Title: Revisiting quantum incompatibility

Date: Jul 26, 2017 02:00 PM

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Abstract:

Contextuality: Conceptual Issues, Operational Signatures, and Applications
Perimeter Institute, July 24-28 2017

Revisiting Quantum Incompatibility

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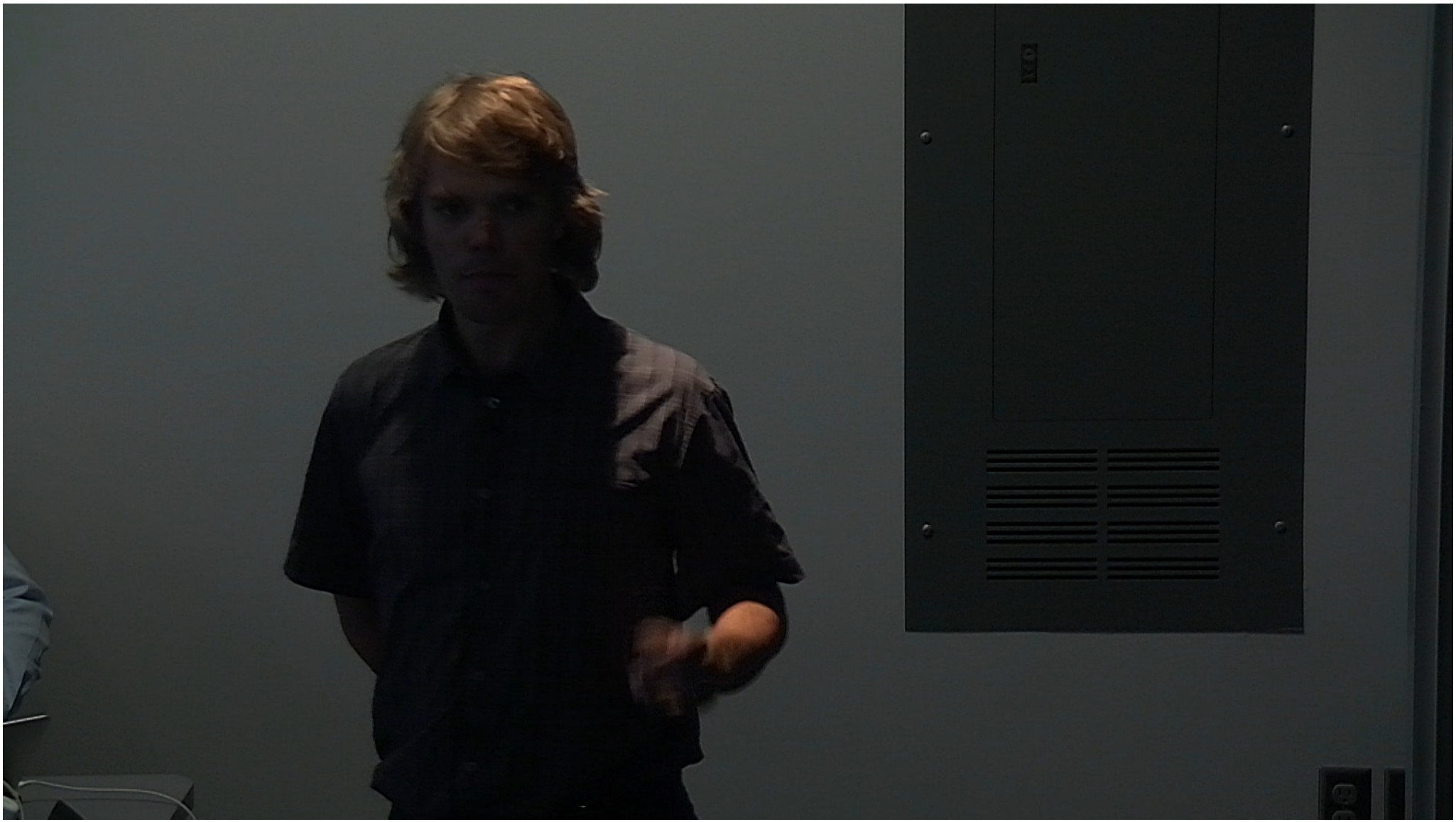
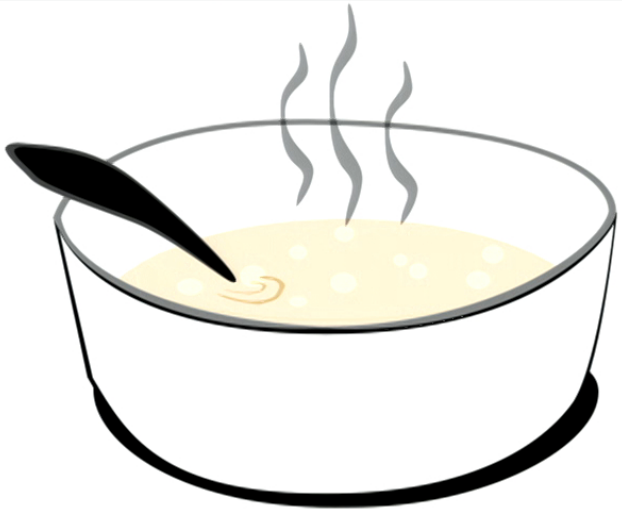
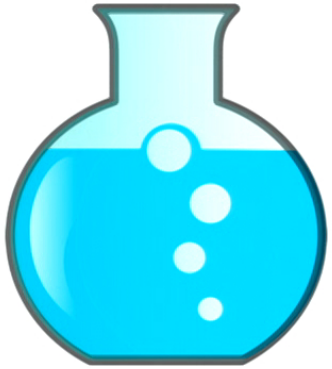
University of Turku



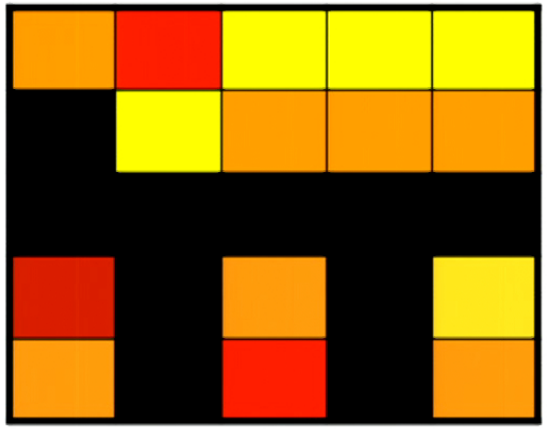


quantum
contextuality



incompatibility

nondisturbance



commutativity

compatibility

Incompatibility

- is a unifying concept
- has many manifestations
- covers several features of quantum theory

this talk

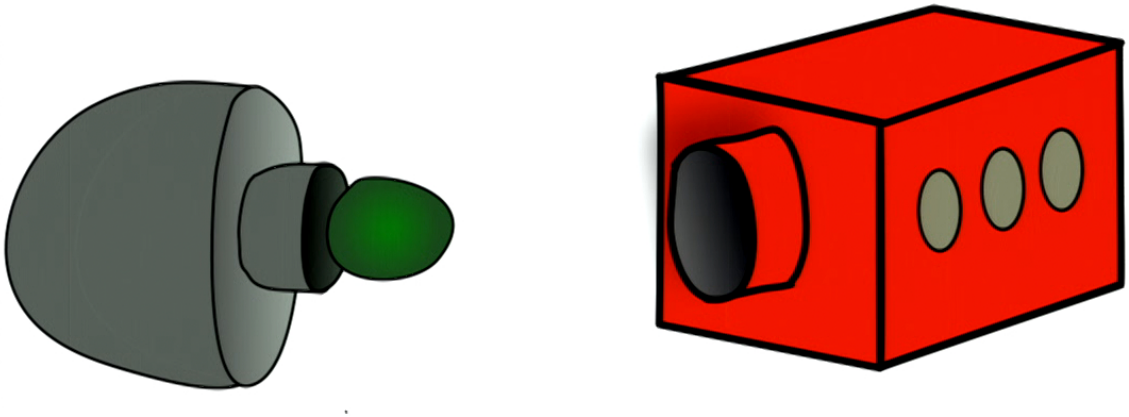
1) simultaneous measurement of quantum observables

2) incompatibility of quantum channels

3) incompatibility of a quantum channel and observable

4) special nature of quantum incompatibility?

quantum observable



quantum observable

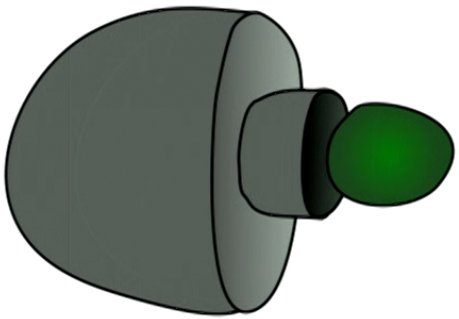
ϱ

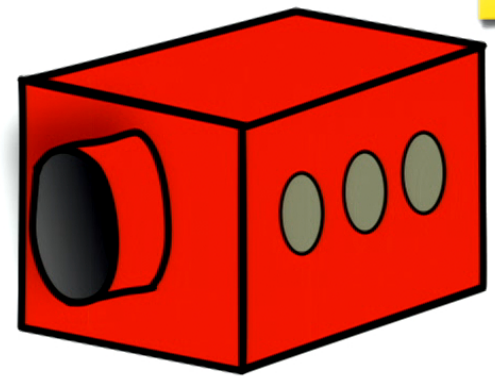
A

POVM

$$A(x) \geq 0$$

$$\sum_x A(x) = I$$





quantum observable

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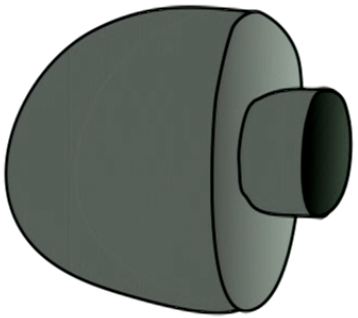
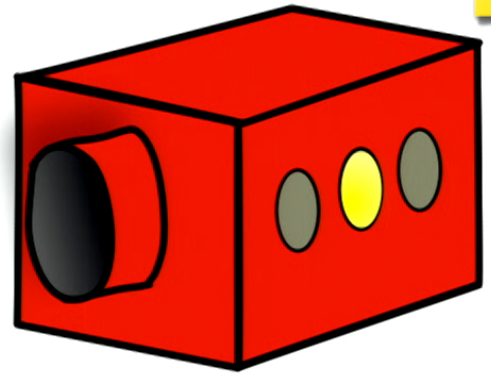
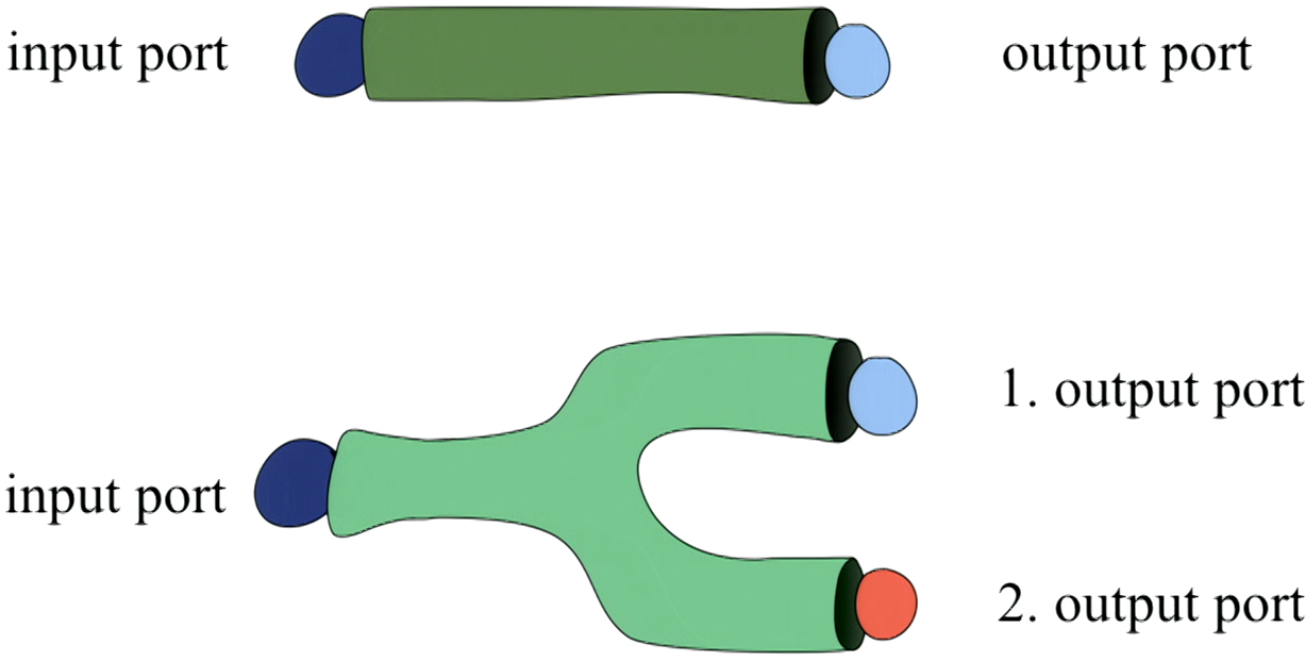


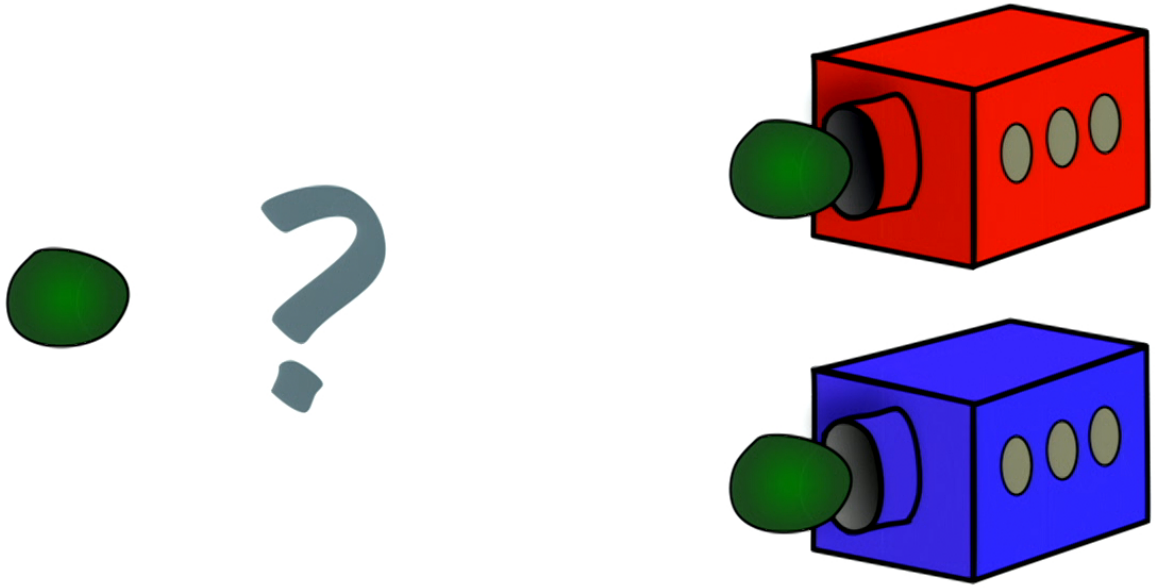


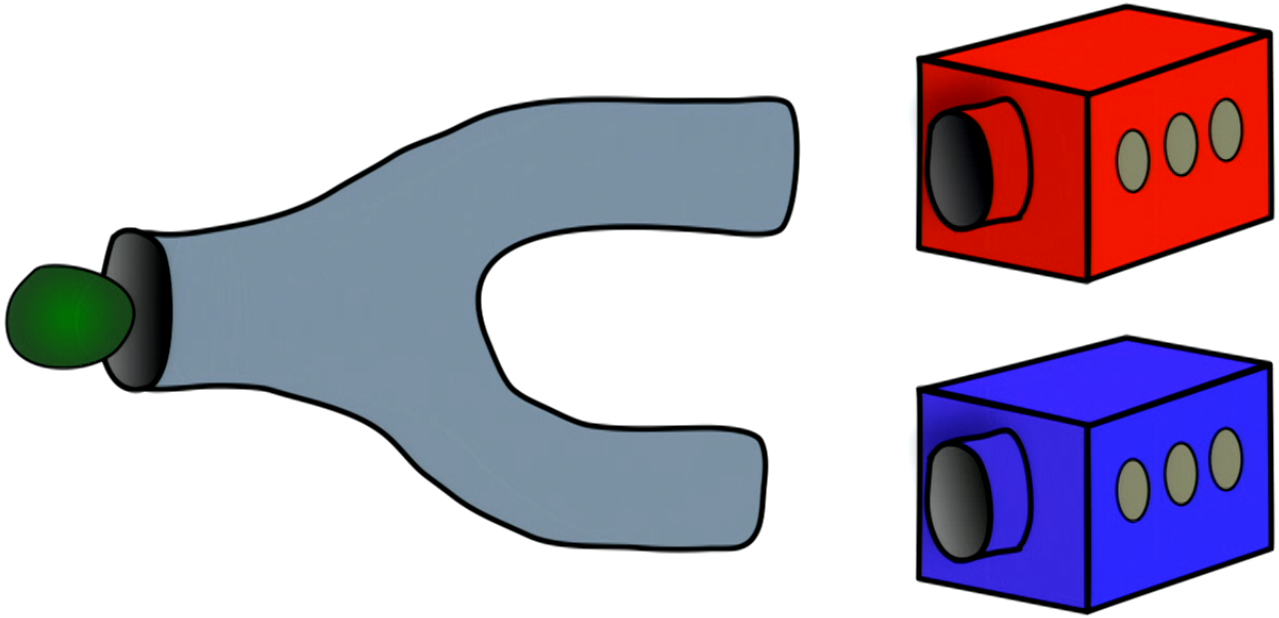
$$\mathrm{tr}[\varrho A(x)]$$

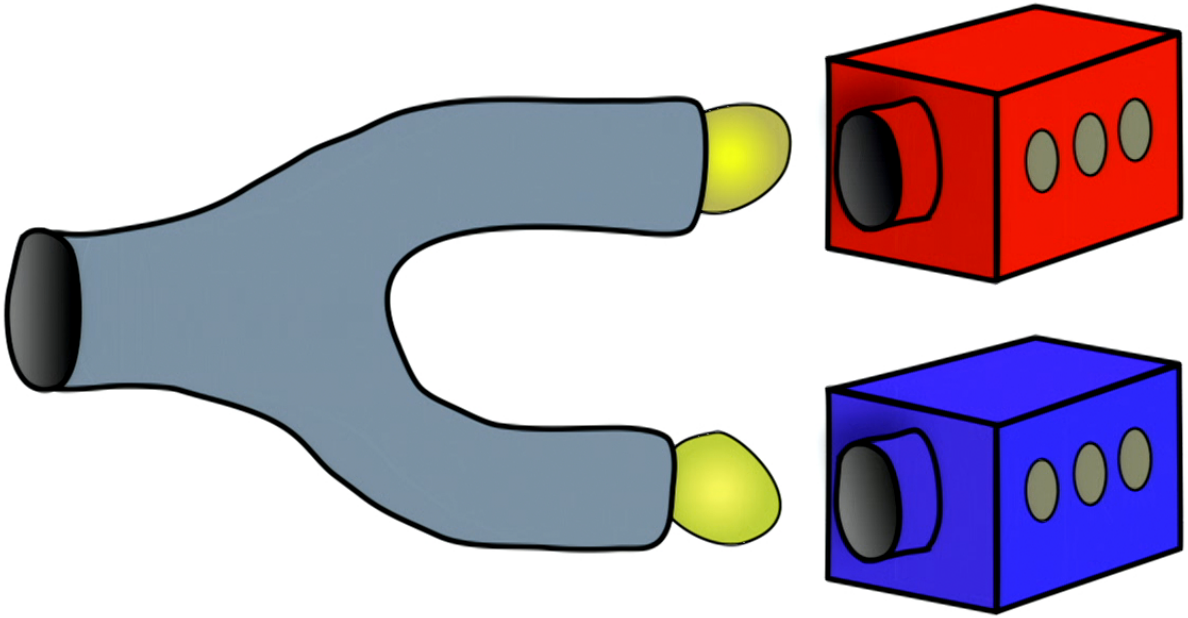
quantum channel



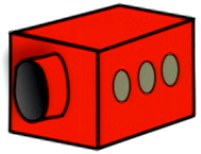
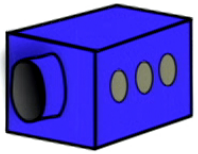
measuring two observables simultaneously

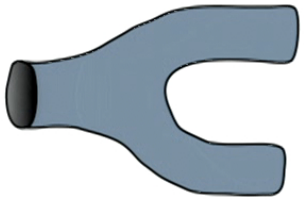


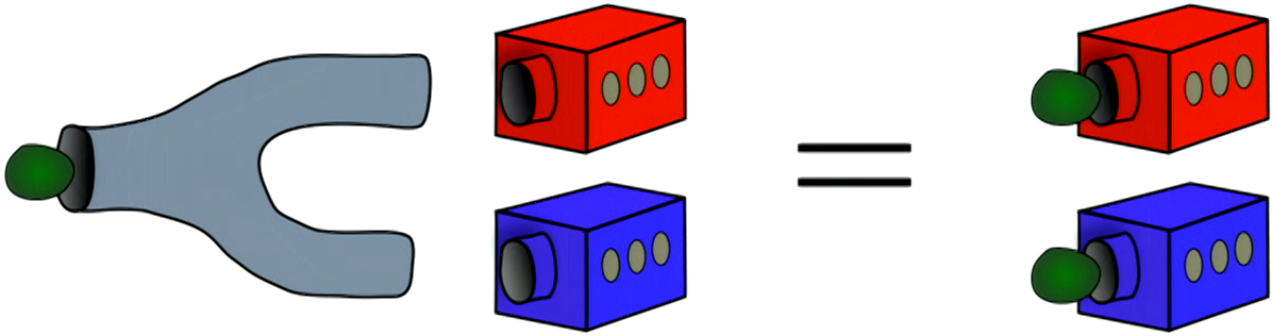




Definition

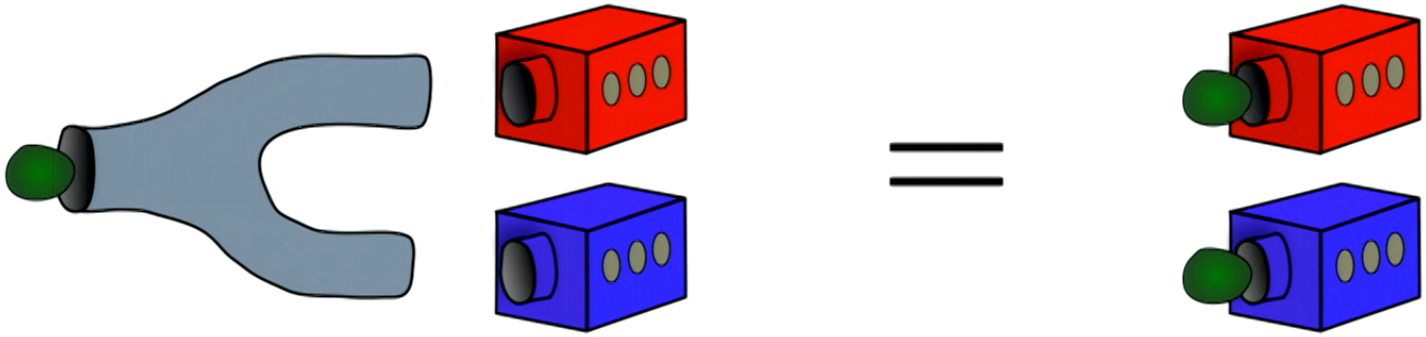
 and  are ***broadcastable***

if there exists  such that

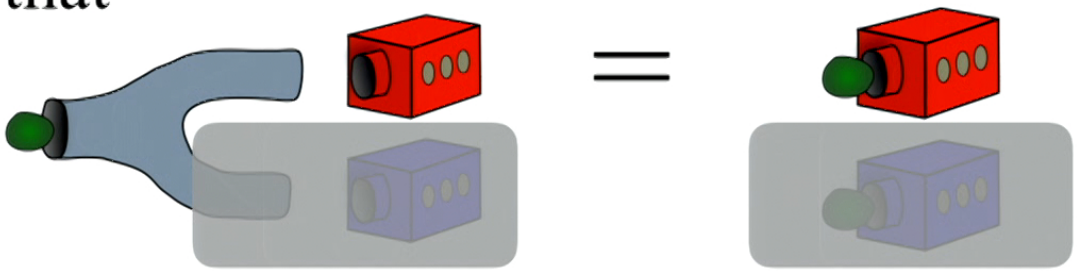


for all .

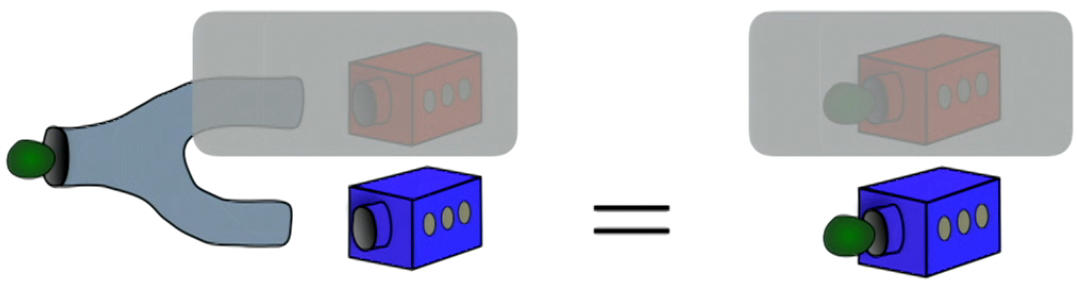
Remark



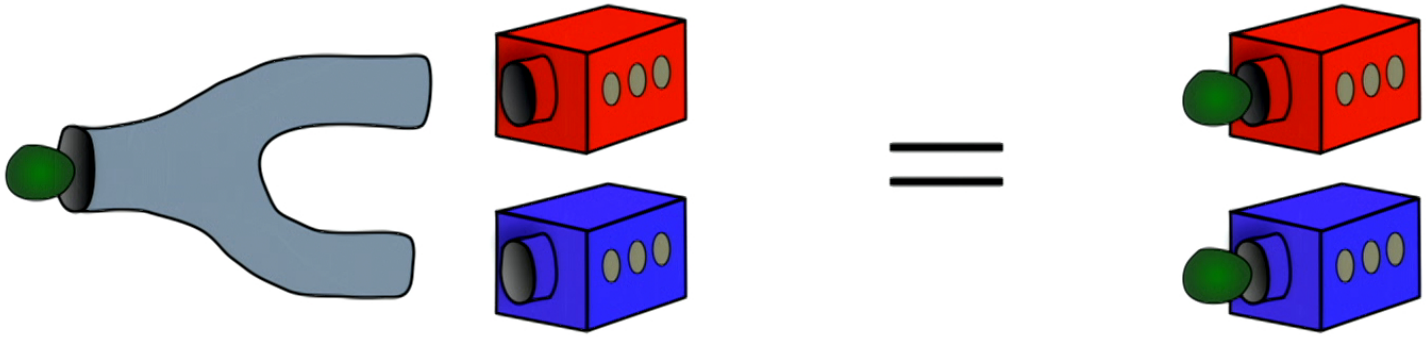
means that



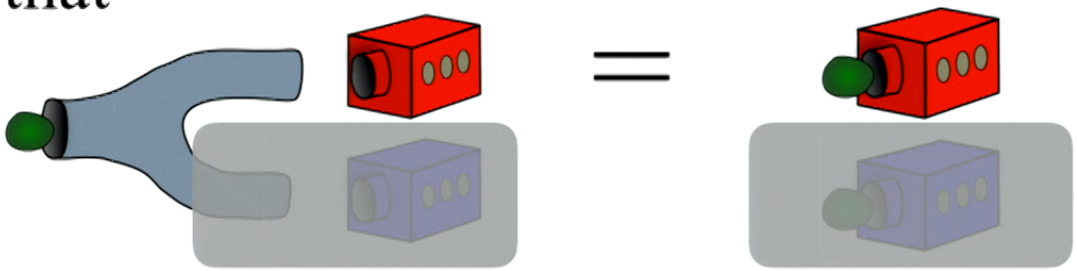
and



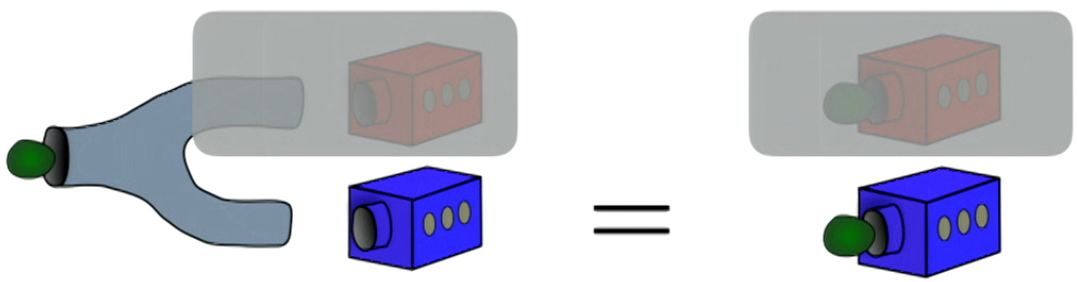
Remark



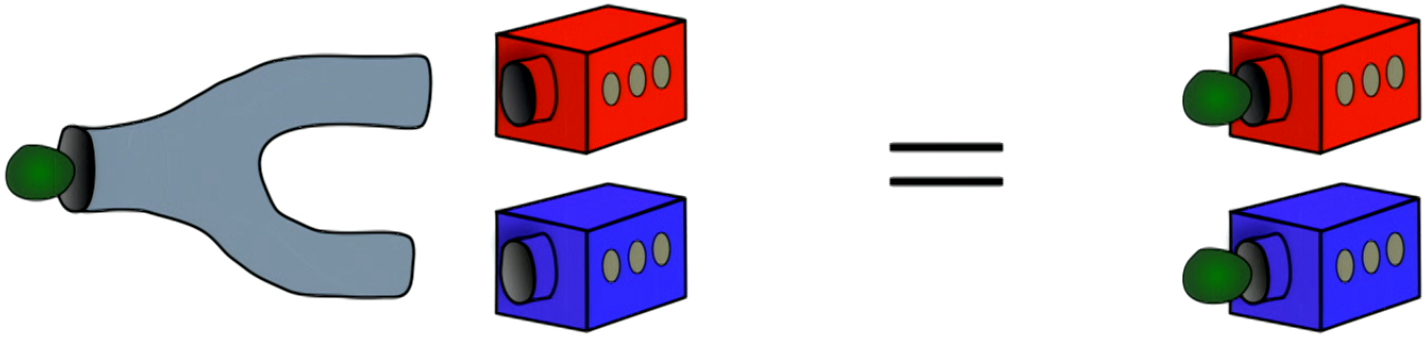
means that



and



Remark



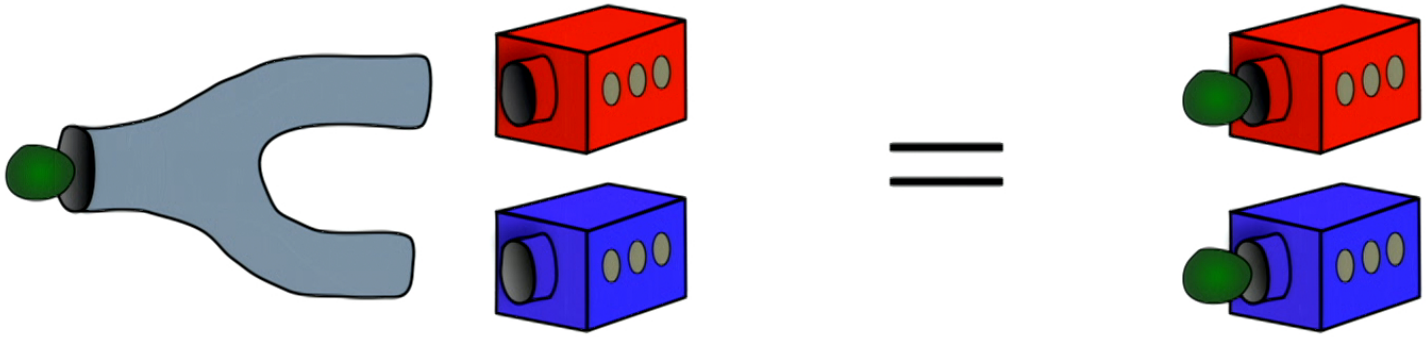
means that

$$\mathrm{tr}[\Lambda(\varrho)A(x)\otimes I] = \mathrm{tr}[\varrho A(x)]$$

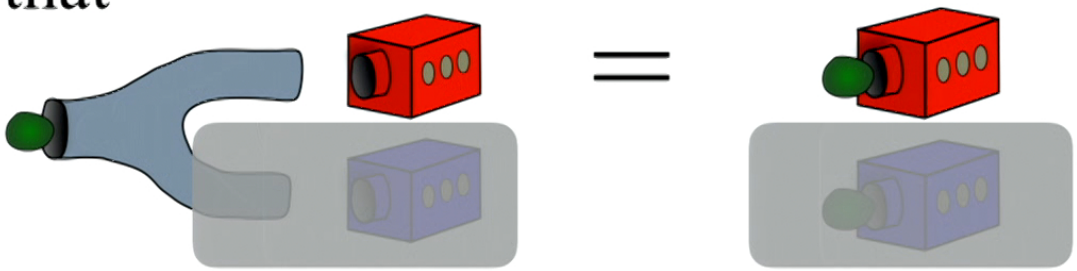
and

$$\mathrm{tr}[\Lambda(\varrho)I\otimes B(y)] = \mathrm{tr}[\varrho B(y)]$$

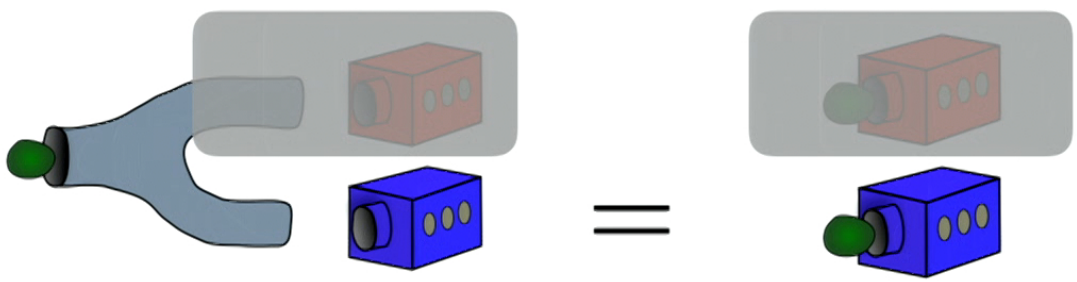
Remark



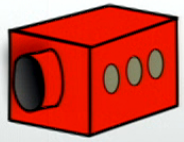
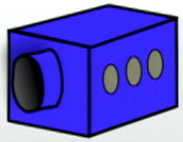
means that



and

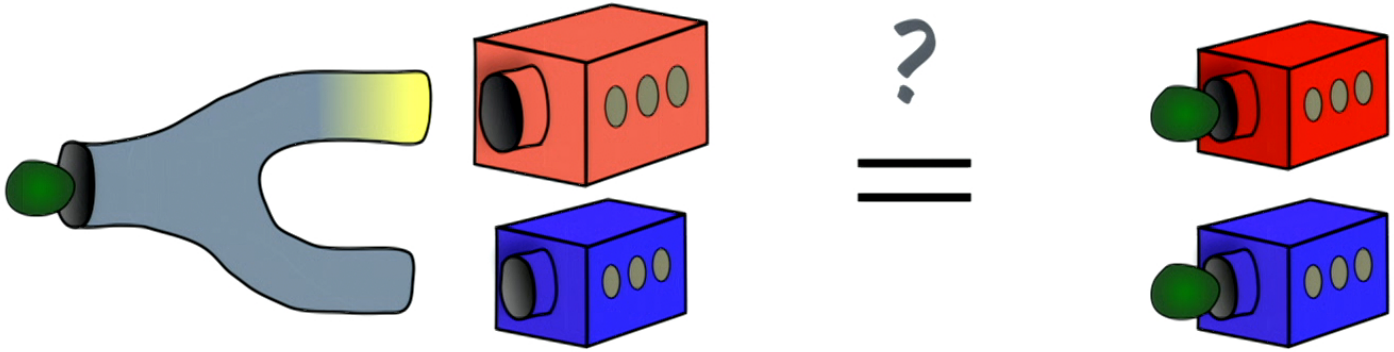


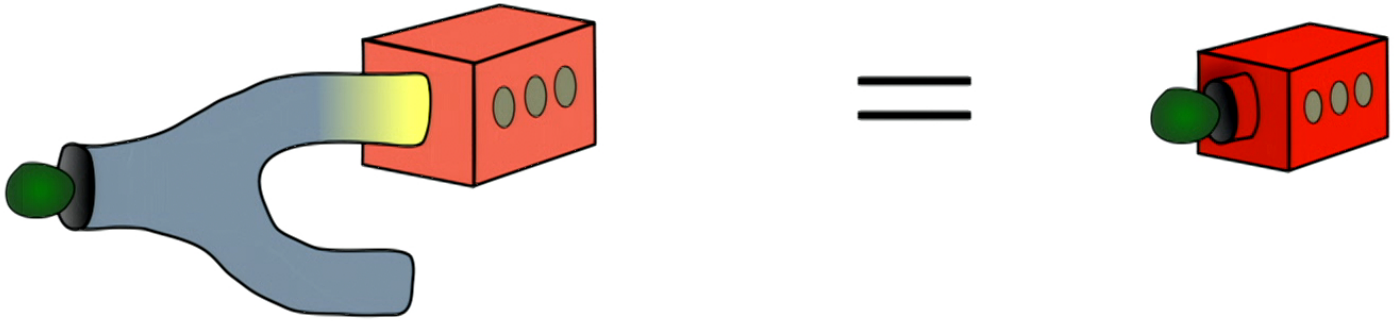
Example:

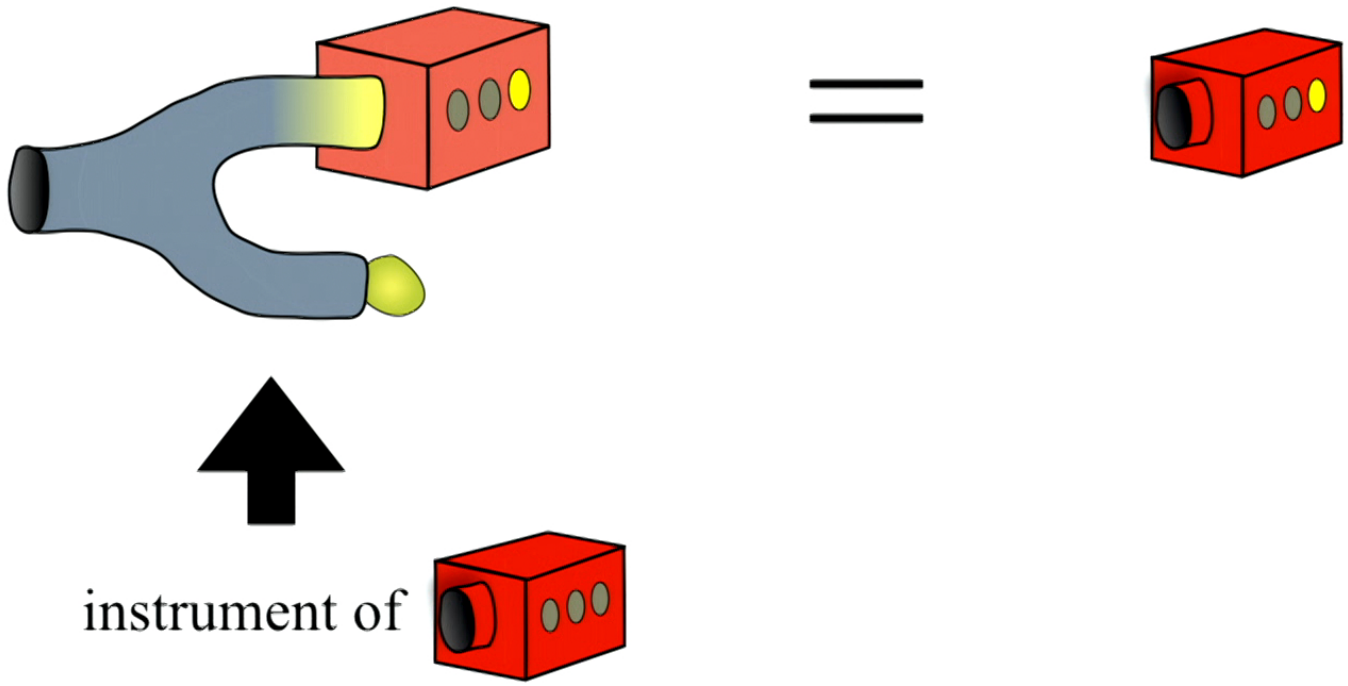
 and  are broadcastable

A B

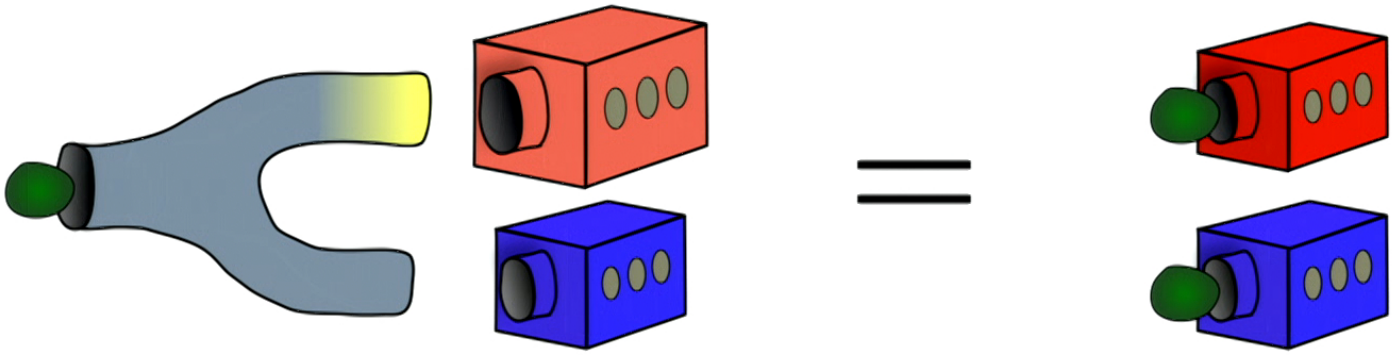
- Fix an orthonormal basis $\{\varphi_j\}$
- Use channel $\Lambda(\varrho) = \sum_{j=1}^{d} \langle \varphi_j | \varrho \varphi_j \rangle |\varphi_j \otimes \varphi_j\rangle\langle \varphi_j \otimes \varphi_j|$
- A and B are broadcastable if all operators $A(x)$,$B(y)$ are diagonal in the fixed basis

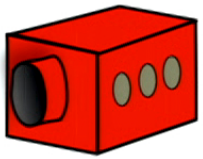


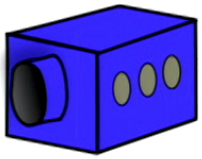



instrument of

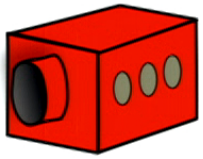
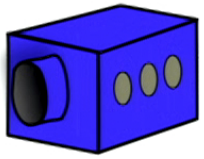
Therefore,



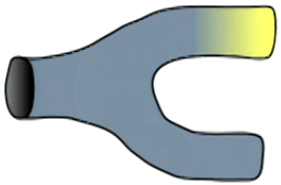
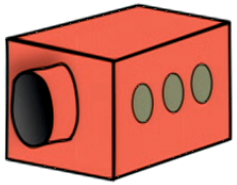
implies that  is measured without

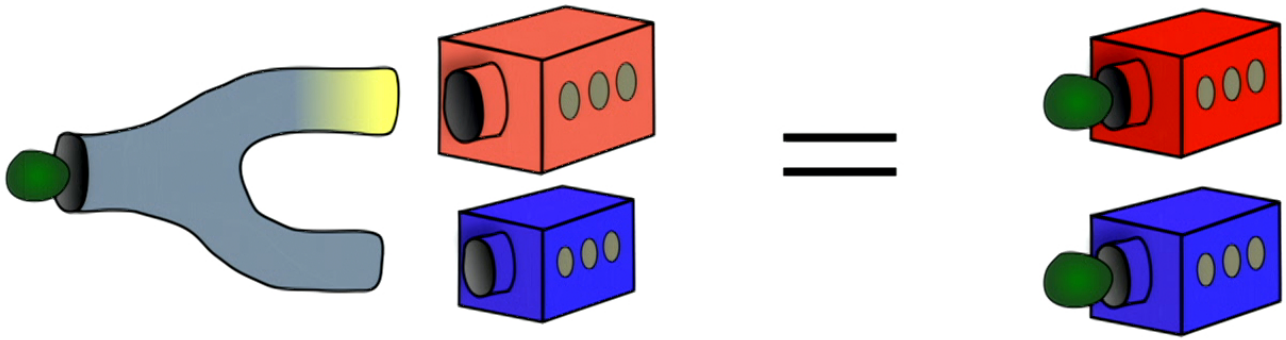
disturbing 

Proposition

 can be measured without disturbing 

if and only if

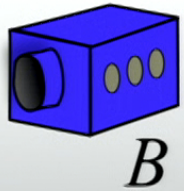
there exist  and  such that



for all

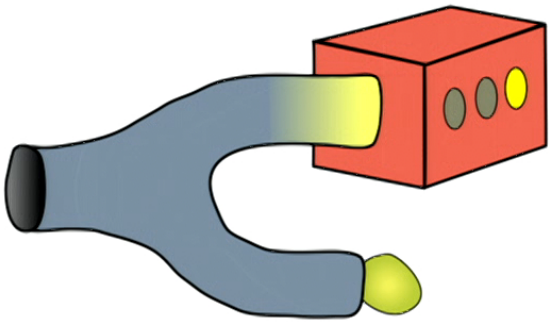
Example:

can be measured without disturbing



when

- $[A(x),B(y)] = 0$ for all outcomes x,y

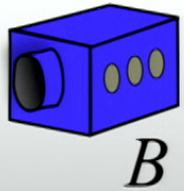


Lüders instrument of A

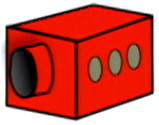
$$\varrho \mapsto \sqrt{A(x)}\varrho\sqrt{A(x)}$$

Example:

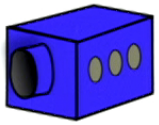
cannot be measured without disturbing



when

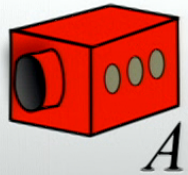
-  is any nontrivial observable

and

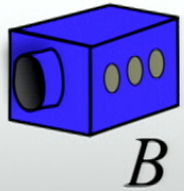
-  is an informationally complete observable

Why?

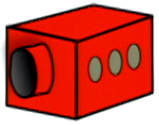
Example:



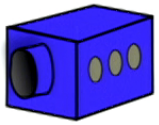
cannot be measured without disturbing



when

-  is any nontrivial observable

and

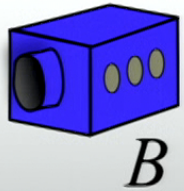
-  is an informationally complete observable

Why?

No information without disturbance

Example:

cannot be measured without disturbing



when

- $[A(x),B(y)] \neq 0$ for some x,y

- $dim\ [span\{B(1),\ldots,B(n)\}] \geq (d\text{-}1)^2 + 1$

(B is informationally complete if $dim\ [span\{B(1),\ldots,B(n)\}] = d^2$)

qubit observables

Proposition

For two qubit observables A and B, the following are equivalent:

(1) A and B are broadcastable

(2) A can be measured without disturbing B

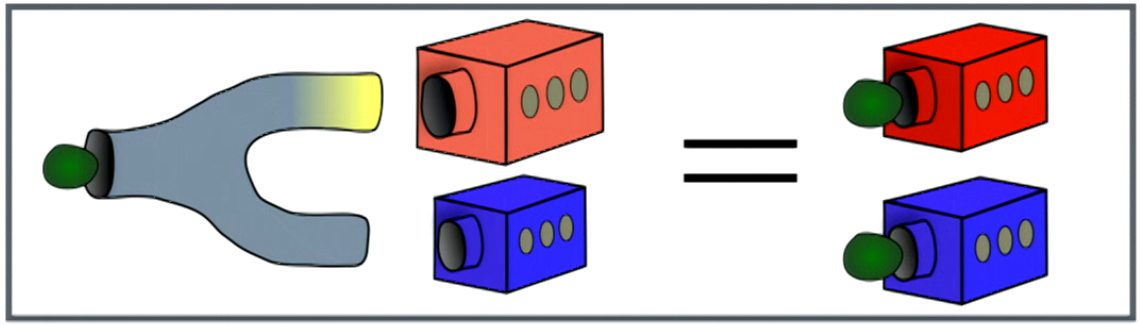
(3) B can be measured without disturbing A

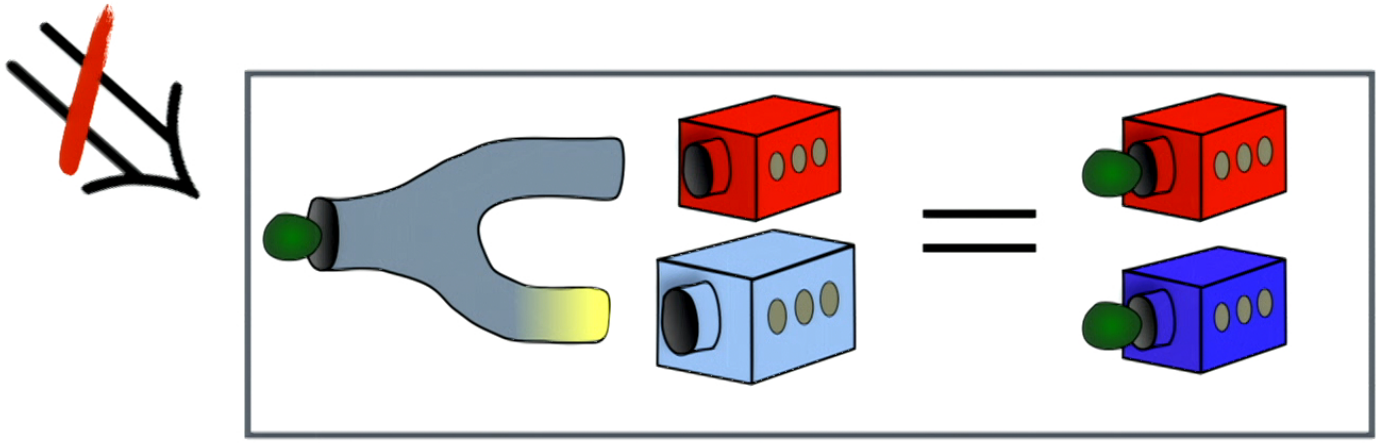
(4) $[A(x),B(y)] = 0$ for all outcomes x,y

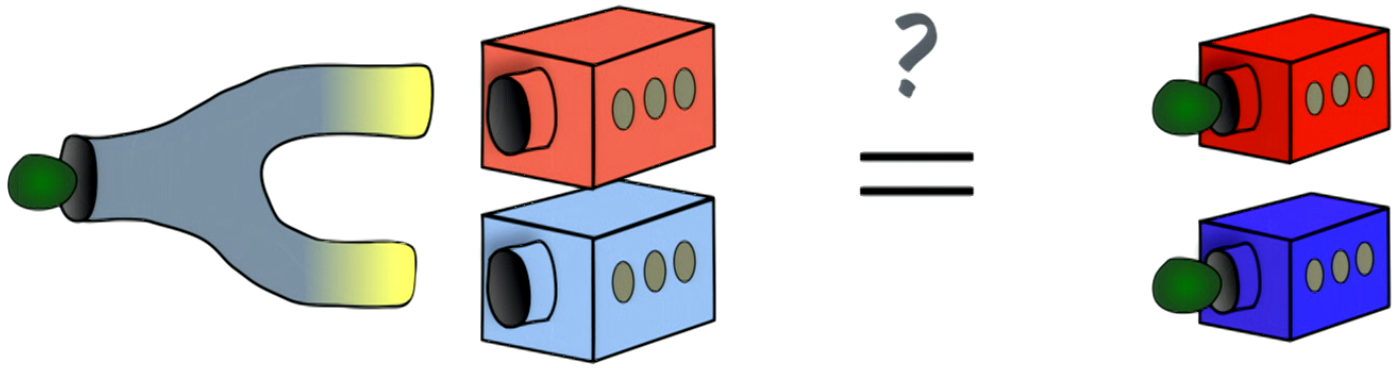
Remarks

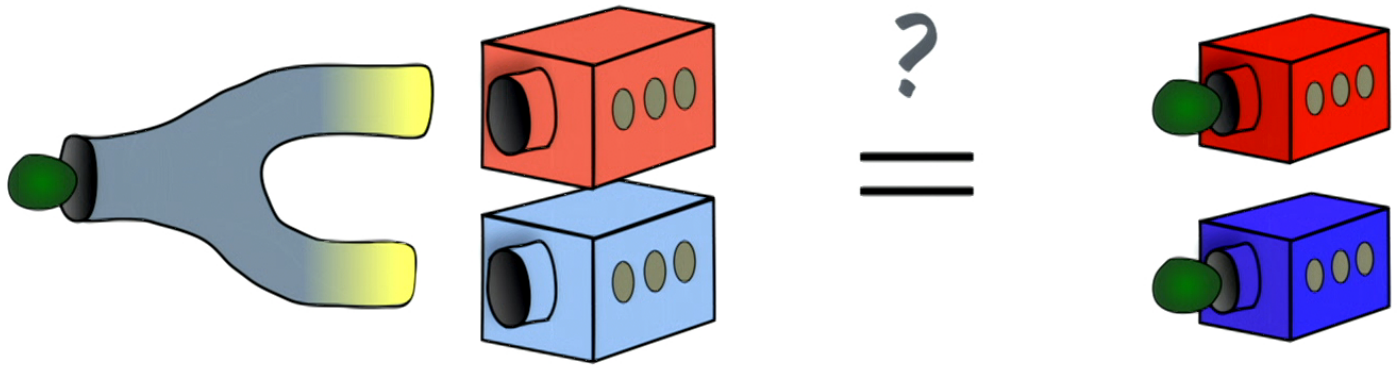
For $d \geq 3$, the nondisturbance relation is

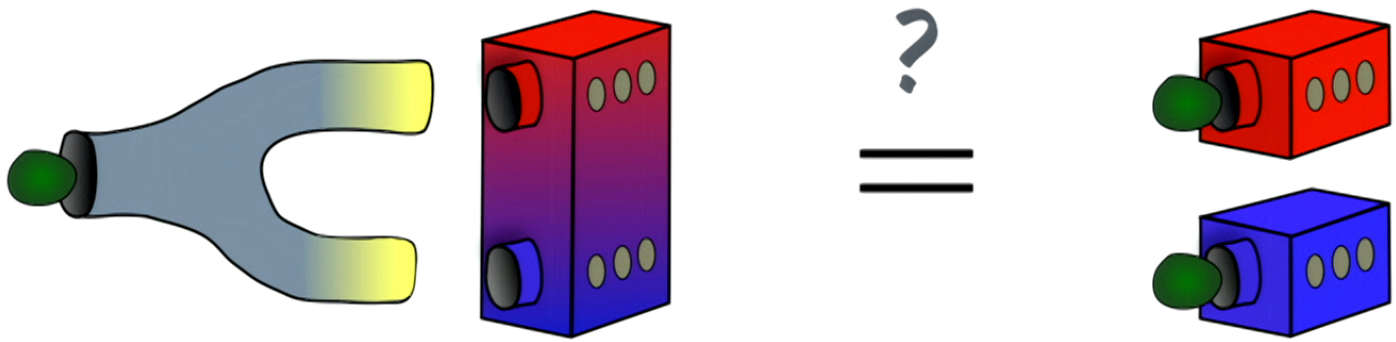
- not equivalent to commutativity
- not symmetric

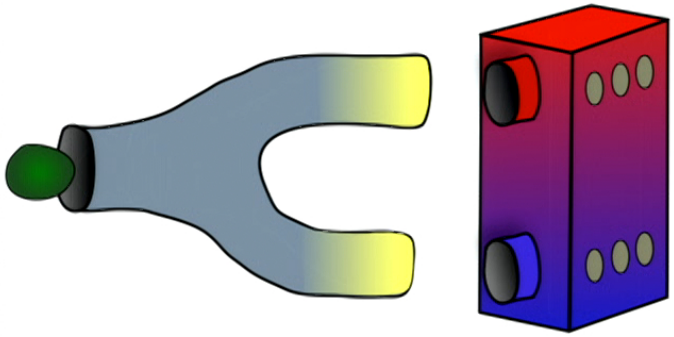


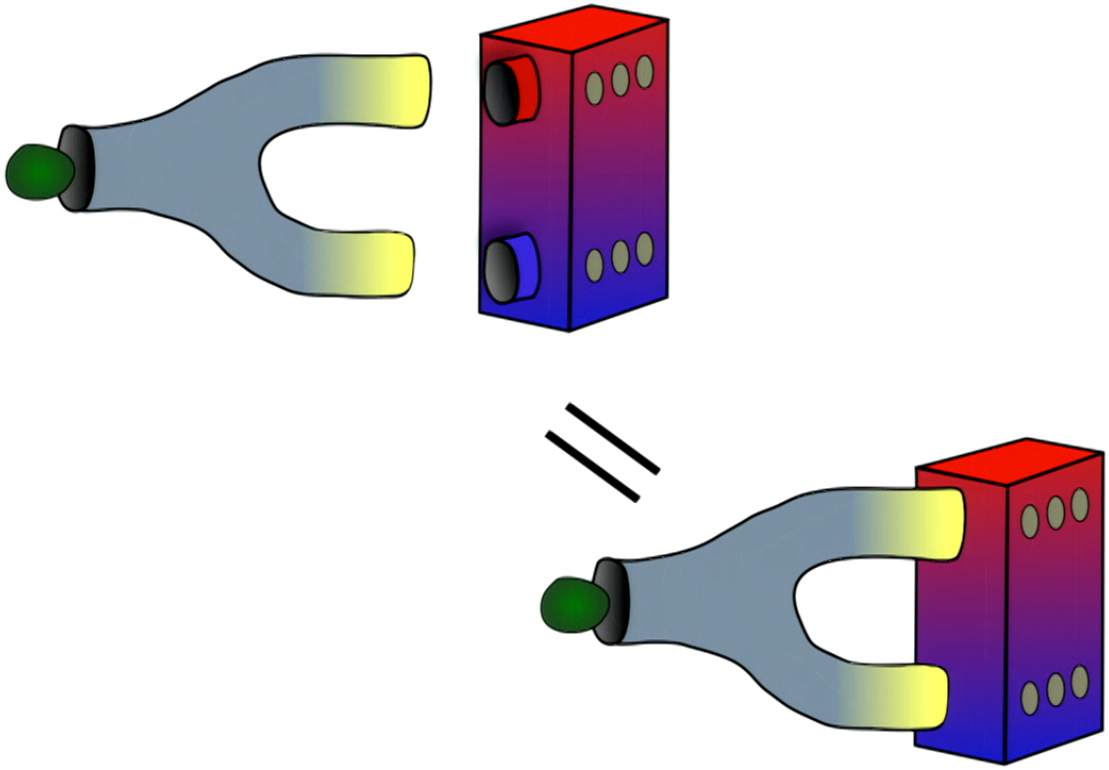




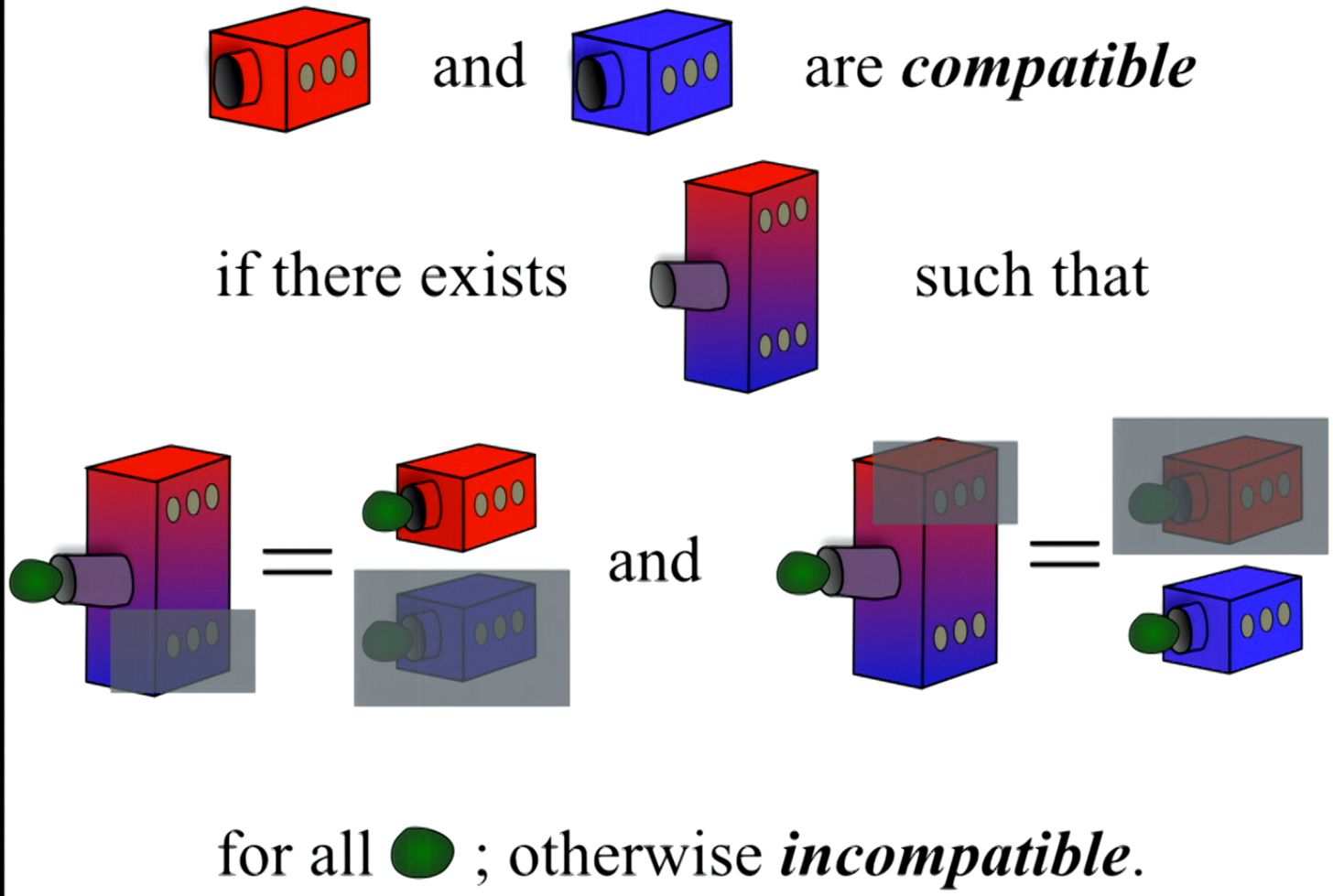








Definition


and
are *compatible*
if there exists
such that
=
and
=
for all ; otherwise *incompatible*.

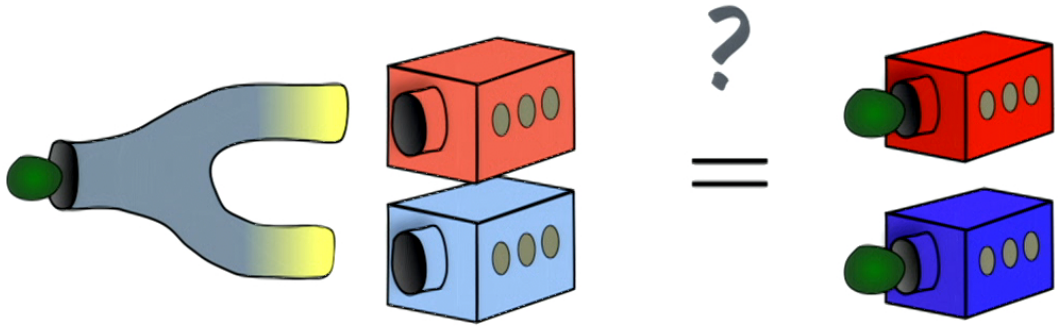
Definition

A and ***B*** are ***compatible***

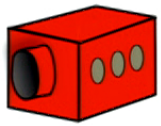
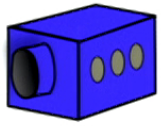
if there exists ***G*** such that

$$\sum_y G(x,y) = A(x)$$

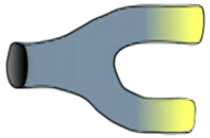
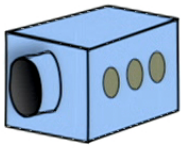
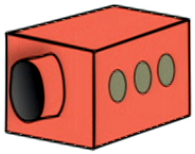
$$\sum_x G(x,y) = B(y)$$

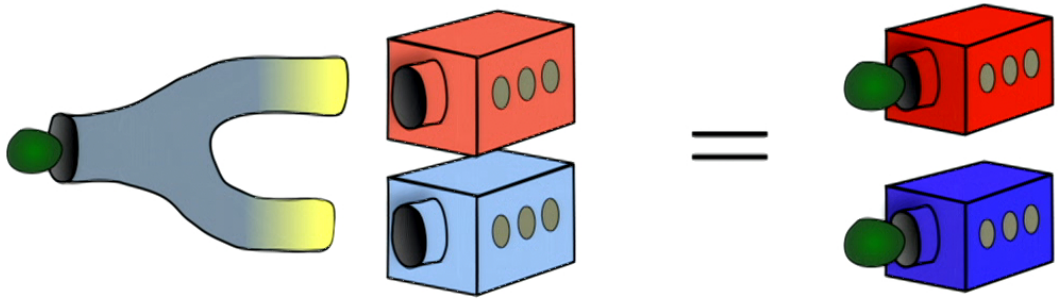


Proposition

 and  are compatible

if and ***only if***

there exist  ,  ,  such that



for all .

Proof

- Assume that A and B are compatible, hence there exists G such that

$$A(x) = \sum_y G(x,y)\,, \quad B(y) = \sum_x G(x,y)$$

- Fix two sets of orthonormal unit vectors:

$$\{\varphi_x\} \quad \text{and} \quad \{\eta_y\}$$

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- Observables:

$$A'(x) = |\varphi_x\rangle\langle\varphi_x| \quad \text{and} \quad B'(y) = |\eta_y\rangle\langle\eta_y|$$

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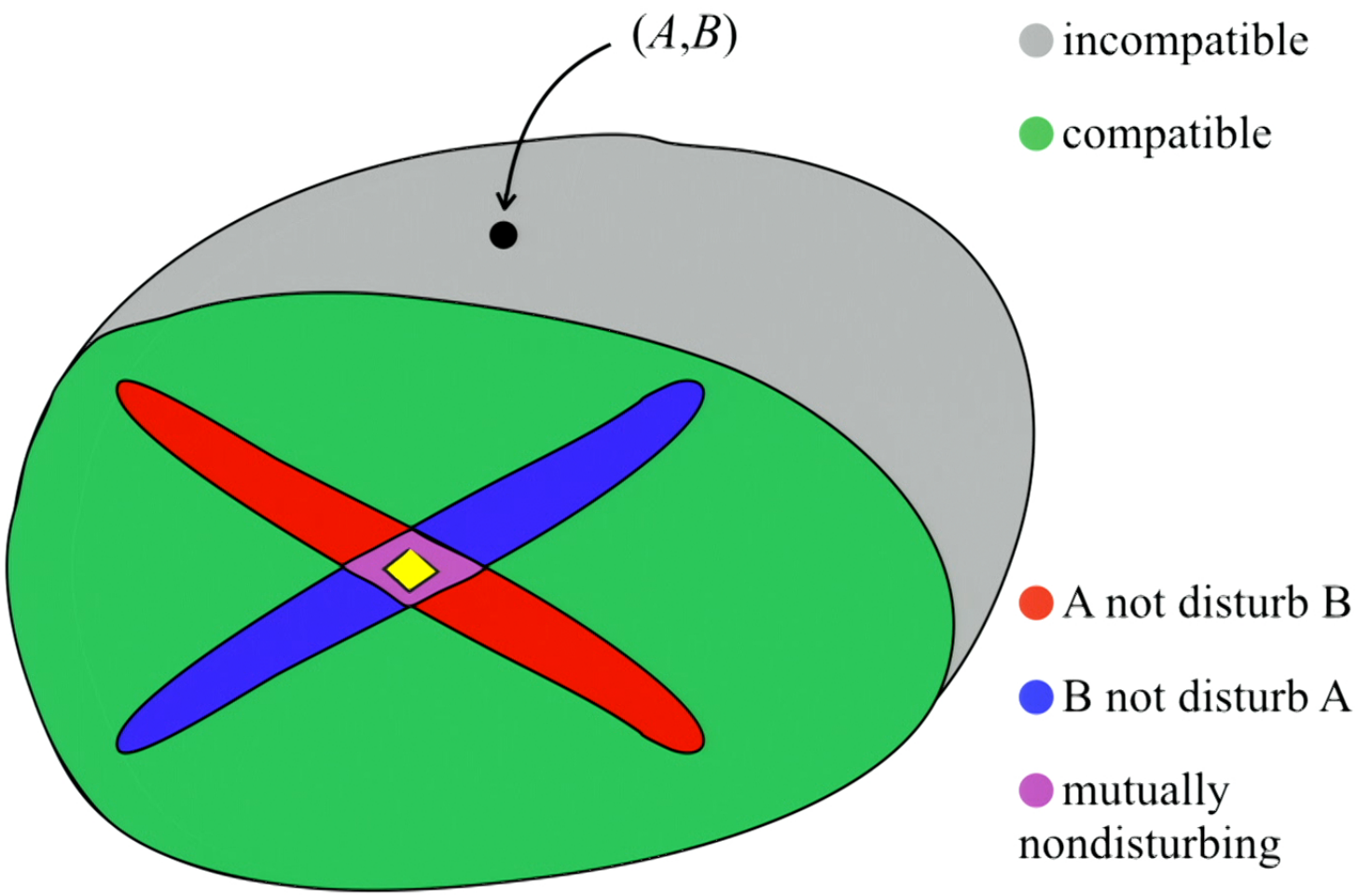
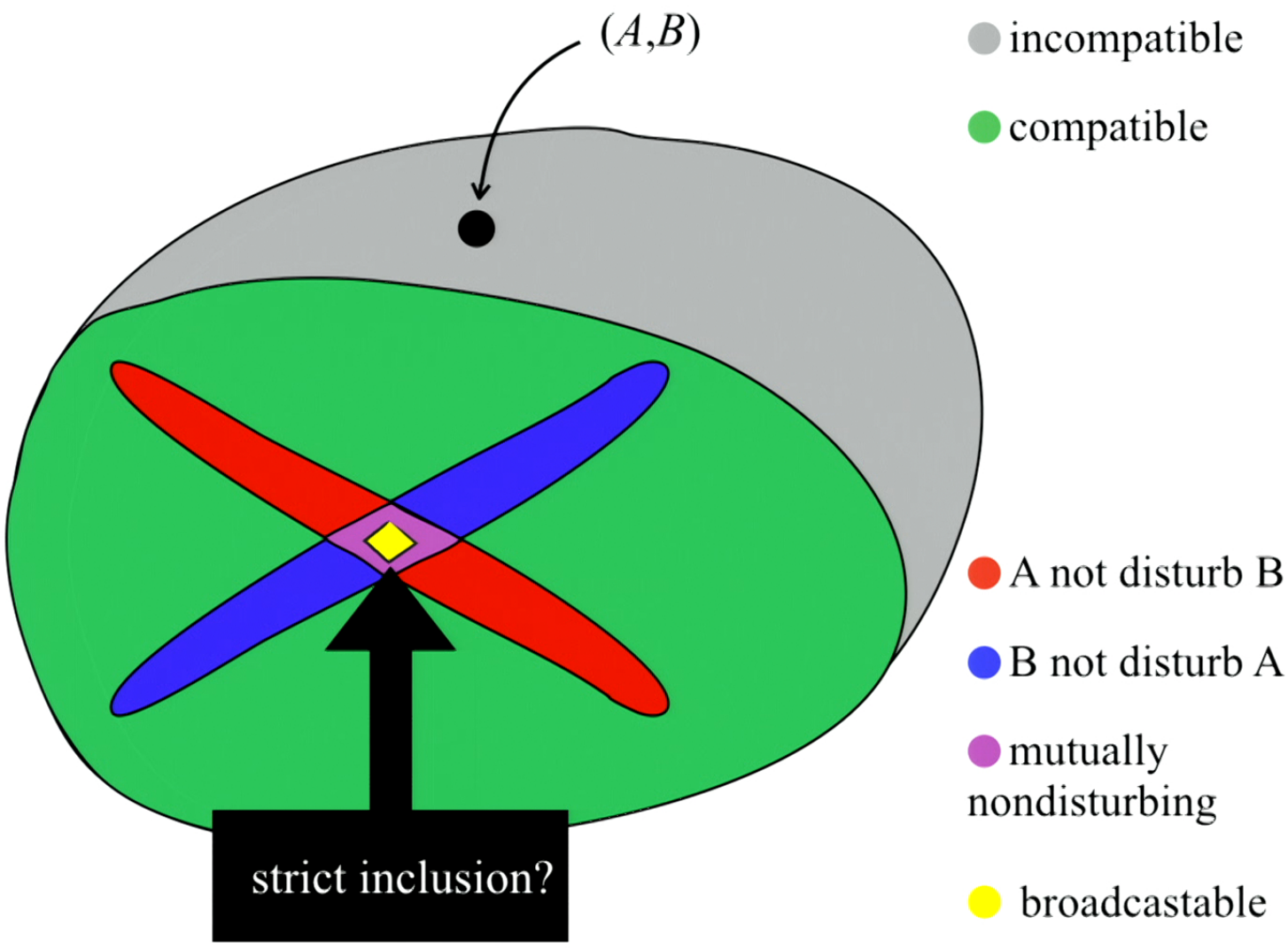
- Observables:

$$A'(x) = |\varphi_x\rangle\langle\varphi_x| \quad \text{and} \quad B'(y) = |\eta_y\rangle\langle\eta_y|$$

- Channel:

$$\Lambda(\varrho) = \sum_{x,y} \mathrm{tr}[\varrho G(x,y)]\ |\varphi_x\rangle\langle\varphi_x| \otimes |\eta_y\rangle\langle\eta_y|$$


(A,B)
incompatible
compatible
A not disturb B
B not disturb A
mutually nondisturbing
broadcastable


(A,B)
incompatible
compatible
A not disturb B
B not disturb A
mutually nondisturbing
broadcastable
strict inclusion?

simple sufficient condition for compatibility

- If A and B are compatible, do they have a joint observable G of some specific "canonical" form?

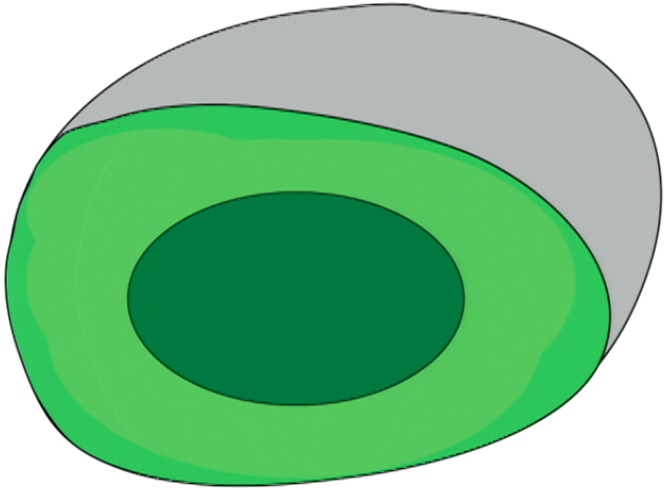
simple sufficient condition for compatibility

$$G(x,y) = \tfrac{1}{2}\{A(x), B(y)\}$$
$$= \tfrac{1}{2}(A(x)B(y) + B(y)A(x))$$

is a valid joint observable of A and B if

$$\{A(x), B(y)\} \geq 0$$

for all outcomes x,y.



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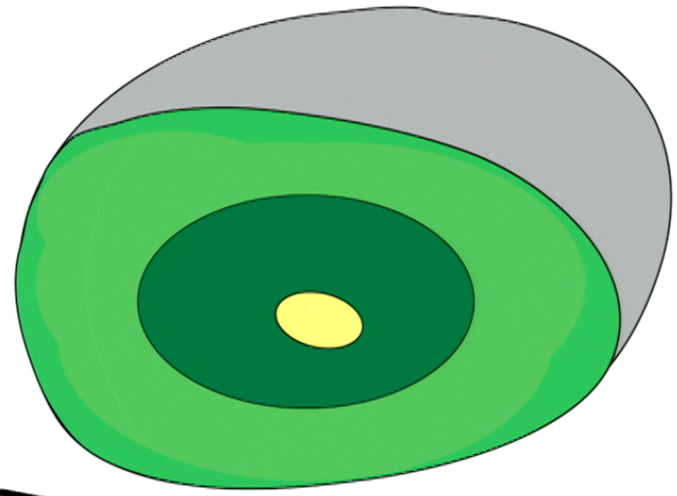


Remark:

$$[A(x), B(y)] = 0 \;\Rightarrow\; \{A(x), B(y)\} \geq 0$$

sharp POVM
$A(x)^2 = A(x)$

Proposition

If A or B is sharp, then the following are equivalent:

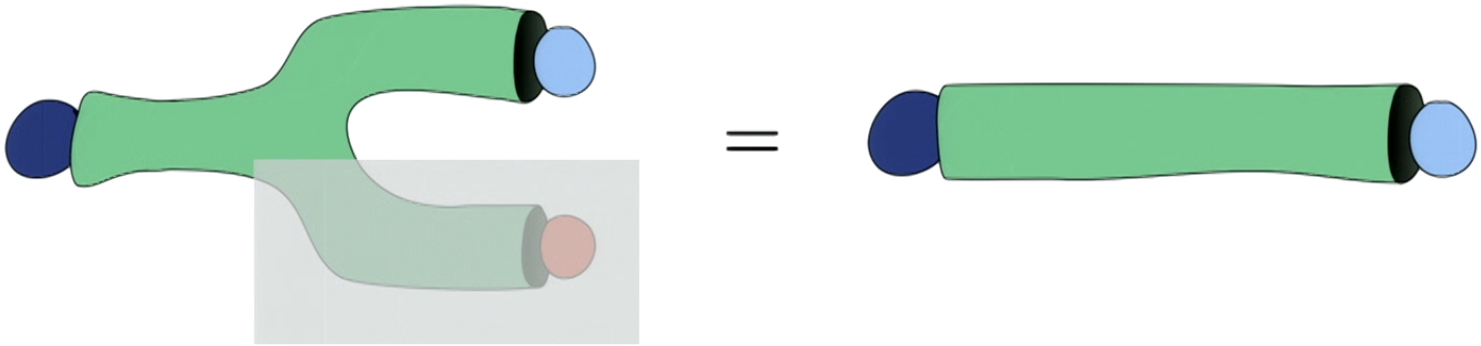
(1) A and B are compatible

(2) A be measured without disturbing B

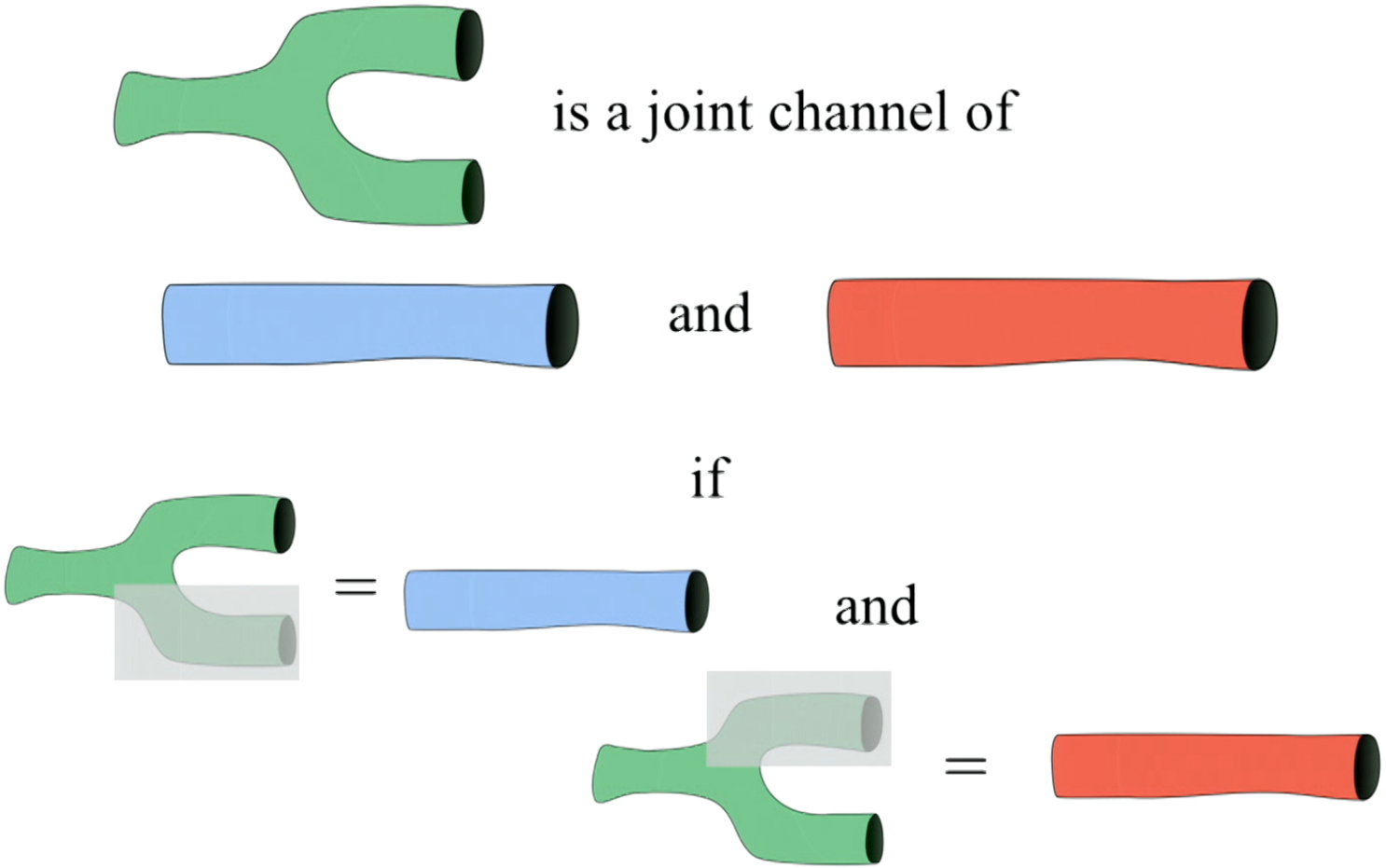
(3) B can be measured without disturbing A

(4) $[A(x),B(y)] = 0$ for all outcomes x,y

ignoring one output port



joint channel



incompatibility

Two channels

and

are **compatible** if they have a joint channel.

Otherwise they are **incompatible**.

incompatibility

Two channels

and

are **compatible** if they have a joint channel.

Otherwise they are **incompatible**.

- For this definition to make sense, channels must have the same input space. However, they can have different output spaces.

concatenation

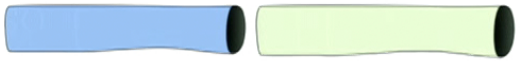
- If and are compatible,

then also and

are compatible.

concatenation

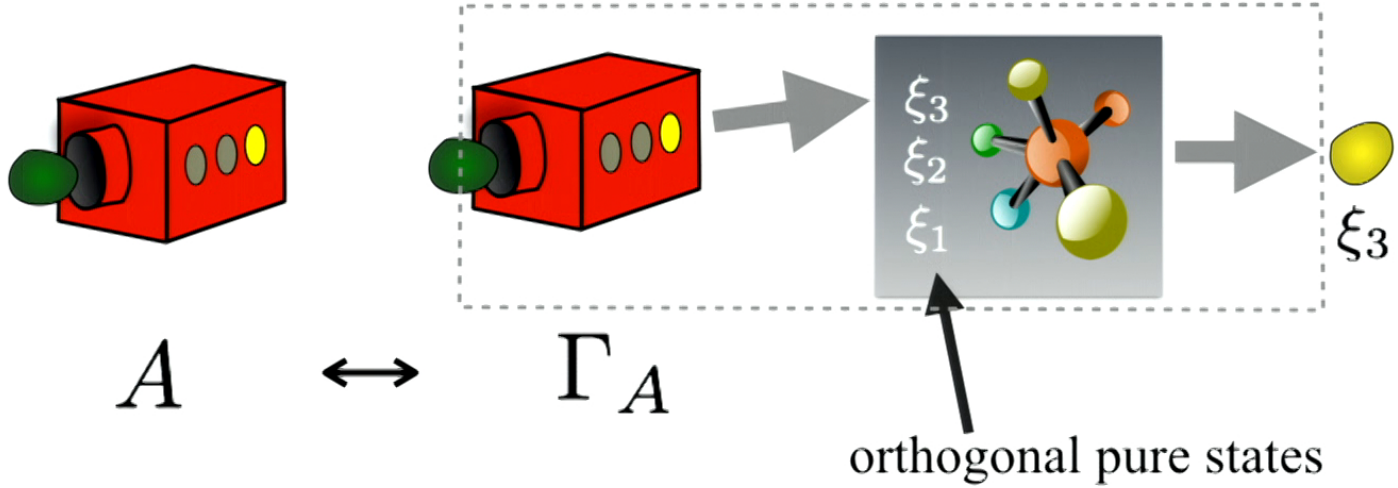
• If and are compatible,

then also  and

are compatible.

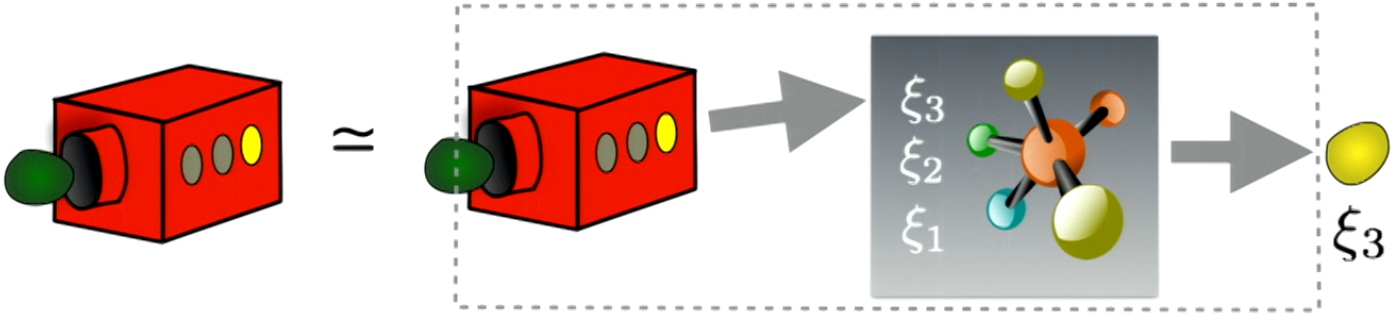
Proof:

observables as channels



$$\Gamma_A(\varrho) = \sum_x \mathrm{tr}[\varrho A(x)]\xi_x$$

observables as channels



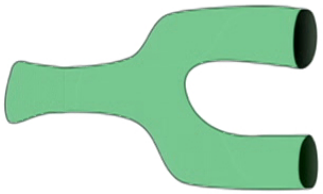
Proposition

Two observables (POVMs) A and B are compatible

if and only if

the corresponding channels Γ_A and Γ_B are compatible.

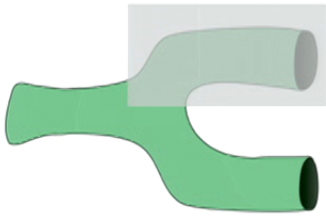
universal broadcasting



is a (perfect and universal) broadcasting channel if

= identity channel

and



= identity channel

no-broadcasting theorem

The following are equivalent:

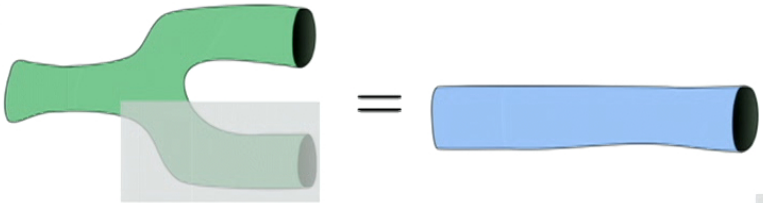
1. Universal broadcasting is possible.
2. Two identity channels are compatible.
3. Any pair of channels are compatible.

No-broadcasting theorem in incompatibility flavor:
There exists a pair of incompatible channels.

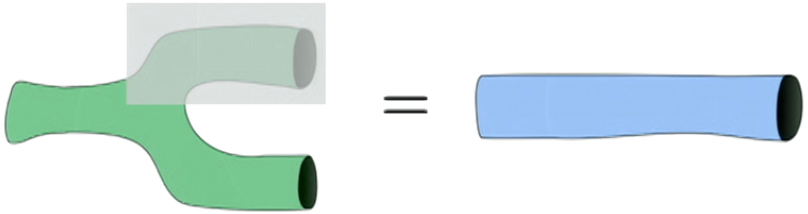
self-compatible channels

is ***self-compatible*** if there exists

such that



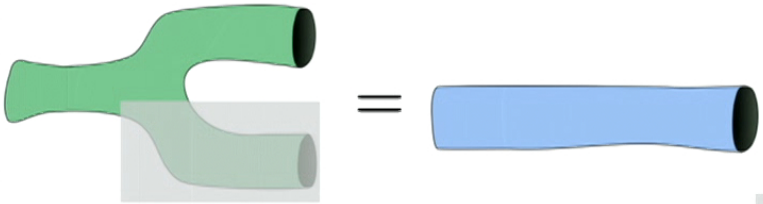
and



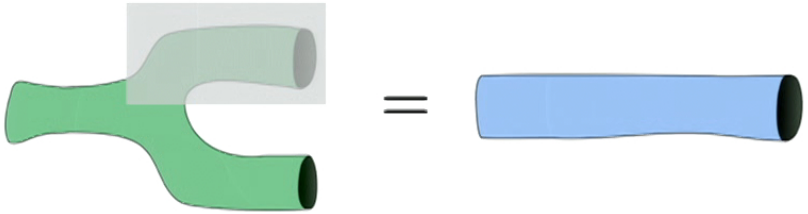
self-compatible channels

is ***self-compatible*** if there exists

such that



and

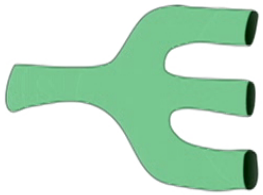


<u>Proposition</u>

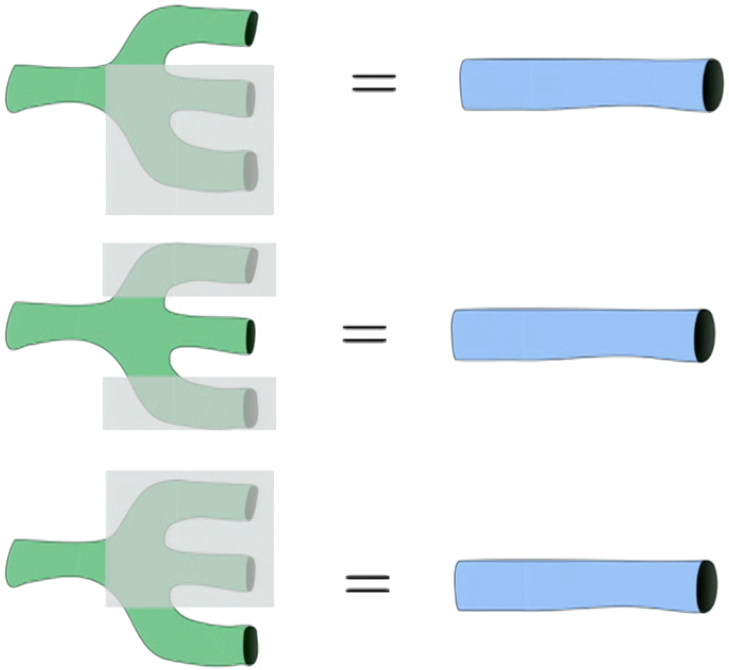
self-compatible channel = anti-degradable channel

n-self-compatible channels

is *3-self-compatible* if there exists



such that

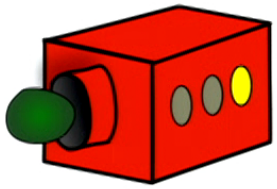


and

and

Every observable is n-self-compatible for every n :

ϱ



2,1,1,3,4,2,1,3,4,..

2,1,1,3,4,2,1,3,4,..

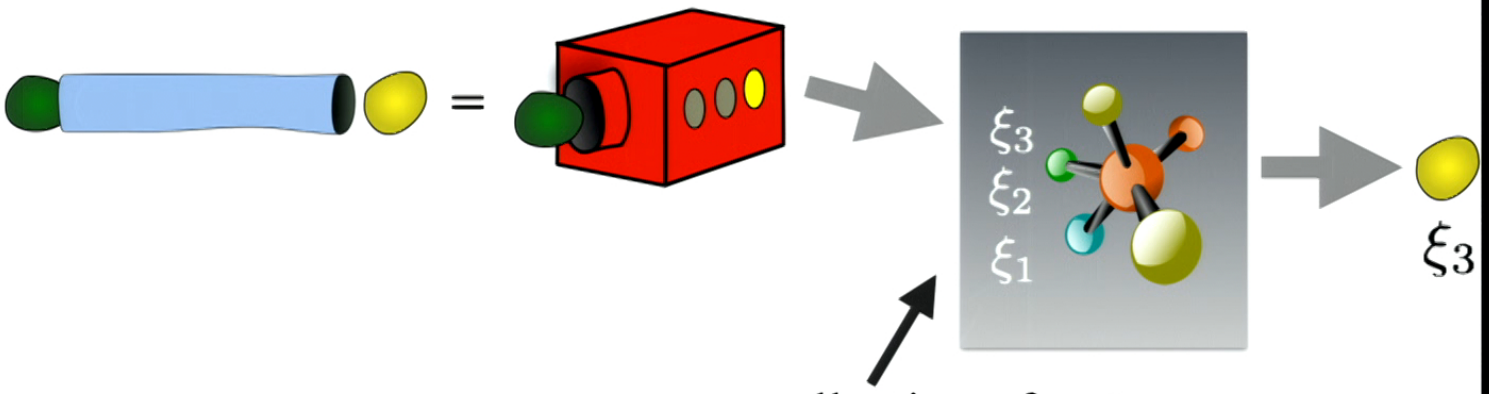
2,1,1,3,4,2,1,3,4,..

2,1,1,3,4,2,1,3,4,..

Theorem

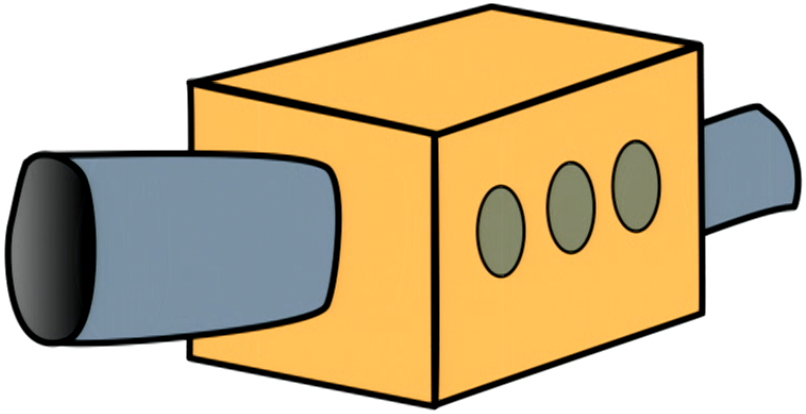
is n-self-compatible for every n=2,3,…

if and only if

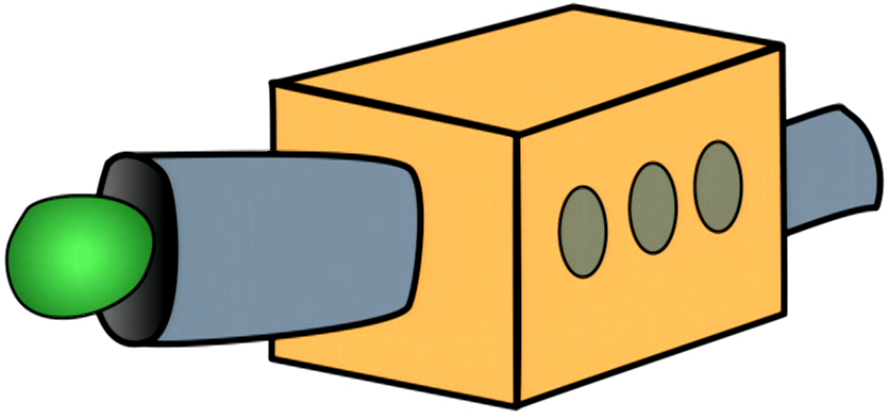


any collection of states

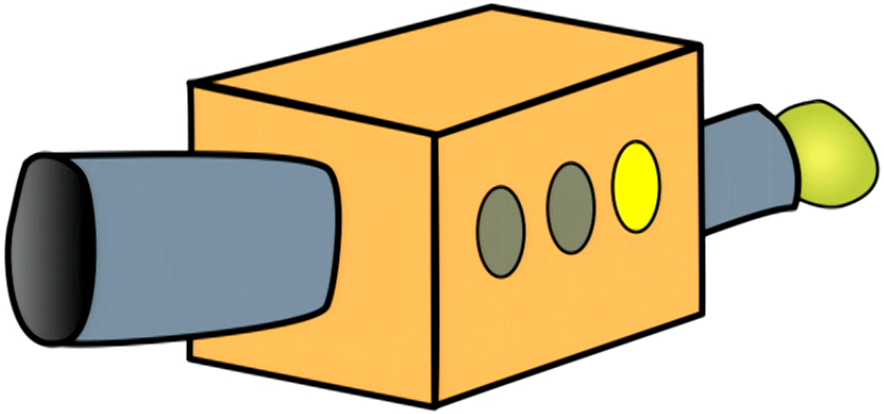
instrument



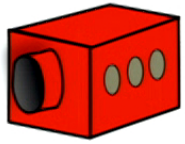
instrument

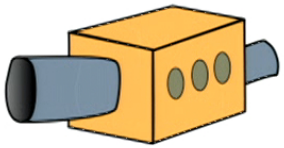


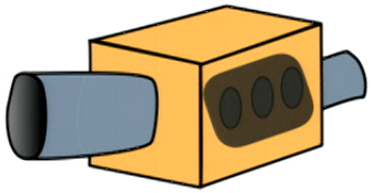
instrument



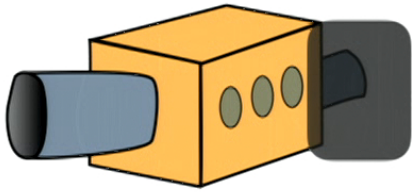
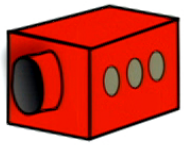
Definition

and  are **compatible** if

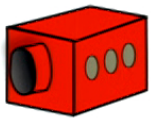
there exists  such that

 =

and

 = 

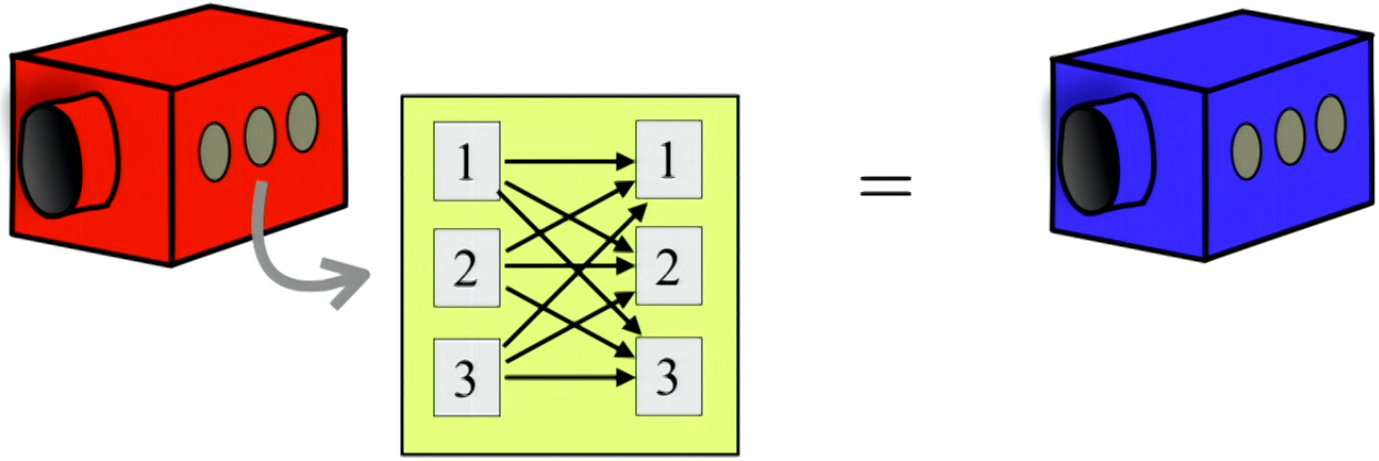
two questions

1) Given  , what are all possible channels compatible with it?

2) How does incompatibility relate to noise and disturbance?

post-processing

$$\begin{pmatrix} 0.4 & 0.2 & 0.4 \\ 0.3 & 0.1 & 0.6 \\ 0.5 & 0.4 & 0.1 \end{pmatrix}$$



Example: binary qubit observables

$$A_1(\pm 1) = \frac{1}{2}(I \pm \sigma_1)$$

sharp observable

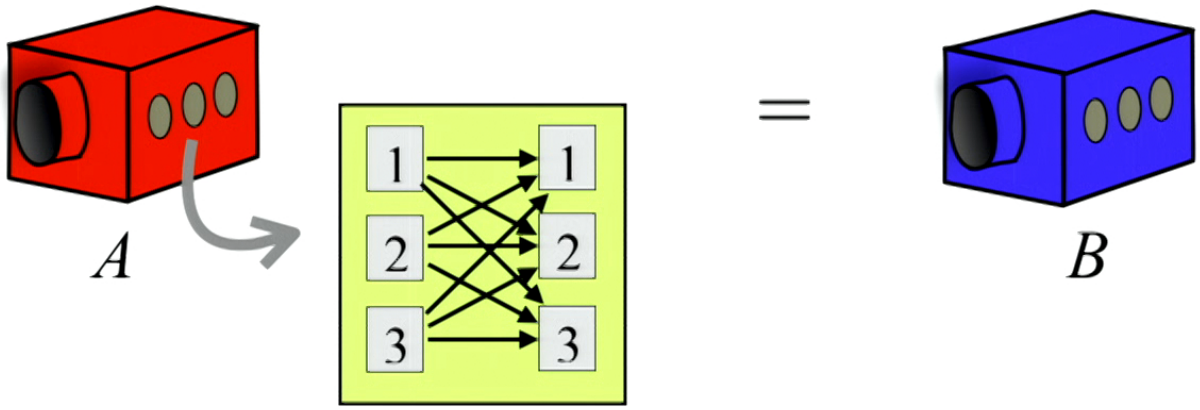
$$A_t(\pm 1) = \frac{1}{2}(I \pm t\sigma_1)$$

post-processing

$$A_0(\pm 1) = \frac{1}{2}I$$

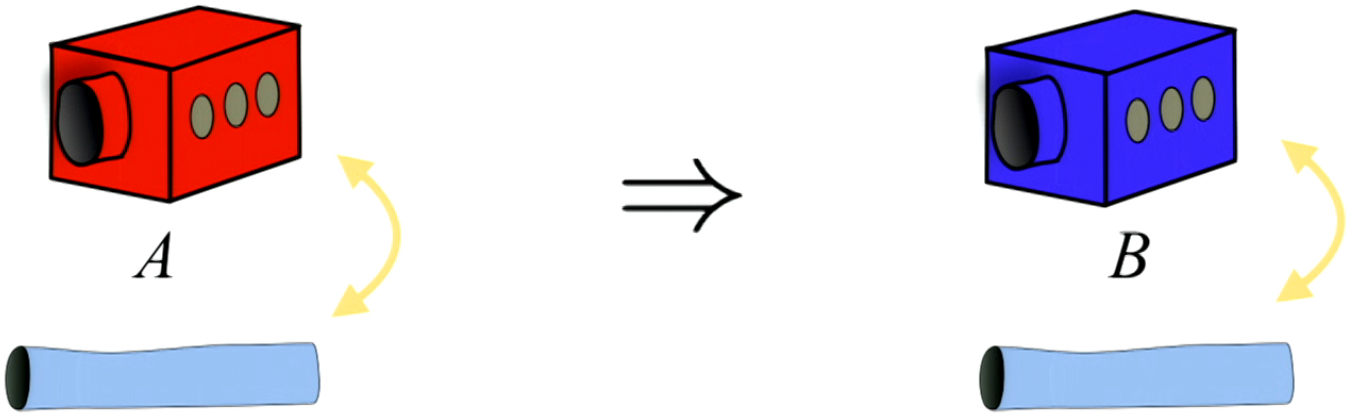
coin tossing observable

noise - disturbance trade-off part 1



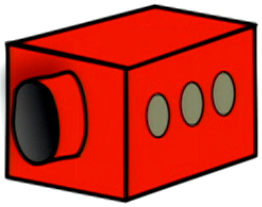
If B is a post-processing of A, then any channel compatible with A is also compatible with B.

noise - disturbance trade-off part 2

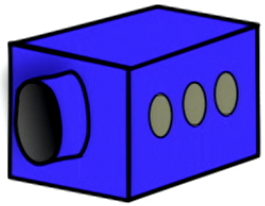


If all channels compatible with A are compatible with B, then B is a post-processing of A.

noise - disturbance trade-off part 3



A

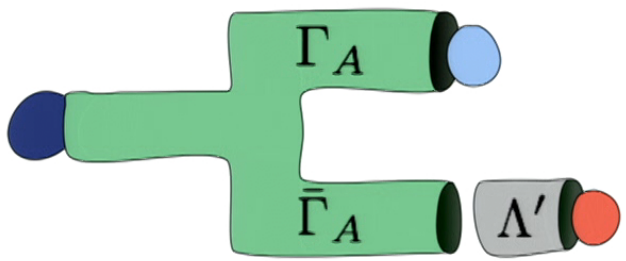


B

- B is a post-processing of A if and only if the **least disturbing channel** of A is compatible with B
- any channel compatible with A can be obtained from its **least disturbing channel** by concatenating it with some other channel

mathematical construction of the least disturbing channel

- think A as channel Γ_A
- take a minimal Stinespring dilation of Γ_A
- form the conjugate channel $\bar{\Gamma}_A$ of Γ_A
- $\bar{\Gamma}_A$ is the least disturbing channel of A in the sense that any channel Λ compatible with A can be written as $\Lambda = \Lambda' \circ \bar{\Gamma}_A$



Example: Pauli channels

- two one-parametric families of binary qubit observables

$$A_t(\pm 1) = \tfrac{1}{2}(I \pm t\sigma_1)$$

$$B_s(\pm 1) = \tfrac{1}{2}(I \pm s\sigma_2)$$

- Pauli channels

$$\Lambda_{\mathbf{p}}(\varrho) = \sum_{j=0}^{3} p_j \sigma_j \varrho \sigma_j$$

$$\mathbf{p} = (p_0, p_1, p_2, p_3)$$

Example: Pauli channels

$$A_t \longleftarrow \boxed{s^2 + t^2 \leq 1} \longrightarrow B_s$$

$$\Lambda_{\mathbf{p}}$$

Example: Pauli channels

$$s^2 + t^2 \leq 1$$

A_t

B_s

$$t \leq 2(\sqrt{p_0 p_1} + \sqrt{p_2 p_3})$$

$\Lambda_{\mathbf{p}}$

Example: Pauli channels

$$s^2 + t^2 \leq 1$$

A_t B_s

completely depolarizing channel

$$p_0 = p_1 = p_2 = p_3 = \frac{1}{4}$$

$$t \leq 2(\sqrt{p_0 p_1} + \sqrt{p_2 p_3})$$

$$s \leq 2(\sqrt{p_0 p_2} + \sqrt{p_1 p_3})$$

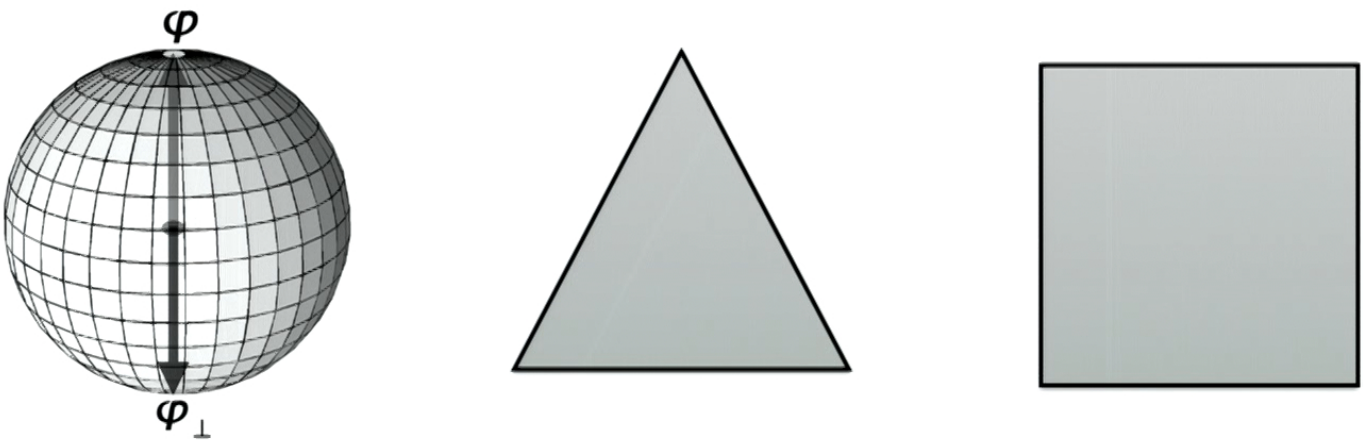
1 1

$\Lambda_{\mathbf{p}}$

Is there anything special about *quantum* incompatibility ?

general probabilistic theory

- set of states S = convex closed subset of a real vector space V



- observable = affine map from S to a set of probability distributions (*no-restriction hypothesis*)

valid in all non-classical theories

No-broadcasting theorem:

There exists an incompatible pair of channels.

[Barnum et al, PRL 99, 240501 (2007)]

valid in all non-classical theories

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[Plávala, PRA 94, 042108 (2016)]

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No-measurement-without disturbance:
There exists an incompatible pair of a channel and observable.

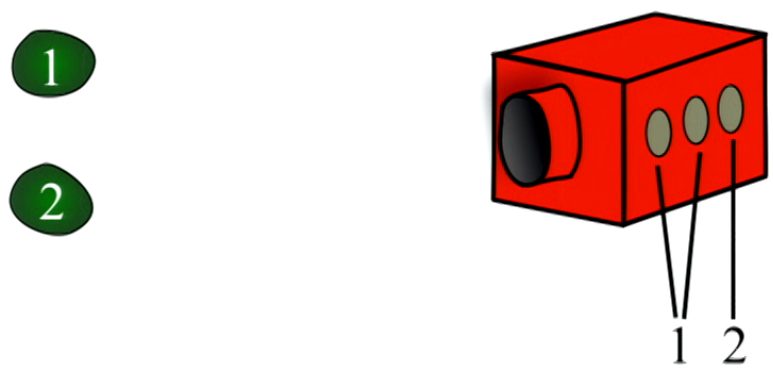
There exists an incompatible pair of observables.

[Plávala, PRA 94, 042108 (2016)]

discrimination of states

Observable A can discriminate two states s_1 and s_2 if there are subsets of outcomes X_1 and X_2 such that

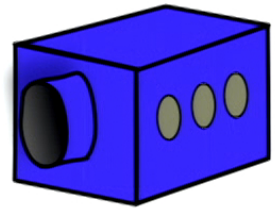
$$prob(A \in X_1 | s_1) = 1 \quad \text{and} \quad prob(A \in X_2 | s_2) = 1$$



pure state tomography

Observable B is **pure state informationally complete** if for any two different pure states s_1 and s_2 ,

$$prob(B|s_1) \neq prob(B|s_2)$$



In quantum theory, two observables A and B are incompatible if A can discriminate (at least) two states and B is pure state informationally complete.

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In the square state space, there exist compatible observables A and B such that A can discriminate two states and B is pure state informationally complete.

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No-broadcasting theorem:
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Something (related to incompatibility) that is valid in quantum theory but not in all general probabilistic theories?

In quantum theory, two observables A and B are incompatible if A can discriminate (at least) two states and B is pure state informationally complete.

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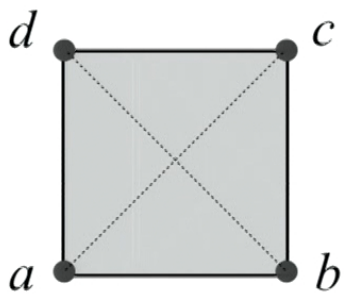
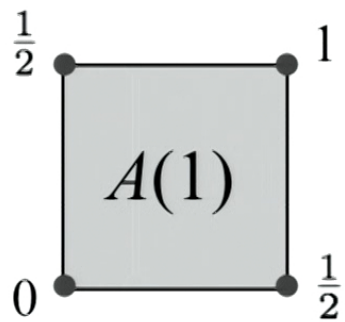
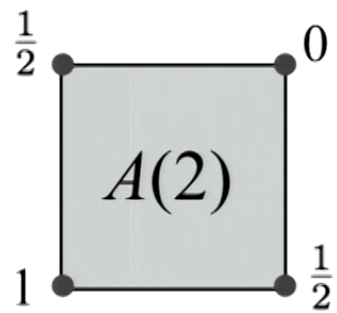


$$a+c = b+d$$

In the square state space, there exist compatible observables A and B such that A can discriminate two states and B is pure state informationally complete.





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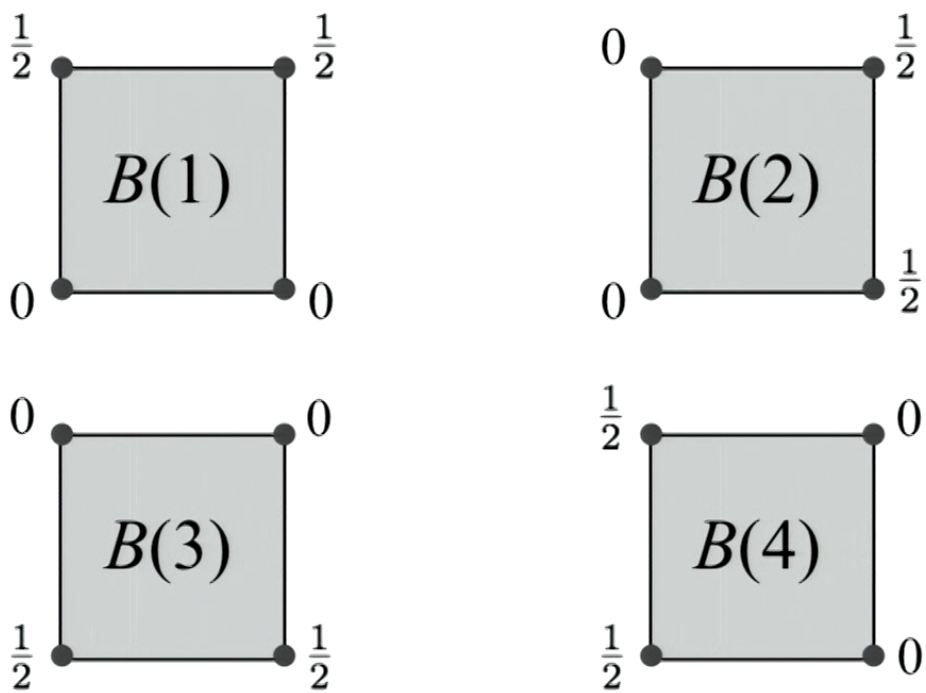

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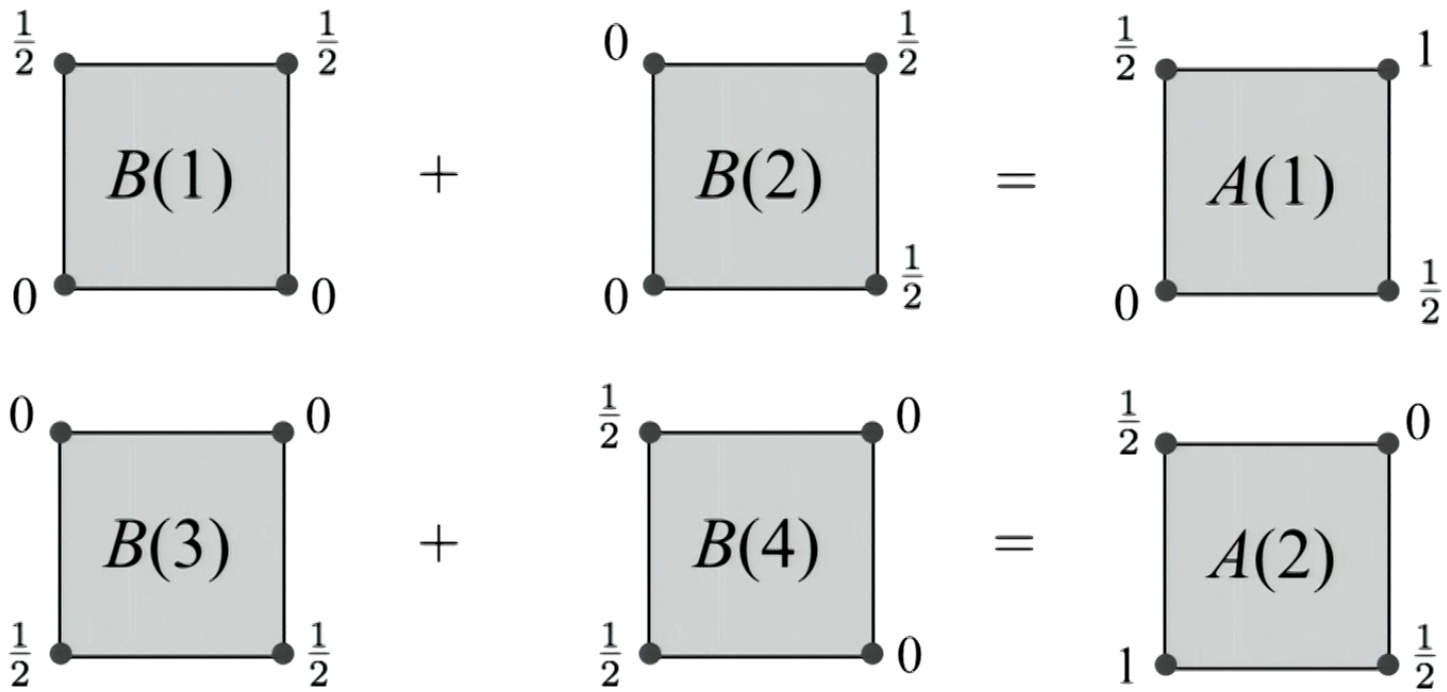

In the square state space, there exist compatible observables A and B such that A can discriminate two states and B is pure state informationally complete.

$$G(1,1) = B(1) \qquad G(2,1) = 0$$

$$G(1,2) = B(2) \qquad G(2,2) = 0$$

$$G(1,3) = 0 \qquad G(2,3) = B(3)$$

$$G(1,4) = 0 \qquad G(2,2) = B(4)$$

In the square state space, there exist compatible observables A and B such that A can discriminate two states and B is pure state informationally complete.

$G(1,1) = B(1)$	$G(2,1) = 0$	$B(1)$
$G(1,2) = B(2)$	$G(2,2) = 0$	$B(2)$
$G(1,3) = 0$	$G(2,3) = B(3)$	$B(3)$
$G(1,4) = 0$	$G(2,2) = B(4)$	$B(4)$
$A(1)$	$A(2)$	

references

- TH, T. Miyadera and M.Ziman, An invitation to quantum incompatibility, JPA 49, 123001 (2016)

1) simultaneous measurement of quantum observables

- TH, Simultaneous measurement of two quantum observables: Compatibility, broadcasting, and in-between, PRA 93, 042118 (2016)
- TH and M.M.Wolf, Nondisturbing quantum measurements, JMP 51, 092201 (2010)

2) incompatibility of quantum channels

- TH and T. Miyadera, Incompatibility of quantum channels, JPA 50, 135302 (2017)

3) incompatibility of a quantum channel and observable

- TH and T. Miyadera, Qualitative noise-disturbance relation for quantum measurements, PRA 88, 042117 (2013)
- TH, D. Reitzner, T. Rybar and M. Ziman, Incompatibility of unbiased qubit observables and Pauli channels, in preparation.

4) incompatibility in GPTs

- S. Filippov, TH and L. Leppäjärvi, Necessary condition for incompatibility of observables in general probabilistic theories, PRA 95, 032127 (2017)