\lambda)}}$$

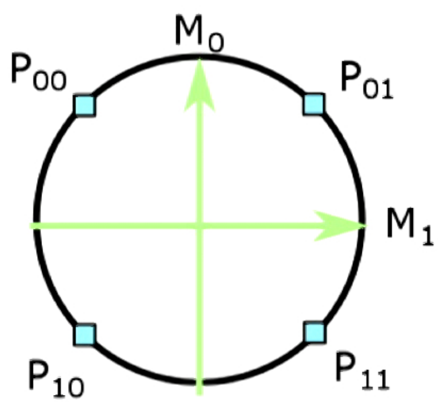
$$p(P_{\text{even}}|\lambda) = p(P_{\text{odd}}|\lambda)$$

$$p(\text{success}) \underset{NCOM}{\leq} \frac{3}{4}$$

[SBKTP, PRL **102**, 010401 (2009)] 99

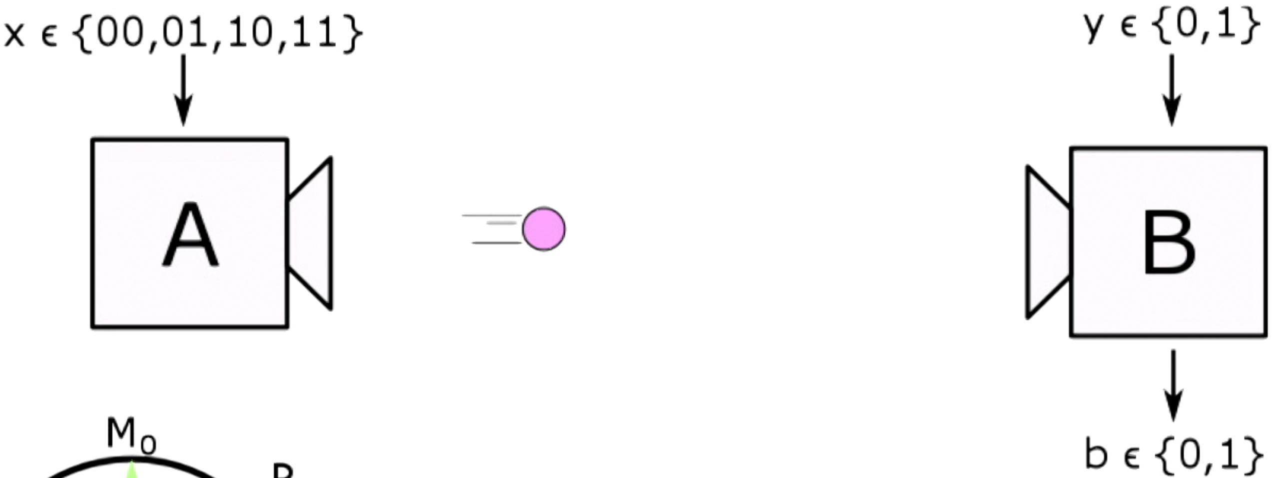
Quantum strategy

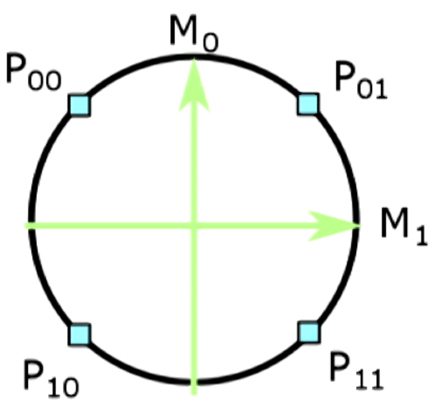




[SBKTP, PRL **102**, 010401 (2009)] 100

Quantum strategy

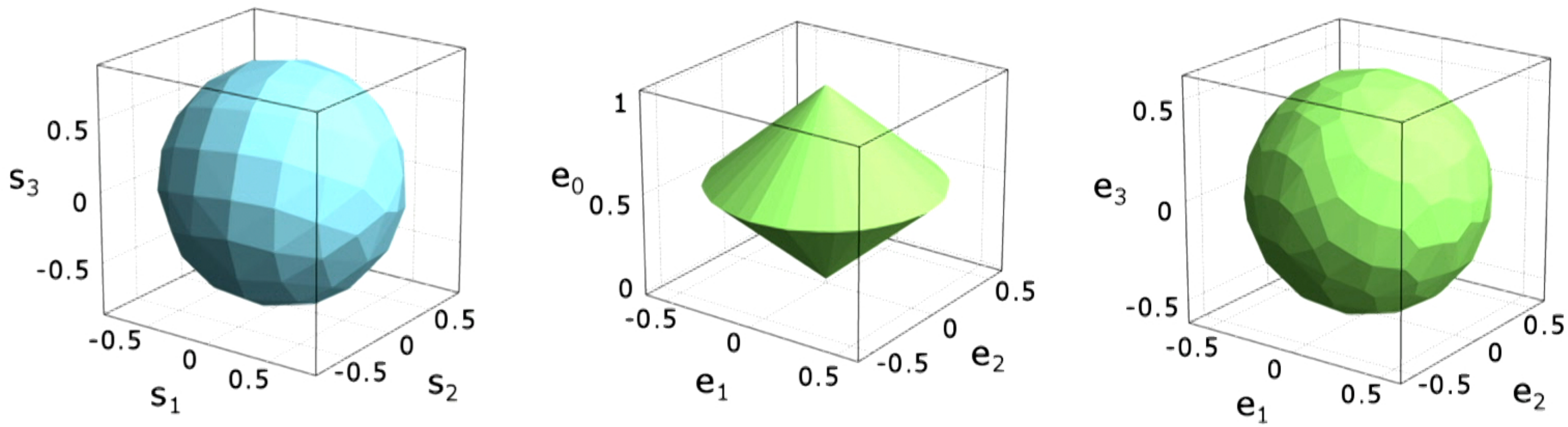




$$p(\text{success}) \underset{QM}{\leq} \cos^2(\pi/8) \approx 0.85355$$

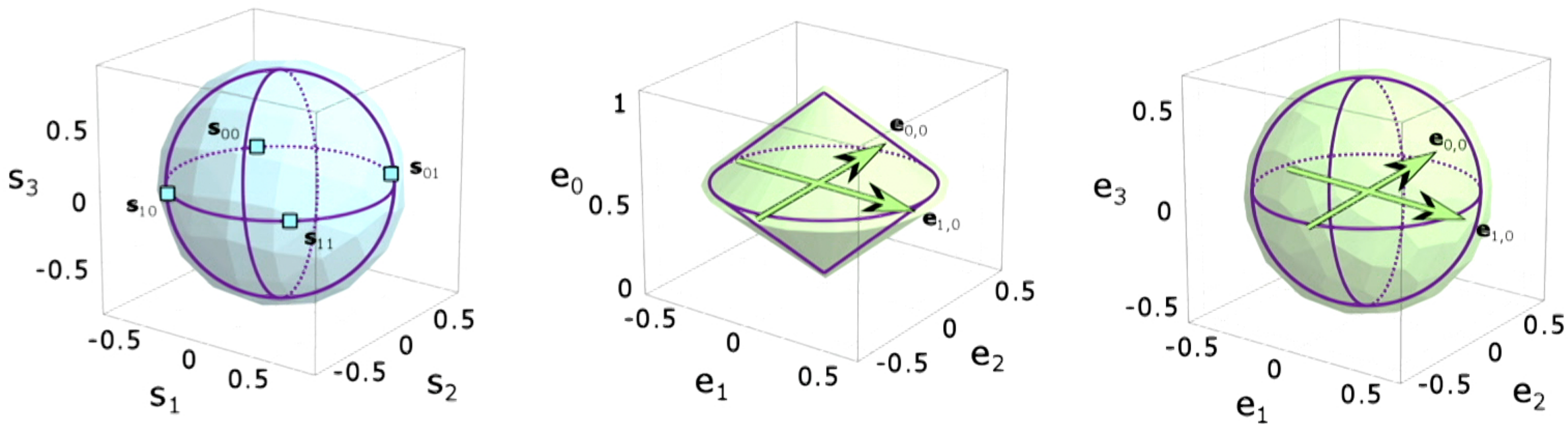
[SBKTP, PRL **102**, 010401 (2009)] 101

Experimental violation of inequality



Search over all GPT states and effects to find maximum inequality violation

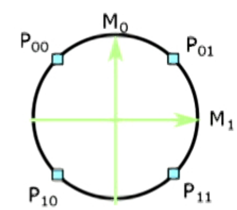
Experimental violation of inequality



Search over all GPT states and effects to find maximum inequality violation

Parity obliviousness $\Longrightarrow \mathbf{s}_{\text{even}} = \mathbf{s}_{\text{odd}}$

Approximate state and effect spaces with spheres



$$p(\text{success})_{\text{qubit}} \approx 0.8536$$

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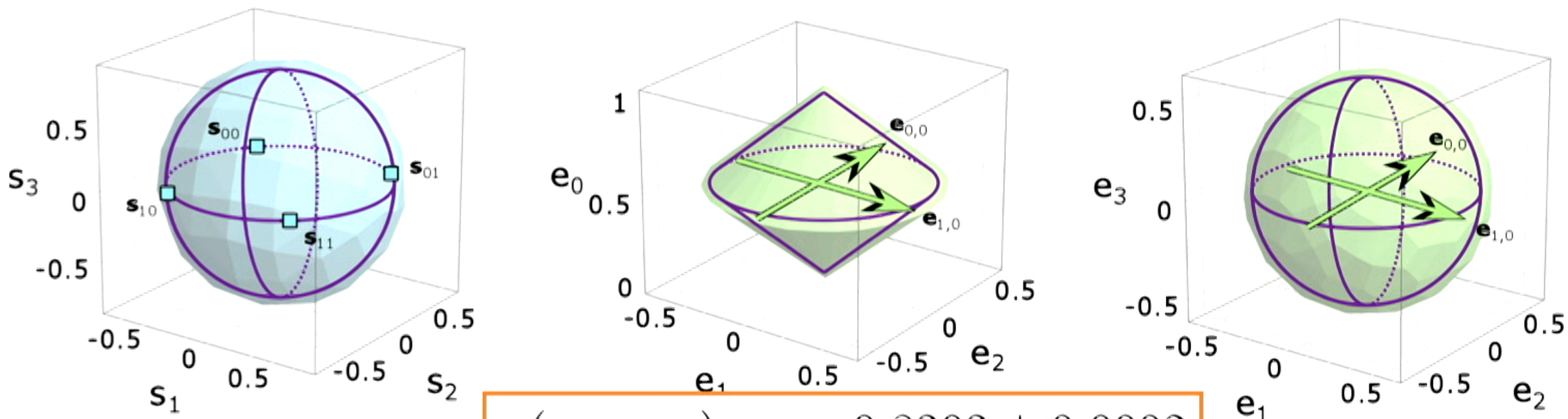
Experimental violation of inequality



$$p(\text{success})_{\text{min}} = 0.8303 \pm 0.0002$$

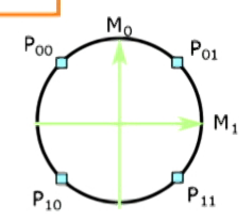
Search over all GPT states and effects to find maximum inequality violation

Parity obliviousness $\Longrightarrow \mathbf{s}_{\text{even}} = \mathbf{s}_{\text{odd}}$

Approximate state and effect spaces with spheres

Get operational equivalence "for free"

Gives an experimental lower bound on "how contextual" our system is



$$p(\text{success})_{\text{qubit}} \approx 0.8536$$

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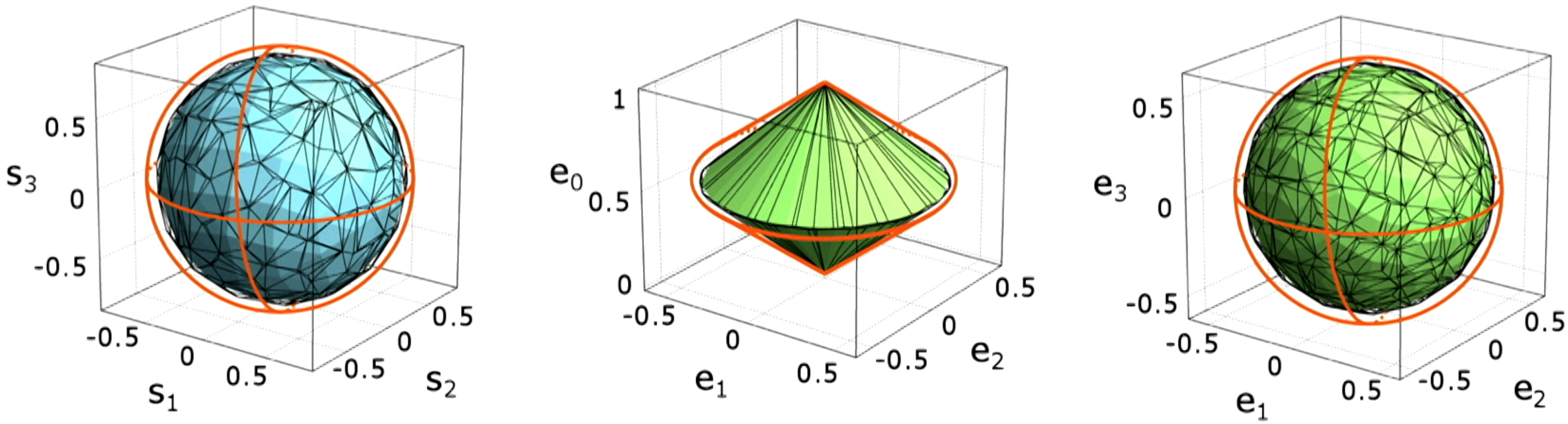
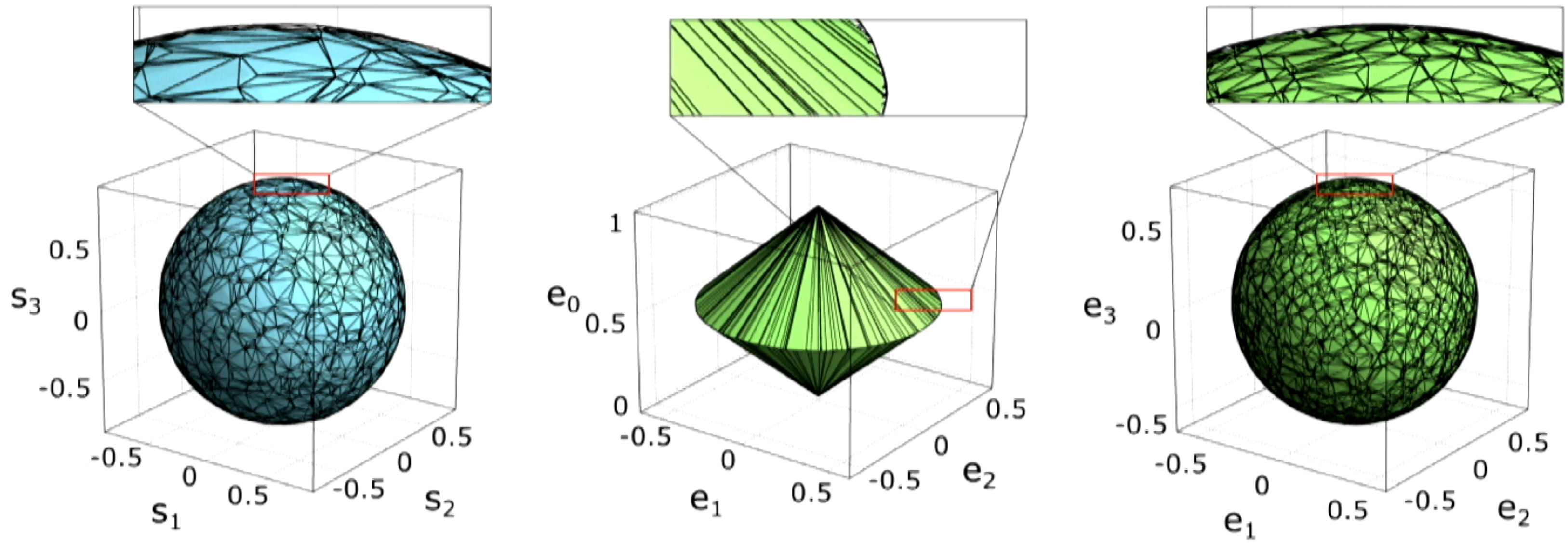
Upper bound



Search over all dual GPT states and effects to find maximum inequality violation

Approximate state and effect spaces with spheres

1000x1000 data



$$p(\text{success})_{\text{min}} = 0.8427 \pm 0.0005$$

$$p(\text{success})_{\text{max}} = 0.8647 \pm 0.0005$$

$$p(\text{success})_{\text{qubit}} \approx 0.8536$$

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GPT tomography

- Reconstruct full state and effect spaces of a system
- Minimize assumptions about the dimensionality of the system
- GPT tomography allowed us to test a noncontextuality inequality
  - Overcame problem of inexact operation equivalence
  - Found both lower and upper bounds on "amount of contextuality" in our system

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Experimental state and measurement tomography for generalised probabilistic theories:

Bounding deviations from quantum theory via noncontextuality inequality violations

Mike Mazurek

Matt Pusey, Rob Spekkens, Kevin Resch

Contextuality: Conceptual Issues, Operational Signatures, and Applications

July 24-28, 2017

UNIVERSITY OF WATERLOO | IQC Institute for Quantum Computing

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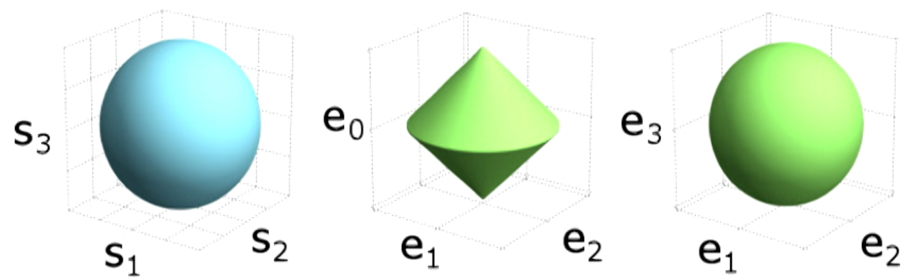
Introduction

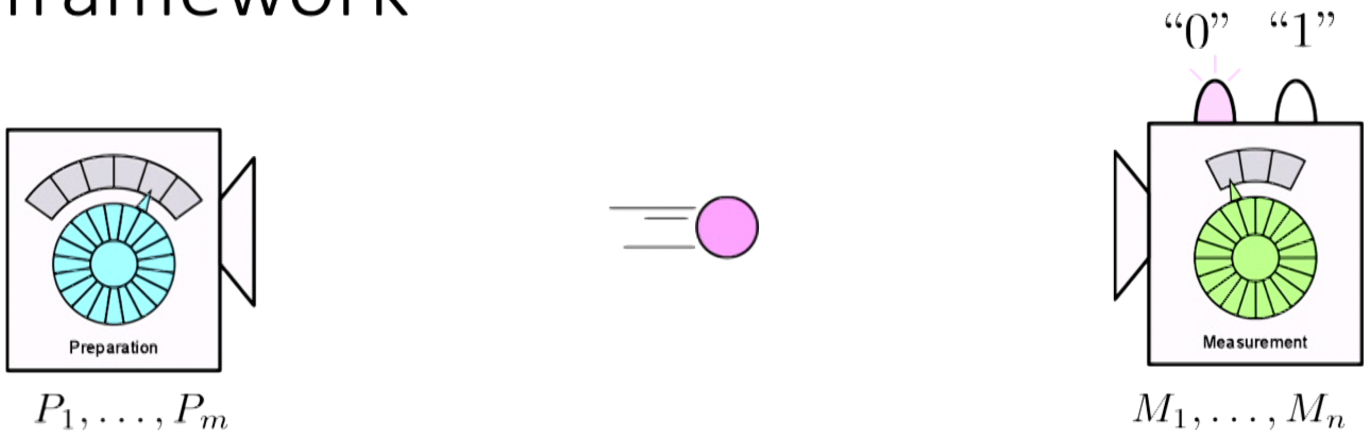
- Contextuality: Conceptual Issues, Operational Signatures, and Applications
- Data analysis tools should not rely on quantum theory for foundational experiments
- Tomography method within generalised probabilistic theory framework
  - Self-consistent
- Application to experimental test of noncontextuality

Outline

- GPT framework
- GPT tomography method and application to an experiment
- Noncontextuality inequality



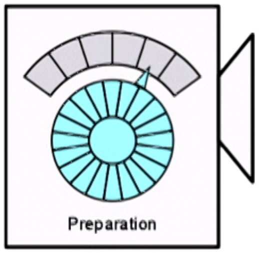
GPT framework



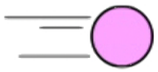
$$p(0|P_i, M_j)$$

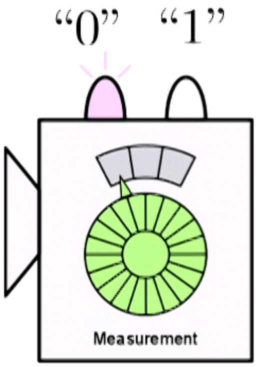
[Hardy, quant-ph/0101012 (2001)]
[Barrett, PRA **75**, 032304 (2007)]

GPT framework



$P_1, \ldots, P_m$





$M_1, \ldots, M_n$

Operational state: $\mathbf{s_i} = \begin{bmatrix} p(0|P_i, M_1) \\ p(0|P_i, M_2) \\ p(0|P_i, M_3) \\ \vdots \end{bmatrix}$

Operational effect: $\mathbf{e_{j,0}} = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$

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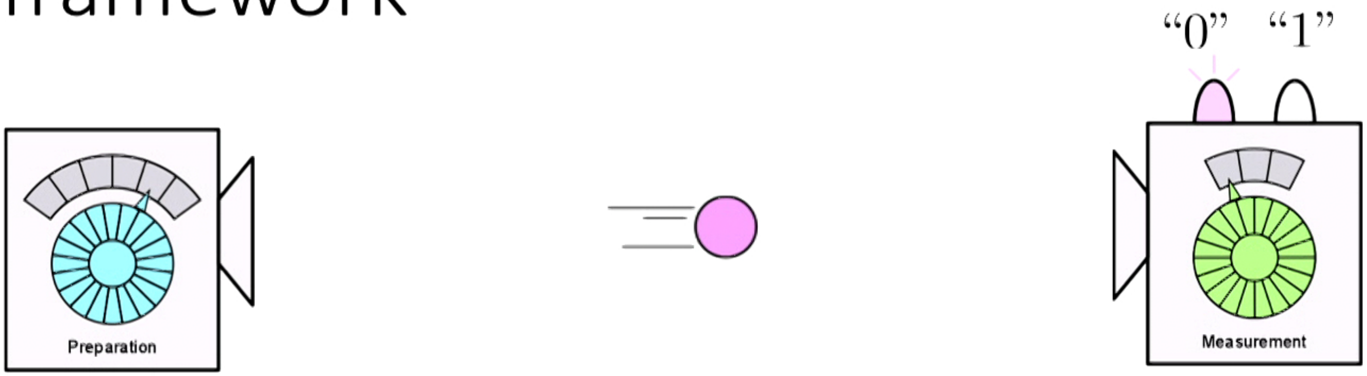
GPT framework



$P_1, \ldots, P_m$ $\qquad$ $M_1, \ldots, M_n$

Operational state: $\mathbf{s_i} = \begin{bmatrix} p(0|P_i, M_1) \\ p(0|P_i, M_2) \\ p(0|P_i, M_3) \\ \vdots \end{bmatrix}$

Operational effect: $\mathbf{e_{j,0}} = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$

$$p(0|P_i, M_j) = \mathbf{s_i} \cdot \mathbf{e_{j,0}}$$

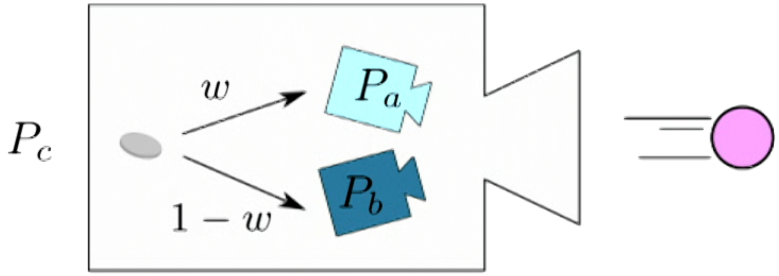
Tomographically complete measurement set

- Number of possible measurements might be very large or infinite
- Will exist some tomographically complete set of *K* measurements which fully determine each state

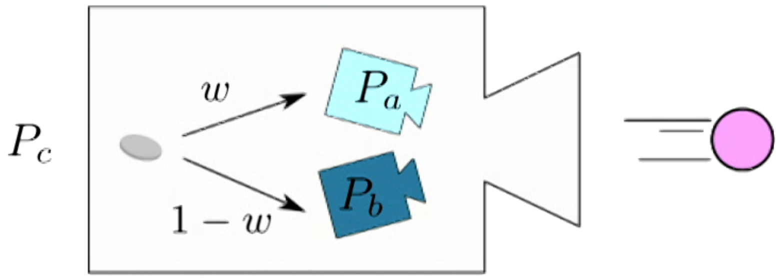
Operational state: $$\mathbf{s_i} = \begin{bmatrix} p(0|P_i, M_1) \\ p(0|P_i, M_2) \\ \vdots \\ p(0|P_i, M_K) \end{bmatrix}$$

$$p(0|P_i, M) = f_{M,0}(\mathbf{s_i})$$

Convex linearity of operational states



Convex linearity of operational states



$$p(0|P_c, M) = wp(0|P_a, M) + (1-w)p(0|P_b, M)$$

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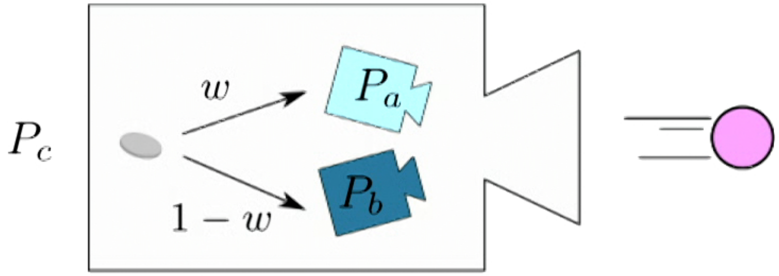
Convex linearity of operational states



$$\mathbf{s_i} = \begin{bmatrix} p(0|P_i, M_1) \\ p(0|P_i, M_2) \\ \vdots \\ p(0|P_i, M_K) \end{bmatrix}$$

$$p(0|P_i, M) = f_{M,0}(\mathbf{s_i})$$

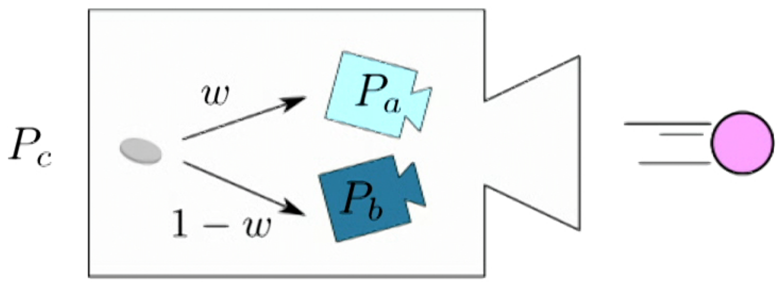
$$p(0|P_c, M) = wp(0|P_a, M) + (1-w)p(0|P_b, M)$$

$$f_{M,0}(\mathbf{s_c}) = wf_{M,0}(\mathbf{s_a}) + (1-w)f_{M,0}(\mathbf{s_b})$$

Also true for *M*s in tomographically complete set

$$\mathbf{s_c} = w\mathbf{s_a} + (1-w)\mathbf{s_b}$$

Convex linearity of operational states



$$\mathbf{s_i} = \begin{bmatrix} p(0|P_i, M_1) \\ p(0|P_i, M_2) \\ \vdots \\ p(0|P_i, M_K) \end{bmatrix}$$

$$p(0|P_i, M) = f_{M,0}(\mathbf{s_i})$$

$$p(0|P_c, M) = wp(0|P_a, M) + (1-w)p(0|P_b, M)$$

$$f_{M,0}(\mathbf{s_c}) = wf_{M,0}(\mathbf{s_a}) + (1-w)f_{M,0}(\mathbf{s_b})$$

Also true for *M*s in tomographically complete set

$$\mathbf{s_c} = w\mathbf{s_a} + (1-w)\mathbf{s_b}$$

$$f_{M,0}(w\mathbf{s_a} + (1-w)\mathbf{s_b}) = wf_{M,0}(\mathbf{s_a}) + (1-w)f_{M,0}(\mathbf{s_b})$$

f(**s**) is convex linear

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Linearity

Convex linearity

$$\mathbf{s} = \sum_i w_i \mathbf{s_i} \Rightarrow f_{M,0}(\mathbf{s}) = \sum_i w_i f_{M,0}(\mathbf{s_i}) \qquad w_i \geq 0, \ \sum_i w_i = 1$$

implies linearity

$$\mathbf{s} = \sum_i \alpha_i \mathbf{s_i} \Rightarrow f_{M,0}(\mathbf{s}) = \sum_i \alpha_i f_{M,0}(\mathbf{s_i}) \qquad \forall \alpha_i \in \mathbb{R}$$

[Hardy, quant-ph/0101012 (2001)]

Linearity

Convex linearity

$$\mathbf{s} = \sum_i w_i \mathbf{s_i} \Rightarrow f_{M,0}(\mathbf{s}) = \sum_i w_i f_{M,0}(\mathbf{s_i}) \qquad w_i \geq 0, \;\; \sum_i w_i = 1$$

implies linearity

$$\mathbf{s} = \sum_i \alpha_i \mathbf{s_i} \Rightarrow f_{M,0}(\mathbf{s}) = \sum_i \alpha_i f_{M,0}(\mathbf{s_i}) \qquad \forall \alpha_i \in \mathbb{R}$$

$$\boxed{\forall M \;\; \exists \mathbf{e}_{M,0} : f_{M,0}(\mathbf{s}) = \mathbf{e}_{M,0} \cdot \mathbf{s}}$$

[Hardy, quant-ph/0101012 (2001)]

2-level classical system (bit)

$$K = 2$$

$$\mathbf{s} = (s_0, s_1) = (1, s_1) \qquad \mathbf{e} = (e_0, e_1)$$

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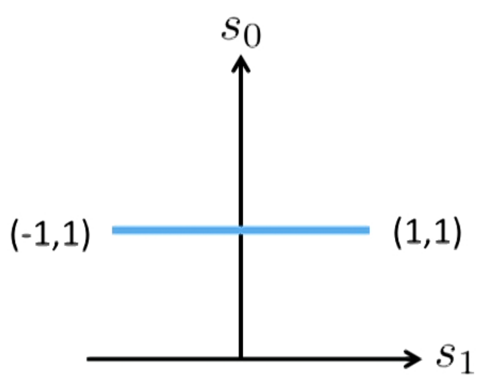

2-level classical system (bit)

$$K = 2 \qquad p(0|M,P) = \mathbf{e}_{\mathrm{M},0} \cdot \mathbf{s}_{\mathrm{P}}$$

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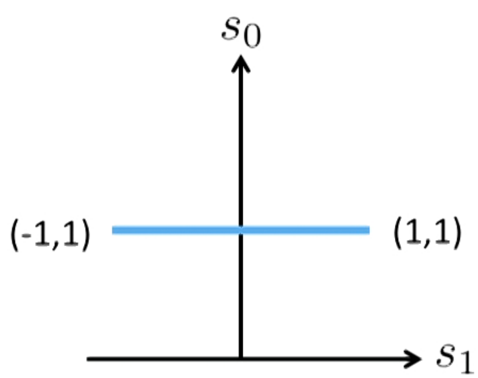

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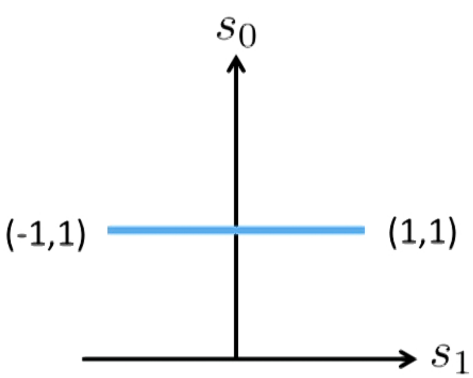

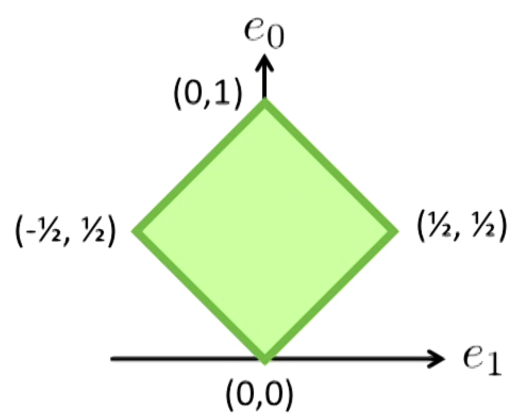

Qubit

$$\rho = \frac{1}{2}(\mathbb{I} + s_{\mathrm{X}}\sigma_{\mathrm{X}} + s_{\mathrm{Y}}\sigma_{\mathrm{Z}} + s_3\sigma_{\mathrm{Z}})$$

$$E = \frac{1}{2}(e_0\mathbb{I} + e_{\mathrm{X}}\sigma_{\mathrm{X}} + e_{\mathrm{Y}}\sigma_{\mathrm{Z}} + e_3\sigma_{\mathrm{Z}})$$

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$$\begin{aligned}\mathrm{Tr}(\rho E) &= \tfrac{1}{2}(e_o + s_{\mathrm{X}}e_{\mathrm{X}} + s_{\mathrm{Y}}e_{\mathrm{Y}} + s_{\mathrm{Z}}e_{\mathrm{Z}}) \\ &= \underbrace{(1, s_{\mathrm{X}}, s_{\mathrm{Y}}, s_{\mathrm{Z}})}_{\mathbf{s}} \cdot \underbrace{\frac{1}{2}(e_0, e_{\mathrm{X}}, e_{\mathrm{Y}}, e_{\mathrm{Z}})}_{\mathbf{e}}\end{aligned}$$

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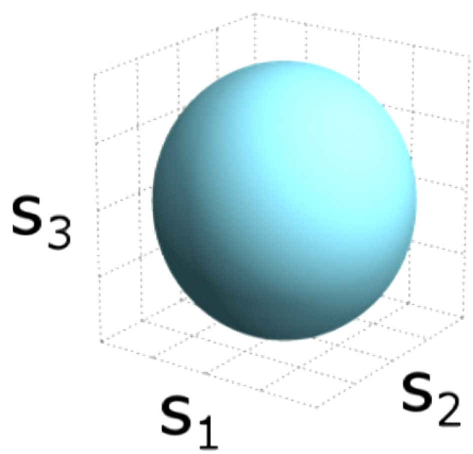

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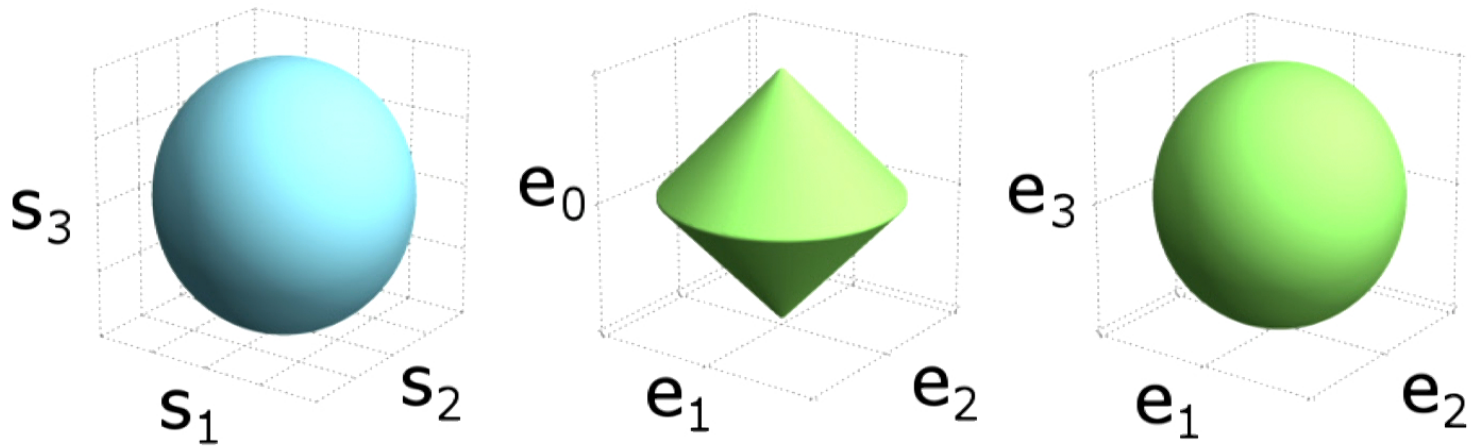

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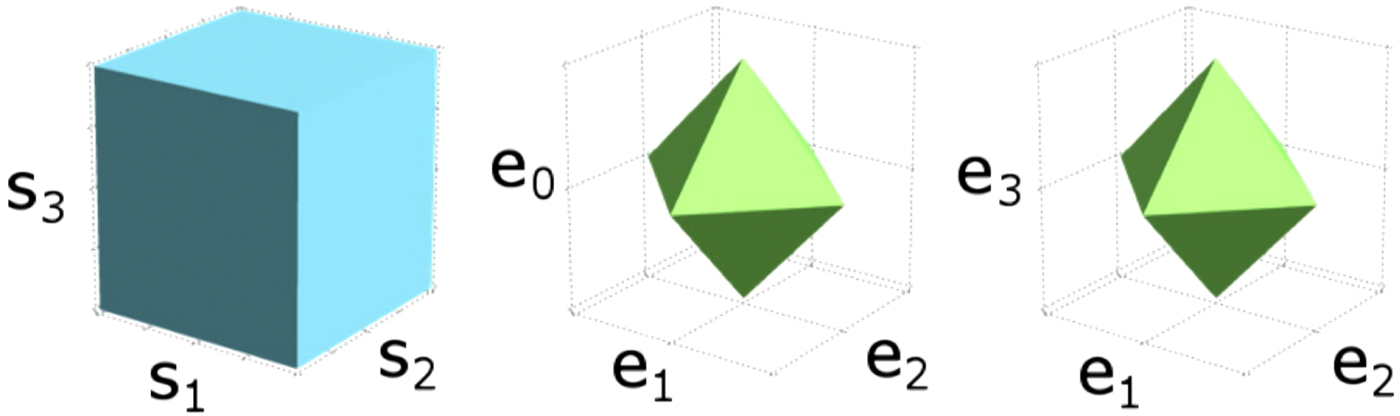
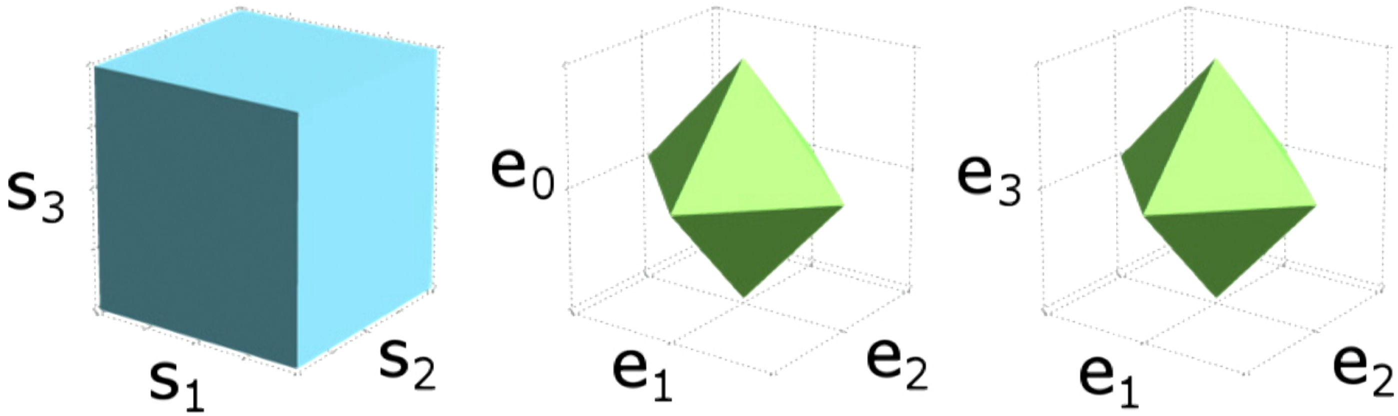
Generalised non-signalling theory

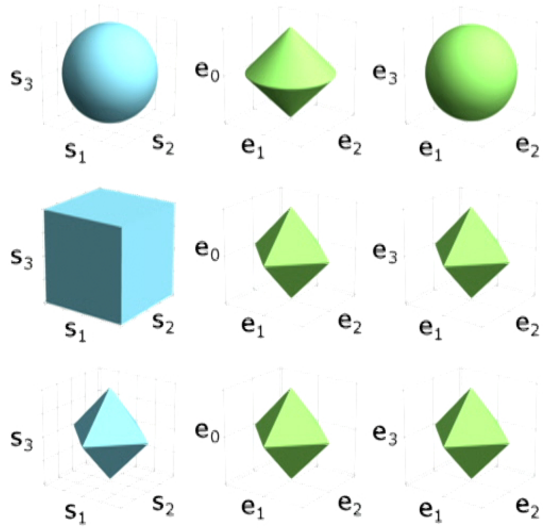


Generalised non-signalling theory



- Some states are deterministic for multiple effects
- Some effects respond deterministically to multiple states

Some example GPTs



Qubit

GNST

Spekkens' toy theory

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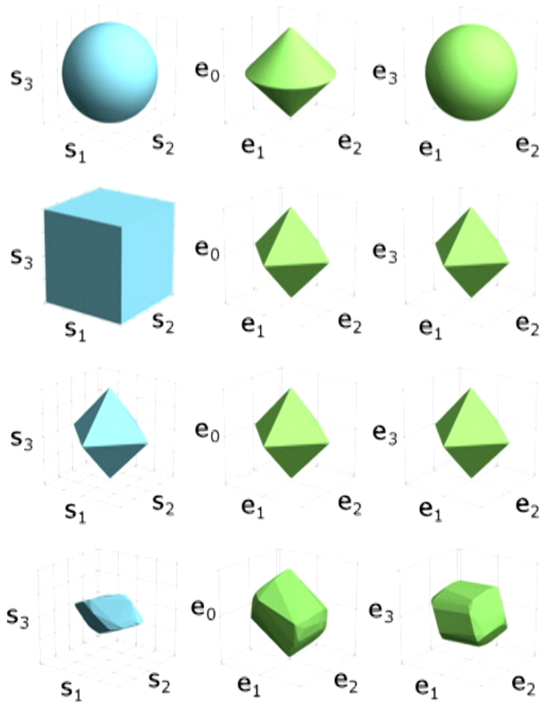
Some example GPTs



Qubit

GNST

Spekkens' toy theory

Random GPT

Dual state and effect spaces

- There might be logically possible states and effects that aren't included in the spaces specified by the GPT

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$$\mathbf{e}_{\text{possible}} : 0 \leq \mathbf{e}_{\text{possible}} \cdot \mathbf{s} \leq 1 \quad \forall\, \mathbf{s} \in S$$

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$$\mathbf{e}_{\text{possible}} \in S_{\text{dual}}$$

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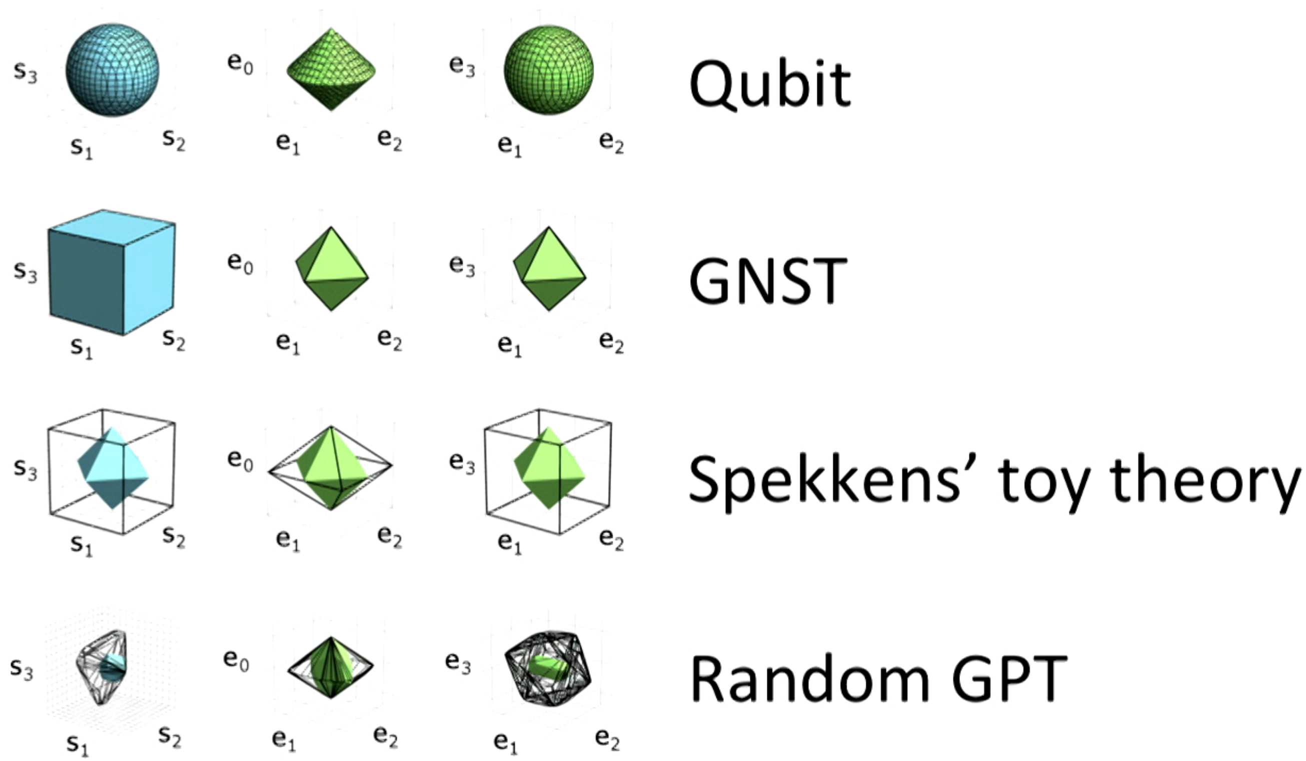
$$\mathbf{e}_{\text{possible}} \in S_{\text{dual}}$$

$$\mathbf{s}_{\text{possible}} \in E_{\text{dual}}$$

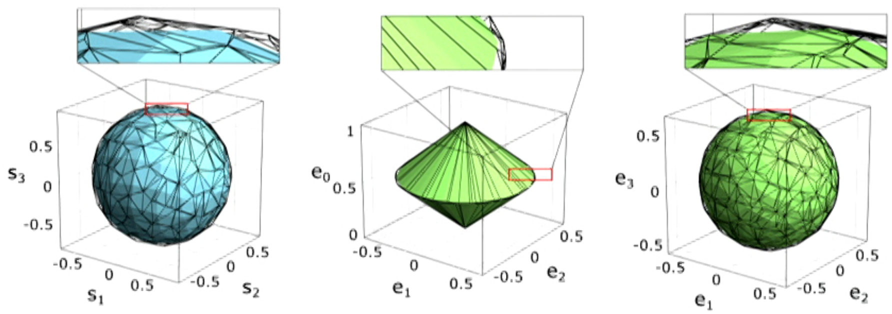
- GPTs in which $S = E_{\text{dual}}$ and $E = S_{\text{dual}}$ are self dual
  - These satisfy no-restriction hypothesis

Some example GPTs

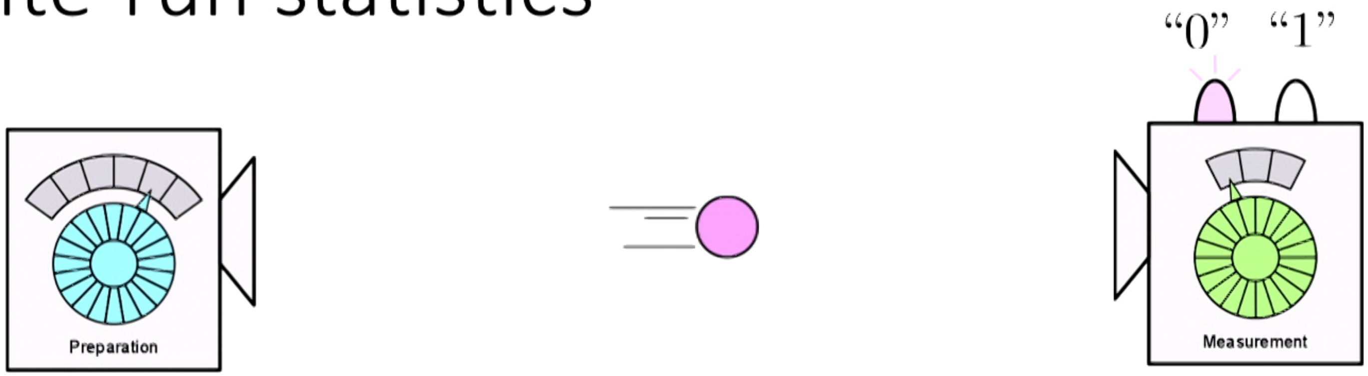


Outline

- GPT framework
- GPT tomography method and application to an experiment
- Noncontextuality inequality



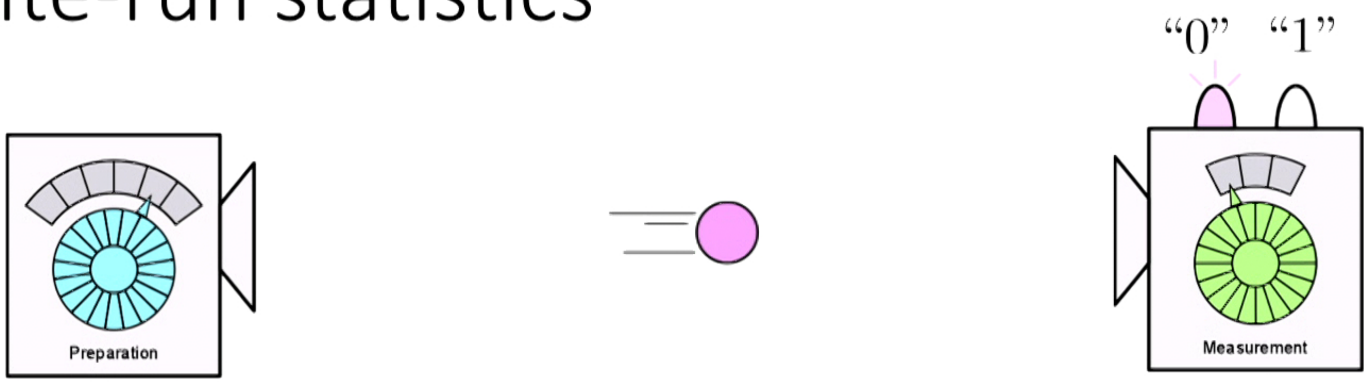
Infinite-run statistics



$P_1, \ldots, P_m$ $\qquad$ $M_1, \ldots, M_n$

$$\begin{pmatrix} 1 & p(0|P_1, M_2) & p(0|P_1, M_3) & p(0|P_1, M_4) & p(0|P_1, M_5) & \cdots \\ 1 & p(0|P_2, M_2) & p(0|P_2, M_3) & p(0|P_2, M_4) & p(0|P_2, M_5) & \cdots \\ 1 & p(0|P_3, M_2) & p(0|P_3, M_3) & p(0|P_3, M_4) & p(0|P_3, M_5) & \cdots \\ 1 & p(0|P_4, M_2) & p(0|P_4, M_3) & p(0|P_4, M_4) & p(0|P_4, M_5) & \cdots \\ 1 & p(0|P_5, M_2) & p(0|P_5, M_3) & p(0|P_5, M_4) & p(0|P_5, M_5) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Infinite-run statistics



$P_1, \ldots, P_m$ $\qquad$ $M_1, \ldots, M_n$

$$\begin{pmatrix} 1 & s_1^{(1)} & \cdots & s_k^{(1)} \\ 1 & s_1^{(2)} & \cdots & s_k^{(2)} \\ 1 & s_1^{(3)} & \cdots & s_k^{(3)} \\ 1 & s_1^{(4)} & \cdots & s_k^{(4)} \\ 1 & s_1^{(5)} & \cdots & s_k^{(5)} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 & e_0^{(2,0)} & e_0^{(3,0)} & e_0^{(4,0)} & e_0^{(5,0)} & \cdots \\ 0 & e_1^{(2,0)} & e_1^{(3,0)} & e_1^{(4,0)} & e_1^{(5,0)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & e_k^{(2,0)} & e_k^{(3,0)} & e_k^{(4,0)} & e_k^{(5,0)} & \cdots \end{pmatrix}$$

GPT states and effects

$$\begin{pmatrix} 1 & s_1^{(1)} & \cdots & s_k^{(1)} \\ 1 & s_1^{(2)} & \cdots & s_k^{(2)} \\ 1 & s_1^{(3)} & \cdots & s_k^{(3)} \\ 1 & s_1^{(4)} & \cdots & s_k^{(4)} \\ 1 & s_1^{(5)} & \cdots & s_k^{(5)} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 & e_0^{(2,0)} & e_0^{(3,0)} & e_0^{(4,0)} & e_0^{(5,0)} & \cdots \\ 0 & e_1^{(2,0)} & e_1^{(3,0)} & e_1^{(4,0)} & e_1^{(5,0)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & e_k^{(2,0)} & e_k^{(3,0)} & e_k^{(4,0)} & e_k^{(5,0)} & \cdots \end{pmatrix}$$

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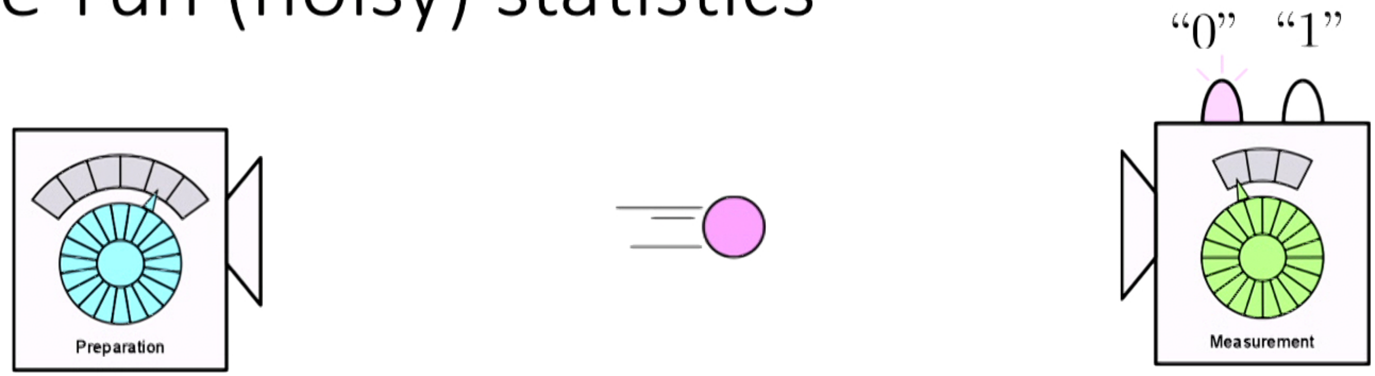
$$p(0|P_i, M_j) = \mathbf{s_i} \cdot \mathbf{e_{j,0}} = (1\, s_1^{(i)} \ \cdots \ s_k^{(i)}) \cdot (e_0^{(j,0)} \ \cdots \ e_k^{(j,0)})$$

Finite-run (noisy) statistics



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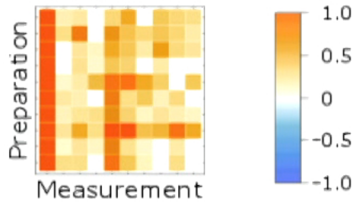
Full rank data matrix -> not factorizable!

Experiment

Raw data



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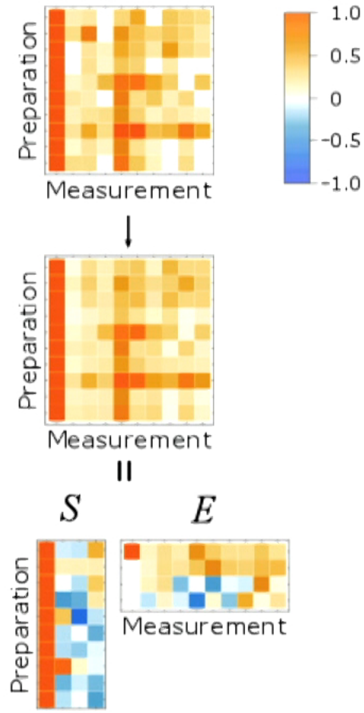
Experiment

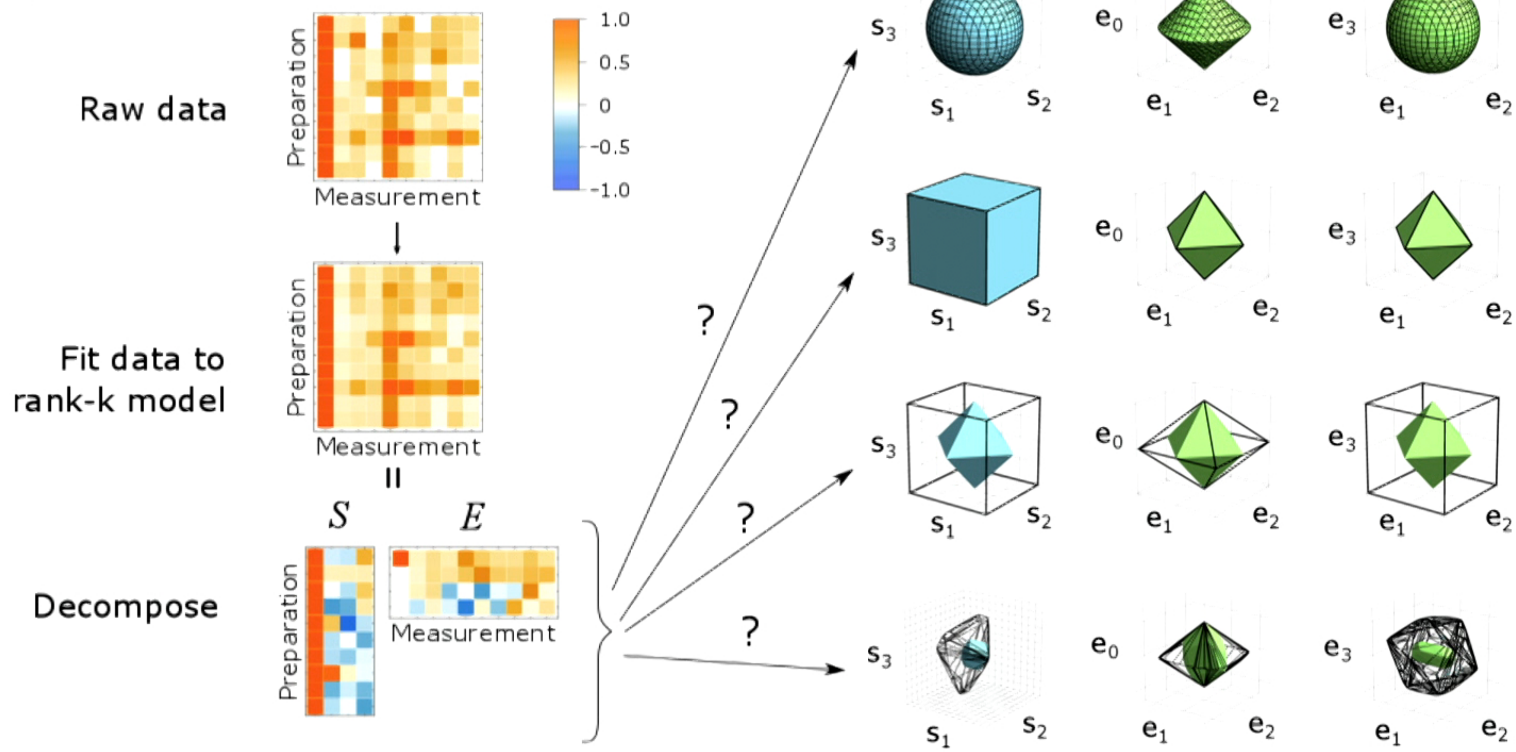
Raw data

Fit data to
rank-k model

Decompose

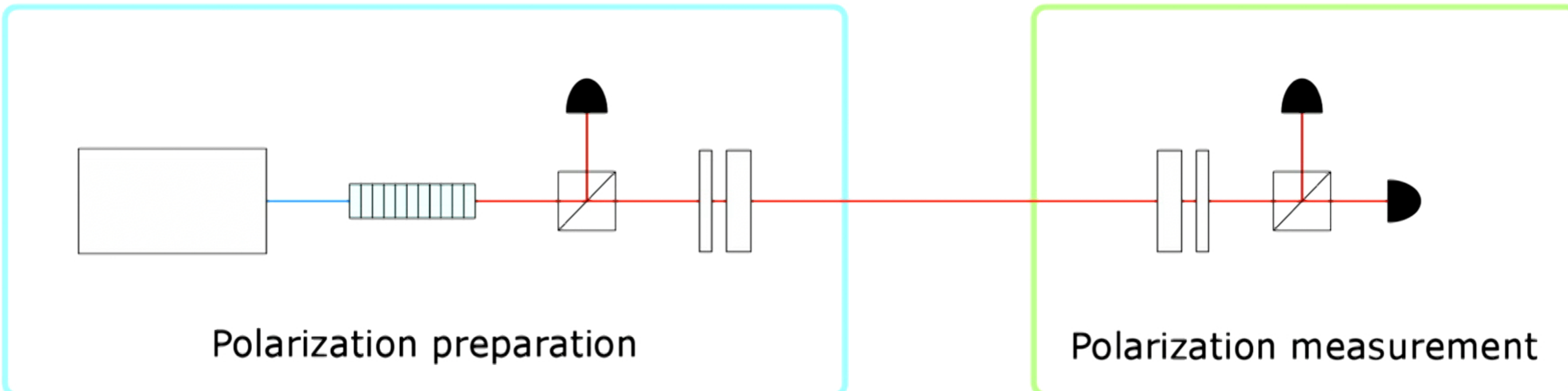


Experiment



Raw data
Preparation
Measurement
1.0
0.5
0
-0.5
-1.0
Fit data to
rank-k model
Preparation
Measurement
=
S
E
Decompose
Preparation
Measurement
?
?
?
?
s3
s1
s2
e0
e1
e2
e3
e1
e2

Experimental set-up



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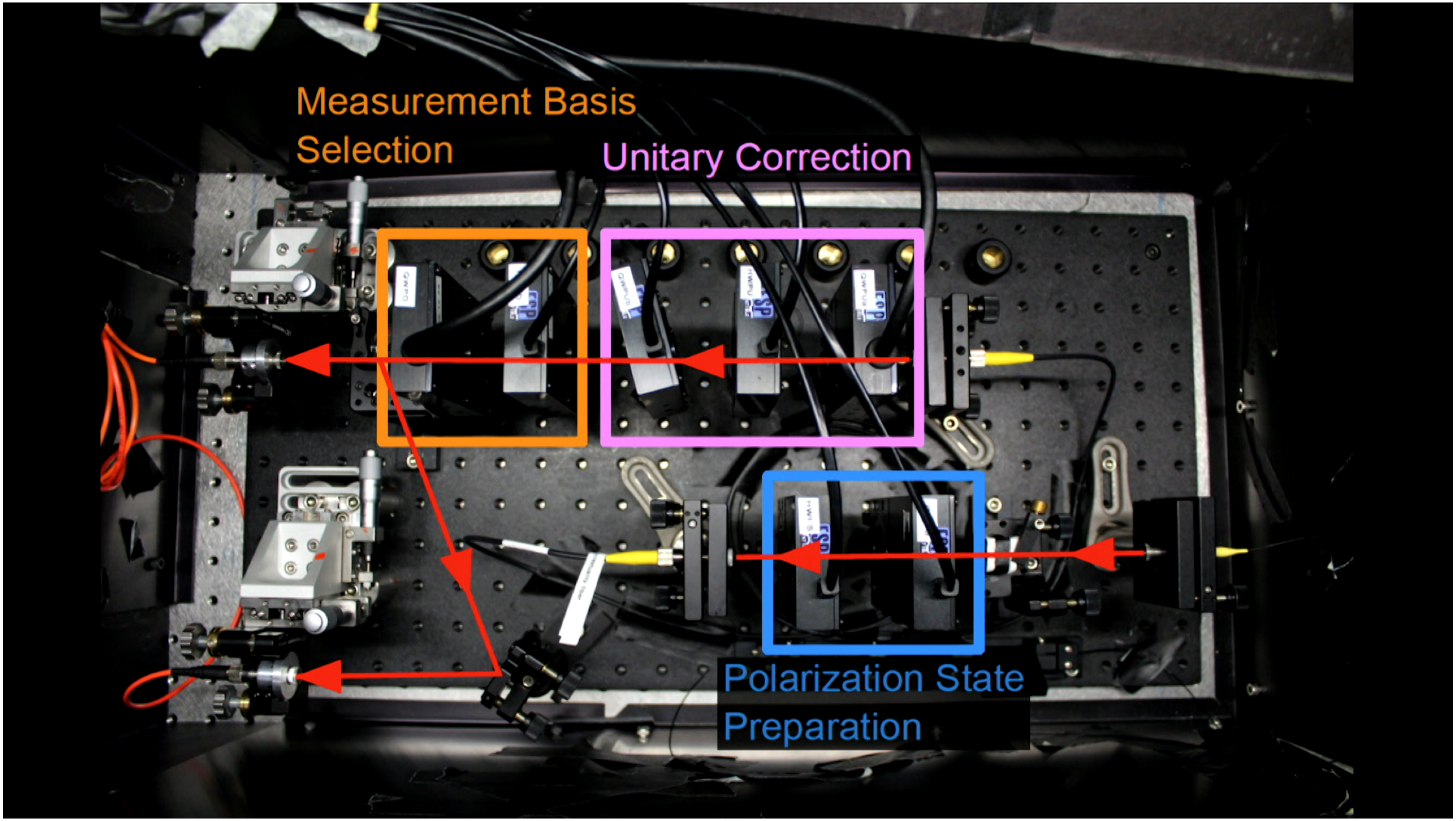
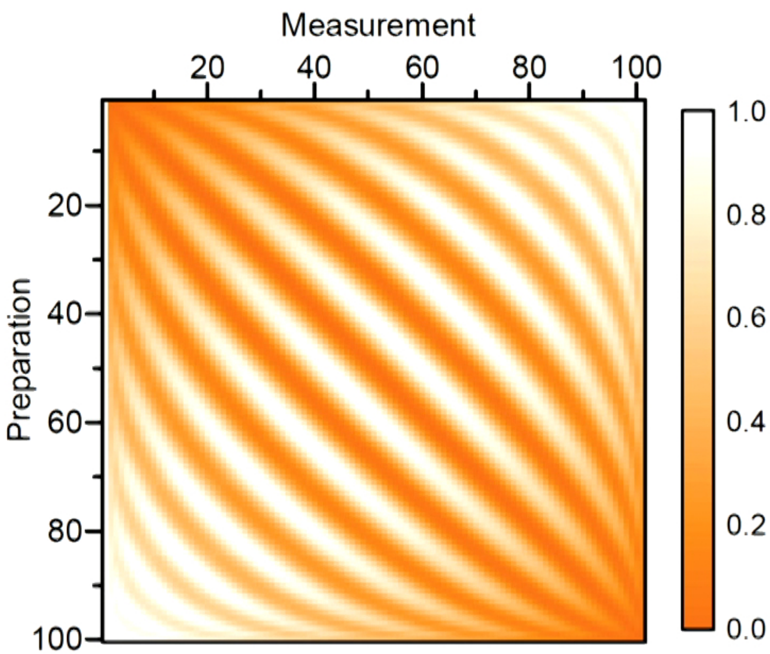

Measurement Basis Selection
Unitary Correction
Polarization State Preparation

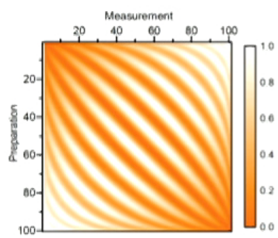
$p(0|P_i,M_j)$



- 100 preparation settings
- 100 measurement settings
- Noise ensures data table is full rank
- Fit data to models of various rank, see which performs best

Fitting to a rank-*k* model

Assume noise in *D* is independent and Poissonian ~ Gaussian

$D =$ 

$$\min_{K} \quad \chi^2 = \sum_{i,j} \frac{(D_{i,j} - K_{i,j})^2}{(\Delta D_{i,j})^2}$$

$$\text{subj. to} \quad \operatorname{rank}(K) \leq k, \quad 0 \leq K_{i,j} \leq 1 \quad \forall\, i,j$$

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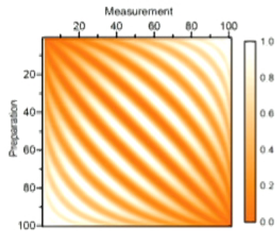
$D =$ 

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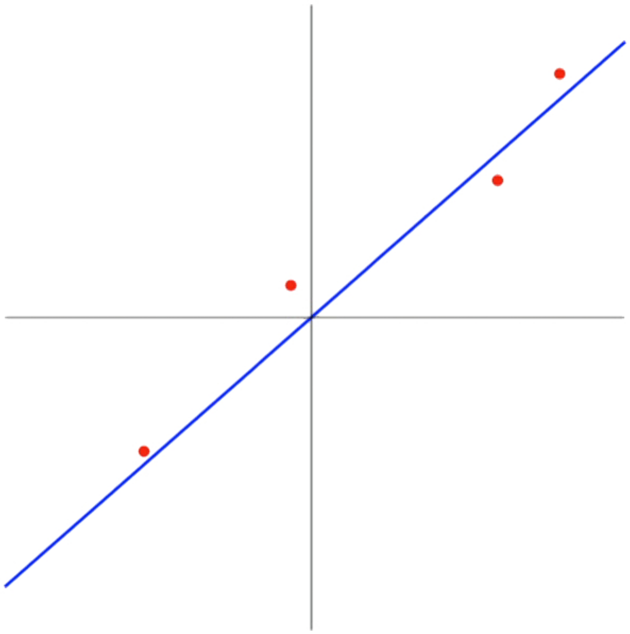

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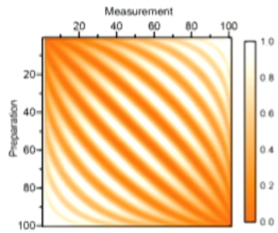
$D =$ 

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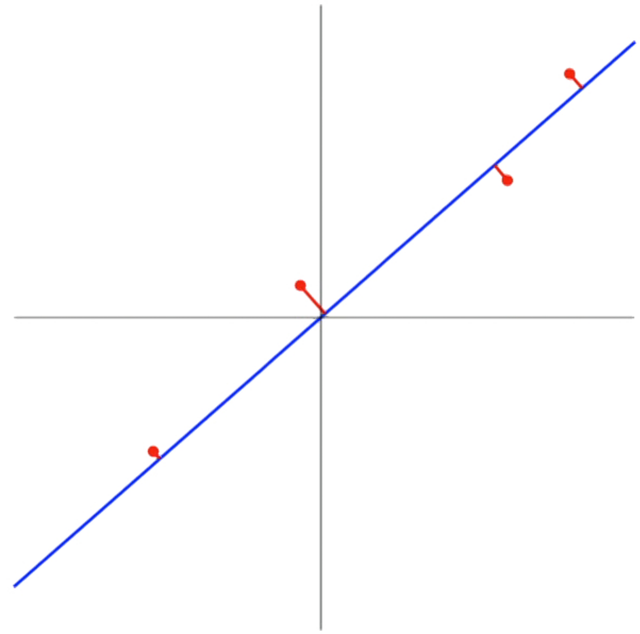

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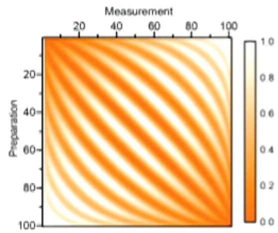
$D =$ 

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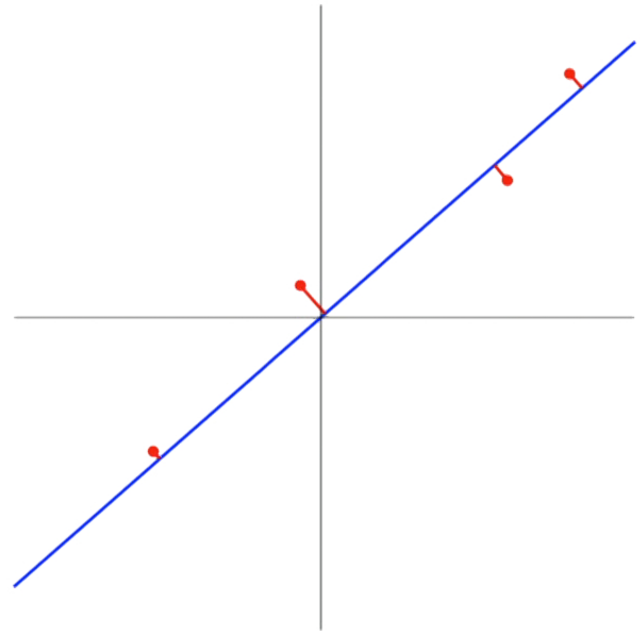
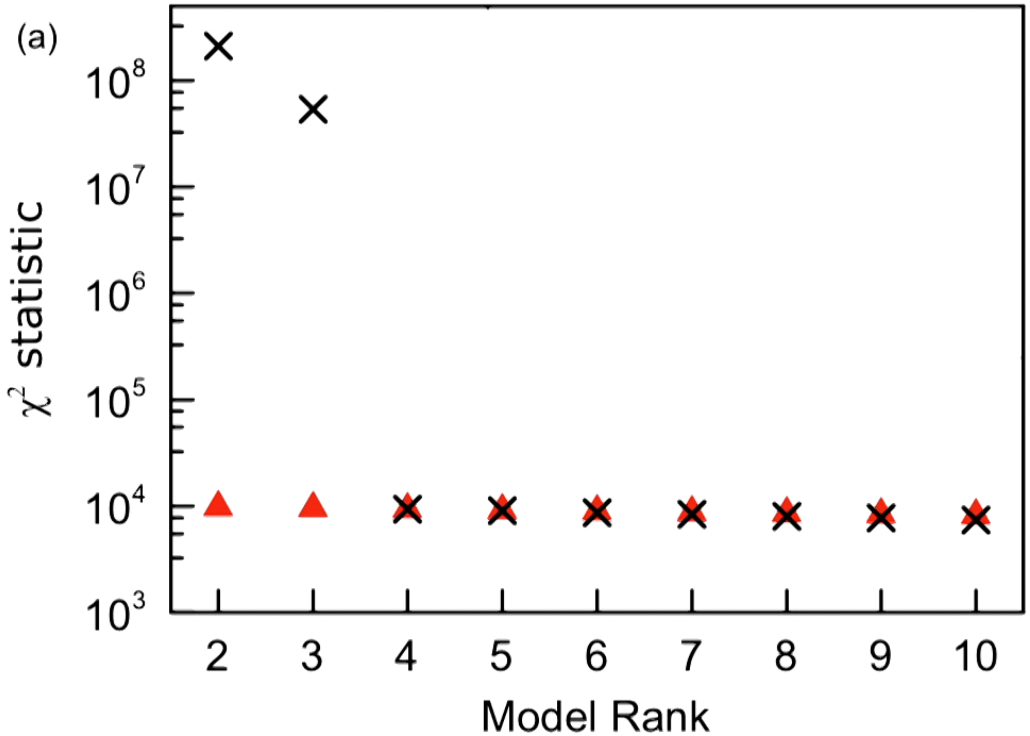


Rank-*k* parameterization of *K*:

$$K = SE$$

Determining *k*

Ranks 2 and 3 underfit the data



Akaike information criterion

- Criterion for model selection

$$AIC = -2\log\mathcal{L} + 2n$$

- Lower AIC value implies higher relative model likelihood
  - Trade-off between not underfitting and overfitting