Title: Contextuality and non-contextuality in (qudit) quantum computation

Date: Jul 27, 2017 02:00 PM

URL: http://pirsa.org/17070053

Abstract:

Contextuality and non-contextuality in (qudit) quantum computing

Dan Browne (University College London)

Joint work with:

Nicolas Delfosse, Cihan Okay, Juan Bermejo-Vega, Robert Raussendorf and **Lorenzo Catani**

arxiv.org/1701.07801
arxiv.org/1610.07093


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One of the following
or
Derives from

Contextuality and non-contextuality in (qudit) stabiliser qm

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Question: Does contextuality play a role in quantum computation?

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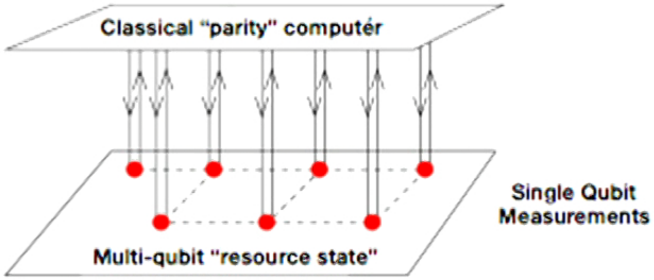
Measurement-based Quantum Computation



Anders and Browne, arXiv:0805.1002
Raussendorf, arXiv:0907.5449

Fault tolerant quantum computing

- Many fault tolerant quantum computing models restricted to **stabilizer quantum mechanics**.
  - Preparation of **stabilizer states**.
  - **Clifford** group unitaries (generated by H, S, CNOT)
  - **Pauli** observable measurements X , Z etc.

Stabilizer quantum mechanics *can be efficiently simulated on a* ***classical computer****.*
– Gottesman-Knill theorem

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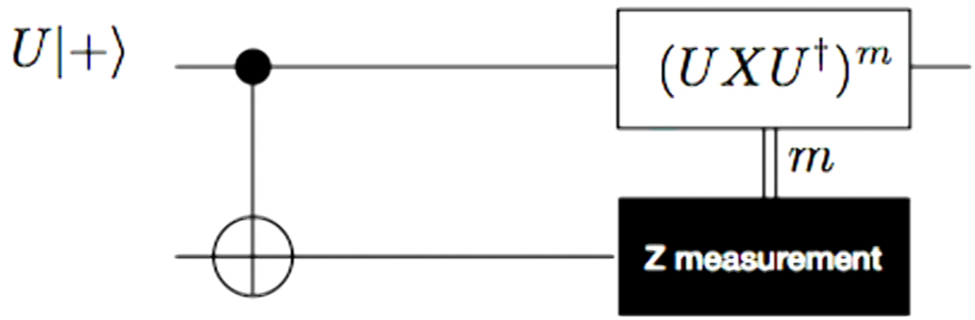
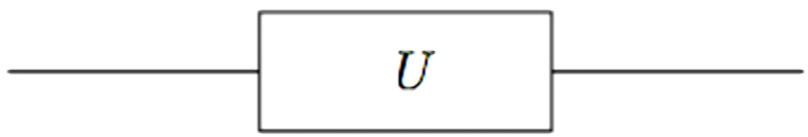
State Injection



equivalent to



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Magic state distillation

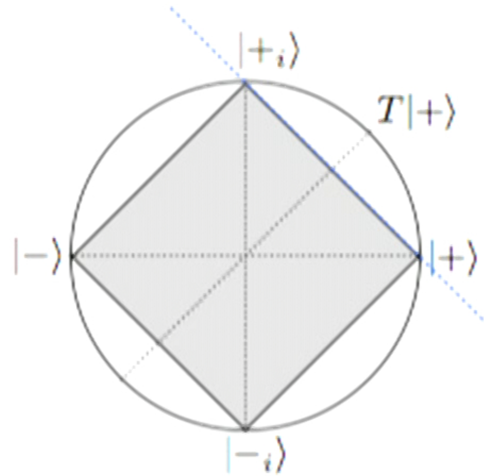
The state $T|+\rangle$ promotes stabilizer qm (via state injection) to full universality.

Given a noisy state ρ, $T|+\rangle$ can be fault tolerantly distilled by a process called **magic state distillation**.[1]

What properties make ρ distillable?

[1] Bravyi and Kitaev 2004, Reichardt 2004

Reichardt: 7-qubit **Steane code**
Noise threshold: 29.2% - tight!



Wigner negativity?

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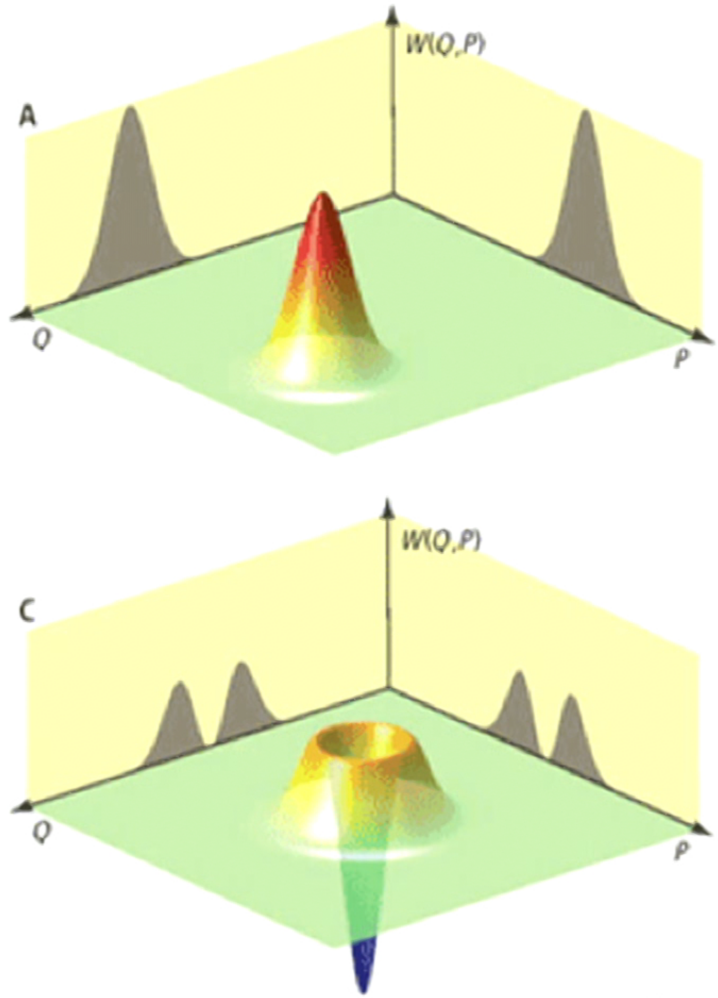
Wigner function

- Wigner (1932)
- A real valued representation of a state's density operator in **phase space**
  - *e.g. position / momentum*



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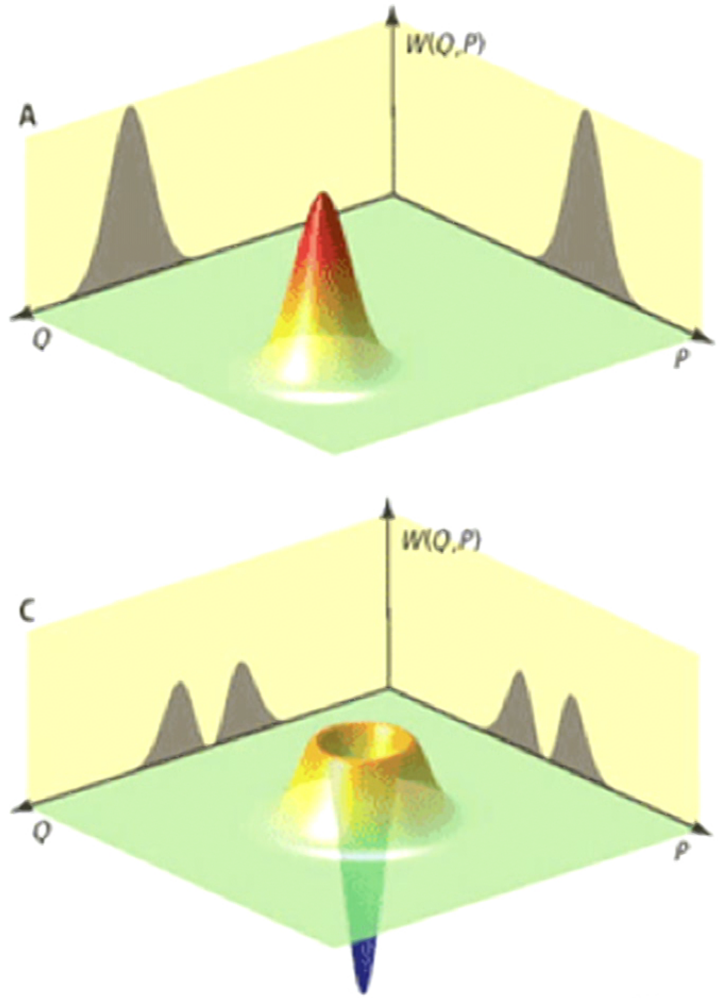
Wigner function

- A **quasi-probability** distribution
- May take **negative values**
- **Integrating out** one variable leaves a **probability distribution**.
- Quantum optics folklore: **Negative Wigner function** is a signature of **non-classicality**.



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Wigner function for qudits

Wootters (1987): Wigner functions for d-dimensional systems (**qudits**).

Gross (2006): Formalism for odd d based on **symplectic structure** of qudit Pauli observables.

All pure **stabilizer states** are non-negative and they are the **only** pure non-negative states.

The Discrete Wigner Function[a]

WILLIAM K. WOOTTERS

Center for Theoretical Physics
The University of Texas at Austin
Austin, Texas 78712
and
Department of Physics and Astronomy
Williams College
Williamstown, Massachusetts 01267[b]

INTRODUCTION

In nonrelativistic quantum mechanics, any spatial quantum state, as opposed to a spin state, can be represented by a real function on classical phase-space known as the Wigner function.[1-4] The Wigner function has a number of special properties that make it a convenient object on which to base a formulation of quantum mechanics, but it is limited in that it applies only to continuous degrees of freedom such as position and not to inherently discrete degrees of freedom such as spin. The main purpose of this paper is to present a generalization of the Wigner function that applies to systems having only a finite number of orthogonal states.

Hudson's Theorem for finite-dimensional quantum systems

D. Gross

Institute for Mathematical Sciences, Imperial College London, London SW7 2BW, UK and
QOLS, Blackett Laboratory, Imperial College London, London SW7 2BW, UK *

(Dated: February 1, 2008)

We show that, on a Hilbert space of odd dimension, the only pure states to possess a non-negative Wigner function are stabilizer states. The Clifford group is identified as the set of unitary operations which preserve positivity. The result can be seen as a discrete version of Hudson's Theorem. Hudson established that for continuous variable systems, the Wigner function of a pure state has no negative values if and only if the state is Gaussian. Turning to mixed states, it might be surmised that only convex combinations of stabilizer states give rise to non-negative Wigner distributions. We refute this conjecture by means of a counter-example. Further, we give an axiomatic characterization which completely fixes the definition of the Wigner function and compare two approaches to stabilizer states for Hilbert spaces of prime-power dimensions. In the course of the discussion, we derive explicit formulas for the number of stabilizer codes defined on such systems.

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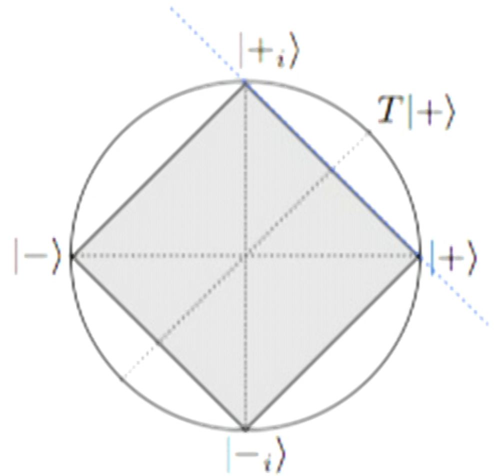


[1] Bravyi and Kitaev 2004, Reichardt 2004

Wigner negativity as a resource?

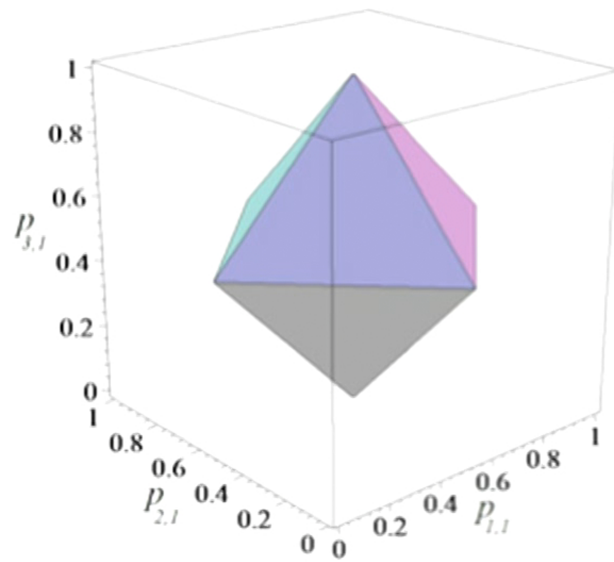
Galvao: 2005

Consider the intersection of positive states of 2 types of Wigner function...

you recover the one-qubit stabilizer states.

(cf. Wallman-Bartlett 8-state model)

Galvao: Wigner negativity necessary for quantum speed-up?



Qudit stabilizer quantum mechanics

d-dimensional state space $|0\rangle, |1\rangle, \ldots, |d-1\rangle$.

Generalised Pauli operators: $X = \sum_j |j+1\rangle\langle j|$ $\quad Z = \sum_j \omega^j |j\rangle\langle j|$

where $\omega = \exp[i2\pi/d]$.

Notation for tensor products:

$$Z^a = Z^{a_1} \otimes Z^{a_2} \otimes \cdots \qquad X^{\mathbf{b}} = X^{b_1} \otimes X^{b_2} \otimes \cdots$$

Commutation rule: $Z^a X^b = \omega^{a \cdot b} X^b Z^a$

In this talk d will **always** be **odd**.

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Phase space

Natural association with points in **phase-space** $\Omega = \mathbb{Z}_d^n \times \mathbb{Z}_d^n$.

For each point $u = u_z u_x \in \Omega$ define a Heisenberg-Weyl operator.

$$T_u = \omega^{-(u_z \cdot u_x)2^{-1}} Z^{u_z} X^{u_x}$$

Note: 2^{-1} is multiplicative inverse of 2 in $\mathbb{Z}_d$.

x →

z ↓

(0,0)	(0,1)	(0,2)
(1,0)	(1,1)	(1,2)
(2,0)	(2,1)	(2,2)

Heisenberg-Weyl operators compose as:

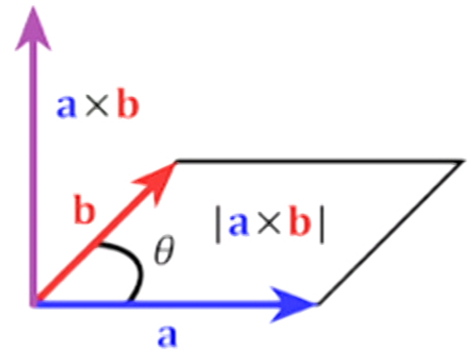
$$T_u T_v = \omega^{[u,v]2^{-1}} T_u T_v$$

where

$$[u, v] = u_z v_x - u_x v_z \mod d$$

is the **symplectic product**.

Note that T_u and T_v **commute** iff $[u, v] = 0$.



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Contextuality in SQM

- Qubit stabiliser quantum mechanics is **contextual**.
  - Peres-Mermin square, GHZ-Mermin, etc.
- But all **odd** d, qudit SQM is **non-contextual**.
  - Folklore: *if a theory has a non-negative Wigner function it has a non-contextual hidden variable model.*

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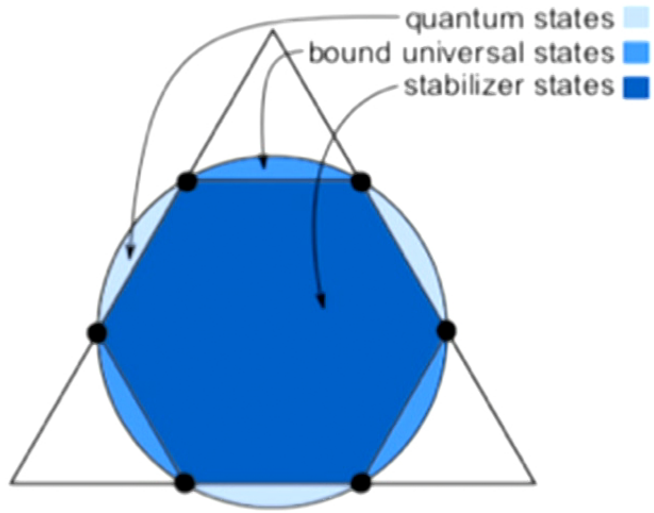
Qudit magic states

Similar to qubits, we can devise **qudit** versions of **fault-tolerant** quantum computing, state injection and **magic state distillation.**

We can ask the same questions.

Is Wigner negativity necessary for magic state distillability?

Veitsch, Ferrie, Gross, Emerson (2012):



Yes it is. Magic state distillation is **impossible** for all odd d states with **non-negative** Wigner functions, **even for non-stabilizer states.**

Gross-Wigner functions

A **phase-space representation** of ρ. Assign a basis of Hermitian operators to each point in $\Omega = \mathbb{Z}_d^n x \mathbb{Z}_d^n$:

$$\rho = \sum_{u \in \Omega} W_\rho(u) A_u$$

We choose the following A_u basis (essentially unique - Gross):

$$A_0 = d^{-n} \sum_{u \in \Omega} T_u \qquad A_u = T_u A_0 T_u^\dagger$$

The **Wigner function** is the set of coefficients wrt this basis:

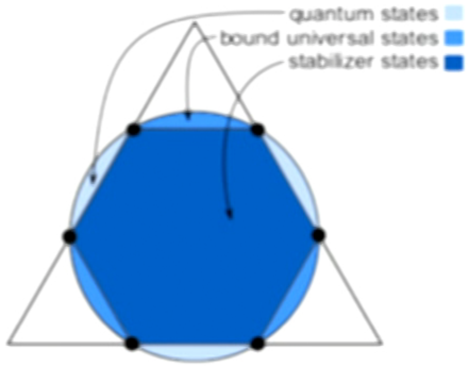
$$W_\rho(u) = d^{-n} \mathrm{Tr}[A_u \rho]$$

Veitsch, Ferrie, Gross, Emerson (2012):

Stabilizer operations **preserve non-negativity**.

Magic state distillation **cannot** distill states with non-negative Wigner functions.

Wigner negativity is a **necessary resource** for (odd qudit) quantum **speedup**.



Enter contextuality...

A hint:

PRL **101**, 020401 (2008) PHYSICAL REVIEW LETTERS week ending 11 JULY 2008

Negativity and Contextuality are Equivalent Notions of Nonclassicality

Robert W. Spekkens

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, United Kingdom CB3 0WA

(Received 25 January 2008; published 7 July 2008)

Two notions of nonclassicality that have been investigated intensively are: (i) negativity, that is, the need to posit negative values when representing quantum states by quasiprobability distributions such as the Wigner representation, and (ii) contextuality, that is, the impossibility of a noncontextual hidden variable model of quantum theory. Although both of these notions were meant to characterize the conditions under which a classical explanation cannot be provided, we demonstrate that they prove inadequate to the task and we argue for a particular way of generalizing and revising them. With the refined version of each in hand, it becomes apparent that they are in fact one and the same. We also demonstrate the impossibility of noncontextuality or non-negativity in quantum theory with a novel proof that is symmetric in its treatment of measurements and preparations.

DOI: 10.1103/PhysRevLett.101.020401 PACS numbers: 03.65.Ta, 03.65.Ud

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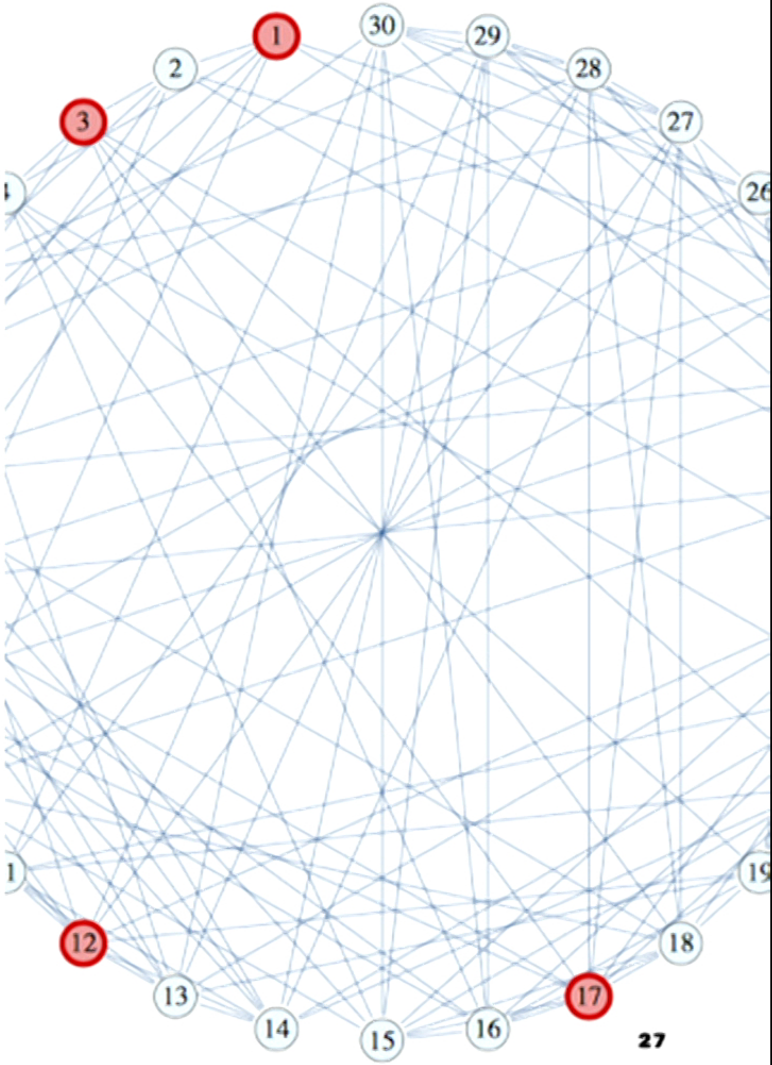
Howard, Wallman, Veitch, Emerson (2014):

Contextuality is necessary for magic state distillation in odd **prime** d.

All **negative-Wigner** single-qudit states violate a CSW contextuality witness (*in a 2-qudit experiment*).

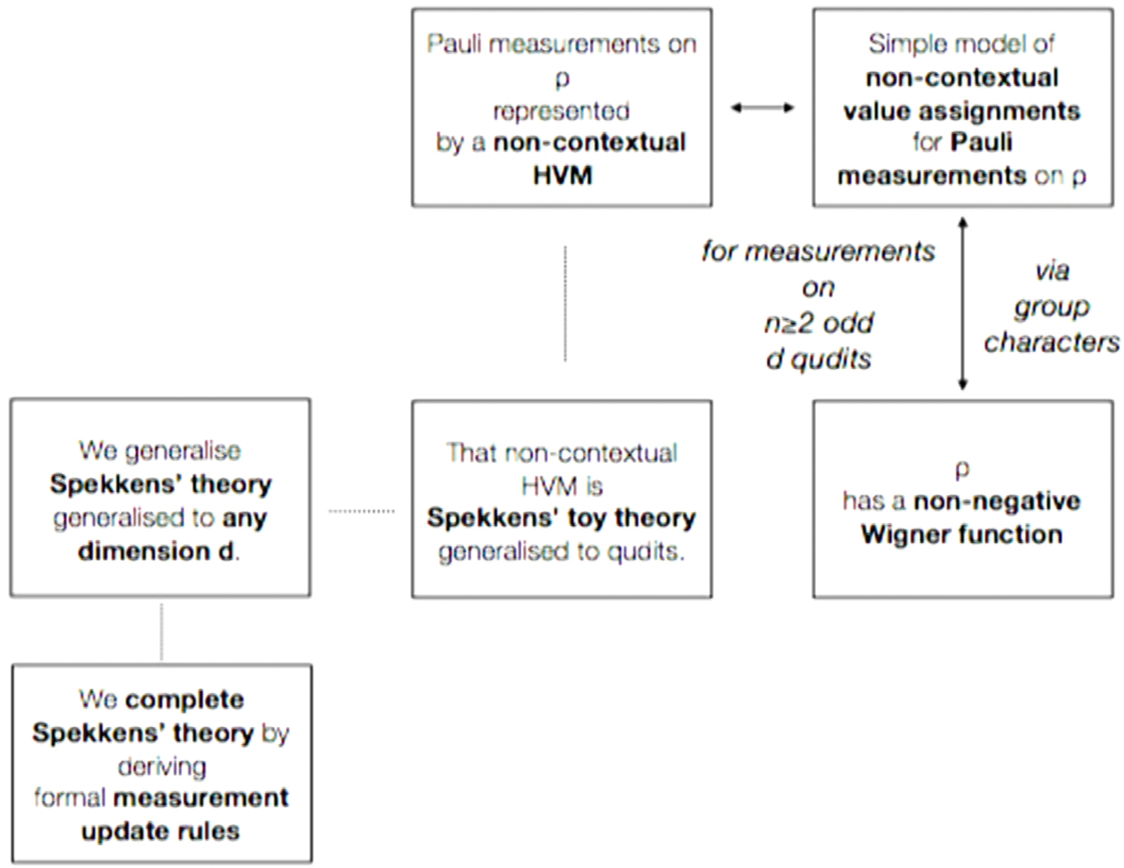


This talk

That non-contextual HVM is **Spekkens toy theory** generalised to qudits.	⟷	Pauli measurements on ρ represented by a **non-contextual HVM**	⟷	ρ has a **non-negative Wigner function**

arxiv.org/ 1610.07093

arxiv.org/ 1701.07801


Pauli measurements on ρ represented by a non-contextual HVM
Simple model of non-contextual value assignments for Pauli measurements on ρ
for measurements on n≥2 odd d qudits
via group characters
We generalise Spekkens' theory generalised to any dimension d.
That non-contextual HVM is Spekkens' toy theory generalised to qudits.
ρ has a non-negative Wigner function
We complete Spekkens' theory by deriving formal measurement update rules

A simple non-contextual model

- We wish to provide a simpler and more general proof than Howard et al.
- Key idea: a simple non-contextual model with minimal assumptions.
- We call it a **non-contextual value assignment**, NCVA.
- c.f. Kochen, Specker 1967 (Thank you Andrew!)

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NCVA for Pauli measurements

We represent Pauli measurements $T_u = \omega^{-(u_z \cdot u_x)2^{-1}} Z^{u_z} X^{u_x}$ by a non-contextual map from ontic state to outcome.

- Label outcomes by corresponding eigenvalues ω^k, $k \in \mathbb{Z}^d$.
- Set of ontic states $\nu \in S$ (no structure or cardinality assumed).
- Non-contextual measurement map $\lambda_\nu(u)$:
  - When we measure T_u, the outcome ω^k depends solely on ontic state ν and the observable u.

$$\omega^k = \lambda_\nu(u)$$

Example: $S = Z_3 \times \mathbb{Z}_3$

Two example $\lambda_\nu(u)$ maps

$$\lambda_\nu(01): \quad \begin{bmatrix} 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 \end{bmatrix}$$

$$\lambda_\nu(10): \quad \begin{bmatrix} 1 & 1 & 1 \\ \omega & \omega & \omega \\ \omega^2 & \omega^2 & \omega^2 \end{bmatrix}$$

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NCVA

We now introduce a probability distribution $q_\rho(\nu)$ over ontic states and impose two conditions:

- Model reproduces quantum statistics:

$$\mathrm{Tr}[T_u \rho] = \sum_{\nu \in S} \lambda_\nu(u) q_\rho(u)$$

- Consistency of commuting sets of observables:
  - For all u, v st $[u, v] = 0$

$$\lambda_\nu(u + v) = \lambda_\nu(u)\lambda_\nu(v).$$

Example

A **non-negative Wigner function** provides an example of an NCVA.

Recall: $W_\rho(u) = d^{-n}\mathrm{Tr}[A_u \rho] = d^{-n}\mathrm{Tr}[T_u A_0 T_u^\dagger \rho]$

We can transform this:

$$T_u A_0 T_u^\dagger = \sum_v T_u T_v T_u^\dagger = \sum_v \omega^{[u,v]} T_v$$

So:

$$W_\rho(u) = d^{-n}\mathrm{Tr}[\sum_v \omega^{[u,v]} T_v \rho]$$

Recall: $$\rho = \sum_{u \in \Omega} W_\rho(u) A_u$$

Hence: $$\mathrm{Tr}[T_u \rho] = \sum_{v \in \Omega} W_\rho(u) \omega^{[u,v]}$$

We can identify:

- $S = \Omega$
- $q_\rho(u) = W_\rho(u)$
- $\lambda_\nu(u) = \omega^{[u,\nu]}$

Check:

$$\lambda_\nu(u+v) = \lambda_\nu(u)\lambda_\nu(v)$$ ✅

The Gross-Wigner function satisfies the NCVA axioms.

In particular,

$$\lambda_\nu(u+v) = \lambda_\nu(u)\lambda_\nu(v)$$

for all u, v st $[u, v] = 0$.

But, this is **not** what we have just shown!

In verifying that last property we did **not** assume $[u, v] = 0$.

As one can readily check:

$$\lambda_\nu(u) = \omega^{[u,\nu]} \text{ satisfies}$$

$$\lambda_\nu(u + v) = \lambda_\nu(u)\lambda_\nu(v)$$

for **all** u, v.

Observation:

$\lambda_\nu(u)$ is one-dimensional group representation of Ω.

It is an **irreducable character** of Ω.

Irreducible Characters

Key properties:

- $\lambda_\nu(u+v) = \lambda_\nu(u)\lambda_\nu(v)$
- One of the following holds:

$$\sum_{v\in\Omega} \lambda_\nu(v) = 0 \qquad \text{or} \qquad \sum_{v\in\Omega} \lambda_\nu(v) = |\Omega|$$

Derives from orthogonality of irreducible characters.

The function $\omega^{[u,*]}$ is an example of a character of Ω.

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Irreducible Characters

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Characters in an NCVA

Recall our definition of the model

- Consistency of commuting sets of observables:
  - For all u, v st $[u, v] = 0$

$$\lambda_\nu(u+v) = \lambda_\nu(u)\lambda_\nu(v).$$

We did **not** assume that $\lambda_\nu(u)$ is a character of Ω.

But we can now **derive** this.

Lemma: In the Pauli NCVA model, for all $n \geq 2$ and all odd $d > 1$, all value assignments $\lambda_\nu(u)$ are **characters** of Ω.

- **Proof sketch:**
- Start with u, v and identify a second pair u', v' such that $[u, v] = [u', v']$.
- Can always do this if $n \geq 2$.
- Decompose:

$$u + v = 2^{-1}((u + v + u' + v') + (u + v - u' - v'))$$

Then successively apply

$$\lambda_\nu(u + v) = \lambda_\nu(u)\lambda_\nu(v) \text{ when } [u, v] = 0.$$

To finally prove that for arbitary u, v:

$$\lambda_\nu(u + v) = \lambda_\nu(u)\lambda_\nu(v)$$

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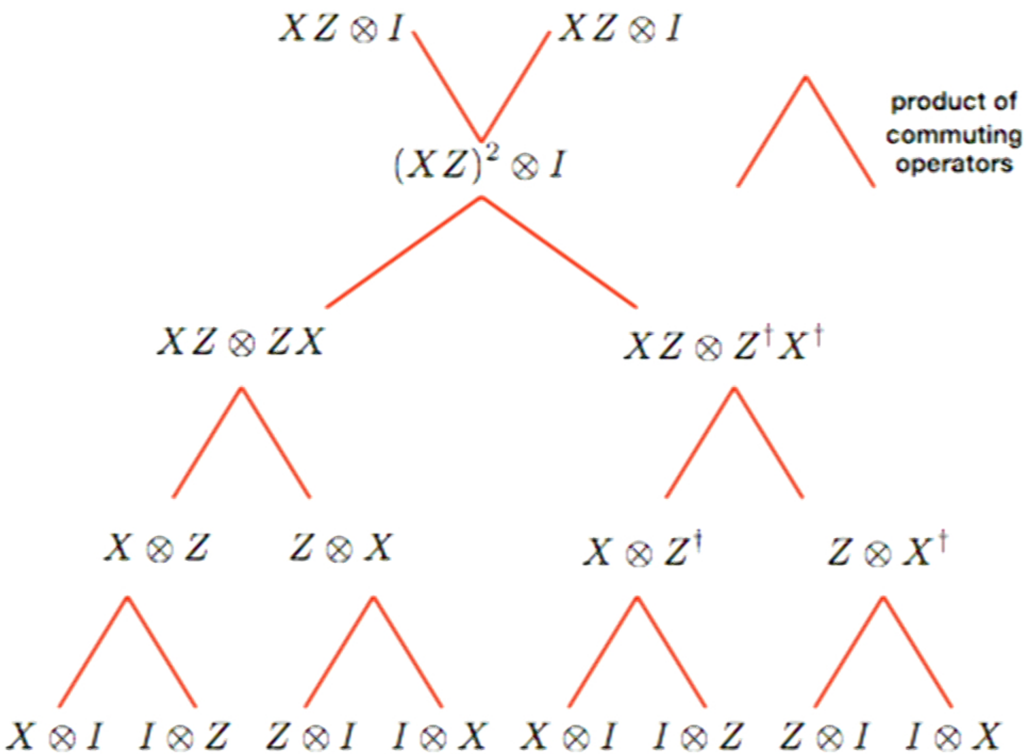
Example: X, Z and XZ.



NCVA for ρ implies non-negative Wigner function

We have already seen:

- Every ρ where $W_\rho \geq 0$ has a NCVA model.

Now we show:

- **Every** ρ with a Pauli-NCVA satisfies $W_\rho \geq 0$.

Proof: Explicit calculation of W_ρ.

We write down the Wigner function...

$$W_\rho(u) = d^{-n}\mathrm{Tr}[A_u\rho] = d^{-2n}\mathrm{Tr}[\sum_{v\in\Omega}\omega^{[u,v]}T_v\rho]$$
$$= d^{-2n}\sum_{v\in\Omega}\omega^{[u,v]}\mathrm{Tr}[T_v\rho]$$

Now we use the NCVA definitions:

$$\mathrm{Tr}[T_u\rho] = \sum_{\nu\in S}\lambda_\nu(u)q_\rho(u)$$

$$W_\rho(u) = d^{-2n}\sum_{\nu\in S}\left(\sum_{v\in\Omega}\omega^{[u,v]}\lambda_\nu(u)\right)q_\rho(u)$$

Finally we note that the term in brackets is a **irreducible character** of Ω, and hence the sum over the character is is 0 or 2^d.

Thus, (for $n \geq 2$ odd d) the existance of a non-negative Wigner function implies the existance of a non-contextual model for Pauli measurements and vice versa.

Interpretation: Contextuality is necessary for magic state distillation in **all odd dimensions**.

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Ontic space is phase space

A more detailed calculation gives:

$$q_\rho(\nu) = W\rho(f(\nu)$$

where $f(\nu)$ is a one-to-one from S to Ω.

Hence the ontic space S is **isomorphic** to phase space Ω.

This is proved, not assumed.

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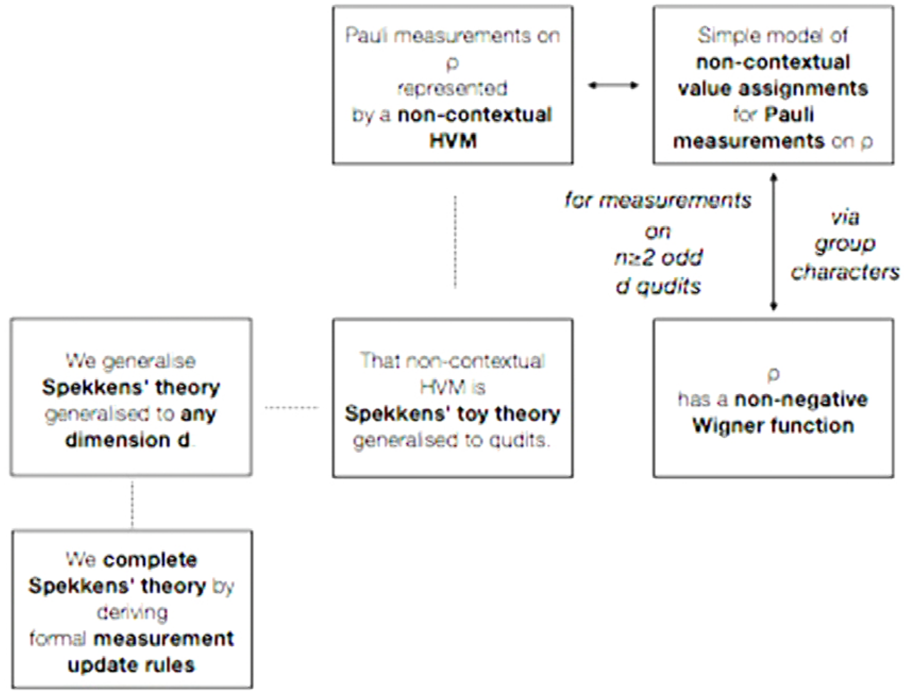
- We have thus derived a unique, up to relabelling, a NCHVM for stabilizer QM based on non-negative Wigner functions or NC value assignments.



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So far, we just have a non-contextual value assignment model for measurements. It gives us statistics for

It has:

- ontic state space Ω.
- A value assignment map satisfying $\lambda_\nu(u+v) = \lambda_\nu(u)\lambda_\nu(v)$

To develop this model into a **full theory**, we'd need to add dynamics and measurement update rules.

Problem: Our model would assign definite values to sets of non-commuting measurements.

Solution: An epistemic restriction.

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Solution: An epistemic restriction.

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In defense of the epistemic view of quantum states: a toy theory

Robert W. Spekkens
Perimeter Institute for Theoretical Physics,
31 Caroline St. North, Waterloo, Canada N2L 2Y5
(Dated: February 1, 2008)

We present a toy theory that is based on a simple principle: the number of questions about the physical state of a system that are answered must always be equal to the number that are unanswered in a state of maximal knowledge. A wide variety of quantum phenomena are found to have analogues within this toy theory. Such phenomena include: the noncommutativity of measurements, interference, the multiplicity of convex decompositions of a mixed state, the impossibility of discriminating nonorthogonal states, the impossibility of a universal state inverter, the distinction between bi-partite and tri-partite entanglement, the monogamy of pure entanglement, no cloning, no broadcasting, remote steering, teleportation, dense coding, mutually unbiased bases, and many others. The diversity and quality of these analogies is taken as evidence for the view that quantum states are states of incomplete knowledge rather than states of reality. A consideration of the phenomena that the toy theory fails to reproduce, notably, violations of Bell inequalities and the existence of a Kochen-Specker theorem, provides clues for how to proceed with this research program.

quant-ph/0401052

Spekken's epistricted theory

- Ontic space: Ω
- Observables are linear functionals on Ω

$$\sum_j a_j X_j + b_j Z_j$$

or represent as an element of $\mathbb{Z}_d^n \times \mathbb{Z}_d^n$, $u = a_1 a_2, \ldots b_1, b_2, \ldots$

$$\sigma = u \cdot \nu$$

Or: represent outcomes as ω^σ and in our NCVA notation we'd write:

$$\lambda_\nu(u) = \omega^{u \cdot \nu}$$

Then the linear functional property implies: $\lambda_\nu(u+v) = \lambda_\nu(u)\lambda_\nu(v)$

Rob left us something to do...

On the other hand, for d an *odd prime*, i.e., any prime besides 2, the quadrature epistricted theory reproduces *precisely* the stabilizer theory for qudits. For such values of d, the epistemic restriction of classical complementarity turns out to be inequivalent to the knowledge-balance principle. The latter specifies only that at most half of the full set of variables can be known, whereas the former picks out particular halves of the full set of variables, namely, the halves wherein all the variables Poisson-commute. Because the restriction of classical complementarity actually reproduces the stabilizer theory for qudits while the knowledge-balance principle does not [7], epistemic restrictions based on the symplectic structure seem to be preferable to those based on a principle of knowledge balance.

A full treatment of measurements would include a discussion of how the epistemic state is updated when the system survives the measurement procedure, but we will not discuss the transformative aspect of measurements in this article.

Spekkens' toy model in all dimensions and its relationship with stabilizer quantum mechanics

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- We derive that measurement update rule
- Extend from prime dimensions to compound dimensions
- For all odd d - prove full equivalence with Stabilizer Quantum Mechanics

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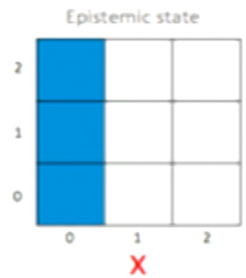
Epistemic states in the theory

X=0

V = {(0,0),(1,0),(2,0)}

$V^{\perp}$ = {(0,0),(0,1),(0,2)}

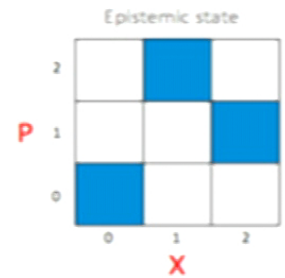


(a)

X+P=0

V = {(0,0),(1,1),(2,2)}

$V^{\perp}$ = {(0,0),(1,2),(2,1)}

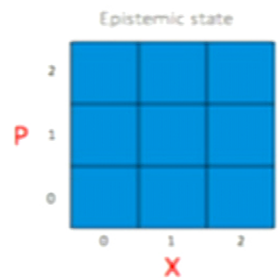


(b)

Nothing known

V = {(0,0)}

$V^{\perp} = \Omega$



(c)

- We are allowed to know a set of **commuting observables**.
- In the phase space formalism, the commuting set is represented by an **isotropic subspace** $V \in \Omega$

Epistemic states take the form of **uniform distributions** over a **shifted sub-space** (Gross 2006).

$$V^{\perp} + w$$

- $V^{\perp}$ is the set of all points in phase space for whom the outcome of all observables of V is zero.
- w is a "representative ontic state ". It encodes the outcomes of all known observables σ_j.

$$\forall j \qquad \sigma_j = \Sigma_j \cdot w$$

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Measurement update rule

Recall:

- V encodes which observables are known.
- w encodes their values.

Need a measurement update rule that

- updates V and w and embodies Spekkens' epistemic restriction

Measurement update rule

In arXiv:1701.07801 we derive updating rules for:

- measurements that **commute** with all previously known observables
- measurements that **do not commute** with all previously known observables

First we assume that d is **prime**.

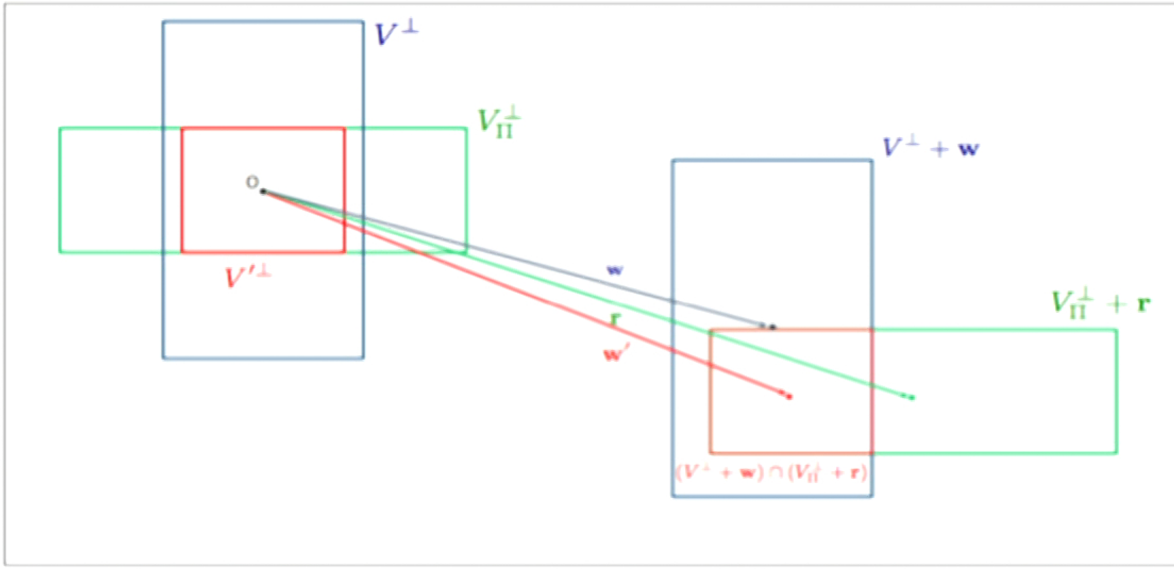

Figure 4: **Updating rules via Venn diagrams.** The figure above schematically shows the subspaces $V^\perp, V_\Pi^\perp, V'^\perp$ and the shifted ones (after applying the corresponding representative ontic vectors $\mathbf{w}, \mathbf{r}, \mathbf{w}'$). In particular this picture explains the expression $V'^\perp = (V^\perp + \mathbf{w} - \mathbf{w}') \cap (V_\Pi^\perp + \mathbf{r} - \mathbf{w}')$ as a result of combining the updating rules for the epistemic subspaces and the representative ontic vectors. It is important to notice that to obtain the correct intersection we have to shift the subspaces $V^\perp + \mathbf{w}$ and $V_\Pi + \mathbf{r}$ back to the same origin (this is the role of $\mathbf{w}'$). Indeed note that $V^\perp \cap V_\Pi^\perp$ is different from $(V^\perp + \mathbf{w}) \cap (V_\Pi^\perp + \mathbf{r})$.

w update rule.

- Let $\Sigma \in \Omega$ represent the measurement, and let the outcome be σ.

Typically w encodes the wrong outcome of new measurement:

$$\Sigma \cdot \boldsymbol{w} = \boldsymbol{\sigma} + \boldsymbol{x}$$

Note that $\Sigma . \Sigma = \boldsymbol{k} \in \mathbb{Z}^{\mathrm{d}}$

Hence set: $\boldsymbol{w}' = \boldsymbol{w} - \boldsymbol{k}^{-1} \boldsymbol{x} \Sigma$.

This shifts the incorrect value while preserving all observables that commute with Σ.

Leaving prime d behind

The Gross-Wigner function is not limited to prime d, why should Spekkens' model?

Problems to overcome:

- In compound d, k^{-1} is not always defined.
  - E.g. in $\mathbb{Z}_4$, $2x = 1$ has no solution.
- In compound d we have a new type of observable.
  - All phase space points are no longer equal!

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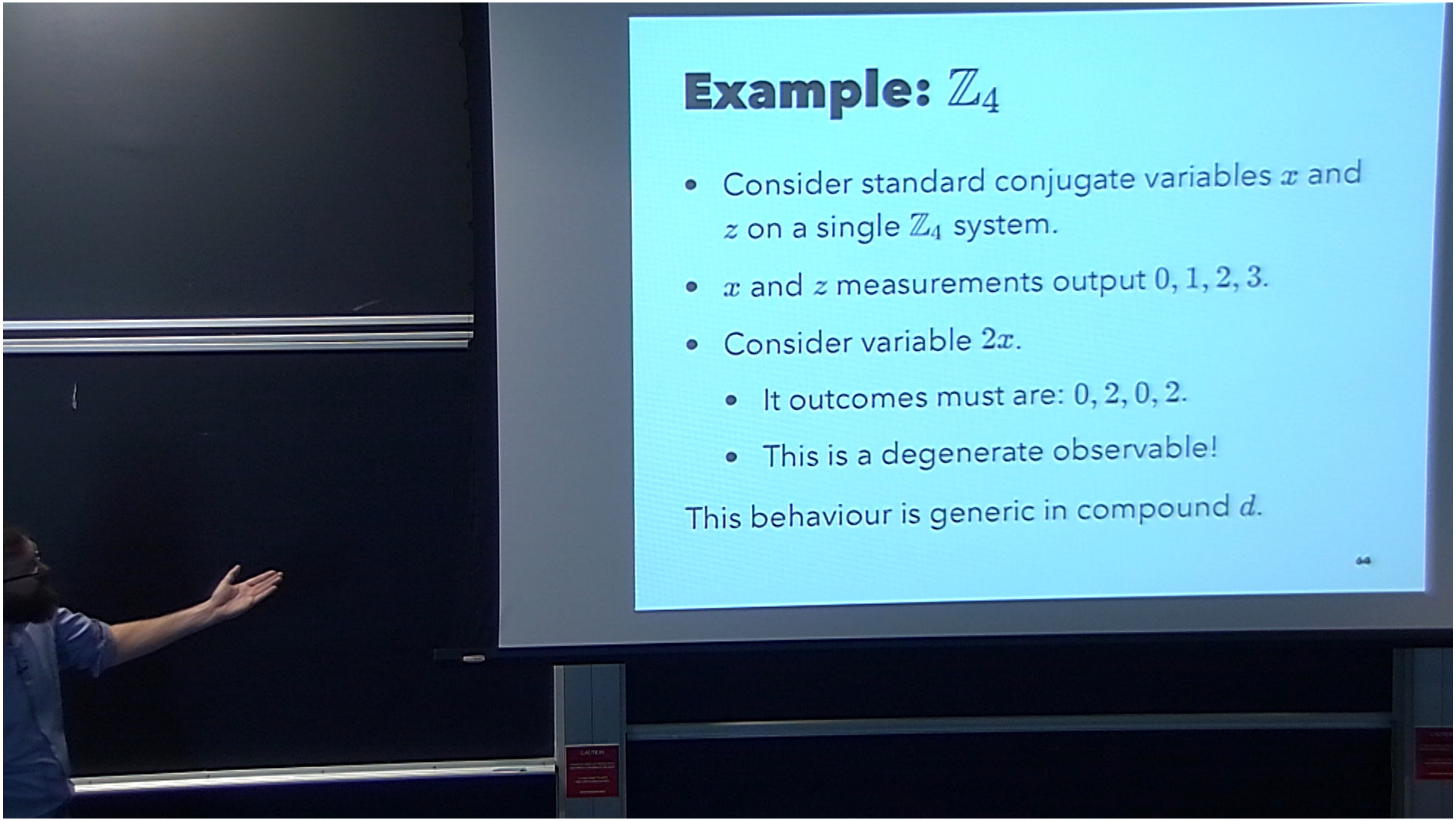
Example: $\mathbb{Z}_4$
Consider standard conjugate variables x and z on a single $\mathbb{Z}_4$ system.
x and z measurements output $0, 1, 2, 3$.
Consider variable $2x$.
It outcomes must are: $0, 2, 0, 2$.
This is a degenerate observable!
This behaviour is generic in compound d.

Example: $\mathbb{Z}_4$

- Consider standard conjugate variables x and z on a single $\mathbb{Z}_4$ system.
- x and z measurements output $0, 1, 2, 3$.
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  - It outcomes must are: $0, 2, 0, 2$.
  - This is a degenerate observable!

This behaviour is generic in compound d.

Course-graining observables

- Let us define any observable whose output spectrum covers all of $\mathbb{Z}_d$ as **fine-graining**.
- All other observables are **course-graining**.

Observation 1: An observable $aX + bP$ is finegraining iff none of the common divisors of a and b are a factor of d.

Observation 2: Dividing out all such common divisors creates a fine-graining observable.

Observation 3: The measurement update rule for prime d applies in compound d for fine-graining observables.

Ω

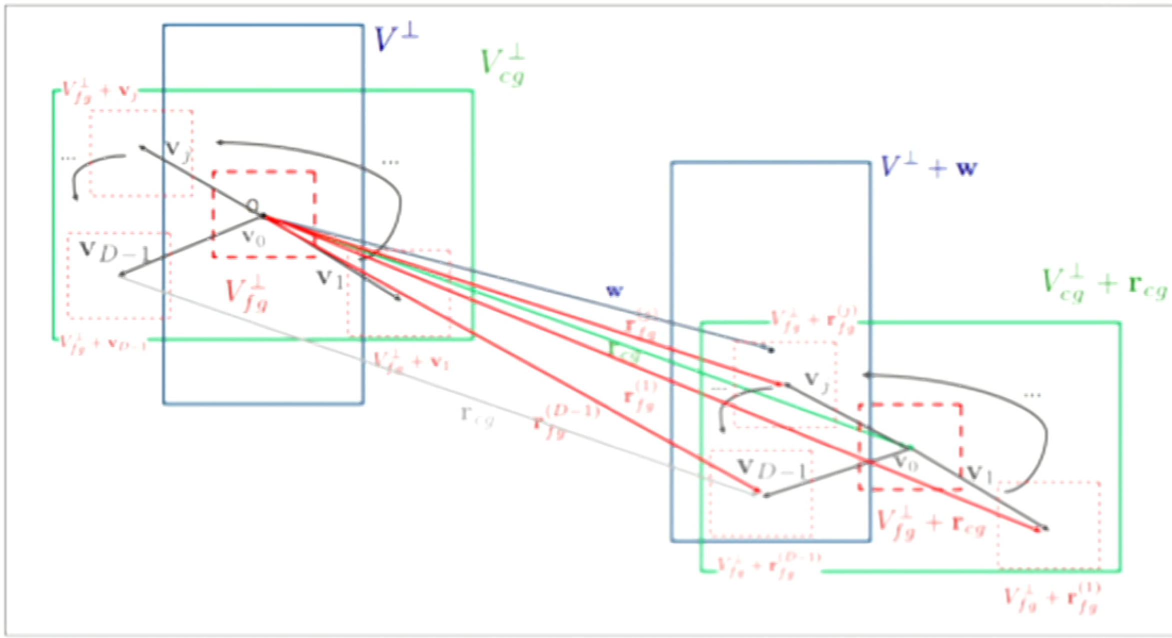
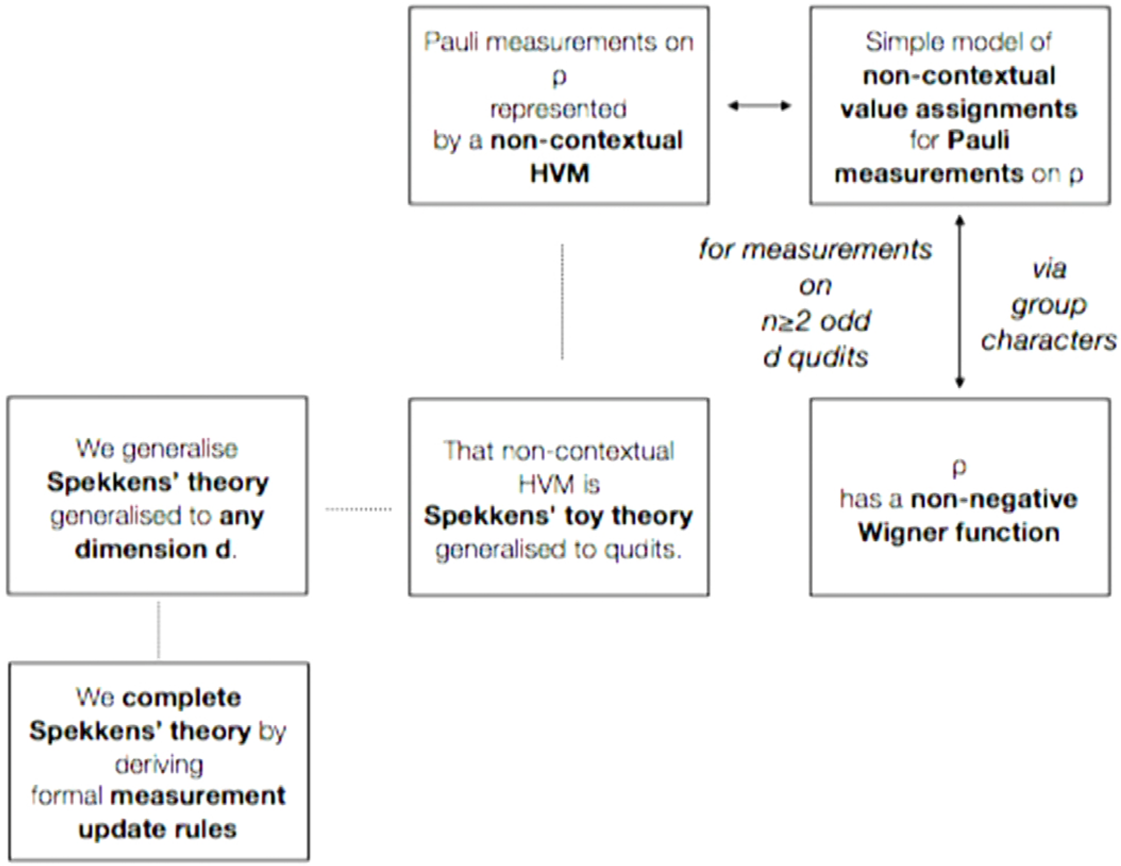


Figure 7: **Schematic representation of Coarse-graining decompositions into fine-graining observables.** The figure above schematically represents the relation between the subspaces $V^\perp, V_{cg}^\perp, V_{fg}^\perp$ and their corresponding shift vectors $\mathbf{w}, \mathbf{r}_{cg}, \mathbf{r}_{fg}^{(j)}$. The green rectangles

Equivalence of Spekkens Theory and SQM for all odd d

	Non-disturbing Measurements (Localization stage) $[\rho, \Pi] = 0$ →	Disturbing Measurements (Localization + randomization stage) $[\rho, \Pi] \neq 0$
Stabilizer Quantum Mechanics	$\rho \to \langle g_1, \ldots, g_N \rangle$ $\Pi \to \langle p_1, \ldots, p_M \rangle$ Add generators ↓ $\rho' \to \langle g_1, g_2, \ldots, g_N, p_1, p_2, \ldots, p_M \rangle$	$\rho \to \langle g_1, \ldots, g_N \rangle$ $\Pi \to \langle p_1, \ldots, p_M \rangle$ Add generators ↓ Remove g_n $\rho' \to \langle g_1, g_2, \ldots, g_{N-1}, p_1, p_2, \ldots, p_M \rangle$
Spekkens Theory	$V' = V \oplus V_\Pi$ $V'^\perp = V^\perp \cap V_\Pi^\perp$ $\mathbf{w}' = \mathbf{w} + \sum_i^n \Sigma_i'^T (\mathbf{r} - \mathbf{w}) \gamma_i$	$V' = V_{commute} \oplus V_\Pi$ $V'^\perp = (V^\perp \oplus V_{other}) \cap V_\Pi^\perp$ $\mathbf{w}' = \mathbf{w} + \sum_i^n \Sigma_i'^T (\mathbf{r} - \mathbf{w}) \gamma_i$
Wigner Functions	$W_{\rho'}(\lambda) = \frac{1}{N} W_\rho(\lambda) R_\Pi(\lambda)$	$W_{\rho'}(\lambda) = \frac{1}{N} \sum_{\mathbf{t} \in V_{other}} W_\rho(\lambda - \mathbf{t}) R_\Pi(\lambda)$

Figure 9: **Equivalence of three theories in odd dimensions in terms of measurement updating rules: Spekkens' toy model, stabilizer quantum mechanics and Gross' theory.**


Pauli measurements on ρ represented by a **non-contextual HVM**
Simple model of **non-contextual value assignments** for **Pauli measurements** on ρ
for measurements on n≥2 odd d qudits
via group characters
We generalise **Spekkens' theory** generalised to **any dimension d**.
That non-contextual HVM is **Spekkens' toy theory** generalised to qudits.
ρ has a **non-negative Wigner function**
We **complete Spekkens' theory** by deriving formal **measurement update rules**

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n=1

What about the $n = 1$ case? Didn't Howard et al cover that?

No! n is the number of qudits being measured. Howard et al require a 2 qudit witness for their construction.

Our results do cover one-qudit states, but a second qudit needs to be present.

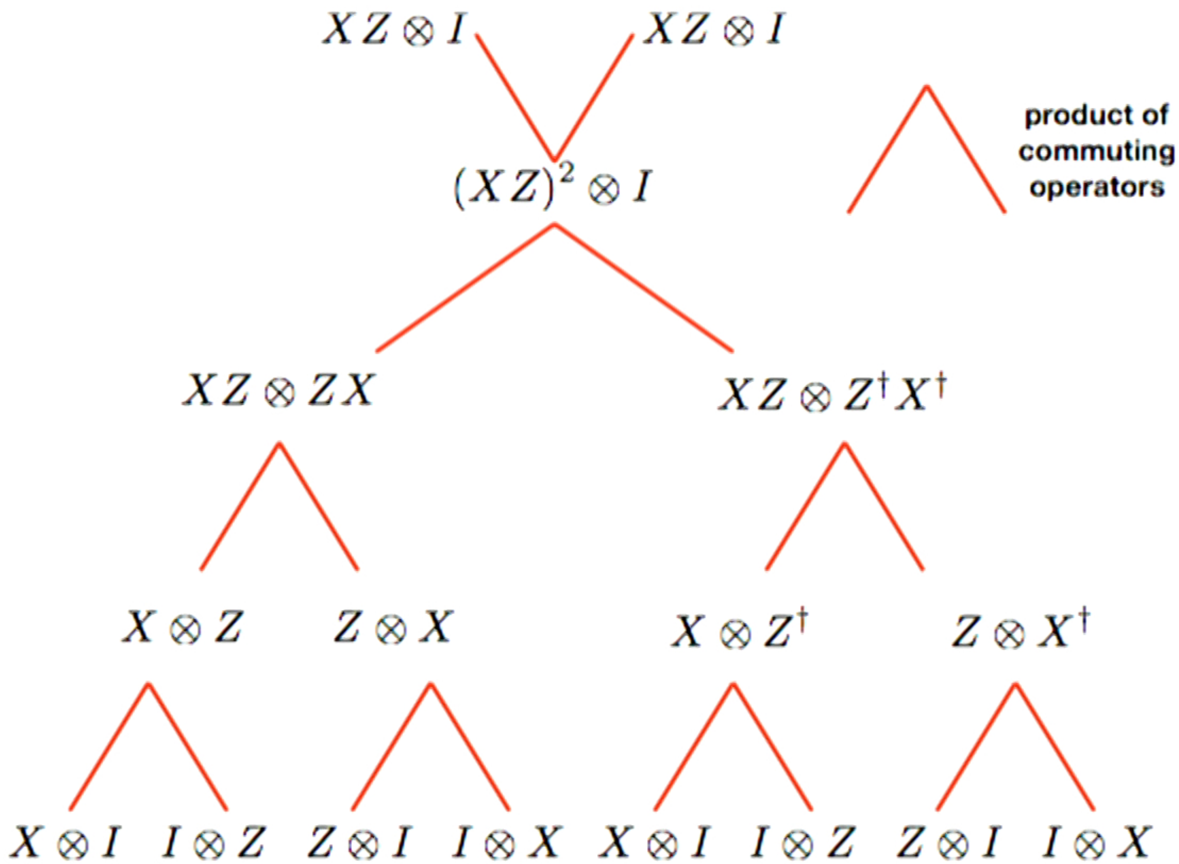
$$\rho \otimes 1/d$$

In state injection, there is **always more than one** qudit!

n=1

What can we say about this case?

- Spekkens model is an example of an NCVA model satisfying the character property.
- But. The proof of the character lemma **fails** in $n = 1$.


$XZ \otimes I$
$XZ \otimes I$
product of commuting operators
$(XZ)^2 \otimes I$
$XZ \otimes ZX$
$XZ \otimes Z^\dagger X^\dagger$
$X \otimes Z$
$Z \otimes X$
$X \otimes Z^\dagger$
$Z \otimes X^\dagger$
$X \otimes I$
$I \otimes Z$
$Z \otimes I$
$I \otimes X$
$X \otimes I$
$I \otimes Z$
$Z \otimes I$
$I \otimes X$

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n=1

Could the character lemma still be true?

No! It is easy to construct counter-examples.

n=1

Does Wigner negativity imply non-contextuality in $n = 1$?

No! It is (slightly trickier but possible) to construct counter-examples.

Outlook

- What do we learn from the anomoly of the $n = 1$ case? A **warning** for other studies of contextuality? An **opportunity** to develop interesting new models?
- Can we derive CWS witnesses from our character proof? (Peres-Mermin "web"?).
- Can we use the Wigner function to link to **Spekkens non-contextuality**?
- Now we have a full hidden variable model for odd d SQM what can we **use** it for?
- These results fail in even dimensions. But how far can we go with such analyses for **qubits**?

[Juan will tell you]

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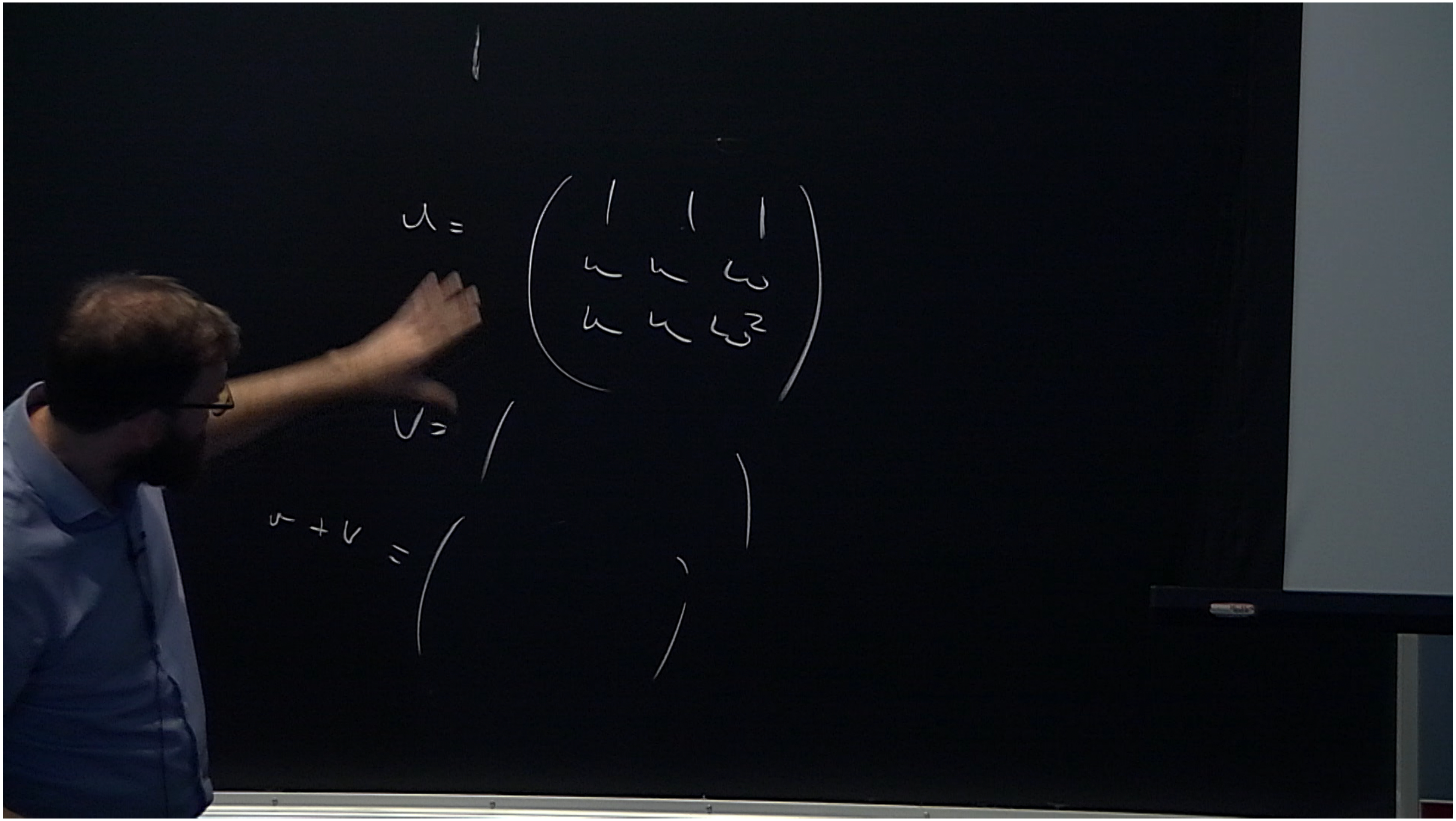
Thank you to my collaborators!

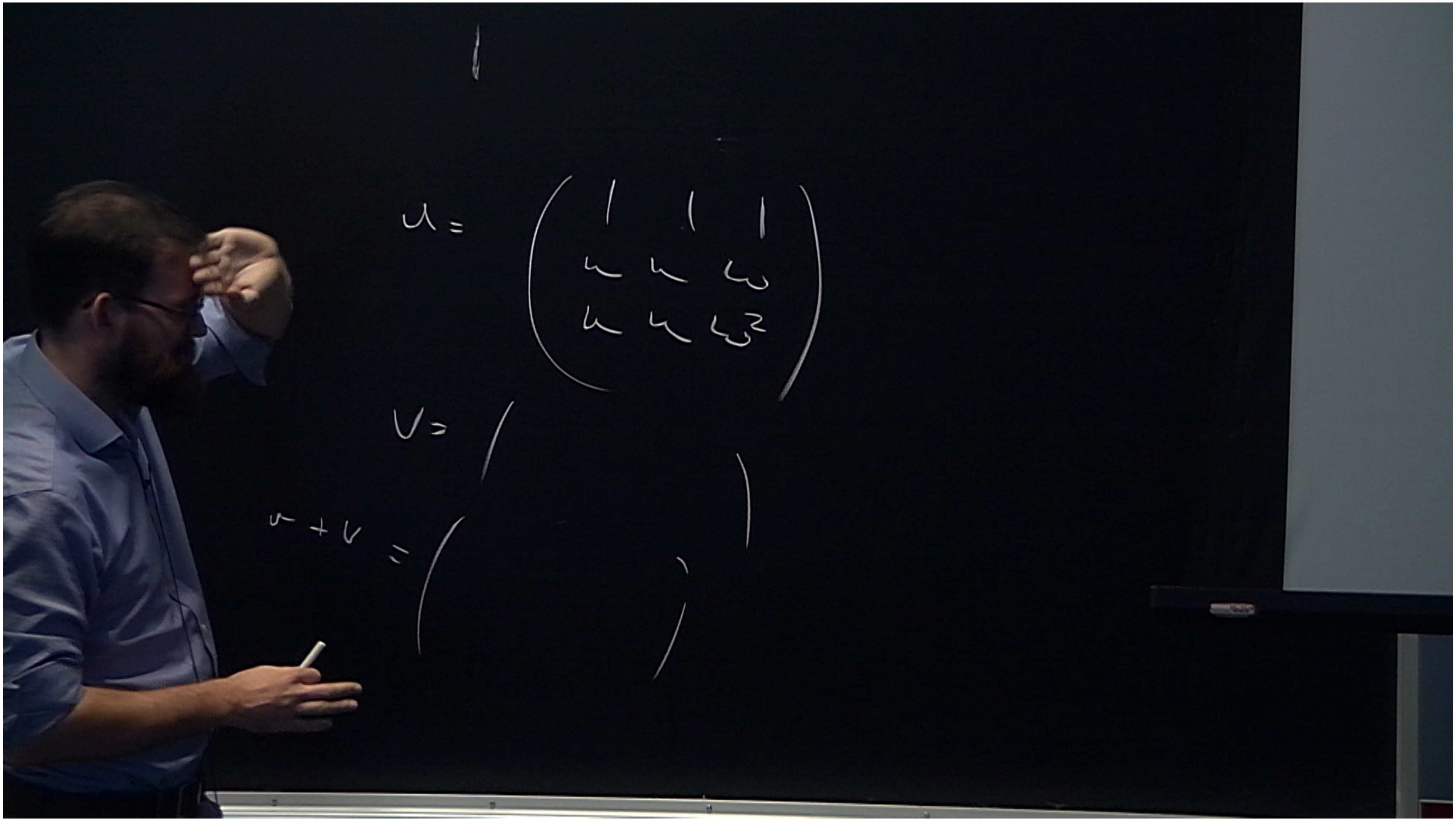
Nicolas Delfosse, Cihan Okay, Juan Bermejo-Vega, Robert Raussendorf and Lorenzo Catani

References:

arxiv.org/1701.07801

arxiv.org/1610.07093


u =
v =
u + v =


u =
V =
u + v =


PP-2R3